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Optimizing Performance Based Internet Advertisement Campaigns

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This study provides an approach to manage an on-going Internet ad campaign that substantially improves the number of clicks and the revenue earned from clicks. The problem we study is faced by an Internet advertising firm (Chitika) that operates in the Boston area. Chitika contracts with publishers to place relevant advertisements (ads) over a specified period on publisher websites. Ad revenue accrues to the firm and the publisher only if a visitor clicks on an ad (i.e., we are considering the cost per click model in this study). This might imply that all visitors to the publisher's website be shown ads. However, this is not the case if the publisher imposes a click-through-rate constraint on the advertising firm. This performance constraint captures the publisher's desire to limit ad clutter on the website and hold the advertising firm responsible for the publisher's opportunity cost of showing an ad that did not result in a click. We develop a predictive model of a visitor clicking on a given ad. Using this prediction of the probability of a click, we develop a decision model that uses a threshold to decide whether or not to show an ad to the visitor. The decision model's objective is to maximize the advertising firm's revenue subject to a click-through-rate constraint. A key contribution of this paper is to characterize the structure of the optimal solution. We study and contrast two competing solutions: (1) a *static* solution, and (2) a *rolling-horizon* solution that re-solves the problem at certain points in the planning horizon. The static solution is shown to be optimal when accurate information on the input parameters to the problem is known. However, when the parameters to the model can only be estimated with some error, the rolling-horizon solution can perform better than the static solution. When using the rolling-horizon solution, it becomes important to choose the appropriate re-solving frequency. The implemented models operate in real time in Chitika's advertising network. Implementation challenges and the business impact of our solution are discussed. In order to present a head-to-head comparison of our implemented approach with the past practice at Chitika, we implemented our solution in parallel to the past practice.

Key words: Internet advertising; performance constraints; visitor profiling; revenue optimization.

1. Introduction

This paper builds on a preliminary investigation of the problem in Mookerjee et al. (2012); a finalist paper for the 2011 Daniel H. Wagner Prize for Excellence in Operations Research Practice. In Mookerjee et al. (2012), a heuristic solution to the problem is provided. In the current study, we present a full and rigorous treatment of the problem, and report on further experience gathered from implementing the solution. More specifically, the current study contributes over Mookerjee et al. (2012) by: (1) finding (and mathematically characterizing) the optimal solution to the problem, (2) demonstrating that the true value of the proposed rolling-horizon approach lies in its ability to cope with incomplete knowledge of the problem's parameters, (3) providing insights into the solution depending on the publisher's volume of traffic and the stringency of the performance constraint imposed on the ad network, and (4) presenting a head-to-head comparison of our implemented approach with the past practice at Chitika.

1.1. Background

In recent times, the Internet advertising revenues in the United States have increased significantly. In 2013, it increased by 17 percent over 2012 to reach $42.78 billion (Interactive Advertising Bureau 2014). This trend is expected to continue: eMarketer (2011) estimates that the Internet advertising revenue in the United States will reach $50 billion by 2015. Another estimate shows that the U.S. Internet advertising will reach $77 billion in 2016, and will comprise 35 percent of all advertising spending, overtaking television advertising (Hof 2011). Finally, Internet advertising is not purely a U.S. phenomenon. In the United Kingdom, the online advertising revenues in 2013 increased by 15.2 percent over 2012 to reach almost £6.3 billion (Interactive Advertising Bureau UK 2014). According to a report by Digital TV Research, the global Internet advertising spending will reach $143 billion in 2017 (Kemp 2012).

Figure 1 describes the main players in the Internet Advertising ecosystem. Other than the visitor, there are 3 main entities involved in Internet advertising: (1) the advertiser (whose advertisement is displayed), (2) the publisher (that provides the real-estate where the advertisement is displayed), and, (3) the advertising firm (usually referred to as an "ad-network"). The ad-network's business is one of monetizing the traffic for publishers (website owners, mobile application providers, etc.). An ad-network gathers data on visitor behavior from its large publisher network and uses this knowledge to target advertisements (hereafter also referred to as "ads") to visitors on a variety of websites. Such firms collect ads from advertisers (or from firms that aggregate ads from many advertisers, namely, *ad-aggregators*) and display these ads to targeted visitors on different publisher sites. When a visitor clicks on an ad on a publisher's website, the advertiser pays the ad-aggregator a contractually agreed amount that corresponds to the cost-per-click for the advertiser (say, $1). A part of this revenue (say, $0.20) is retained by the ad-aggregator. The rest ($0.80) is paid to the ad-network. The ad-network shares a portion of this amount (say, $0.50) with the publisher and keeps the rest for itself ($0.30). If there is no click and the contract with the advertiser is on a cost-per-click basis, then no money exchanges hands.

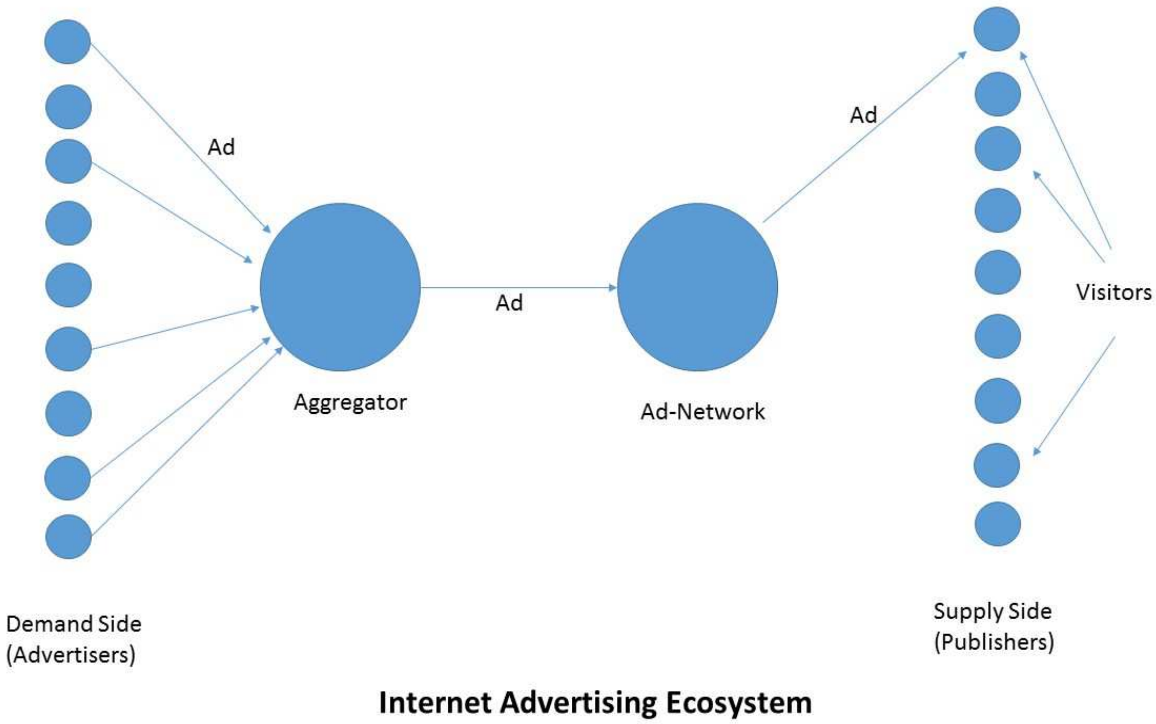

Figure 1 The Main Players in the Internet Advertising Ecosystem

The attractiveness of the Internet as a medium for advertising arises, in large part, from the ability to track and measure the performance of an ad campaign. On the Internet, the display of an ad (or an *impression*) can be associated to a click or sometimes, even a final business outcome such as a sale or a signup (or more generally, a *conversion*). As a result, performance based pricing (where payment often depends user *clicks* generated for an ad) is the leading ad pricing model, accounting for approximately 65 percent of the total ad revenues in 2013 (Interactive Advertising Bureau 2014, Atkinson 2014).

1.2. Problem and Motivation

Despite the obvious attractiveness (to advertisers) of a performance based payment scheme (such as a cost-per-click model), at the end, it is the publisher that must bear the opportunity cost of an ad display that does not result in a click. To address this incentive misalignment, an emerging trend in the ad industry is to require the ad-network to manage ad display at a publisher's site such that an efficiency (or a click-through-rate) constraint is respected. Typically, the publisher and the ad-network enter into a contract. A publisher usually contracts with a single or a small number of ad-networks. This is because the more the traffic seen by an ad-network, the better the ad-network is able to accurately estimate important characteristics of the traffic. The better the knowledge of the traffic available to an ad-network, the better its ad targeting algorithms get, and hence lead to increased clicks. Publishers may contract with more than one ad-network to create some competition among these partners.

At the end of each month, the ad-network pays a publisher a (contractually agreed upon) fraction of revenue accrued from clicks that were generated on the ads displayed on the publisher's site. In addition, the publisher enforces a click-through-rate constraint on the ad-network, which requires that the monthly average click-through-rate (over the contract period that could be several months) is above a specified value. Such a constraint ensures that the publisher's space for ads on its website is used efficiently. If too many ads placed did not result in clicks, this is an indication of a wasted opportunity for the publisher. Either the publisher could have placed more relevant ads, or used the space for additional content. The click-through-rate constraint essentially balances two opposing goals of the publisher: (1) generate as much revenue as possible from ads, and (2) keep the website content interesting so that visitors continue to patronize the website. If a publisher becomes too greedy and shows an excessive number of ads, this could come at the expense of the content; the main driver of traffic to the website. In this case, the publisher would likely suffer in the long run. On the other hand, if a visitor clicks on an ad, it often indicates that she likes it. In such cases, the ad could be considered to be almost indistinguishable from content.

1.3. Overview of Solution

An ad-network usually manages multiple publishers. However, the problem for each publisher can be considered a separate problem as long as there is no constraint from the advertisers that links two publishers. In the problem setting considered in this paper, the advertisers do not impose any such constraint. Therefore, we analyze the problem for each publisher separately.

Essentially, any solution to the user profiling problem can only address the problem by some variant of an ad *filter*: using this filter, only some users are selected to be shown ads. In past studies, this filter is also referred to as *targeting* (Gerken 2008, Goldfarb and Tucker 2011). The actionable information associated with a user includes the available details of the *impression*. Based on these details, a decision needs to be made to show or not to show an ad. One convenient way to use the impression details is to map them to a *click-probability*. That is, given these details, what is the probability of a click for the ad associated with the impression? Next, for this click-probability, should we show the ad? If the decision is to show an ad, the best ad is picked given what the ad-network knows about the impression's details and the visitor. In cases where the ad-network decides not to show an ad, there are usually two possibilities. First, the ad unit could simply collapse and the publisher could use the extra space for displaying additional content, or expand its display of existing content. Second, the

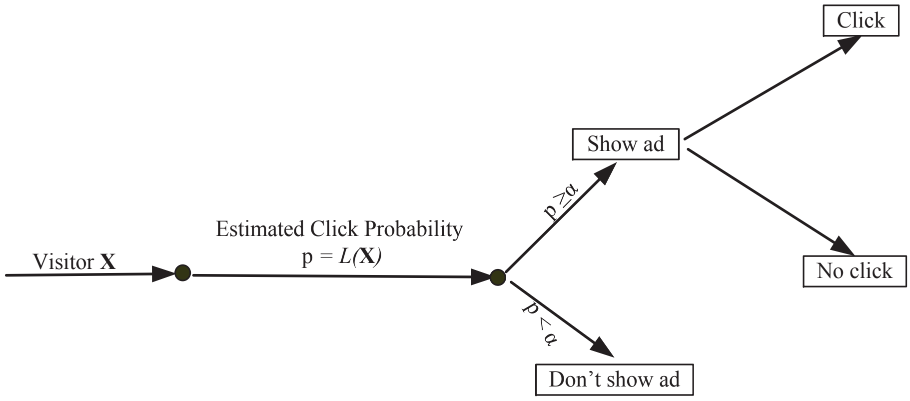

Figure 2 Overall Solution Process

ad-network could make a programmed call to another ad-network (designated by the publisher, e.g., Chitika could call Google) that could potentially use the available slot for displaying another ad.

The above discussion suggests the use of a *threshold policy.* This policy can be stated as follows. *If the click-probability of an impression is greater than a threshold, then show the ad, else do not show it.* We later show that such a threshold policy is optimal. Figure 2 illustrates the overall solution process. The figure depicts that the solution to the user profiling problem consists of two parts: (1) a step involving data analytics (to predict the click-probability, p), and (2) a follow-up step involving decision analytics (to choose the threshold, α).

With regard to the problem of choosing the correct threshold, the natural question arises: should the threshold be held constant over the planning horizon, or should the value of the threshold be varied depending on the current *state* of the problem, i.e., the impressions and clicks that have been observed so far and the time left in the planning horizon (typically, this horizon is one month)? Put differently, should the display criterion (the threshold) be relaxed or tightened, depending on how much time is left in the planning horizon and whether the current click-through-rate is above or below the target level that has to be achieved at the end of the month (on average). If we are ahead (i.e., the current click-through-rate is above the target value), then should the threshold be lowered (so more ads can be shown) to potentially earn more ad revenue? On the other hand, if we are behind (i.e., the current click-through-rate is below the target value), then should the threshold be increased (so that ads are only shown to more interested visitors), in an attempt to meet the click-through-rate constraint? Increasing the threshold sacrifices ad revenue, but we may need to take this action to achieve the target click-through-rate at the end of the planning horizon.

1.4. Genesis of the Problem

The study in this paper is motivated by the user profiling problem at the Internet ad-network, Chitika (`www.chitika.com`). In early 2010, in an effort to grow its publisher base and to attract highly visible publishers (e.g. The Wall Street Journal), Chitika launched *Chitika Premium*, a new service offering publishers an innovative value proposition. In Chitika Premium, a publisher could (for the first time) control the average click-through-rate of Chitika's ads. To solve this problem, it was necessary to find a way to respect a publisher's click-through-rate constraint while collecting as much revenue as possible for Chitika and the publisher. More generally, it was necessary for Chitika to come up with a way to target only those users with ads if they had a reasonable chance of clicking on them.

1.5. Contributions of the Study

The main contribution of this study is to provide an approach to manage an Internet ad campaign that substantially improves the number of clicks and the revenue earned from clicks. Past research on managing ad campaigns has focused on solving the problem of choosing the best ad to a given visitor. The implicit assumption in most extant studies on campaign management is that ads are shown to every arriving visitor. Given the emphasis on efficiency (namely, the click-through-rate constraint), our approach decides when *not* to show an ad to a visitor. This filtering of ads must occur in a manner to balance the opposing goals of revenue and efficiency. If too many ads are filtered, the efficiency constraint would be met (or even exceeded), but opportunities to generate clicks would likely be lost. On the other hand, if not enough filtering is done, more clicks would likely be generated at the cost of not meeting the target efficiency constraint. A key contribution of this paper is to characterize the structure of the optimal solution to the problem of maximizing revenue subject to a click-through-rate constraint. Here, we study and contrast two competing solutions: (1) a *static* solution, and (2) a *rolling-horizon* solution. The static solution is shown to be optimal when accurate information on the input parameters to the problem is known. However, when the parameters to the model can only be estimated with some error, the rolling-horizon solution can perform better than the static solution. When using the rolling-horizon solution, it becomes important to choose the appropriate re-solving frequency. The solution approach proposed in this paper has been implemented in Chitika. The implementation details of the solution (together with the revenue impact on the company) have been presented.

2. Related Work

We briefly review related work in three areas: (1) optimization of Internet ad campaigns, (2) consumer choice models, and (3) multi-stage decision making.

2.1. Optimization of Internet Ad Campaigns

Several studies in this stream have focused on the display of ads on a variety of platforms (e.g., websites, mobile phones, Internet-enabled game consoles, etc.) to optimize a goal (e.g., clicks, revenue, etc.) over a given planning horizon. In addition to incorporating the characteristics of the ads to be displayed (such as size, location, etc.) and the challenges associated with fitting a set of ads in the available display space, ad-servers usually consider advertiser constraints that arise from ad saturation and competition between ads (Kumar et al. 2007, Turner et al. 2011, Turner 2012). In previous studies, the underlying optimization problem is cast as one of *scheduling* ads over a planning horizon while respecting a set of constraints that reflect advertiser interests (Dawande et al. 2003, 2005, Kumar et al. 2006). Ghosh et al. (2009b) design a bidding agent that acquires a given number of impressions for the advertiser with a given target budget. Unlike these studies, our problem is motivated from the joint perspective of the publisher and the ad-network. While the guarantees provided by the models in these studies are directed toward advertisers, our problem requires that we provide a guarantee to the *publisher*, namely, one of meeting or exceeding a specified click-through-rate constraint.

