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Chapter 1

Introduction

We study the pricing of cardinality bundles, where firms set prices that depend only on the size of the purchased bundle. The cardinality bundling (CB) problem we study was originally proposed by Hitt and Chen (2005) and it involves consumers having a specific preference structure called Spence-Mirrlees Single Crossing Property (SCP).

In Chapter 2, we show that the optimal prices to the problem can be obtained in strongly polynomial time. The solution approach we developed is useful in developing an algorithm to solve the quantity-discount problem proposed by Spence (1980).

In Chapter 3, we studies the pricing for cardinality bundles (CB) when bundling involves complex costs. We first extend the existing CB model to allow fixed costs in adding additional bundles. We show that CB problem with fixed costs can be solved as a shortest-path problem. We then extend the CB model in another way to solve CB problem with submodular cost structure. Such analysis is especially useful when there exists economies of scale in production.

The existing analytical framework lacks sub-additivity constraints on bundle pricing, which limits its application in reality. In Chapter 4, we solve the CB problem with additional constraints on bundle prices. We first study the CB problem with marginal decreasing prices and prove that it is a shortest-path problem. Second, we propose a dynamic programming algorithm to solve the CB problem with unit decreasing prices. Third, we analyze the CB problem with sub-additive prices and convert its MINLP formulation to a mixed-integer programming (MIP) one. Finally, we provide analytical and numerical analysis on the gaps between different CB models.

Chapter 2

Cardinality Bundling with Spence-Mirrlees Reservation Prices

2.1 Introduction

Bundling and its benefits have been studied extensively in the literature. For example, Bakos and Brynjolfsson (1999) show that when products are synergistic, offering bundles of products can yield higher profits than selling them separately. The earliest work on bundling (e.g., Stigler, 1963, Adams and Yellen, 1976, McAfee et al., 1989) focused on *mixed bundling*, wherein every combination of goods is sold at a possibly different price. However, because the number of combinations quickly increases with the number of goods, the pricing problem becomes intractable except for a small number of goods (Hanson and Martin, 1990). So, alternate bundling schemes – such as component pricing, where only the components are sold; or pure bundling, where only the bundle is sold – have also been studied and deployed. The focus of this work is to study another bundling scheme called *cardinality bundling* or, in short, CB.

In CB, bundles of equal cardinality or size are sold at the same price. That is, for a firm that sells J goods, consumer may purchase any one good for a listed price, a bundle of any two goods for a different price, and so on and so forth. In contrast to mixed bundling (which requires pricing $2^J - 1$ bundles), CB only requires prices for J bundles. Perhaps because of the simplicity of the pricing scheme, CB has been adopted in practice. Pricing for theme parks within entertainment complexes such as Disney World are based on CB. Consumers can purchase multi-day (2, 3, 4 or 5 day) passes and can choose to visit any of the four theme parks each day. Similarly, Eastlink cable TV allows its consumers to choose their channel combinations within the cardinality bundles (12 or 20 channels) purchased.

The current literature on CB is relatively sparse and we review it briefly here. Most relevant to the current paper is Hitt and Chen (2005), where they study the pricing of cardinality bundles assuming that each consumer is restricted to buy at most one bundle. They explore conditions under which CB can attain the same profit as mixed bundling. Further assuming that consumers' reservation price satisfy Spence-Mirrlees Single Crossing Property (SCP), they propose and analyze a readily computable pricing strategy. Wu et al. (2008) also restrict the consumer to purchase at most one bundle and seek to solve the CB pricing problem as a nonlinear mixed-integer program. They use Lagrangian relaxation, subgradient ascent, and heuristic methods to derive bounds for the problem. Chu et al. (2011) consider a CB model where unit prices for bundles decrease with increasing size. They use computations and real data to argue that profit from their CB model is almost the same as that from mixed bundling.