Some recent studies have also examined the problem from the perspective of publishers. For example, Najafi-Asadolahi and Fridgeirsdottir (2014) present a revenue model for the publisher to optimize the pricing of display ads. Further, Balseiro et al. (2014) analyze a trade-off for the publisher between the short-term revenue from an Ad Exchange and the long-term benefits of delivering good quality impressions to the advertisers. Ghosh et al. (2009a) model the publisher's problem of fulfilling guaranteed advance contracts by bidding in the spot market. Finally, Balseiro et al. (2015) study the competitive landscape that arises in Ad Exchanges and the implications for publishers' decisions. Although these studies are related to our research, their focus is clearly different.

In addition to the practical issues concerning ad scheduling addressed in previous work, there are a growing number of patents that deal with inventions to measure the effectiveness of an ad campaign (Harvey et al. 2010, Gerken 2008, Lindsay et al. 2010, Srinivasan and Shamos 2010). Finally, beyond the research on optimizing Internet ad campaigns, there are numerous examples of academic studies that examine Internet advertising at a more micro-level: (1) impact of ad position on profitability (Agarwal et al. 2011), (2) targeting strategies, including privacy concerns (Evans 2009, Goldfarb and Tucker 2011), and (3) wear-in, wear-out effects of Internet ads (Chatterjee et al. 2003).

2.2. Consumer Choice Models

There is a vast body of work in the Marketing area that deals with predicting a consumer's choice among a set of discrete alternatives. The predictive technique we employ in this study draws from the stream of literature on consumer choice models. Unlike previous studies, we employ Logistic regression for predicting – in *real-time* – a visitor's probability of a click. The reason that prediction must occur in real-time is that a real-time decision (to show or not to show an ad) is contingent on this prediction. In previous studies, the use of Logit is usually made to infer consumer trends (e.g., how display ads affect subsequent choices consumers make to consume brand-specific content (Bucklin and Sismeiro 2009)). The key difference in real-time prediction arises, of course, from the fact that the prediction must be done quickly. This time constraint naturally limits the set of independent variables that real-time predictive models can employ. We will discuss such limitations later in Section 6 where we describe actual implementation details.

2.3. Multi-stage Decision-Making

A central feature of the problem in our study is that we need to manage an on-going ad campaign, i.e., one that evolves over time. The two problem-solving approaches we propose in this paper — a static model, and a rolling-horizon model — are inspired by models of multi-stage decision-making that are often solved using dynamic programming techniques (Baldacci et al. 2013, Hwang et al. 2013, Lai et al. 2010, Moallemi and Saglam 2013). Typically, in past research, rolling-horizon solutions are considered to be a good way to solve complex, multi-stage optimization problems where analytical solutions prove to be intractable. On the other hand, we use the rolling-horizon approach as a way to cope with the presence of incomplete (inaccurate) knowledge of the problem's parameters. Although some of the past studies have used the rolling-horizon approach in a similar manner (e.g., Ghosh et al. 2009b, Besbes and Maglaras 2012, Wang et al. 2014), both our context and implementation are different from those studies.

As mentioned earlier, this paper builds on a preliminary investigation of the problem where a rolling-horizon solution is provided as a heuristic (Mookerjee et al. 2012). The current study contributes by finding (and mathematically characterizing) the optimal solution to the problem. In the current study, we also demonstrate that the true value of the rolling-horizon approach lies in its ability to cope with incomplete knowledge of the problem's parameters. We also provide insights into the manner in which the rolling-horizon approach should best be employed; the key is to carefully choose the update frequency of this approach. Essentially, in the presence of noisy parameters, there is a subtle trade-off between two effects that occur in the rolling-horizon solution: (1) the positive effect of being able to react to the actual state of the problem (and hence, being able to better cope with noise), (2) the negative force that arises because this solution departs from the optimal policy that is shown to be static. The positive force is more influential when the problem parameters are sufficiently inaccurate, but the negative force hurts the rolling-horizon approach if the problem parameters are known more accurately. In order to present a head-to-head comparison of our implemented approach with the past practice at Chitika, we implemented our solution in parallel to the past practice. The results show that our approach improves the number of clicks significantly.

3. Model and Solution

3.1. Description of the Data Analytic Model

Here we describe how the probability of a click and the distribution of these probabilities in the visitor population for a publisher is estimated.

3.1.1. Predicting the Probability of a Click: The prediction method uses a vector of observations collected from the visitor's cookie as well as meta data available from the `http` header (Kolluri et al. 2013). These observations include variables such as the visitor's search string, Internet browser, operating system, previous click data, and so on. We use Logistic regression (Logit) to combine information associated with the visitor with other information about the publisher's website. One of the major strengths of an ad-network lies in its ability to develop a large publisher network. This enables the firm to have repeated interactions with visitors across different websites, thus enabling the firm to develop a *profile* of each visitor. The Logit model uses this profile information to estimate the chances that a visitor will click on a given ad. This model can be expressed as a function $p = L(\mathbf{X})$, where p is the estimated click-probability, and $\mathbf{X}$ is the vector of variables used for the prediction. The model uses over 50 different variables; some of the important ones are listed in Table 1.

3.1.2. The Click-probability Distribution: Using the above Logit model, the value of p for any given visitor to the publisher's site can be estimated. We can estimate the click-probability distribution for a given publisher using a sample of these probabilities. The click-probability distribution provides us with vital information. As we show later, the optimal policy requires us to show an ad to a visitor only if the click-probability (p) meets or exceeds a given threshold value (say, α). That is, show the ad only if $p \geq \alpha$, where $0 \leq \alpha \leq 1$. For any given threshold, the click-probability distribution allows us to estimate the probability that an ad will be shown to a randomly selected visitor and the conditional probability that a visitor will click on an ad given that the click-probability exceeds the threshold. Conceptually, the probability that an ad will be shown is the upper tail of the distribution (i.e., the area under the density curve above α), and the click-probability that a visitor will click on an ad is the conditional expectation of the upper tail of the distribution.

Let $f(p)$ denote the probability density of p and $\beta(\alpha)$ denote the probability that an ad is shown to a visitor for a given threshold α. Then, $\beta(\alpha)$ can be calculated as

$$\beta(\alpha) = \int_{\alpha}^{1} f(p)dp. \tag{1}$$

Variable	**Description**	**Possible Values**
operating_system	Which operating system is being used by the visitor?	Linux, Android, Mac OS, Microsoft Windows, etc.
browser	Which browser is being used by the visitor?	Internet Explorer, Firefox, Chrome, Safari, Opera, etc.
search_engine	Which search engine is used by the visitor?	Google, Yahoo, Bing, etc.
screen_resolution	Pixel density in visitor's screen	One among a few possibilities
bad_speller	Is the visitor a bad speller?	Yes, No
search-string_type	Does the search string have local intent?	Yes, No
keyword_interest	Total number of clicks by this visitor in the past for ads on similar search strings	Integer
domain_keyword_interest	Total number of clicks by all the visitors in the past for ads on similar search strings	Integer
ad_keyword_fit	Match between a given ad and keyword	Numeric
day	Day of the visit	Monday, Tuesday, ..., Sunday
time	Time of the visit	Morning, Afternoon, Evening, Night
height	Height of the ad unit	Numeric
width	Width of the ad unit	Numeric
loc_x	x-location of the ad	Numeric
loc_y	y-location of the ad	Numeric
user_clicks	Total number of clicks by this visitor in the past	Integer
user_imps	Total number of impressions for this visitor in the past	Integer
CLICK	**Dependent Variable** Did the visitor click on the ad shown?	Yes, No

Table 1 Important Variables in the Logit Model

It is clear from the expression for $\beta(\alpha)$ that the probability of showing an impression decreases with the threshold, α. Let $\delta(\alpha)$ denote the conditional probability that a visitor will click on an ad, given that the click-probability exceeds the threshold α. This probability is given by

$$\delta(\alpha) = \frac{\int_{\alpha}^{1} pf(p)dp}{\beta(\alpha)}. \tag{2}$$

Now, in the following proposition, we present two important properties of $\delta(\alpha)$. The proofs are provided in the Electronic Companion.

PROPOSITION 1. *The conditional probability of a click (i.e., $\delta(\alpha)$) increases with α, i.e.,* $\frac{d\delta(\alpha)}{d\alpha} > 0, \alpha < 1$.

PROPOSITION 2. *A threshold of α, $\alpha \in [0,1)$, ensures that the click-through-rate will be greater than α, or $\delta(\alpha) > \alpha$, $\alpha \in [0,1)$.*

The result in Proposition 1 highlights the trade-off between the revenue and the click-through-rate. As α increases, the probability of showing an impression (i.e., $\beta(\alpha)$) decreases. Therefore, the expected number of clicks (hence, the revenue of the ad-network) also decreases with α. However, as shown in Proposition 1, the conditional probability of a click increases with α. Hence, as α increases, the click-through-rate increases. Proposition 2, provides a clue of how to set the threshold to ensure a certain click-through-rate.

SYMBOL	DEFINITION
K	Number of periods in the planning horizon. Depending on the situation being considered, each period could correspond to the arrival of one visitor or a certain number of visitors
$\tilde{m}$	Random variable for the number of impressions in the planning horizon
$\tilde{r}$	Random variable for the number of clicks in the planning horizon
$\tilde{p}, p$	Random variable for the click-probability, realization of the click probability
$f(p)$	Density function for the click-probability
α	Click-probability Threshold in a Static Policy
$\beta(\alpha)$	Probability of impression for threshold α
$\delta(\alpha)$	Conditional probability of click, given that $p \geq \alpha$
η	Publisher's click-through-rate constraint

Table 2 Model Parameters and Variables

3.2. Description of the Decision Analytic Model

Table 2 summarizes the main notation in the decision analytic model. We divide the planning horizon into K periods where each period corresponds to an arrival of a visitor to the publisher's website. For each arrival, we need to decide whether or not to show an ad. As discussed earlier, the objective is to maximize the expected number of clicks subject to the following click-through-rate constraint:

$$\mathbb{E}\left[\frac{\tilde{r}}{\tilde{m}}\right] \geq \eta. \tag{3}$$

In equation (3), $\tilde{r}$ (respectively, $\tilde{m}$) represents the random variable for the number of clicks (respectively, impressions) that arise from an underlying click-probability distribution $f(p)$ and a given policy to show ads. An analytical expression for this constraint is difficult to obtain. However, it is possible to approximate the constraint very accurately using the ratio of expectations, rather than finding the expectation of the ratio. We present this result below.

PROPOSITION 3. *As the number of arrivals (i.e., K) approaches ∞, $\mathbb{E}\left[\frac{\tilde{r}}{\tilde{m}}\right]$ approaches $\frac{\mathbb{E}[\tilde{r}]}{\mathbb{E}[\tilde{m}]}$.*

We therefore consider the revised optimization problem with the left-side of Constraint (3) suitably replaced. Using this revised constraint, the optimization problem can be developed as follows.

Let $I[p, S_i^I]$ be the 1/0 (for show ad and not show ad, respectively) decision variable associated with the i^{th} arrival, under the policy I. Here, S_i^I is the *state* of the system (a vector) just before the i^{th} arrival. The state represents the entire history of events (impression and click events at each prior arrival) that have occurred under the policy I, $i = 1, 2, \ldots, K$. Further, p represents a realization from the distribution $f(p)$, the click-probability of an arriving visitor. Note that since the click-probability distribution $f(p)$ is the same for all arrivals, the realization p_i (for the i^{th} arrival) can simply be written as p.

Then, we have

$$\mathbb{E}[\tilde{r}] = \mathbb{E}_{\tilde{p},\tilde{\mathbf{S}}^I}\left[\sum_{i=1}^{K} I[p, S_i^I]p\right], \quad \text{and}$$
$$\mathbb{E}[\tilde{m}] = \mathbb{E}_{\tilde{p},\tilde{\mathbf{S}}^I}\left[\sum_{i=1}^{K} I[p, S_i^I]\right].$$

In the above, the expectation is over the random click probability $\tilde{p}$ and the random state *matrix* $\tilde{\mathbf{S}}^I$. Row i of this matrix is a random vector that can take values represented as a vector of states, S_i^I, $i = 1, 2, \ldots, K$. The optimization problem can now be formally expressed as

Problem P

$$\max_I \mathbb{E}_{\tilde{p},\tilde{\mathbf{S}}^I}\left[\sum_{i=1}^{K} I[p, S_i^I]p\right],$$

s.t.

$$\mathbb{E}_{\tilde{p},\tilde{\mathbf{S}}^I}\left[\sum_{i=1}^{K} I[p, S_i^I]p\right] - \eta\mathbb{E}_{\tilde{p},\tilde{\mathbf{S}}^I}\left[\sum_{i=1}^{K} I[p, S_i^I]\right] \geq 0. \tag{4}$$

3.3. Solution

The Lagrange Dual Function for this problem can be written as

$$L(\psi) = \max_I\left\{\mathbb{E}_{\tilde{p},\tilde{\mathbf{S}}^I}\left[\sum_{i=1}^{K} I[p, S_i^I]p\right] + \psi\left(\mathbb{E}_{\tilde{p},\tilde{\mathbf{S}}^I}\left[\sum_{i=1}^{K} I[p, S_i^I]p\right] - \eta\mathbb{E}_{\tilde{p},\tilde{\mathbf{S}}^I}\left[\sum_{i=1}^{K} I[p, S_i^I]\right]\right)\right\}, \tag{5}$$

where ψ ($\psi \geq 0$) is the Lagrange multiplier. In the above, I represents a policy to decide the show/no-show decisions. Note that the dual function $L(\psi)$ is an upper bound for the optimal solution to Problem P. If $\psi = 0$, then the objective functions for the dual function and Problem P are the same, but the constraint can be ignored for the dual function. If $\psi > 0$, the value of the dual function for any feasible solution for Problem P will be greater than or equal to the value of the objective in Problem P. The dual function can be rewritten as

$$L(\psi) = \max_I \mathbb{E}_{\tilde{p},\tilde{\mathbf{S}}^I}\left[\sum_{i=1}^{K} I[p, S_i^I]\,(p(1+\psi) - \psi\eta)\right].$$

Let I^* be an optimizer of $L(\psi)$, assuming it exists. Later, we demonstrate its existence. Thus,

$$Ł(\psi) = \mathbb{E}_{\tilde{p},\tilde{\mathbf{S}}^{I^*}} \left[\sum_{i=1}^{K} I^*[p, S_i^{I^*}] \, (p(1+\psi) - \psi\eta) \right].$$

If we take the maximization step inside the summation, we will get an upper bound for $L(\psi)$ because we are maximizing each component of this expression. Hence, we get.

$$Ł(\psi) \leq \mathbb{E}_{\tilde{p},\tilde{\mathbf{S}}^{I^*}} \left[\sum_{i=1}^{K} \max_{I_i} I_i[p, S_i^{I^*}] \, (p(1+\psi) - \psi\eta) \right].$$

In the above, we are allowing ourselves to choose the best policy I_i, associated with the i^{th} arrival. This is a relaxation because although each component I_i is being individually maximized, we are still taking an expectation over the matrix $\tilde{\mathbf{S}}^{I^*}$, corresponding to the optimal policy. To maximize each component of the summation in the above, set $I_i[p, S_i^{I^*}] = 1$ if $p(1+\psi) - \psi\eta \geq 0$, otherwise, set $I_i[p, S_i^{I^*}] = 0$. Note that $I_i^*[p, S_i^{I^*}]$ does not depend on the state $S_i^{I^*}, \forall i$. This policy provides an upper bound for $L(\psi)$. Substituting this policy, we get

$$Ł(\psi) \leq \sum_{i=1}^{K} \mathbb{E}_{\tilde{p}:(p(1+\psi)-\psi\eta)\geq 0} \left[(p(1+\psi) - \psi\eta) \right].$$

In words, the expression inside the summation in the above inequality is the expectation (over $\tilde{p}$) of $(p(1+\psi) - \psi\eta)$, in the region where this quantity is greater than or equal to zero. The random variable $\tilde{p}$ represents the click-probability of any given arrival, and follows the distribution $f(p)$. The inequality $(p(1+\psi) - \psi\eta) \geq 0$ implies that p is greater than or equal to a threshold, i.e., $p \geq \frac{\psi\eta}{1+\psi}$. Denote this threshold as α ($\alpha = \frac{\psi\eta}{1+\psi}$). In the range $\psi \geq 0$, $\alpha \in [0, \eta)$. Hence, we get

$$\mathbb{E}_{\tilde{p}:p\geq\alpha} \left[(p(1+\psi) - \psi\eta) \right] = \int_{\alpha}^{1} (p(1+\psi) - \psi\eta) f(p) dp = \left(\frac{\beta(\alpha)\eta}{\eta - \alpha} \right) (\delta(\alpha) - \alpha).$$

Thus,

$$L(\psi) \leq \left(\frac{K\beta(\alpha)\eta}{\eta - \alpha} \right) (\delta(\alpha) - \alpha).$$

We know that an upper bound for the optimal solution to Problem P is

$$\min_{\psi \geq 0} L(\psi).$$

We can rewrite it as

$$\min_{\psi \geq 0} L(\psi) \leq \min_{\alpha \in [0,\eta)} \left(\frac{K\beta(\alpha)\eta}{\eta - \alpha} \right) (\delta(\alpha) - \alpha).$$

If we drop the constant $K\eta$, in order to minimize the expression on the right-hand-side, we need to find the minimum value of

$$\zeta(\alpha) = \left(\frac{\beta(\alpha)}{\eta - \alpha} \right) (\delta(\alpha) - \alpha), \alpha \in [0, \eta).$$

Using the definitions of $\beta(\alpha)$ and $\delta(\alpha)$, we can write,

$$\zeta(\alpha) = \frac{1}{\eta - \alpha}\left(\int_{\alpha}^{1} pf(p)dp - \alpha \int_{\alpha}^{1} f(p)dp\right).$$

It can be shown that the derivative of $\zeta(\alpha)$ with respect to α is

$$\frac{\beta(\alpha)}{(\eta-\alpha)^2}\left(\delta(\alpha) - \eta\right).$$

Observe that the sign of this derivative depends only on the sign of the term $(\delta(\alpha) - \eta)$. There are two cases to consider. For both of these cases, we will first assume that $\eta < 1$.

- $\delta(0) < \eta$:

We need to find a solution for $\delta(\alpha) - \eta = 0$, $\alpha \in [0, \eta)$. From Proposition 2, we know that for any $\alpha < 1$, $\delta(\alpha) > \alpha$. Hence, $\delta(\eta) > \eta$. Thus, there is a solution (α^*) and it is unique since $\delta(\alpha)$ increases in α. Also, note that if $\delta(\alpha) < \eta$, $\zeta(\alpha)$ decreases in α. Thus, α^* is a global and unique minimum. To show that α^* is optimal for Problem P, we need to check the above solution for feasibility, or

$$K\beta(\alpha)\left(\delta(\alpha) - \eta\right) \geq 0.$$

As can be seen, the solution $\delta(\alpha) = \eta$ is feasible for Problem P.

- $\delta(0) \geq \eta$:

First, let $\delta(0) > \eta$. In this case, since $\delta'(\alpha) > 0$, there is no solution for $\delta(\alpha) = \eta$, in the range $\alpha \in (0, \eta)$. Then, in this case, the minimizer α^* should be set to 0 because $\zeta(\alpha)$ is increasing in α if $\delta(0) > \eta$. Note that $\alpha^* = 0$ is feasible for Problem P. If $\delta(0) = \eta$, we also have $\alpha^* = 0$.

Finally, if $\eta = 1$, then the only feasible solution is to set $\alpha = 1$, implying that no visitor is shown ads. However, the constraint will be satisfied since $\beta(1) = 0$.

Based on the above discussion, we present the following proposition.

PROPOSITION 4. *The optimal solution to Problem P is:*

(a) If $\eta < 1$ and $\delta(0) \geq \eta$, $\alpha^* = 0$.

(b) If $\eta < 1$ and $\delta(0) < \eta$, then α^* is the solution of $\delta(\alpha) = \eta$.

(c) If $\eta = 1$, $\alpha^* = 1$.

Observe that the above proposition states that it is optimal to use a *static, threshold* policy. That is, a threshold policy is optimal and the threshold value is the same for different states. In the Electronic Companion, we provide the intuition for this result using a simple example of a problem with exactly two arrivals.

Finally, we present the following remark.

REMARK 1. An alternate way to express the constraint in Problem P is to require that the probability of obtaining a required click-through-rate is more than a certain specified value ω. With this revised constraint, a static threshold policy is no longer optimal.

4. Impact of Inaccurate Problem Parameters

We now examine the impact of inaccurate problem parameters on the solution presented in Proposition 4. The inaccuracy concerns the estimation of the click-probability distribution $f(p)$. Specifically, we focus our attention on the impact of inaccurately estimating the parameters of this distribution. With inaccurate parameters, let us assume that the true functions β and δ are wrongly estimated as $\hat{\beta}$ and $\hat{\delta}$ respectively. To address inaccurate parameters, we propose a *rolling-horizon* approach. In a rolling-horizon approach, Problem P (presented in section 3.2) is re-solved at certain points during the planning horizon. Formally, after s arrivals (or $(K-s)$ remaining arrivals), $s \in (1,2,\dots,K-1)$, given that the impressions and the clicks that have occurred are m_s and r_s, respectively, we need to maximize the expected number of clicks in the $(K-s)$ remaining arrivals. The constraint requires that r_s plus the expected number of clicks in the remaining arrivals is greater than η times m_s plus the expected number of impressions in the remaining $(K-s)$ arrivals. Then, as shown earlier, the threshold that should be used for the $(s+1)^{th}$ arrival is given by the solution of

$$\frac{r_s + (K-s)\hat{\beta}(\alpha)\hat{\delta}(\alpha)}{m_s + (K-s)\hat{\beta}(\alpha)} = \eta. \tag{6}$$

The rolling-horizon approach can be expected to cope well in presence of inaccurate model parameters because it uses actual state information to update the threshold in each period. We explain the basic idea behind varying the thresholds as follows. Imagine that we are at the beginning of the planning horizon, i.e., the first period of the planning horizon. We set the threshold at the smallest level such that the target click-through-rate level should just be achieved at the end of the month (on average). We keep this threshold value constant for the current period. At the end of the period, we collect data on the number of impressions and the number of clicks. The current click-through-rate value is the number of clicks divided by the number of impressions. If this value is better than the required click-through-rate at the end of the month, we can afford to lower the threshold and show more ads. Conversely, if the current click-through-rate value is below the threshold, we must set the next period's threshold higher. At the beginning of each period, we set the threshold to a value that, if kept constant for the rest of the planning horizon, would just achieve, on an expected basis, the revised target threshold for the rest of the planning horizon.

In Figure 3, we consider a simple example to illustrate how the rolling-horizon approach works. Imagine that there are 3 days in the planning horizon and that the value of the threshold (α) is updated on a daily basis. Let the click-through-rate constraint be 0.01. At the beginning of the first day, we find the lowest value of α that (if held constant) would just achieve the required click-through-rate. Suppose this value is α_1. We set the threshold to α_1 for the first day. At the end of the first day, we observe the number of clicks (say r_1) and the number of impressions (say m_1).

We next use these values (r_1 and m_1) to find the lowest value of α that, if held constant for the remaining 2 days, would just achieve the required click-through-rate of 0.01. Suppose this value is α_2. Intuitively, this value should be higher than α_1 if the click-through-rate in the first day was below the required target (0.01), otherwise it should be lower. After considering the fact that we are behind (ahead) on the constraint, we set the value of α to α_2 for the second day. Similarly, at the beginning of the third day, we use the actual clicks and impressions that have occurred so far to calculate the lowest value of α needed to achieve the click-through-rate constraint (say, α_3). For this problem, the static approach would have used a constant threshold value (α_1) for the entire duration, that is, the threshold value used by the rolling-horizon approach in the first period.

	Day 1	Day 2	Day 3
Impressions	m_0	m_1	m_2
Clicks	r_0	r_1	r_2
Decision Variables	α_1	α_2	α_3

Figure 3 The Mechanics of the Rolling-Horizon Approach

The rolling-horizon approach has the advantage that it can increase or reduce the threshold to incorporate state information. In any particular state, we may be better or worse than what we need to be with respect to the target click-through-rate constraint. If we are doing better, we can afford to use a lower value of α. Conversely, if real-world feedback is such that we are behind on the constraint, we need to tighten (increase) the threshold to catch up. Although some of the past studies have used the rolling-horizon approach in a similar paper (e.g., Ghosh et al. 2009b, Besbes and Maglaras 2012, Wang et al. 2014), both our context and implementation are different from those studies.

5. Experimental Results

The primary goal of the numerical experiments in this section is to provide answers to two questions that are of practical interest: (1) What update frequency should be used in the rolling-horizon approach? (2) In the presence of inaccurate problem parameters (or *noise*), which method (static or rolling-horizon) should be used? The first question essentially boils down to asking: within practical constraints, should the most frequent update frequency be used? The danger of updating too frequently (more importantly, if updating is based on the outcomes of a small number of arrivals) is that the threshold could be changed based upon a unlikely random draw (of impressions and clicks). Thus, in an attempt to cope with noise, an excessively frequent update policy could lead to poor decisions, and hurt overall performance. The second question may be especially relevant when we expect there to be a moderate or small amount of noise. In such situations, there is a tension between the benefits of using an optimal approach (i.e., static) versus the virtues of using a sub-optimal one that is able to cope with noise. For these experiments, we first estimate the click-probability distribution using the data from different publishers.

5.1. Estimation of the Click Probability Distribution

Figure 4 shows a frequency plot of the click-probability (p) estimated using the Logit model discussed in Section 3.1.1. This figure indicates that the probability increases rapidly from zero to a maximum frequency. After the maximum frequency, the frequency sharply reduces. We can estimate the click-probability distribution for a given publisher using a sample of these probabilities. Using data collected for several different publishers, we were able to estimate that the click-probability follows a `Gamma` distribution, with shape and scale parameters specialized for each publisher. In addition to the `Gamma` distribution, there are other distributions that also fit reasonably well (albeit, to a lesser extent than `Gamma`) with the data. These include `Exponential`, `Logistic`, and `Lognormal`.

In Figure 4, we depict $f(p)$ for a publisher with shape and scale parameters 2.25 and 0.005, respectively. To estimate this distribution, we need not concern ourselves with the censoring impact of the threshold policy that shows ads to only a subset of arriving visitors (namely, those that have $p \geq \alpha$). For the purposes of estimating $f(p)$, it would be sufficient to compute the value of the click-probability (p) for each arriving visitor. This is needed anyway since we need to check if $p \geq \alpha$ for every visitor. In the next subsection, we discuss the experimental setup.

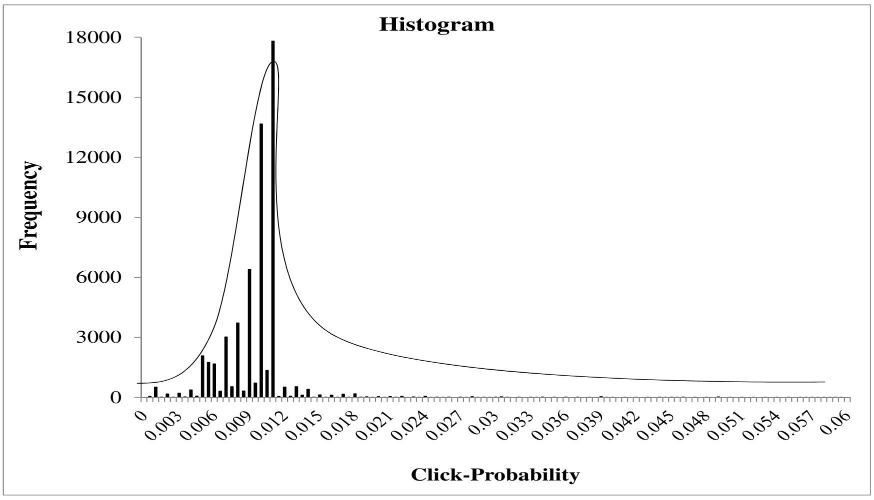

Figure 4 Click-Probability Distribution for a Particular Publisher

5.2. Experimental Setup

We describe the findings from our experiments corresponding to the two questions discussed above: how to choose update frequency, and which method to use when there is noise. To study noise, we consider the use of an inaccurate shape parameter corresponding to the click probability distribution. The factors varied are the click-through-rate constraint (η), the wrong shape parameter (k), and the update frequency (μ). The correct shape parameter is denoted by k' and the scale parameter is denoted by q. Therefore, the expected value of the click probability p is $k'q$, i.e., $\mathbb{E}[p] = k'q$. Before discussing the details of experiments, we present the following analytical result regarding the performance of the static approach using an incorrect shape parameter.

PROPOSITION 5. *In presence of the wrong shape parameter, the performance of the static approach is described as follows:*

(a) The static approach provides an optimal solution when either (i) $\eta \leq \min[k'q, kq]$*, or (ii)* $k = k'$.

(b) The static approach provides a sub-optimal solution when either (i) $kq < \eta \leq k'q$*, or (ii)* $kq < k'q \leq \eta$.

(c) The static approach provides an infeasible solution when either (i) $k'q < \eta \leq kq$*, or (ii)* $k'q < kq \leq \eta$.

In the experiments, we examine all the conditions presented in this proposition. Clearly, the rolling-horizon approach may perform better than the static approach for the conditions presented in parts (b) and (c) of the proposition. We now discuss the parameter values used for the experiments. Based on the result presented in Figure 4, the value of correct shape parameter (k') in these experiments is 2.25. The values of the wrong shape parameter are chosen (in steps of 0.10) in the interval [1.75, 2.75]. The values of the click-through-rate constraint are chosen (in steps of 0.005) from the interval [0.005,0.02], and the update frequency is chosen from the set {1,2,3,5,6,10,15,30}. Each element (μ) of this set corresponds to an update made after $\frac{K\lambda}{\mu}$ arrivals, where K is the number of periods (= 30 in our experiments), and λ is the number of arrivals in each period. Hence, when $\mu = 1$, there is only one value of the threshold chosen for the entire problem, thus implying that under this setting, the static and the rolling-horizon approaches are equivalent. Finally, based on the result presented in Figure 4, the scale parameter (q) for these experiments is 0.005.

In all the experiments reported in the section, care is taken to ensure that the performance of the rolling-horizon approach indeed converges to its expected value before this value is compared with the performance of the static approach. The performance of the static approach, of course, can be calculated analytically. However, in order to correctly estimate the performance of the rolling-horizon approach for a given treatment (i.e., a particular choice of the values of the parameters,

η, k, and μ), we need to run a sufficient number of replications and use the average performance across these replications as a basis for comparison. In order to guarantee convergence for a specific level precision, we use a convergence procedure proposed by Law et al. (1981).

Before we describe the convergence procedure, it is worth noting that the issue of updating too frequently (and hence running the danger of using a small sample to make an update decision) is fundamentally different from the issue of convergence. For any update frequency, it is possible to run a sufficient number of replications to ensure convergence and obtain an accurate estimate of performance. However, with an excessively high update frequency, it is possible that performance could suffer. We describe the convergence procedure in the next subsection followed by a discussion of the method to choose an appropriate update frequency.

5.3. Convergence Procedure

In the convergence procedure used in our experiments, we iteratively increase the number of random replications until a specified precision is obtained. Let us denote the average number of clicks in n replications by $\bar{r}(n)$ and the sample variance by $S^2(n)$. Hence, given that r_i is the number of clicks in i^{th} replication, we have

$$\bar{r}(n) = \frac{r_1 + r_2 + \ldots + r_n}{n}, \quad \text{and} \tag{7}$$

$$S^2(n) = \frac{\sum_{i=1}^{n} \left[r_i - \bar{r}(n)\right]^2}{n-1}. \tag{8}$$

After n replications, an approximate $100(1-\nu)$ percent confidence interval for the expected number of clicks (i.e., $\mathbb{E}[\tilde{r}]$) is given by

$$\bar{r}(n) \pm t_{n-1,1-\nu/2} \sqrt{\frac{S^2(n)}{n}}. \tag{9}$$

Let us define half-length of the confidence interval given in Equation (9) as

$$\zeta(n,\nu) = t_{n-1,1-\nu/2} \sqrt{\frac{S^2(n)}{n}}. \tag{10}$$

If the estimate $\bar{r}(n)$ is such that $\frac{\bar{r}(n) - \mathbb{E}[\tilde{r}]}{\mathbb{E}[\tilde{r}]} = \xi$, then we say that $\bar{r}(n)$ has a *relative error* of ξ, or that the *percentage error* in $\bar{r}(n)$ is 100ξ percent. If we keep increasing replications until $\frac{\zeta(n,\nu)}{\bar{r}(n)}$ is less than or equal to ξ, then $\bar{r}(n)$ has a relative error of at most $\frac{\xi}{1-\xi}$ with a probability of approximately $1-\nu$ (Law and Kelton 2000, p. 513). Hence, in order to obtain an estimate of $\mathbb{E}[\tilde{r}]$ with a relative error of ξ and a confidence level of $100(1-\nu)$ percent, we should keep increasing the replications until

$$\frac{\zeta(n,\nu)}{\bar{r}(n)} \leq \frac{\xi}{1+\xi}. \tag{11}$$

Before discussing the formal steps of this procedure, we present the following result.

Remark 2. The simulation converges faster when the arrival rate (i.e., λ) is higher.