We begin by considering the model and the proposed pricing strategy of Hitt and Chen (2005) for cardinality bundles assuming that reservation prices follow SCP. We show that the optimal prices can be obtained in polynomial time, by solving a linear programming (LP) problem. In contrast, the techniques proposed in Hitt and Chen (2005) may not generate optimal prices. The LP reformulation provides many insights into cardinality bundling. It paves the way for developing useful approximation schemes for the continuous case (see Spence (1980) and Section 2.3), allows us to extend our analysis to models with complex cost structures, such as fixed costs for bundle setup or variable costs with economies of scale (as discussed in Chapter 3), and reveals valid inequalities that help determine prices that disincentivize consumers from purchasing more than one bundle .

2.2 CB Discrete Case: Model & Analysis

A customized cardinality bundling strategy models a situation where a vendor offers a menu of products that may be purchased in a bundle, whose price is determined by its size. The consumer is free to choose any products as long as the number of goods she chooses matches the bundle size for which she has paid. This model was originally proposed by Hitt and Chen (2005), where they assume that the consumers can be ordered such that a consumer of higher type not only assigns a higher value to bundles of a given size but also derives higher marginal value from increasing the bundle size. When the consumers can be ordered this way, their reservation prices are said to satisfy the Spence-Mirrlees Single Crossing Property (SCP).

In this section, we consider the cardinality bundling problem, which is modeled to optimally choose the sizes and prices of the bundles a vendor should offer in the market. Our basic model is the same as that in Hitt and Chen (2005) and we review it here for the sake of completeness. Consider a vendor who sells J products and assume that there are I consumers in the market. In the following, we denote the bundle of size j as Bundle j. We assume WLOG that all bundles, $1, \dots, J$ are offered in the market and the vendor decides their prices. We denote the price of Bundle j as p_j. Obviously, the consumer does not pay anything for Bundle 0, whose price is therefore fixed at 0. We assume that the cost of the Bundle j for vendor is c_j and that the total cost to the vendor is the sum of the costs for all the bundles sold. Clearly, c_0 is 0. The model makes a reasonable assumption that a consumer's willingness-to-pay (WTP) is non-decreasing with the bundle size,[1] which would be trivially true if extra units can be freely

[1] Hitt and Chen (2005) imposes WTP for each consumer to be concave in j, which we relax in our model.

disposed. The model further assumes that each consumer can purchase at most one bundle.

Let $w_{ij} \geq 0$ denote the WTP of Consumer i for Bundle j. For every i, we set w_{i0} to zero to denote that consumers, who do not purchase anything, do not derive any value out of the vendor's products. Since WTPs are non-decreasing with bundle size, $w_{ij} \geq w_{ij'}$ for $j \geq j'$. Since the choice of the bundle rests with the consumer, if Consumer i purchases Bundle j_i, this bundle must maximize her consumer surplus, *i.e.*, $j_i \in \arg\max_j \{w_{ij} - p_j\}$. Let J_i be the set of bundles Consumer i prefers with price vector p. If $|J_i| > 1$, we assume that Consumer i purchases a Bundle j_i that belongs to $\arg\max_j \{p_j - c_j \mid j \in J_i\}$, *i.e.*, the surplus-maximizing bundle that yields the most profit to the vendor. This assumption is typical in the literature and is without loss of generality.[2]

Let x_{ij} be 1 if Consumer $i \in \{1, 2, \ldots, I\}$ buys Bundle $j \in \{0, 1, 2, \ldots, J\}$ and 0 otherwise. Then, CBP can be formulated as follows (see Hitt and Chen, 2005):

$$\text{CBP1}: \quad \underset{x_{ij}, p_j}{\text{Max}} \quad \sum_{i=1}^{I} \sum_{j=0}^{J} x_{ij}(p_j - c_j)$$

$$\text{s.t.} \quad \sum_{j'=0}^{J} (w_{ij'} - p_{j'}) x_{ij'} \geq w_{ij} - p_j \ \forall i, \ \forall j \tag{2.1}$$