Intuitively, this result is expected: the simulation should converge faster when the arrival rate is higher because the estimate for the mean number of clicks should be better when the estimate is being constructed with a higher sample size (number of arrivals). The steps of this procedure can now be formally defined as follows (Law and Kelton 2000, pp. 513-514):

Step 1: Make $n_0 \geq 2$ random replications and set $n = n_0$.

Step 2: Compute $\bar{r}(n)$ and $\zeta(n,\nu)$ from $r_1, r_2, \ldots, r_n$.

Step 3: If $\frac{\zeta(n,\nu)}{\bar{r}(n)} \leq \frac{\xi}{1+\xi}$, then use $\bar{r}(n)$ as the point estimate for $\mathbb{E}[\tilde{r}]$ and stop. Otherwise, replace n by $n+1$, make an additional replication of the simulation, and go to Step 2.

Law et al. (1981) show that this procedure obtains an estimate of $\mathbb{E}[\tilde{r}]$ with a confidence level arbitrary close to $100(1-\nu)$ percent provided that $\mathbb{E}[\tilde{r}] \neq 0$ and the desired relative error is sufficiently close to 0. Hence, in our experiments, we set $\xi = \nu = 0.005$. Further, in order to ensure that the simulation indeed converges, we set $n_0 = 50$.

5.4. Choice of Update Frequency

For some publishers that have relatively low traffic, the rolling-horizon could suffer from having to update the threshold after a relatively small sample of arrivals (impressions, clicks). These updates may not be as reliable as the ones that were made with a large sample of impressions and clicks. Because of the inherent instability of inferences drawn from small samples, the rolling-horizon approach may sometimes be led astray and its performance could suffer vis-a-vis the static approach that does not rely on actual events to choose the threshold. It is here that we could expect to observe the malicious side of frequent updating. These speculations are borne out in our experiments that are discussed below.

Let us consider a relatively extreme case of a publisher with approximately 10 arrivals per period ($=\lambda$). We find that, when $\eta = 0.005$ or 0.0075, both the static and the rolling-horizon methods show ads to all the visitors. Hence, at these levels of η, there is no benefit of using the rolling-horizon approach. For all other values of η, the static approach performs better than the rolling-horizon approach. Further, the solution of rolling-horizon approach does not even meet the click-through-rate constraint as the update frequency increases. Thus, to summarize, the rolling-horizon approach indeed suffers when the update decisions are based on relatively small samples.

To avoid this problem with the rolling-horizon approach, we use an update period that ensures that the number of arrivals is large enough such that the update decision is not based on an unlikely draw of random events. Let the number of arrivals in one period be λ. We need to find a threshold ($\hat{\lambda}$) such that it is safe to update every period if $\lambda > \hat{\lambda}$. Further, let α be the threshold at the beginning of a given period. The update decision at the end of the period depends on the number of impressions and clicks in that period, as well as the total clicks and impressions observed until the beginning of that period.

Recall that (using Equation (6)) the new value of the threshold at the end of a period is calculated using

$$\frac{R+\tilde{r}+(K-s)\hat{\beta}(\alpha)\hat{\delta}(\alpha)}{M+\tilde{m}+(K-s)\hat{\beta}(\alpha)}=\eta, \tag{12}$$

where $\tilde{r}$ and $\tilde{m}$ are the number of clicks and impressions observed in the current period and R and M are the total clicks and impressions observed until the beginning of the period. The randomness in the quantities $\tilde{r}$ and $\tilde{m}$ is what affects the quality of the update decision. If these quantities are close to their mean values, then the update decision will likely be of good quality. However, if there is large variation in these quantities, then the update decision can be poor and would likely hurt the performance of the rolling-horizon approach.

From the above equation, it is clear that the distribution of the quantity $(\eta\tilde{m}-\tilde{r})$ is of key interest. The mean number of impressions is $\mathbb{E}[\tilde{m}]=\lambda\beta(\alpha)$ and the mean number of clicks is $\mathbb{E}[\tilde{r}]=\lambda\beta(\alpha)\delta(\alpha)$. Using Wald's theorem for variance, it can be shown that the variance of the number of impressions is $\mathbb{V}[\tilde{m}]=\lambda\beta(\alpha)(1-\beta(\alpha))$ and the variance of the number of clicks is $\mathbb{V}[\tilde{r}]=\lambda\beta(\alpha)\delta(\alpha)(1-\beta(\alpha)\delta(\alpha))$, where the functions β and δ are defined in Equations (1) and (2). The variance of the quantity $(\eta\tilde{m}-\tilde{r})$ is given by $\eta^2\mathbb{V}[\tilde{m}]+\mathbb{V}[\tilde{r}]-2\mathbb{COV}[\eta\tilde{m},\tilde{r}]$. Further, we can approximate $(\eta\tilde{m}-\tilde{r})$ to be Normally distributed[1] with mean $\mu=\eta\mathbb{E}[\tilde{m}]-\mathbb{E}[\tilde{r}]$ and variance $\sigma^2=\eta^2\mathbb{V}[\tilde{m}]+\mathbb{V}[\tilde{r}]-2\mathbb{COV}[\eta\tilde{m},\tilde{r}]$.

We have, $\mathbb{COV}[\tilde{m},\tilde{r}]=\mathbb{E}[\tilde{m}\tilde{r}]-\mathbb{E}[\tilde{m}]\mathbb{E}[\tilde{r}]$. Also,

$$\begin{aligned}
\mathbb{E}[\tilde{m}\tilde{r}] &= \sum_{m=0}^{\lambda}\sum_{r=0}^{m} mr\mathbb{P}[r|m]\mathbb{P}[m] \\
&= \sum_{m=0}^{\lambda} m\mathbb{P}[m]\sum_{r=0}^{m} r\mathbb{P}[r|m] \\
&= \sum_{m=0}^{\lambda} m\mathbb{P}[m]\sum_{r=0}^{m} r\frac{m!}{(m-r)!(r!)}\delta(\alpha)^r(1-\delta(\alpha))^{m-r} \\
&= \sum_{m=0}^{\lambda} m\mathbb{P}[m]\cdot m\delta(\alpha) \\
&= \delta(\alpha)\sum_{m=0}^{\lambda} m^2\mathbb{P}[m] \\
&= \delta(\alpha)(\lambda\beta(\alpha)(1-\beta(\alpha))+(\lambda\beta(\alpha))^2).
\end{aligned} \tag{13}$$

[1] The `Normal` distribution is a standard model that fits here based on the large number of trials (i.e., the number of arrivals). In addition, we have empirically verified the distribution of this random quantity and found that the `Normal` distribution fits well. Some other distributions that agree with the data are `Weibull`, `Beta`, and `Logistic`.

Thus, $\mathbb{COV}[\tilde{m},\tilde{r}] = \lambda\beta(\alpha)\delta(\alpha)(1-\beta(\alpha))$.

Using the above expression, the variance of the quantity $(\eta\tilde{m}-\tilde{r})$ is

$$\sigma^2 = \lambda\beta(\alpha)\left(\eta^2(1-\beta(\alpha)) - 2\eta\delta(\alpha)(1-\beta(\alpha)) + \delta(\alpha)(1-\beta(\alpha)\delta(\alpha))\right). \tag{14}$$

Hence, in order to ensure that any random draw of the quantity $\eta\tilde{m}-\tilde{r}$ is relatively *stable*, we can derive a minimum sample size that ensures that the quantity stays within an error of χ around its mean with a probability of $100(1-\phi)$ percent. Using standard procedures for constructing confidence intervals, the lower bound on the number of arrivals in each period (i.e., $\hat{\lambda}$) can be calculated as follows:

$$\hat{\lambda} = \frac{Z^2_{1-\phi/2}\left[\eta^2(1-\beta(\alpha)) + \delta(\alpha)(1-\beta(\alpha)\delta(\alpha)) - 2\eta\delta(\alpha)(1-\beta(\alpha))\right]}{\beta(\alpha)\chi^2(\eta-\delta(\alpha))^2}. \tag{15}$$

We use the above expression for the minimum number of arrivals before which an update can be safely performed. When $\delta(\alpha)=\eta$, the above expression is not valid. This condition usually satisfies only in the first period. After the first period, because of randomness, $\delta(\alpha)$ is usually not equal to η. When $\delta(\alpha)=\eta$, since the mean of the random quantity is zero, we use the notion of an absolute error (e) and calculate the threshold using

$$\frac{1}{\eta^2(1-\beta(\alpha)) + \delta(\alpha)(1-\beta(\alpha)\delta(\alpha)) - 2\eta\delta(\alpha)(1-\beta(\alpha))}\left(\frac{e^2}{\beta(\alpha)Z^2_{1-\phi/2}}\right).$$

Below the threshold, the update could be based on an unlikely draw of events and hurt performance. For example, when $\eta = 0.0175, \alpha = 0, \chi = \phi = 0.05$, $k' = 2.25$, and $q = 0.005$, $\hat{\lambda} = \frac{(1.96)^2\times 2.25\times 0.005[1-(2.25\times 0.005)]}{(0.05)^2[0.0175-(2.25\times 0.005)]^2} = 437,573$, or 0.5 million (approximately). Thus, for a publisher with the above characteristics, and a traffic of about 100,000 hits per day, a daily update policy is not appropriate. That is, we need to wait for about 5 days to get a reasonable sample size to perform an update. To provide a sense for the above traffic volumes, we observe that at Chitika, *enterprise* level publishers have a traffic in excess of 1 million hits per day. Many publishers, however, are in the range of 50,000 to 100,000 hits per day. There is also a long tail; a significant number of publishers have a daily traffic of about 5,000 hits per day.

We will refer to a publisher as a *high-volume* publisher if the total number of arrivals per month exceeds the update threshold for that publisher $(\hat{\lambda})$. Note that we need to calculate the threshold for all possible values of $\alpha \in [0,1]$ and use the maximum threshold value across these values of α. If the monthly traffic is not sufficient to warrant at least one update, we will refer to such a publisher as a *low-volume* one.

5.5. High-Volume Publishers

For these experiments, we consider a publisher with 1 million arrivals per day ($=\lambda$). This high-volume of traffic easily permits daily updates for the rolling-horizon approach. When there is noise, we expect the rolling-horizon approach to do better.

In Table 3, we begin with comparing the static solution with the rolling-horizon solution when the shape parameter (and, therefore the click-probability) is *correct*. In this table, the first column presents the value of shape parameter ($k = k' = 2.25$). When there is no noise, the static approach provides the optimal solution at any value of η (as indicated in Proposition 4 and Proposition 5(a)). Further, in this case, the rolling-horizon approach does not update the threshold at any value of η. Therefore, both approaches have the same performance (for each η). The last column in the table represents the percentage improvement of the rolling-horizon solution over the static solution, which is clearly zero for all values of η.

Shape Parameter ($k = k'$)	Click-Through-Rate Constraint (η)	Number of Clicks		
		Rolling-Horizon Solution	Static Solution	Percentage Improvement
2.25	0.005	337500	337500	0
2.25	0.0075	337500	337500	0
2.25	0.01	337500	337500	0
2.25	0.0125	327865	327865	0
2.25	0.015	287997	287997	0
2.25	0.0175	238305	238305	0
2.25	0.02	189474	189474	0

Table 3 Comparison between Static and Rolling-Horizon Solutions for Correct Shape Parameter

Now, in Table 4, we compare the static solution with the rolling-horizon solution when the shape parameter (and, therefore the click-probability) is *underestimated*. In this table, the first column presents the value of wrong shape parameter (k). As mentioned earlier, the correct shape parameter is 2.25. Hence, $k = 2.15$ represents a small underestimation, whereas $k = 1.75$ represents a high underestimation. When $\eta \leq k'q$, the correct choice is $\alpha = 0$ (based on Proposition 4). Further, when $\eta \leq kq$, the static approach chooses $\alpha = 0$. Therefore, as shown in Proposition 5(a), when $\eta \leq \min[k'q, kq]$, the static approach provides the optimal solution. In this case, all visitors are shown ads (i.e., $\alpha = 0$). Hence, of course, the rolling-horizon approach also uses the threshold $\alpha = 0$ and obtains the optimal solution. For $\eta \leq 0.0075$ (i.e., 0.005 or 0.0075), the condition $\eta \leq \min[k'q, kq]$ is satisfied even for the lowest value of k considered in the experiment (i.e., $k = 1.75$). Hence, for $\eta \leq 0.0075$, there is no benefit in using the rolling-horizon approach (at any value of k).

In our experiment, the correct value of $\mathbb{E}[p] = k'q = (2.25)(0.005) = 0.01125$. Hence, for $\eta \leq 0.01125$, the optimal choice is $\alpha = 0$ (based on Proposition 4). Further, the static approach chooses $\alpha = 0$ when $\eta \leq 0.005k$ (according to Proposition 4). For example, when $\eta = 0.01$, the static approach chooses $\alpha = 0$ for $k \geq 2$. However, when $\eta = 0.01$ and $k < 2$, the static approach chooses

Wrong Shape Parameter (k)	Click-Through-Rate Constraint (η)	Number of Clicks		
		Rolling-Horizon Solution	Static Solution	Percentage Improvement
1.75	0.005	337500	337500	0
1.75	0.0075	337500	337500	0
1.75	0.01	336850	333688	0.95
1.75	0.0125	322111	307764	4.66
1.75	0.015	285144	266585	6.96
1.75	0.0175	236717	219986	7.61
1.75	0.02	188779	174961	7.90
1.85	0.005	337500	337500	0
1.85	0.0075	337500	337500	0
1.85	0.01	337131	335633	0.45
1.85	0.0125	323399	311473	3.83
1.85	0.015	285736	270303	5.71
1.85	0.0175	236990	223273	6.14
1.85	0.02	188854	177558	6.36
1.95	0.005	337500	337500	0
1.95	0.0075	337500	337500	0
1.95	0.01	337470	337128	0.10
1.95	0.0125	324802	315371	2.99
1.95	0.015	286538	274284	4.47
1.95	0.0175	237323	226733	4.67
1.95	0.02	189139	180172	4.98
2.05	0.005	337500	337500	0
2.05	0.0075	337500	337500	0
2.05	0.01	337500	337500	0
2.05	0.0125	326187	319344	2.14
2.05	0.015	286962	278506	3.04
2.05	0.0175	237829	230366	3.24
2.05	0.02	189233	182966	3.43
2.15	0.005	337500	337500	0
2.15	0.0075	337500	337500	0
2.15	0.01	337500	337500	0
2.15	0.0125	327055	323508	1.10
2.15	0.015	287958	283093	1.72
2.15	0.0175	237938	234167	1.61
2.15	0.02	189337	186110	1.73

Table 4 Comparison between Static and Rolling-Horizon Solutions for Underestimation

$\alpha > 0$, whereas the correct value of α (based on the correct shape parameter) is zero. Therefore, in this case (where $kq < \eta \le k'q$), the static approach provides a sub-optimal solution (as shown in Proposition 5(b)). In this case, the rolling-horizon chooses a lower value of the threshold and this value steadily decreases. As a result, for $\eta = 0.01$ and $k < 2$, the rolling-horizon approach outperforms the static approach in Table 4. At the higher values of the click-through-rate constraint, either $kq < \eta \le k'q$ or $kq < k'q \le \eta$ satisfies. Therefore, according to Proposition 5(b), the static approach provides a sub-optimal solution at the higher values of the click-through-rate constraint. Hence, as seen in Table 4, even when there is a small underestimation (i.e., $k = 2.15$), the rolling-horizon approach performs better for some (relatively high) values of the click-through-rate constraint.

Next, in Table 5, we compare the static solution with the rolling-horizon solution when the shape parameter (and, therefore the click-probability) is *overestimated*. In the case of overestimation, both static and rolling-horizon approaches would expect more clicks to happen than the actual number of

Wrong Shape Parameter (k)	Click-Through-Rate Constraint (η)	Number of Clicks		
		Rolling-Horizon Solution	Static Solution	Percentage Improvement
2.35	0.005	337500	337500	0
2.35	0.0075	337500	337500	0
2.35	0.01	337500	337500	0
2.35	0.0125	**Infeasible (0.012489)**	**Infeasible (0.012103)**	**(3.19)**
2.35	0.015	**Infeasible (0.014992)**	**Infeasible (0.014723)**	**(1.83)**
2.35	0.0175	**Infeasible (0.017491)**	**Infeasible (0.017283)**	**(1.20)**
2.35	0.02	**Infeasible (0.019992)**	**Infeasible (0.019824)**	**(0.85)**
2.45	0.005	337500	337500	0
2.45	0.0075	337500	337500	0
2.45	0.01	337500	337500	0
2.45	0.0125	**Infeasible (0.012480)**	**Infeasible (0.011601)**	**(7.58)**
2.45	0.015	**Infeasible (0.014983)**	**Infeasible (0.014421)**	**(3.90)**
2.45	0.0175	**Infeasible (0.017483)**	**Infeasible (0.017055)**	**(2.51)**
2.45	0.02	**Infeasible (0.019984)**	**Infeasible (0.019628)**	**(1.81)**
2.55	0.005	337500	337500	0
2.55	0.0075	337500	337500	0
2.55	0.01	337500	337500	0
2.55	0.0125	**Infeasible (0.012450)**	**Infeasible (0.011250)**	**(10.67)**
2.55	0.015	**Infeasible (0.014965)**	**Infeasible (0.014074)**	**(6.33)**
2.55	0.0175	**Infeasible (0.017466)**	**Infeasible (0.016803)**	**(3.95)**
2.55	0.02	**Infeasible (0.019967)**	**Infeasible (0.019434)**	**(2.74)**
2.65	0.005	337500	337500	0
2.65	0.0075	337500	337500	0
2.65	0.01	337500	337500	0
2.65	0.0125	**Infeasible (0.012387)**	**Infeasible (0.011250)**	**(10.11)**
2.65	0.015	**Infeasible (0.014913)**	**Infeasible (0.013687)**	**(8.96)**
2.65	0.0175	**Infeasible (0.017426)**	**Infeasible (0.016529)**	**(5.43)**
2.65	0.02	**Infeasible (0.019934)**	**Infeasible (0.019213)**	**(3.75)**
2.75	0.005	337500	337500	0
2.75	0.0075	337500	337500	0
2.75	0.01	337500	337500	0
2.75	0.0125	**Infeasible (0.012303)**	**Infeasible (0.011250)**	**(9.36)**
2.75	0.015	**Infeasible (0.014827)**	**Infeasible (0.013236)**	**(12.02)**
2.75	0.0175	**Infeasible (0.017359)**	**Infeasible (0.016232)**	**(6.94)**
2.75	0.02	**Infeasible (0.019877)**	**Infeasible (0.018984)**	**(4.70)**

Table 5 Comparison between Static and Rolling-Horizon Solutions for Overestimation

clicks. Hence, we find that both approaches are not able to meet the click-through-rate constraint in several instances of overestimation. For infeasible cases, we report the click-through-rate (numbers in parenthesis) that was achieved by an approach. Also, in these cases, we report (in the last column) the percentage improvement (in terms of the click-through-rate) of the rolling-horizon approach over the static approach. For $\eta = 0.005$, 0.0075, and 0.01, the condition $\eta \leq \min[k'q, kq]$ is always satisfied. Therefore, according to Proposition 5(a), the static approach provides an optimal solution. In fact, in these scenarios, both approaches show ad to all the visitors, and the corresponding solutions meet the click-through-rate constraint. Therefore, in these cases, the performances of two approaches are statistically the same. However, for $\eta = 0.0125$, 0.015, 0.0175, and 0.02, either $k'q < \eta \leq kq$ or $k'q < kq \leq \eta$ satisfies. Therefore, according to Proposition 5(c), the static approach provides an infeasible solution. In these scenarios, Table 5 shows that the rolling-horizon solution also fails to meet the click-through-rate constraint. However, for these values of η, as the update frequency increases, the solution of rolling-horizon approach comes closer to the click-through-rate constraint. To summarize, in the case of overestimation (as in the case of underestimation), there is no need to use the rolling-horizon approach at low values of η (specifically, when $\eta \leq kq$). However, at higher values of η, the rolling-horizon approach should be preferred.

5.6. Low-Volume Publishers

For these experiments, we consider a publisher with 10 arrivals per period ($=\lambda$). The results are presented in Table 6. Based on the result of Proposition 5, when $\eta = 0.005$ or 0.0075 for underestimation, both static and rolling-horizon approaches show ad to all the visitors. Hence, at these levels of η in underestimation, both approaches have the same performance. These results are similar to those for the high-volume publisher. However, unlike that in the case of high-volume publisher, even for all other values of η in underestimation, the rolling-horizon approach does not update the threshold. Hence, in underestimation, both approaches perform the same for all values of η.

The results for the correct estimation case and the overestimation case are similar to those for the high-volume publisher with only exception: unlike that in the high-volume publisher case, at higher values of η in overestimation, the solution of rolling-horizon approach does not come closer to the click-through-rate constraint. Hence, in summary, when the monthly arrival volume is low, even a single update is not warranted at any level of η or any level of noise.

A natural question arises: since a low-volume publisher is not likely to survive, is it important to study such a publisher? Thus we need to clarify why a low-volume publisher is of practical interest. Sometimes, a publisher may choose to divide its traffic across different ad-networks and benefit from them having to compete for a larger share of the traffic. In such cases, a new or small ad-network may command only a small fraction of a publisher's traffic. Thus, while a publisher may attract a large body of visitors, the *share* of the traffic handled by a particular ad-network may be small. For such publishers, updating is not feasible during the month, and hence, the static approach becomes the only available method.

5.7. Bayesian Updating of Parameters

At the end of each period (say, a month), the parameters (say κ_1, κ_2, for the purposes of this discussion) of the click-probability distribution $f(p)$ could potentially be updated. We present below a Bayesian framework for this process (Berger 1985).

Let us denote the prior joint density for the parameters κ_1, κ_2 as $\pi(\kappa_1, \kappa_2)$. We observe a set of data and would like to update the prior to a posterior as $\pi'(\kappa_1, \kappa_2|\texttt{data})$. As we know, the data

Wrong Shape Parameter (k)	Click-Through-Rate Constraint (η)	Number of Clicks		
		Rolling-Horizon Solution	Static Solution	Percentage Improvement
1.75	0.005	3.37	3.37	0
1.75	0.0075	3.37	3.37	0
1.75	0.01	3.31	3.31	0
1.75	0.0125	2.96	2.96	0
1.75	0.015	2.47	2.47	0
1.75	0.0175	1.95	1.95	0
1.75	0.02	1.49	1.49	0
1.85	0.005	3.37	3.37	0
1.85	0.0075	3.37	3.37	0
1.85	0.01	3.34	3.34	0
1.85	0.0125	3.01	3.01	0
1.85	0.015	2.52	2.52	0
1.85	0.0175	1.99	1.99	0
1.85	0.02	1.52	1.52	0
1.95	0.005	3.37	3.37	0
1.95	0.0075	3.37	3.37	0
1.95	0.01	3.36	3.36	0
1.95	0.0125	3.06	3.06	0
1.95	0.015	2.57	2.57	0
1.95	0.0175	2.04	2.04	0
1.95	0.02	1.55	1.55	0
2.05	0.005	3.37	3.37	0
2.05	0.0075	3.37	3.37	0
2.05	0.01	3.37	3.37	0
2.05	0.0125	3.11	3.11	0
2.05	0.015	2.62	2.62	0
2.05	0.0175	2.07	2.07	0
2.05	0.02	1.58	1.58	0
2.15	0.005	3.37	3.37	0
2.15	0.0075	3.37	3.37	0
2.15	0.01	3.37	3.37	0
2.15	0.0125	3.16	3.16	0
2.15	0.015	2.67	2.67	0
2.15	0.0175	2.12	2.12	0
2.15	0.02	1.61	1.61	0
2.25	0.005	3.37	3.37	0
2.25	0.0075	3.37	3.37	0
2.25	0.01	3.37	3.37	0
2.25	0.0125	3.22	3.22	0
2.25	0.015	2.73	2.73	0
2.25	0.0175	2.18	2.18	0
2.25	0.02	1.66	1.66	0
2.35	0.005	3.37	3.37	0
2.35	0.0075	3.37	3.37	0
2.35	0.01	3.37	3.37	0
2.35	0.0125	**Infeasible (0.012155)**	**Infeasible (0.012155)**	**(0)**
2.35	0.015	**Infeasible (0.014796)**	**Infeasible (0.014796)**	**(0)**
2.35	0.0175	**Infeasible (0.017402)**	**Infeasible (0.017402)**	**(0)**
2.35	0.02	**Infeasible (0.019881)**	**Infeasible (0.019881)**	**(0)**
2.45	0.005	3.37	3.37	0
2.45	0.0075	3.37	3.37	0
2.45	0.01	3.37	3.37	0
2.45	0.0125	**Infeasible (0.011620)**	**Infeasible (0.011620)**	**(0)**
2.45	0.015	**Infeasible (0.014502)**	**Infeasible (0.014502)**	**(0)**
2.45	0.0175	**Infeasible (0.017139)**	**Infeasible (0.017139)**	**(0)**
2.45	0.02	**Infeasible (0.019750)**	**Infeasible (0.019750)**	**(0)**
2.55	0.005	3.37	3.37	0
2.55	0.0075	3.37	3.37	0
2.55	0.01	3.37	3.37	0
2.55	0.0125	**Infeasible (0.011287)**	**Infeasible (0.011287)**	**(0)**
2.55	0.015	**Infeasible (0.014152)**	**Infeasible (0.014152)**	**(0)**
2.55	0.0175	**Infeasible (0.016894)**	**Infeasible (0.016894)**	**(0)**
2.55	0.02	**Infeasible (0.019455)**	**Infeasible (0.019455)**	**(0)**
2.65	0.005	3.37	3.37	0
2.65	0.0075	3.37	3.37	0
2.65	0.01	3.37	3.37	0
2.65	0.0125	**Infeasible (0.011287)**	**Infeasible (0.011287)**	**(0)**
2.65	0.015	**Infeasible (0.013749)**	**Infeasible (0.013749)**	**(0)**
2.65	0.0175	**Infeasible (0.016554)**	**Infeasible (0.016554)**	**(0)**
2.65	0.02	**Infeasible (0.019357)**	**Infeasible (0.019357)**	**(0)**
2.75	0.005	3.37	3.37	0
2.75	0.0075	3.37	3.37	0
2.75	0.01	3.37	3.37	0
2.75	0.0125	**Infeasible (0.011287)**	**Infeasible (0.011287)**	**(0)**
2.75	0.015	**Infeasible (0.013293)**	**Infeasible (0.013293)**	**(0)**
2.75	0.0175	**Infeasible (0.016243)**	**Infeasible (0.016243)**	**(0)**
2.75	0.02	**Infeasible (0.019073)**	**Infeasible (0.019073)**	**(0)**

Table 6 Comparison between Static and Rolling-Horizon Solutions for Low-Volume Publisher

generating process for the click-probability distribution ($f(p)$) follows a `Gamma`(κ_1, κ_2) distribution. Under these conditions, a very general choice of the prior joint density (Miller 1980)

$$\pi(\kappa_1, \kappa_2) = \frac{1}{C} \frac{\kappa_2^{\rho_1 \kappa_1 - 1} \rho_2^{\kappa_1 - 1} e^{-\rho_3 \kappa_2}}{(\Gamma(\kappa_1))^{\rho_4}}$$

will form a conjugate pair, where $\rho_1, \rho_2, \rho_3, \rho_4$ are the parameters of the prior, and C is a normalizing constant. That is, the posterior density $\pi^{'}(\kappa_1, \kappa_2 | \texttt{data})$ is the same as the prior density but with new parameters (denoted by $\rho_1^{'}, \rho_2^{'}, \rho_3^{'}, \rho_4^{'}$). The above density has the property that the conditional density of κ_1 for a fixed value of κ_2 is a `Gamma` distribution, and the conditional density of κ_2 for a fixed value of κ_1 is a `Log-Concave` distribution. Both `Gamma` and `Log-Concave` are very flexible distributions (Pradhan and Kundu 2011). Hence these distributions adapt well to specific levels of a decision-maker's knowledge of the parameters, i.e., from being almost uniformed to being well-informed (Son and Oh 2006).

Next consider a set of observations $p_1, p_2, \ldots, p_k$ (denoted as $\mathbf{p}$) for the click-probability, collected by applying the Logit model to all (k) arrivals in a month. Our job is to find the posterior density $\pi^{'}(\kappa_1, \kappa_2 | \mathbf{p})$. It can be shown that the updating rules for obtaining the new set of parameters are (Miller 1980)

$$\rho_1^{'} = \rho_1 + k,$$

$$\rho_2^{'} = \rho_2 \prod_{i=1}^{k} p_i,$$

$$\rho_3^{'} = \rho_3 + \sum_{i=1}^{k} p_i,$$

$$\rho_4^{'} = \rho_4 + k.$$

Having obtained the posterior density $\pi^{'}(\kappa_1, \kappa_2 | \mathbf{p})$, we can find, using numerical search, point estimators $\hat{\kappa}_1$ and $\hat{\kappa}_2$ as the values that maximize the posterior density. Note that since we are only interested in obtaining new point estimates for the shape and scale parameters, the value of the normalizing constant need not be explicitly calculated. These new point estimates become the parameters for the click-probability distribution `Gamma`$(\hat{\kappa}_1,\ \hat{\kappa}_2)$ used to optimize the click-probability threshold for the next month.

The above process can be repeated for each month, i.e., at the beginning of each month, we could use the past month's data to update the click-probability distribution and use this updated distribution to calculate the optimal threshold to be employed for the rest of the month. The above process of updating is quite general: While practical considerations may limit the updation of parameters to a monthly exercise, there is nothing to prevent the above process being executed at the end of each new arrival.

5.8. Final Recommendations

For low-volume publishers, the static approach is recommended to avoid the danger of updating based on small samples. Also, when η is low, the static approach and the rolling-horizon approach have similar performance for any level of error. Here, it may be better to use the static approach to avoid the overhead of implementing an approach that attempts to update the threshold, but ends up not needing to since the value of η is low. For high-volume publishers and high levels of η, the rolling-horizon approach outperforms the static approach for most error levels. In the case of overestimation errors, neither approach meets the click-through-rate constraint but the rolling-horizon approach comes closer to meeting this constraint. Since overestimation errors often lead to infeasible solutions, it is better to be conservative while estimating the parameters of the click-probability distribution for a new publisher.

To start off, for a new publisher, we use one month as a guess period. Beyond the first month, we estimate the model parameters using the past data. In the initial months for a new publisher, the rolling-horizon approach is appropriate, unless the click-through-rate constraint or the traffic volume is low. Then, if the click-probability distribution parameters for the publisher stabilize, the static approach can be used. Regardless of the approach used (static or rolling-horizon), the different traffic parameters associated with a publisher (click-probability distribution parameters and traffic volume) are checked at the end of each month to see if these parameters have changed. If so, the publisher can be regarded as a new one (with a new set of parameters) from the standpoint of the approach that should be employed. The recommendations for a new publisher are summarized in Figure 5. We may consider updating the traffic parameters more frequently using a Bayesian learning process. However, more frequent updating of parameter values will create another level of noise. Hence, we choose to update traffic parameters only at the end of each month.

6. Implementation Experience at Chitika

We begin this section with some background on Chitika's existing network architecture and some essential details of the ad data flow, i.e., the process of how an ad is served. Figure 6 shows the architecture and some details of the ad data flow. Currently, Chitika delivers ads using ad-servers in 5 data centers across the country. Each data center handles the ad traffic from a specific geographical region, e.g., Midwest, South, East, and so on. A geo-balancer placed at the head of the network ensures that ad-servers in the correct geographical are contacted for ad delivery. Once the ad request goes to a data center in a particular geographical area, a load balancer within each data center ensures that no particular ad-server in the data center gets over-loaded. There are several hundred ad-servers in each data center and each ad-server in a data center typically gets about 30-40 ad-requests per second. This request originates at a script that executes on the publisher's webpage at the time the page is being rendered for the incoming visitor.