$$\sum_{j=0}^{J} x_{ij} = 1 \ \forall i \tag{2.2}$$

$$p_0 = 0 \tag{2.3}$$

$$x_{ij} \in \{0, 1\} \ \forall i, \ \forall j. \tag{2.4}$$

[2]To see this, let $J'(j) = \{j' \mid p_{j'} - c_{j'} < p_j - c_j\}$ be the set of bundles that provides less profit to vendor than j. Observe that since the number of consumers and bundles is finite, there exists an $\epsilon > 0$ such that even if the price of a bundle that a consumer does not prefer is reduced by $J\epsilon$, the consumer continues to prefer the bundles in J_i after the change. Now, consider a new pricing scheme p', where the price of Bundle j is set to $p'_j = p_j - |J'(j)|\epsilon$. Then, it is easy to verify that, when the prices are p', Consumer i prefers the Bundle $j_i \in \arg\max\{p_j - c_j \mid j \in J_i\}$ over other bundles in J_i and, since $|J'(j)| < J$, this preference is also over bundles not in J_i. Further, the vendor does not lose more than $JI\epsilon$ in the profit when he prices the bundles using p' instead of p. Since ϵ can be chosen to be arbitrarily small, this yields a sequence of solutions for which vendor's profit converges to the one obtained under our assumption.

Let (x^*, p^*) be a solution that generates the maximum profit for the vendor. Assuming (3.2), Constraints (3.1) enforce incentive compatability (IC) and individual rationality (IR) for Consumer i. The left hand side models the consumer surplus from the purchase decision and the right hand side models the consumer surplus from the purchase of alternate bundles. The case with $j = 0$ ensures that consumer only purchases bundles with non-negative surplus. Constraints (3.2) enforce that each consumer purchases only one bundle. Observe that $\mathrm{CBP1}$ is a mixed integer nonlinear program (MINLP) since the price vector p_j and consumer decisions x_{ij} are variables and their products appear in the objective and in Constraint (3.1).

Like in other nonlinear pricing problems, Hitt and Chen (2005) assume that consumer valuations satisfy the Spence-Mirrlees Single Crossing Property (SCP) (see Spence, 1980). We also make the same assumption, which imposes the following ordering on the consumers' WTP for the bundles:

$$w_{ij} \geq w_{i'j} \; \forall i > i', \tag{2.5}$$

$$w_{ij} - w_{ij'} \geq w_{i'j} - w_{i'j'} \; \forall i > i', \; \forall j > j'. \tag{2.6}$$

The interpretation of these conditions is straightforward. A consumer with a higher index has a (weakly) higher WTP for any bundle. Also, the WTP exhibits increasing differences, *i.e.*, as bundle size increases, the WTP for a higher-indexed consumer increases more rapidly than the WTP for a lower-indexed consumer. Essentially, this assumption states that consumers can be ordered by types, with higher type consumers valuing the products and marginal changes in bundle sizes more than the lower type ones. Before we develop an efficient solution for this problem, we review the currently available approaches using examples.

Example 1 *Consider a scenario with $I = 4$ consumers, $J = 4$ bundle sizes, and costs $c_j = 0$*

Table 2.1: Willingness-to-pay for Example 1

Bundle size	Consumers' WTP I_1	I_2	I_3	I_4
0	0	0	0	0
1	26	36	58	120
2	47	62	91	180
3	58	77	113	221
4	62	83	123	240

for all j. Suppose the WTP for the consumers are as given in Table 3.1. It can be verified easily that they satisfy SCP. We use BARON (Tawarmalani and Sahinidis, 2002) to solve the MINLP formulation of CBP1. *(Note that BARON guarantees that it finds the global optimal solution at termination.) The optimal solution thus found is to set $p_1^* = p_2^* = 47$, $p_3^* = 62$, and $p_4^* = 72$. It is easy to check that, with these prices, Consumer 1, 2, 3, and 4 buy Bundles 2, 3, 4, and 4 respectively. The optimal profit for the vendor is 253.*[3]

We now make a small change to the setting of Example 1 and illustrate that the optimal assignment for a consumer depends on the WTP of all other consumers.