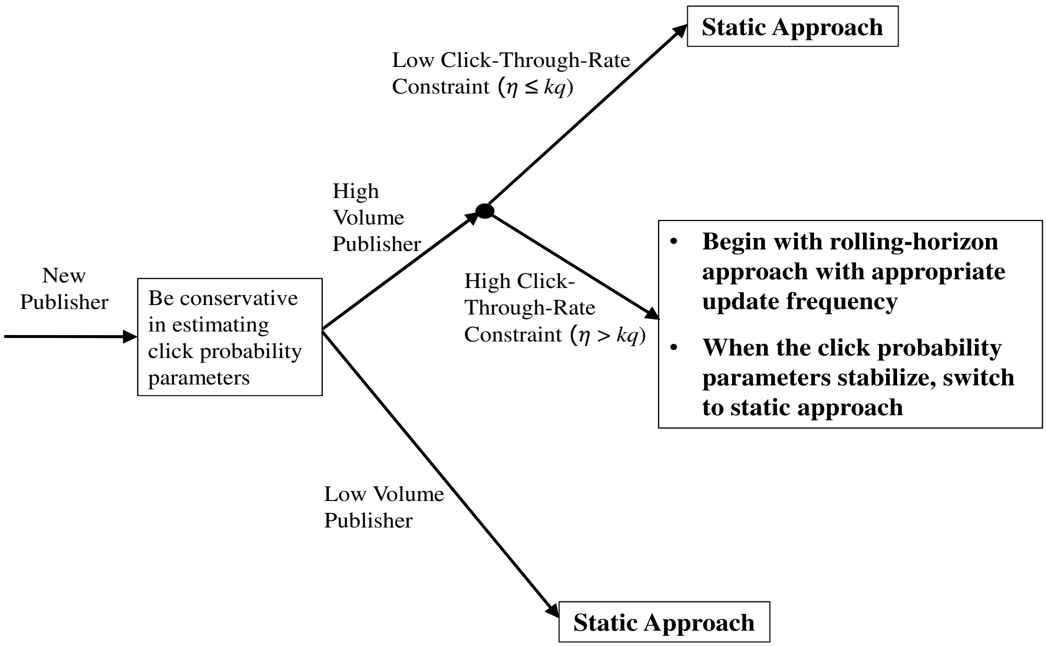

Figure 5 Final Recommendations

6.1. The Ad Data Flow

When a visitor comes to a publisher's site, a request is made for an ad to one of the ad aggregators that partner with Chitika to serve ads. This ad aggregator returns the ad as an ad-unit (in about 200 milliseconds) from one of the many advertisers that it serves. This ad is rendered on the publisher's page. Sometimes, it is necessary to get an image associated with the ad. In such cases, the ad-server obtains the image by calling an Akamai (`www.Akamai.com`) edge-server that stores the image. After displaying the ad, a log of this impression is created. This log records detailed data on the specifics of the ad shown, time shown, and other information concerning the visitor. If the visitor clicks on the ad, the ad-server logs this event. It is important to note that the ad-server that records the impression may be different from the one that records the click (although both are likely to be in the same data center). Once every hour, all the ad-servers at a data center purge their logs to a local log processor at the data center. These local log processors (one for each data center), in turn, broadcast their contents to a master log processor that collects the entire log for the past hour. At the end of each hour, it takes about 10–15 minutes to collect the log data from each data center to get a consolidated picture of the events that transpired during the last hour. Thus at the end of each hour, we must process the raw log data to match impressions with clicks. The entire process, starting from the call to Chitika by a script on the publisher's page, to the rendering of the ad on the page typically takes less than 0.5 second. If the ad rendering takes more than a second, the impression is likely to be "wasted," i.e., a click is unlikely to be generated for such a delayed ad.

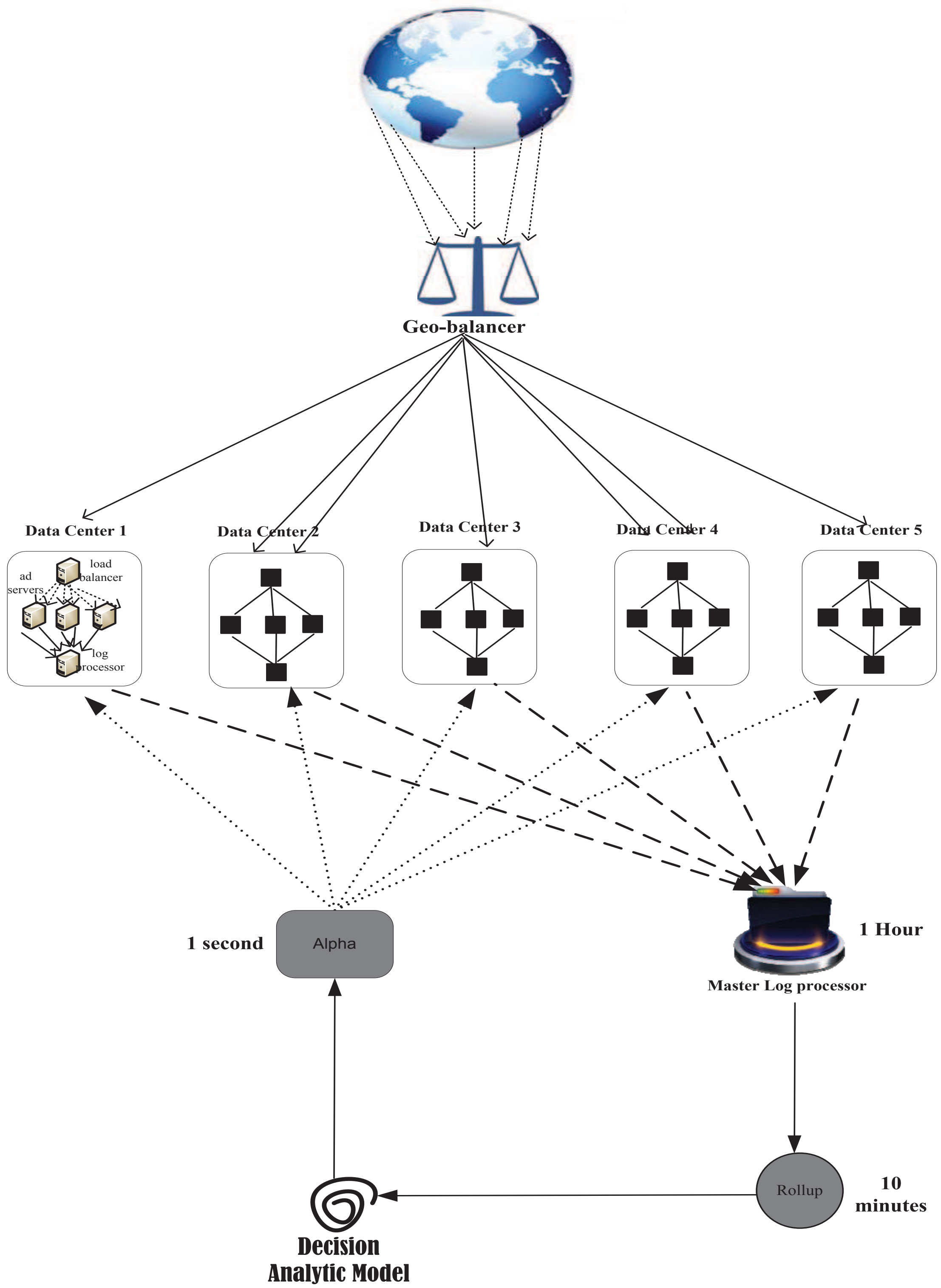

Figure 6 The Ad-Delivery Architecture at Chitika

6.2. Implementation Challenges

The above architecture poses several implementation challenges. With the new implementation, not all ad-requests result in an ad display. This is because an ad-server must first decide whether or not to call the ad-partner for an ad. In order to make this decision, the ad-server calculates the probability of a click for that specific visitor and compares this value with the threshold for ad display for that publisher. If the click-probability is higher than the threshold, the ad-sever makes a call to the ad partner to supply an ad. One of the main challenges is to decide how to implement the logic to calculate the value of the click-probability so that we can compare it against the threshold α. While the value of α stays constant for a period (say, for a day), we need to calculate the value of the click-probability (p) in real-time for every visitor. Recall that we need to calculate the click-probability using $p = L(\mathbf{X})$. This function can be quickly computed using the knowledge of $\mathbf{X}$ for every visitor. A database lookup (for the historical information in $\mathbf{X}$) is infeasible, given the time constraints. Fortunately, we can extract historical contents from the visitor's cookie and other (real-time) details from the `http` header and calculate the value of p in about 50 milliseconds. Thus the evaluation of the rule $p \geq \alpha$ is feasible given the constraints of the problem. However, we need to implement the entire logic at the front-end, implying that every ad-server in Chitika's network must replicate this logic. For the rolling-horizon approach, we must consider the time taken for replication in deciding how frequently we wish to update the value of α. Note that to find a new value of α, we need to know the actual events (impressions and clicks) that occurred in the last period. As mentioned above, Chitika processes the logs of these events only once every hour. Thus it is not feasible to update the value of α more than once every hour. For practical reasons, however, updating more frequently than once a day was deemed impractical by Chitika engineers.

We also encountered another implementation issue concerning the click-through-rate constraint. As mentioned earlier, the threshold chosen in a given period depends on the click-through-rate constraint set by the publisher. In practice, we found that publishers were not very good at setting a reasonable value for this constraint. Clearly, the constraint has revenue implications. If the constraint is set too high, we will only show ads to interested visitors and while the click-through-rate will likely be high, the total revenue from ads will suffer. To help publishers balance the trade-off between ad revenue and ad clutter, we provided publishers with a "revenue slider." The revenue slider allows the publisher to slide a button and observe the revenue impact of a candidate value of the click-through-rate constraint. At each point along the slider (representing different values of the click-through-rate constraint), the slider application shows the expected revenue that the publisher will get at the end of a month, if the current value of the click-through-rate constraint is enforced. This revenue calculation uses the specific details of the publisher's traffic and the revenue sharing contract between Chitika and the publisher. For more details, please refer to Mookerjee et al. (2012).

6.3. Benefits and Impact

The implementation began in early 2010 and initially resulted in an increase in revenue for Chitika at the rate of about $3,000 per day. Based on the data collected between March 2010 and September 2010, the total increase in revenue was estimated to be in the order of $1.2 million per year. This revenue increase came from Chitika being able to sign up more publishers under the Chitika Premium program. Chitika was also able to use its Premium program to partner with a very large ad aggregator to show ads in the United Kingdom. As part of the trial process, Chitika was asked to demonstrate a click-through-rate of 0.015, or 1.5%. Our methodology was able to provide a click-through-rate of 0.0151 or 1.51%. This accuracy won Chitika the contract and contributed to a huge revenue increase for the company.

In December 2010, Chitika offered another service called Chitika *Select*. Most of Chitika's publishers came on board to use Chitika Premium with the expectation that Chitika would show ads only to visitors coming to the site from search engines (i.e., search traffic). This was a good starting point. Although Chitika had ads for visitors who came to the site from other sources (e.g., by directly typing in the url), Chitika was not able to show these ads to such visitors as it could dilute the click-through-rate. However, with Select, Chitika offered publishers the chance to expand the usage of ads (and hence drive more revenue) with the assurance that the expanded coverage would not dilute the Premium click-through-rate by more than 25%. Without a way to control the dilution in the click-through-rate, the expanded coverage of ads could have seriously hurt the click-through-rate and run the risk losing some publishers completely. But, with our solution, Chitika was able to guarantee a certain level of click-through-rate, and hence able to give the publishers this option with assurance. The Select offering expanded usage of Chitika's service across a large percentage of the network traffic. Although with Chitika Premium, Chitika only took search traffic and collapsed the ads for non search traffic, the Select service could show ads to a much larger traffic base. The Chitika Select offering gave the firm an additional 25% boost in revenue.

In the years 2012 and 2013, Chitika developed a patented product called *Prophet* that uses the ideas developed in this paper (Kolluri et al. 2013). Because many publishers have begun to sell some of their impressions on ad exchanges,[2] it has become possible to observe a publisher's characteristics without even entering into a performance contract with the publisher. Having observed the traffic characteristics of a publisher, Chitika is able to offer the publisher an attractive, customized contract for displaying ads. This approach is similar to the marketing strategy used by credit card companies (pioneered by Capital One): an offer is made to a potential customer rather than a customer approaching the company for a credit card. This reduces the problem of adverse selection.

[2] Ad exchanges are supply-side agents that publishers use to sell their inventory of ad space on a real-time basis, i.e., ad space is auctioned one-by-one in real-time when a visitor comes to the website.

Prophet has been extremely successful and has contributed to a 15% increase in Chitika's revenue in each year following 2011. While all this success cannot be attributed to the methods developed in this paper, they have surely acted as a platform upon which new innovation has become possible. In order to isolate the impact of the proposed solution, we implemented our solution in parallel to the past practice at Chitika. Below, we present the results of our experiment. We begin with presenting the details of past practice at Chitika.

6.3.1. Chitika's Past Approach: Before our solution was devised, Chitika used a greedy approach to ensure that their solution satisfied the click-through-rate constraint. Before each arrival, a check was made to see if the click-through-rate achieved so far was above or below the required click-through-rate. When click-through-rate achieved so far was found to be above the constraint, an ad was shown to the incoming visitor. However, if the click-through-rate was below the constraint, the ad display decision was made as follows. Based on certain easily measurable attributes of the visitor (such as the number of past clicks on similar ads), each visitor was considered to be in one of two categories: *clicker* and *non-clicker*. The ad was shown to clickers but not to non-clickers. Over time, based on past click performance, a clicker could become a non-clicker and vice versa. Before the first arrival, the click-through-rate was assumed to be below the constraint.

6.3.2. Experimental Design for Head-to-Head Comparison: This experiment was conducted for one month with 30 different publishers. These publishers were selected from three different categories: (i) 10 large publishers with arrival rates around 1,000,000 per day, (ii) 10 medium publishers with arrival rates around 100,000 per day, and (iii) 10 small publishers with arrival rates around 10,000 per day. Initially, Chitika allowed us to select 5% of the traffic randomly for each of the publishers to conduct experiments. To conduct a head-to-head comparison between our proposed approach and Chitka's past approach, we decided to use our approach for half of the randomly selected traffic (i.e., 2.5% of the traffic for each publisher) and Chitika's past approach for the rest of the traffic. However, for the last category of publishers (i.e., with arrival rates around 10,000 per day), it resulted in small number of arrivals per day in each group. Therefore, we requested Chitika to experiment with 10% of the traffic for just the last category of publishers. Hence, for this category, we used our approach for 5% of the randomly selected traffic and Chitika's past approach for the rest of the traffic. The results are presented in the next subsection.

6.3.3. Implementation Results: In Table 7, we present the results of our experiment. Both approaches achieve the click-through-rate constraint for each publisher. Therefore, we do not report the final click-through-rate in the table. As shown in the table, our proposed approach improves the number of clicks significantly. The maximum percentage improvement is more than 68%. Interestingly, our proposed approach provides inferior solution for 4 publishers (out of 30 publishers).

Arrival Rate	Shape Parameter	Scale Parameter	Click-Through-Rate Constraint (η)	Number of Clicks		
				Chitika's Past Approach	Proposed Solution	Percentage Improvement
1000000	2.0	0.0045	0.011	4982	6264	25.73
1000000	2.0	0.0045	0.011	5151	6193	20.23
1000000	2.1	0.0046	0.012	5367	6591	22.81
1000000	2.1	0.0046	0.012	5198	6659	28.11
1000000	2.2	0.0048	0.013	5447	7209	32.35
1000000	2.2	0.0048	0.013	5874	6958	18.45
1000000	2.3	0.0050	0.014	6793	7782	14.56
1000000	2.3	0.0050	0.014	6808	7764	14.04
1000000	2.4	0.0050	0.015	6683	7923	18.55
1000000	2.4	0.0050	0.015	6626	8022	21.07
100000	2.0	0.0045	0.011	556	599	7.73
100000	2.0	0.0045	0.011	549	648	18.03
100000	2.1	0.0046	0.012	560	675	20.54
100000	2.1	0.0046	0.012	554	612	10.47
100000	2.2	0.0048	0.013	563	676	20.07
100000	2.2	0.0048	0.013	650	707	8.77
100000	2.3	0.0050	0.014	766	767	0.13
100000	2.3	0.0050	0.014	532	807	51.69
100000	2.4	0.0050	0.015	785	824	4.97
100000	2.4	0.0050	0.015	692	754	8.96
10000	2.0	0.0045	0.011	96	130	35.42
10000	2.0	0.0045	0.011	128	114	-10.94
10000	2.1	0.0046	0.012	144	130	-9.72
10000	2.1	0.0046	0.012	121	114	-5.79
10000	2.2	0.0048	0.013	92	155	68.48
10000	2.2	0.0048	0.013	139	127	-8.63
10000	2.3	0.0050	0.014	128	151	17.97
10000	2.3	0.0050	0.014	131	162	23.66
10000	2.4	0.0050	0.015	144	159	10.42
10000	2.4	0.0050	0.015	124	168	35.48

Table 7 Improvement in the Number of Clicks for the Experiments Conducted at Chitika

All of these publishers are from the category of publishers with arrival rates around 10,000 per day (i.e., the category with lowest arrival rates). Note that we implemented each approach for only 5% of the traffic for these publishers. Hence, for these publishers, the number of arrivals per day for each approach is just around 500. The inferior results may be attributed to the fact that the number of arrivals is too low to illustrate the true benefit of the proposed approach.

Finally, Table 8 presents the results for improvement within a category of publishers and over all the publishers. In this table, we also provide the t-statistics to assess the significance of improvement. These results indicate that the improvements are statistically significant both within each category of publishers and over all the publishers.

Arrival Rate	Number of Clicks	
	Improvement Within a Category of Publishers	Overall Improvement
1000000	1243.6**	448.7**
100000	86.2**	
10000	16.3*	

* $p < 0.05$, ** $p < 0.01$

Table 8 Overall Improvement and Statistical Significance

7. Extensions and On-going Work at Chitika

We have begun to apply the idea developed in this study to other, related problems faced by ad-networks. We briefly discuss two extensions here. The first extension that we are considering is to use a slightly modified objective function for the user profiling problem, whereas the second extension deals with including advertiser constraints in the user profiling problem. For details of some other extensions, such as *real-time media buying* and *fading ads*, the readers can refer to Mookerjee et al. (2012). We briefly describe two extensions below.

7.1. Maximizing Weighted Sum of Clicks

In this extension, we use the expected *revenue* from the clicks, instead of using the expected number of clicks. Thus, in the modified objective function, we use a weighted sum of clicks, where the weights are the revenue-per-click values associated with each click. Here, similar to the base model, the publisher first sets the click-through-rate constraint. After that, the ad-network determines the threshold for the click-probability that is a function of the revenue-per-click, which is determined by the advertiser. The threshold policy remains optimal for the new objective function considered in this problem.

7.2. Inclusion of Advertiser Constraints

In this extension, we include advertiser constraints in the user profiling problem. This problem is similar to the one described in the base model except that the solution must respect the performance constraints of both the publisher and the advertisers that advertise on the publisher's site. In addition to the publisher's constraint of exceeding a given click-through-rate, advertiser interests motivate an additional constraint, that can be referred to as a *conversion* constraint. This constraint requires that the conversion ratio be above a certain specified constant. The conversion ratio is defined as the number of conversions that are generated from clicks divided by the number of clicks. To keep advertisers happy, we require that the conversion ratio (of ads shown at the publisher's site) be above a specified fraction. To solve this problem, we plan to use the data analytics step to predict, for a given ad, both the probability of a click and the probability of a conversion. The decision to show an ad would depend on both these probabilities. In other words, we will use two thresholds (one for the click-probability and the other for the conversion ratio) to control ad display. The readers can refer to Figure 7 of Mookerjee et al. (2012) for the overall solution process.

8. Concluding Remarks

The main contribution of this study is to provide an approach to manage an on-going Internet ad campaign that substantially improves the number of clicks and the revenue earned from clicks.

The basic idea is to not show ads to every visitor, but show ads to only those visitors that have a reasonable chance of generating a click. Most publishers would like to maximize the revenue earned from ads (via clicks), but not clutter the website with too many ad impressions that do not generate a click. Thus, for the ad-network, which does not directly suffer from un-clicked ad impressions, it is necessary to optimize the revenue earned from the ad campaign while respecting a click-through-rate constraint that is supplied by the publisher. We formulate this optimization problem and find the optimal value of the decision variable (a probability threshold that governs the display of ads). Since an ad campaign usually lasts several weeks (or even months), it raises the potential to dynamically manage the campaign (i.e., change the probability threshold across periods during the planning horizon). We present a rolling-horizon approach to vary the threshold and find that it is useful when the model parameters are not known accurately.

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Electronic Companion
to
Optimizing Performance Based Internet Advertisement Campaigns

Proofs

EC.1. Proof of Proposition 1

Proposition 1. The conditional probability of a click (i.e., $\delta(\alpha)$) increases with α, i.e., $\frac{d\delta(\alpha)}{d\alpha} > 0, \alpha < 1$.

From Equation (2), we have

$$\frac{d\delta(\alpha)}{d\alpha} = \frac{-\left(\int_\alpha^1 f(p)dp\right)\alpha f(\alpha) + f(\alpha)\left(\int_\alpha^1 pf(p)dp\right)}{(\beta(\alpha))^2}. \tag{EC.1}$$

By definition, for $\alpha < 1$,

$$\int_\alpha^1 \alpha f(p)dp < \int_\alpha^1 pf(p)dp.$$

Using this inequality in Equation (EC.1), we get

$$\frac{d\delta(\alpha)}{d\alpha} > 0.$$

EC.2. Proof of Proposition 2

Proposition 2. A threshold of α, $\alpha \in [0,1)$, ensures that the click-through-rate will be greater than α, or $\delta(\alpha) > \alpha$, $\alpha \in [0,1)$.

From Equation (2), we have

$$\delta(\alpha) = \frac{\int_\alpha^1 pf(p)dp}{\int_\alpha^1 f(p)dp}. \tag{EC.2}$$

By definition, for $\alpha < 1$,

$$\int_\alpha^1 \alpha f(p)dp < \int_\alpha^1 pf(p)dp.$$

Using this inequality in Equation (EC.2), we get

$$\delta(\alpha) > \frac{\int_\alpha^1 \alpha f(p)dp}{\int_\alpha^1 f(p)dp} = \alpha.$$

■

EC.3. Proof of Proposition 3

`Proposition 3. As the number of arrivals (i.e., ` K `) approaches` ∞, $\mathbb{E}\left[\frac{\tilde{r}}{\tilde{m}}\right]$ `approaches` $\frac{\mathbb{E}[\tilde{r}]}{\mathbb{E}[\tilde{m}]}$.

For this proof, we will use the second order Taylor's expansion as an approximation for a function of interest. Later, we discuss the accuracy of this approximtion. Using Taylor's expansion for two variables, the second order expansion for $f(\tilde{r},\tilde{m})$ about the point $\theta=(\theta^r,\theta^m)$ (denoted by $g(\tilde{r},\tilde{m})$) is given by

$$\begin{aligned} g(\tilde{r},\tilde{m}) = &\ f(\theta)+f_r(\theta)(\tilde{r}-\theta^r)+f_m(\theta)(\tilde{m}-\theta^m)+ \\ &\frac{1}{2}\left[f_{rr}(\theta)(\tilde{r}-\theta^r)^2+2f_{rm}(\theta)(\tilde{r}-\theta^r)(\tilde{m}-\theta^m)+f_{mm}(\theta)(\tilde{m}-\theta^m)^2\right]. \end{aligned}$$

We set $f(\tilde{r},\tilde{m})=\frac{\tilde{r}}{\tilde{m}}$ and $\theta=(\mathbb{E}[\tilde{r}],\mathbb{E}[\tilde{m}])$. Taking expectations, the above simplifies to the following equation. Here, we use the fact that $\mathbb{E}[\tilde{r}-\mathbb{E}[\tilde{r}]]=0$ and $\mathbb{E}[\tilde{m}-\mathbb{E}[\tilde{m}]]=0$. Also, $f_{rr}=0$.

$$\begin{aligned} \mathbb{E}\left[g(\tilde{r},\tilde{m})\right] = &\ \mathbb{E}\left[\frac{\mathbb{E}[\tilde{r}]}{\mathbb{E}[\tilde{m}]}+\frac{1}{\mathbb{E}[\tilde{m}]}(\tilde{r}-\mathbb{E}[\tilde{r}])-\frac{\mathbb{E}[\tilde{r}]}{(\mathbb{E}[\tilde{m}])^2}(\tilde{m}-\mathbb{E}[\tilde{m}])+\right. \\ &\left.\frac{1}{2}\left\{0-\frac{2}{(\mathbb{E}[\tilde{m}])^2}(\tilde{r}-\mathbb{E}[\tilde{r}])(\tilde{m}-\mathbb{E}[\tilde{m}])+\frac{2\mathbb{E}[\tilde{r}]}{(\mathbb{E}[\tilde{m}])^3}(\tilde{m}-\mathbb{E}[\tilde{m}])^2\right\}\right] \\ = &\ \frac{\mathbb{E}[\tilde{r}]}{\mathbb{E}[\tilde{m}]}+\left[\frac{-\mathbb{COV}[\tilde{r},\tilde{m}]}{(\mathbb{E}[\tilde{m}])^2}+\frac{\mathbb{E}[\tilde{r}]\mathbb{V}[\tilde{m}]}{(\mathbb{E}[\tilde{m}])^3}\right]. \end{aligned} \tag{EC.3}$$

Let K be the number of arrivals, and β_i denote the probability of showing an ad to the i^{th} visitor in a solution. Below, we show that the term $\mathbb{COV}[\tilde{r},\tilde{m}]$ in Equation (EC.3) is strictly less than $\sum_{i=1}^{K}\beta_i$ for any solution. In the following analysis, δ_i denotes the conditional probability that the i^{th} visitor will click on an ad, given that an ad is shown. Further, we define r_i such that $r_i=1$ if the i^{th} visitor clicks on an ad; otherwise it is zero. Similarly, m_i is defined such that $m_i=1$ if an ad is shown to the i^{th} visitor; otherwise it is zero.

Hence,

$$\tilde{r} = r_1+r_2+\ldots r_K, \text{and}$$
$$\tilde{m} = m_1+m_2+\ldots m_K.$$

Further,

$$\mathbb{COV}[\tilde{r},\tilde{m}]=\mathbb{E}[\tilde{r}\tilde{m}]-\mathbb{E}[\tilde{r}]\mathbb{E}[\tilde{m}].$$

Here,

$$\mathbb{E}[\tilde{r}\tilde{m}] = \mathbb{E}\left[(r_1+r_2+\ldots r_K)(m_1+m_2+\ldots m_K)\right]$$

$$= \sum_{i=1}^{K}(\beta_i\delta_i) + \sum_{i=1}^{K}\sum_{j=1;j\neq i}^{K}(\beta_i\beta_j\delta_j), \text{and}$$

$$\mathbb{E}[\tilde{r}]\mathbb{E}[\tilde{m}] = \sum_{i=1}^{K}(\beta_i^2\delta_i) + \sum_{i=1}^{K}\sum_{j=1;j\neq i}^{K}(\beta_i\beta_j\delta_j).$$

Therefore,

$$\mathbb{COV}[\tilde{r},\tilde{m}] = \sum_{i=1}^{K}(\beta_i\delta_i) - \sum_{i=1}^{K}(\beta_i^2\delta_i) = \sum_{i=1}^{K}(\beta_i\delta_i)(1-\beta_i) < \sum_{i=1}^{K}\beta_i.$$

Hence,

$$\frac{\mathbb{COV}[\tilde{r},\tilde{m}]}{(\mathbb{E}[\tilde{m}])^2} < \frac{\sum_{i=1}^{K}\beta_i}{\left(\sum_{i=1}^{K}\beta_i\right)^2} = \frac{1}{\sum_{i=1}^{K}\beta_i}.$$

Thus, the above can be neglected in Equation (EC.3) for large values of K.

Also, $\mathbb{V}[\tilde{m}] = \mathbb{E}[\tilde{m}^2] - (\mathbb{E}[\tilde{m}])^2 = \sum_{i=1}^{K}\beta_i - (\sum_{i=1}^{K}\beta_i)^2 < \sum_{i=1}^{K}\beta_i$. In addition, $\mathbb{E}[\tilde{r}] = \sum_{i=1}^{K}\beta_i\delta_i < \sum_{i=1}^{K}\beta_i$. Thus, $(\mathbb{E}[\tilde{r}]\mathbb{V}[\tilde{m}]) < (\sum_{i=1}^{K}\beta_i)^2$. But, we know that $(\mathbb{E}[\tilde{m}])^3 = (\sum_{i=1}^{K}\beta_i)^3$. Hence,

$$\frac{\mathbb{E}[\tilde{r}]\mathbb{V}[\tilde{m}]}{(\mathbb{E}[\tilde{m}])^3} < \frac{1}{\sum_{i=1}^{K}\beta_i},$$

which can also be neglected in Equation (EC.3) for large values of K.

Hence, as the number of arrivals (i.e., K) approaches ∞, the approximation $\mathbb{E}[g(\tilde{r},\tilde{m})]$ for $\mathbb{E}\left[\frac{\tilde{r}}{\tilde{m}}\right]$ approaches $\frac{\mathbb{E}[\tilde{r}]}{\mathbb{E}[\tilde{m}]}$ in Equation (EC.3). Note that, in general, $\frac{\mathbb{E}[\tilde{r}]}{\mathbb{E}[\tilde{m}]}$ may be an overestimate or an underestimate of $\mathbb{E}\left[\frac{\tilde{r}}{\tilde{m}}\right]$ depending on the sign of $-\frac{\mathbb{COV}[x,y]}{(\mathbb{E}[y])^2} + \frac{\mathbb{E}[x]\mathbb{V}[y]}{(\mathbb{E}[y])^3}$ in Equation (EC.3).

We also verify the above approximation using a simulation setup to show that the maximum error is usually much less than 0.10%. As we show in the main paper, the optimal policy uses a threshold α to decide whether or not to show ads to visitors. In Figure EC.1, we show the impact of changing the threshold α on the approximation error. A higher value of α increases the error but as Figure EC.1 shows, it still stays extremely small. Another factor that affects the error is the number of arrivals (K). As K increaases, the error shrinks. We can see that when $K = 2000$, the error becomes even smaller. In most real-world publishers, the number of arrivals in a month is quite large (usually, at least 100,000 arrivals). Thus, the approximation can be used with confidence. ∎

EC.4. Proof of Proposition 4

`Proposition 4. The optimal solution to Problem P is:`

(a) If $\eta < 1$ and $\delta(0) \geq \eta$, $\alpha^* = 0$.

(b) If $\eta < 1$ and $\delta(0) < \eta$, then α^* is the solution of $\delta(\alpha) = \eta$.

(c) If $\eta = 1$, $\alpha^* = 1$.

The proof directly follows from the discussion provided in the main body of the paper. ∎

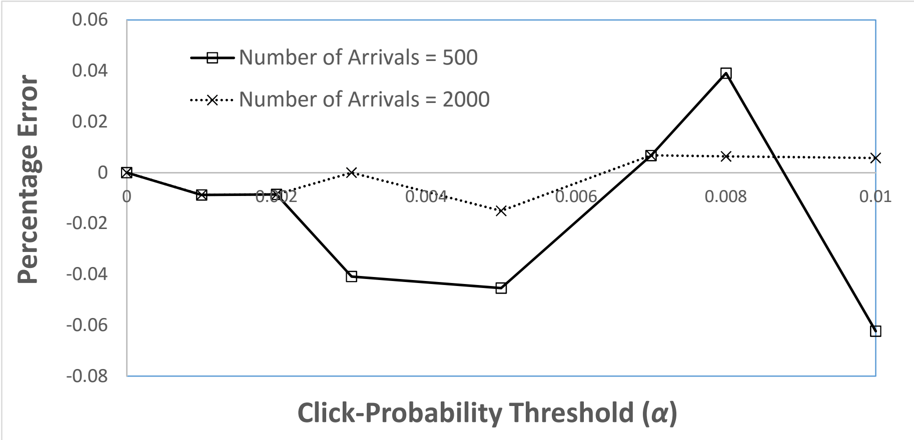

Figure EC.1 Simulation Study of the Approximation

EC.5. Intuition for the Result in Proposition 4

Consider a simple example of a problem with exactly two arrivals. For such a problem, the value of the threshold can potentially be changed after the first arrival. There are three possible states that can be reached after the first arrival (and before the second arrival). These states are (2,0,0), (2,1,0), and (2,1,1), where in the ordered triplet (i,j,l), j and l are, respectively, the numbers of impressions and clicks, just before the i^{th} arrival. Thus, for a problem with two arrivals, there are four possible thresholds, $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, where the threshold α_k corresponds to the state (i,j,l) such that $k = i + j + l$. Note that α_1 corresponds to the threshold used for the state (1,0,0), that is, the state just before the first arrival. We next discuss why the result $\alpha_k = \alpha$ makes sense for the two arrival problem.

Let β_k be the probability of an impression, and δ_k be the conditional probability of a click, associated with the threshold α_k. Using this notation, it is easy to show that the expected number of impressions is given by

$$\beta_1 + \beta_2(1-\beta_1) + \beta_1(\beta_3(1-\delta_1) + \beta_4\delta_1),$$

and the expected number of clicks is given by

$$(1-\beta_1)\beta_2\delta_2 + \beta_1(1-\delta_1)\beta_3\delta_3 + \beta_1\delta_1 + \beta_1\delta_1\beta_4\delta_4.$$

Next, using these expressions for the expected number of impressions and clicks, the click-through-rate constraint for the problem can be written as

$$\beta_1(\delta_1-\eta) + (1-\beta_1)\beta_2(\delta_2-\eta) + \beta_1(1-\delta_1)\beta_3(\delta_3-\eta) + \beta_1\delta_1\beta_4(\delta_4-\eta) \geq 0.$$

It is intuitive that we must never exceed the above constraint, i.e., it is sufficient to just meet it. This implies that the greater than or equal to sign can be replaced by an equal to sign. Further, in each of the terms $(\delta_k - \eta)$, where k = 1,2,3,4, are chosen such that they all equal to zero. Thus, $\delta_k = \eta$ is one solution, implying that all the thresholds are equal. Otherwise, if $\delta(0) > \eta$, then all the thresholds are set to zero, and once again, the thresholds are equal. Therefore, the optimal solution is static.