Example 2 *In the setting of Example 1, change w_{41} from 120 to 100, so that WTPs still satisfy SCP. If* CBP1 *is now solved using BARON, the optimal solution assigns Consumer 1 to Bundle*

[3]Result 3 in Hitt and Chen (2005) claims that the following approach optimally solves CBP1, which we show later isn't always the case. Consumer i is assigned to the largest bundle size j that satisfies the following condition:

$$(I - i + 1)(w_{ij} - w_{i,j-1}) - (I - i)(w_{i+1,j} - w_{i+1,j-1}) \geq c_j - c_{j-1}. \quad (2.7)$$

We remark that, when Consumer i is assigned a bundle, the WTP of consumers other than i and $i+1$ are ignored. Here, the right hand side is 0 since we assume $c_{j'} = 0$ for all $1 \leq j' \leq J$. The left hand side values are shown in Table 2.2.

Bundle size	LHS values I_1	I_2	I_3	I_4
0				
1	-4	-8	16	120
2	6	12	-14	60
3	-1	1	3	41
4	-2	-2	1	19

Table 2.2: Left hand side values of Equation (2.7)

For Example 1, the above approach yields the same solution as the optimal solution found earlier using BARON.

0 *yielding a profit of* 256.[4] *There is no optimal allocation that assigns Bundle* 2 *to Consumer* 1.[5] *Any allocation that ignores the WTP of Consumer* 4 *while allocating bundle to Consumer* 1 *will thus not yield optimal profit.* [6,7]

The only available approaches to solve $\mathrm{CBP1}$ use either an MINLP solver or a MIP solver on a linearization of $\mathrm{CBP1}$ that does not use a global solver on $\mathrm{CBP1}$ directly. The MINLP-/MIP-based approach is, however, not amenable to comparative statics because global optimality certificates are typically neither small nor easy to obtain. In this section, we develop an alternate solution approach that is efficient, guarantees optimality, and is amenable to comparative statics.

2.2.1 Properties of the Optimal Solution

First, we identify some properties of the optimal solution.[8]

Proposition 3 *There exists an optimal pricing scheme that is nondecreasing with bundle size.*

Proposition 4 *There exists an optimal solution to* $\mathrm{CBP1}$ *that satisfies:*

$$\sum_{j'=j}^{J} x_{i+1j'} \geq \sum_{j'=j}^{J} x_{i,j'} \; i = 1, \ldots, I-1, \; \forall j. \tag{2.8}$$

That is, there exists an optimal solution where the mapping from consumer types to bundle sizes is non-decreasing, i.e., for any $i < I$*, if Consumer* i *buys Bundle* j*, then Consumer* $i+1$ *buys a Bundle* j' *such that* $j' \geq j$*. Further, for any given price vector, there exists a feasible allocation of bundle sizes to consumer types that is non-decreasing.*

[4] The optimal assignment of Consumer 1, 2, 3, and 4 is to Bundles 0, 0, 1, and 4 respectively. The corresponding prices are $p_1^* = 58$ and $p_2^* = p_3^* = p_4^* = 198$.

[5] In fact, if Consumer 1 is restricted to purchase Bundle 2, the vendor cannot obtain a profit more than 253.

[6] Hitt and Chen (2005) claims that it is optimal to assign Consumer 1 to Bundle 2 even in this case. This is so, because for $i = 1$, Equation (2.7) is independent of w_{41}. However, as shown above, this is not an optimal assignment.

[7] In the proof of Result 3, Hitt and Chen (2005) modify the procedure when higher type consumers do not buy larger sized bundles. This modification does not apply here.

[8] All the proofs are provided in the appendix.

Proposition 5 *There exists an optimal pricing scheme such that if two bundle sizes j and j' are bought by some consumers and $j' > j$ then $p_{j'} - c_{j'} > p_j - c_j$.*

Proposition 6 *Among the consumers purchasing a non-zero bundle size, the lowest indexed one is charged at her WTP in every optimal solution.*

Proposition 4 is particularly interesting, since it provides redundant, yet rather important, constraints that facilitate the solution of $\mathrm{CBP1}$. Further, Proposition 4 applies to other bundling problems where WTPs follow SCP, including those where consumers may purchase more than one bundle (Kannan et al., 2014a). Propositions 3, 4, and 5 imply that prices are higher for larger-sized bundles purchased; the higher type consumers purchase weakly larger-sized bundles; and the profits also increase with the purchased bundle sizes.

2.2.2 A Solution Approach

In this section, we refromulate $\mathrm{CBP1}$ so as to develop a solution approach. A key step in reformulating the problem is that optimal profit satisfies a substructure optimality condition that is totally unimodular. Then, we show that there exists a simple approach to solve the dual of the reformulation.

Reformulating the MINLP to a 0-1 IP

We first provide some intuition into what makes it possible to solve $\mathrm{CBP1}$ quickly. First, assume that the vendor fixes a certain bundle size that the first consumer will purchase. Since the first consumer must purchase one of Bundles $0, \ldots, J$, this yields $J + 1$ problems for the vendor to solve. The key property that enables the vendor to solve the problem is that once the first consumer is allocated Bundle j, the remaining problem can be solved by solving a smaller cardinality bundling problem, *i.e.*, one which has Consumers $2, \ldots, I$ and Bundles $j, \ldots, J$. This subproblem can then be solved recursively using the same technique. Before we provide a formal proof of our algorithm, we build some intuition into the problem structure.

Consider the cardinality bundling problem where the vendor only considers Consumers $i', \ldots, I$ and prices the bundles so that each of these consumers buys one of the Bundles $j', \ldots, J$. To accomplish this, by Proposition 4, it suffices to restrict i' to purchase a bundle of size at least j' and to remove Consumers $1, \ldots, i'-1$. More generally, assume that the vendor wishes to ensure that i' buys one of the options from a set of bundle sizes, say J'. Then, the corresponding problem can be formulated by adding the constraint, $\sum_{j \in J'} x_{i'j} = 1$, to $\mathrm{CBP1}$. We denote this problem as CBP$(i', j' \mid J')$ and the corresponding optimal profit as $\Pi^{\text{CBP}(i',j'|J')}$. Obviously, $\Pi^{\text{CBP}(i',j'|\{j',\ldots,J\})} = \max_{j \geq j'} \Pi^{\text{CBP}(i',j'|\{j\})}$.[9] Therefore, it suffices to find a way to solve CBP$(i', j' \mid \{j'\})$, whose solution can in turn be obtained by solving CBP$(i'+1, j' \mid \{j', \ldots, J\})$. As it turns out, this is because the purchasing decision of Consumers $i'+1, \ldots, I$ are the same in the two problems. If we denote the set $\{j', \ldots, J\}$ as $j'_{\geq}$:

$$\Pi^{\text{CBP}(i',j'|\{j'\})} = \Pi^{\text{CBP}(i'+1,j'|j'_{\geq})} + \underbrace{(w_{i'j'} - c_{j'})}_{\text{sale of } j' \text{ to } i'} + \underbrace{(I - i')(w_{i'+1j'} - w_{i'j'})}_{\text{restrictions on prices}}. \tag{2.9}$$

The first adjustment is because of the revenue and cost from selling j' to i' and the second is because the price of Bundle j' is constrained to the WTP of Consumer i' in CBP$(i', j' \mid \{j'\})$ whereas it is constrained to the WTP of Consumer $i'+1$ in CBP$(i'+1, j' \mid j'_{\geq})$. In order to make the result also apply to the case when $i' = I$, we define $w_{I+1j} = w_{Ij}$. To capture this difference succinctly, we let $v_{i'j'}$ denote $w_{i'j'} - (I - i')(w_{i'+1j'} - w_{i'j'})$ and rewrite (2.9) as: $\Pi^{\text{CBP}(i',j'|\{j'\})} = \Pi^{\text{CBP}(i'+1,j'|j'_{\geq})} + (v_{i'j'} - c_{j'})$.

Now, we formally show that the cardinality bundling problem can be linearized into a 0-1 integer program using the above notation.

[9] Further, by Proposition 4, $\Pi^{\text{CBP}(i',j'|\{j\})} = \Pi^{\text{CBP}(i',j|\{j\})}$ because if i purchases j, then every higher type consumer purchases a bundle j or, higher and $j \geq j'$.