EC.6. Proof of Remark 1

`Remark 1. An alternate way to express the constraint in Problem P is to require that the probability of obtaining a required click-through-rate is more than a certain specified value` ω`. With this revised constraint, a static threshold policy is no longer optimal.`

As discussed earlier, we divide the planning horizon into K periods where each period corresponds to an arrival of a visitor to the publisher's website. In practice, a period may be chosen to represent any convenient epoch, such as the arrival of a given number of visitors. At the end of each period, there is the potential to change the threshold used to control the display of ads. Clearly, the most granular choice of a period is one where the threshold is potentially updated at the end of each arrival. Let S_{ijl} denote a state where the number of impressions is j and the number of clicks is l just before the i^{th} arrival, $i = 1, 2, \ldots, K$. The threshold level (used to determine whether or not to show an ad) chosen in this state is denoted by α_{ijl}. Based on these thresholds, we obtain a random number of impressions $(\tilde{m})$, and random number of clicks $(\tilde{r})$ over the planning horizon. The optimization problem is to maximize the expected number of clicks, subject to the constraint that the probability of obtaining a required click-through-rate is more than a certain specified value ω.

We use $\beta(\alpha_{ijl})$ to denote the probability that a randomly chosen impression has a click-probability greater than or equal to α_{ijl}. Also, $\delta(\alpha_{ijl})$ is the conditional probability that a click will occur for a randomly chosen impression given that the click-probability associated with the impression is greater than or equal to α_{ijl}. Let $\tilde{C}(\alpha_{ijl})$ and $\tilde{D}(\alpha_{ijl})$ be the random variables for clicks (1 for click and 0 for no click) and impressions (1 if an ad is shown and 0 if an ad is not shown), respectively, in state S_{ijl}. Then, we have

$$\mathbb{E}\left[\tilde{C}(\alpha_{ijl})\right] = \beta(\alpha_{ijl})\delta(\alpha_{ijl}), \quad \text{and}$$

$$\mathbb{E}\left[\tilde{D}(\alpha_{ijl})\right] = \beta(\alpha_{ijl}).$$

If we define $\Pi_{ijl} = \mathbb{P}[S_{ijl}]$ as the probability of state S_{ijl}, the expected number of clicks is given by

$$\mathbb{E}[\tilde{r}] = \sum_{i=1}^{K}\sum_{j=0}^{i-1}\sum_{l=0}^{j} \beta(\alpha_{ijl})\delta(\alpha_{ijl})\Pi_{ijl}.$$

The set of feasible states at the end of the horizon (i.e., after K arrivals have occurred) is $y \in Y$. A state y is feasible if $l \geq \eta j$. Also, $\Omega_y = 1$ for all feasible states, 0 otherwise. The constraint in this problem represents the condition that the probability of ending in a feasible state must be greater than or equal to ω. Hence, given that $\vec{\alpha} = < \alpha_{ijl} >$ is the click-probability threshold vector, this problem can be formally expressed as

$$\max_{\vec{\alpha}} \sum_{i=1}^{K}\sum_{j=0}^{i-1}\sum_{l=0}^{j} \beta(\alpha_{ijl})\delta(\alpha_{ijl})\Pi_{ijl},$$

subject to

$$\sum_{j=0}^{K}\sum_{l=0}^{j}\sum_{y\in Y} \Pi_{(K+1)jl}\Omega_y - \omega \geq 0, \tag{EC.4}$$

$$0 \leq \alpha_{ijl} \leq 1; \quad i = 1, 2, \ldots, K; \quad j = 0, 1, \ldots, i-1; \quad l = 0, 1, \ldots, j.$$

The Lagrangian for this problem can be written as

$$L = \sum_{i=1}^{K}\sum_{j=0}^{i-1}\sum_{l=0}^{j} \beta(\alpha_{ijl})\delta(\alpha_{ijl})\Pi_{ijl} + \varrho\left(\sum_{j=0}^{K}\sum_{l=0}^{j}\sum_{y\in Y} \Pi_{(K+1)jl}\Omega_y - \omega\right), \tag{EC.5}$$

where ϱ is the Lagrange multiplier.

In this problem, there are two kinds of states: *pure* and *mixed*. A pure state is one that always ends in either a feasible state (called *pure-feasible*) or an infeasible state (called *pure-infeasible*). A state S_{ijl} is pure-feasible if $\frac{l}{K} \geq \eta$, and is pure-infeasible if $\frac{l+(K-i+1)}{j+(K-i+1)} < \eta$. We begin by analyzing the pure state.

EC.6.1. Pure State

Here, we show that it is optimal to set $\alpha_{ijl} = 0$ for all pure states. For this, let R_{ijl} denote the set of all possible states that are reachable from state S_{ijl}. Note that these states include all the states that are one or more steps away from S_{ijl}, not just the states adjacent to S_{ijl}. We also refer to the members of R_{ijl} as the child states of S_{ijl}. We first present the following lemma.

Lemma EC.1. *For a given state S_{ijl}, the sum of the probabilities of child states is the same as the probability of state S_{ijl}. In other words, $\sum_{S_\Phi \in R_{ijl}} \mathbb{P}[S_\Phi] = \Pi_{ijl}$.*

Proof of Lemma EC.1:

Let us consider r^{th} and $(r+1)^{st}$ arrivals. If the state just before the r^{th} arrival is S_{rjl}, then there are three possible states just before the $(r+1)^{st}$ arrival: (i) $S_{(r+1)jl}$ with probability $1-$

$\beta(\alpha_{rjl})$, (ii) $S_{(r+1)(j+1)l}$ with probability $\beta(\alpha_{rjl})(1-\delta(\alpha_{rjl}))$, and (iii) $S_{(r+1)(j+1)(l+1)}$ with probability $\beta(\alpha_{rjl})\delta(\alpha_{rjl})$. Therefore, we have

$$\begin{aligned}\Pi_{(r+1)jl} &= \Pi_{rjl}(1-\beta(\alpha_{rjl})),\\ \Pi_{(r+1)(j+1)l} &= \Pi_{rjl}\beta(\alpha_{rjl})(1-\delta(\alpha_{rjl})),\\ \Pi_{(r+1)(j+1)(l+1)} &= \Pi_{rjl}\beta(\alpha_{rjl})\delta(\alpha_{rjl}).\end{aligned}$$

Addition of these probabilities gives

$$\Pi_{(r+1)jl}+\Pi_{(r+1)(j+1)l}+\Pi_{(r+1)(j+1)(l+1)}=\Pi_{rjl}.$$

In words, the sum of the probabilities of immediately reachable states from any given state is the probability of that state. Recursively, we can also show that the sum of all future state probabilities is the same as the probability of given state. ■

We now present the following lemma for all pure states.

LEMMA EC.2. *It is optimal to set* $\alpha_{ijl}=0$ *for all pure states.*

Proof of Lemma EC.2:

The values of β and δ in Equation (EC.5) depend only on the threshold value in the current state (i.e., α_{ijl}). Further, by definition, the value of Π_{ijl} does not depend on $\alpha_{i'j'l'}$, where $i'\geq i, j'\geq j, l'\geq l$ (it depends only on $\alpha_{i''j''l''}$, where $i''<i, j''\leq j, l''\leq l$). Hence, according to the result of Lemma EC.1, the *sum* of the probabilities of the ending states that are reachable from state S_{ijl} (which is equal to the probability Π_{ijl}) does not depend on the threshold values at state S_{ijl} or at any of its descendant states. Therefore, for a pure state, the choice of the threshold used at the current state (and onwards till the end of the horizon) does not have any influence on the constraint. For the objective function, however, it is best to use $\alpha_{ijl}=0$ for state S_{ijl} and all descendant states. The formal proof is as below.

From Equation (EC.5), for a pure state (feasible or infeasible), $\frac{\partial L}{\partial \alpha_{ijl}}=\Pi_{ijl}(\beta(\alpha_{ijl})\delta(\alpha_{ijl}))'$. This is because only the objective function (and not the constraint) is a function of α_{ijl}. The Kuhn-Tucker conditions require that for a solution to be optimal, it is necessary that $\alpha_{ijl}\frac{\partial L}{\partial \alpha_{ijl}}=0$. For any state S_{ijl}, $\Pi_{ijl}>0$, unless the only feasible solution to the problem is never to show ads, i.e., $\alpha_{ijl}=1$ for all states. Thus, $\frac{\partial L}{\partial \alpha_{ijl}}>0$, implying that $\alpha_{ijl}=0$ for any pure state. ■

EC.6.2. Mixed State

Coming to mixed states, let us first consider a mixed state just before the K^{th} arrival, i.e., S_{Kjl}. Further, assume that $\frac{l}{j}\geq\eta$, implying that this state is *currently* feasible. This also implies that the states $S_{(K+1)(j+1)(l+1)}$ and $S_{(K+1)jl}$ are both feasible, but the state $S_{(K+1)(j+1)l}$ is not (otherwise, the state S_{Kjl} would be pure and not mixed). Thus, from Equation (EC.5), the first-order condition is:

$$\frac{\partial L}{\partial \alpha_{ijl}} = (1+\varrho)\Pi_{ijl}(\beta(\alpha_{ijl})\delta(\alpha_{ijl}))' + \varrho(\Pi_{ijl}(1-\beta(\alpha_{ijl}))') = 0.$$

Hence,

$$\frac{(\beta(\alpha_{ijl})\delta(\alpha_{ijl}))'}{(1-\beta(\alpha_{ijl}))'} = -\frac{\varrho}{1+\varrho}.$$

The above result implies that, for all mixed states just before the K^{th} arrival that are currently feasible, the optimal choice of the threshold is a constant, and does not depend on the exact values of j and l.

For all mixed states just before the K^{th} arrival that are currently infeasible, it is easy to show that the threshold value should be set to zero. For such states, the only feasible descendant is $S_{(K+1)(j+1)(l+1)}$, and the corresponding partial derivative of the Lagrangian is

$$(1+\varrho)\Pi_{ijl}(\beta(\alpha_{ijl})\delta(\alpha_{ijl}))'.$$

Since the above quantity is strictly negative, the necessary conditions imply that α_{Kjl} must be set to zero for all mixed states that are currently infeasible.

The arguments above clearly show that the optimal policy for this problem is not static, i.e., the threshold values are not the same for all the states.

EC.6.3. Solution Methodology

To solve the problem completely, we need to use a backward solution procedure. First, as discussed earlier, the threshold value at a given state S_{ijl} does not depend on the probability of the state, Π_{ijl}, implying that the threshold values used before the current state do not matter. On the other hand, the threshold values used in future states do matter for the choice of the threshold at the current state. Therefore, if we can solve for the optimal threshold values at the K^{th} arrival, we can use these values to solve for the threshold values at the $(K-1)^{th}$ arrival, and so on. For any mixed state, we need to identify the values of the threshold that will be used in all descendant states that lie on a feasible path, i.e., a path that ends in a feasible state. Since we are working backwards, the future thresholds have already been chosen. Thus, the value of the threshold chosen at a particular stage will only influence the probabilities of descendants that lie on a feasible path.

For any mixed state, we can write

$$(1+\varrho)\Pi_{ijl}(\beta(\alpha_{ijl})\delta(\alpha_{ijl}))' + \varrho\Pi_{ijl}(1-\beta(\alpha_{ijl}))' c_{ijl} = 0, \quad \text{or}$$

$$(1+\varrho)\Pi_{ijl}(\beta(\alpha_{ijl})\delta(\alpha_{ijl}))' c_{ijl} + \varrho\Pi_{ijl}(1-\beta(\alpha_{ijl}) + \beta(\alpha_{ijl})(1-\delta(\alpha_{ijl})))' d_{ijl} = 0.$$

The values of the constants c_{ijl} and d_{ijl} do not depend on α_{ijl}. However, it is clear that the optimal choice of α_{ijl} does depend on the particular state S_{ijl} via these constants. ■

EC.7. Proof of Proposition 5

Proposition 5. In presence of the wrong shape parameter, the performance of the static approach is described as follows:

(a) The static approach provides an optimal solution when either (i) $\eta \leq \min[k'q, kq]$, or (ii) $k = k'$.

(b) The static approach provides a sub-optimal solution when either (i) $kq < \eta \leq k'q$, or (ii) $kq < k'q \leq \eta$.

(c) The static approach provides an infeasible solution when either (i) $k'q < \eta \leq kq$, or (ii) $k'q < kq \leq \eta$.

According to Proposition 4, when $\eta < 1$ and $\delta(0) \geq \eta$, the optimal value is $\alpha = 0$ (i.e., the ad is shown to all the visitors). Note that $\delta(0) = \mathbb{E}[\tilde{p}]$. For a Gamma distribution, the correct value of $\mathbb{E}[\tilde{p}]$ is $k'q$, where k' is the correct shape parameter and q is the scale parameter. Therefore, when there is no noise, the static approach chooses $\alpha = 0$ for $\eta \leq k'q$. Further, when the shape parameter is underestimated, the static approach still chooses $\alpha = 0$ when $\eta \leq kq$. Hence, when $\eta \leq \min[k'q, kq]$, the static approach provides an optimal solution. Also, when there is no noise (i.e., $k = k'$), the static approach provides an optimal solution (according to Proposition 4).

Let us now consider part (b) of the proposition. When $\eta \leq k'q$ and $\eta > kq$, the static approach chooses $\alpha > 0$, whereas the correct value of α (based on the correct shape parameter) is zero. From Proposition 1, the conditional probability of a click increases with α. Hence, the static approach provides a feasible solution. However, clearly, the expected number of clicks decreases with α. Hence, for $kq < \eta \leq k'q$, the static approach provides a sub-optimal solution. When $kq < k'q \leq \eta$, both the optimal solution and the static approach choose $\alpha > 0$. However, the smallest value of α that satisfies $\delta(\alpha) = \eta$ is higher in the static solution compared to that in the optimal solution. Hence, according to Proposition 4, the static solution chooses a higher value of α than the optimal one based on the correct value of the shape parameter. Therefore, based on an argument similar to the above, the static approach provides a sub-optimal solution.

Finally, in part (c) of the proposition, when $k'q < kq$ and $\eta > k'q$, the smallest value of α that satisfies $\delta(\alpha) = \eta$ is lower in the static solution than what it is in the optimal solution. Hence, according to Proposition 4, the static solution choose a lower value of α than the optimal one based on the correct value of the shape parameter. Since the conditional probability of a click increases with α (see Proposition 1), the click-through-rate in the static solution is lower than that in the optimal solution. Further, as shown in Proposition 4, the optimal solution chooses α such that it just meets the click-though-rate constraint. Hence, when $k'q < kq$ and $\eta > k'q$ (that is equivalent to the conditions presented in part (c)), the static approach provides an infeasible solution. ■

EC.8. Proof of Remark 2

`Remark 2. The simulation converges faster when the arrival rate (i.e.,` λ`) is higher.`

Let $\tilde{r}_i$ be the random variable for the number of clicks in i^{th} replication. We first consider a scenario where the click-probability threshold in replication i remains the same in each period of that replication. In this scenario, $\tilde{r}_i$ is the sum of independent and identically distributed $\tilde{m}_i$ Bernoulli trials, where $\tilde{m}_i$ is the random variable for the number of impressions in i^{th} replication. Let the probability of success in each of these Bernoulli trials (which is the conditional probability of click given that the ad is shown) be δ. On the other hand, $\tilde{m}_i$ is a binomial random variable with parameters λ and β (i.e., $\tilde{m}_i \stackrel{d}{=} B(\lambda, \beta)$), where β is the probability of impression. Then, $\tilde{r}_i$ is a compound random variable with (Ross 2009, pp. 119-120, 166-167)

$$\mathbb{E}\left[\tilde{r}_i\right] = \delta\mathbb{E}\left[\tilde{m}_i\right] = \lambda\delta\beta, \quad \text{and}$$
$$\mathbb{V}\left[\tilde{r}_i\right] = \delta(1-\delta)\mathbb{E}\left[\tilde{m}_i\right] + \delta^2\mathbb{V}\left[\tilde{m}_i\right] = \lambda\delta\beta\left[(1-\delta)+\delta(1-\beta)\right].$$

For large n, we have

$$\bar{r}(n) \approx \mathbb{E}\left[\tilde{r}_i\right] = \lambda\delta\beta, \quad \text{and} \tag{EC.6}$$
$$S^2(n) \approx \mathbb{V}\left[\tilde{r}_i\right] = \lambda\delta\beta\left[(1-\delta)+\delta(1-\beta)\right], \tag{EC.7}$$

where $\bar{r}(n)$ and $S^2(n)$ are defined in Equations (7) and (8), respectively. Since δ and β are independent of λ, Equations (EC.6) and (EC.7) show that both $\bar{r}(n)$ and $S^2(n)$ are functions of λ. Hence, using the expression in Equation (10), it is easy to observe that the numerator of the left-side in (11) is a function of $\sqrt{\lambda}$, whereas the denominator is a function of λ. Hence, the left-side of the ratio in (11) decreases in the arrival rate λ. Since the ratio also decreases with n (the number of replications), the inequality satisfies for a lower value of n when λ is higher. Therefore, the simulation converges faster for higher value of λ. Using the similar arguments, we can show that the simulation converges faster for higher value of λ even when the click-probability threshold in a replication does not remain same across periods. ■

References

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