Proposition 7 CBP1 *can be reformulated as the following 0-1 integer linear problem:*

$$\text{CBP2} : \text{Max}_{x_{ij}} \left\{ \sum_{i=1}^{I} \sum_{j=0}^{J} (v_{ij} - c_j) x_{ij} \;\middle|\; (3.2), (3.4), (3.7) \right\}.$$

Let x^* be an optimal solution to CBP2. *Let $\{i_0, \dots, i_k\}$ be the lowest type consumers that purchase a certain bundle size and, for any j, let $r'(j) = \arg\min_r \left\{ i_r \mid \sum_{j'=j}^{J} x^*_{ij'} = 1 \right\}$. If there is no feasible solution, set $p_j = w_{IJ} + \epsilon$ for an arbitrary $\epsilon > 0$. Otherwise, let $j(i) = \sum_{j=0}^{J} j x_{ij}$ and*

$$p_j = w_{i_0 j(i_0)} + \sum_{r=1}^{r'(j)} \left(w_{i_r j(i_r)} - w_{i_r j(i_{r-1})} \right). \tag{2.10}$$

Converting CBP1 into CBP2 is possible because $\sum_{i=1}^{I} \sum_{j=0}^{J} v_{ij} x_{ij}$ captures the total revenue for any feasible x_{ij}. Thus, v_{ij} is the incremental revenue from selling Bundle j to Consumer i.

We return to the setting of Example 2 to illustrate the application of Proposition 7 and compute the maximum profit for the vendor in this case. Table 2.3 shows v_{ij} values for Example 2. So, to compute the profit, the appropriate v_{ij} values are summed up. For example, if a vendor tries to serve Consumers 1, 2, 3, 4 with Bundles 1, 2, 3, 4 respectively, then the total vendor profit is $v_{11} + v_{22} + v_{33} + v_{44} = 245$. The maximum profit is the summation of v_{ij} that yields the maximum value and is such that x_{ij} satisfy Constraints (3.2), (3.4), and (3.7). In particular, this implies that the only admissible strategies are such that higher type consumers are served larger-sized bundles. In this case, the maximum profit evaluates to $v_{1,0} + v_{2,0} + v_{3,1} + v_{4,4} = 256$.

In fact, CBP2 can be solved without the binary restrictions (3.4) because its constraint matrix is totally unimodular.

Table 2.3: Computing v_{ij} for Example 2

Bundle size	v_{ij} I_1	I_2	I_3	I_4
0	0	0	0	0
1	-4	-8	16	100
2	2	4	2	180
3	1	5	5	221
4	-1	3	6	240

Proposition 8 *The constraint matrix of* CBP2 *is totally unimodular.*

Since the constraint matrix of CBP2 is totally unimodular, we can relax its binary restrictions.

Linear Program

Next, we reformulate CBP2 as a linear program (LP).

Proposition 9 *Let* $a_{ij} = \sum_{j'=j}^{J} x_{i+1,j'} - \sum_{j'=j}^{J} x_{ij'}$*, where* $x_{I+1,j}, \forall j \neq J$ *is understood to be* 0 *and* $x_{I+1,J}$ *is understood to be* 1*. Then,* CBP2 *is equivalent to the following* CBP2a*:*

$$\text{CBP2a}: \quad \underset{x_{ij},a_{ij}}{\textit{Max}} \quad \sum_{i=1}^{I}\sum_{j=0}^{J}(v_{ij} - c_j)x_{ij}$$

$$\textit{s.t.} \quad a_{ij} - a_{i,j+1} + x_{ij} - x_{i+1,j} = 0 \; \forall (i,j) \neq (I,J) \tag{2.11}$$

$$a_{IJ} + x_{IJ} = 1 \tag{2.12}$$

$$a_{iJ+1} = 0 \; \forall i \tag{2.13}$$

$$a_{ij} \geq 0; \; x_{ij} \geq 0 \; \forall i, \; \forall j. \tag{2.14}$$

Recall that the original cardinality bundling problem appeared to be an MINLP problem, which has now been transformed into an LP, CBP2a. Therefore, we are now able to draw upon the general comparative static results from the LP literature and apply them to the CBP context to generate managerial insights. In the following subsection, we develop a few related insights.

However, before we proceed, we formally show next that $\mathrm{CBP2a}$ can be solved "fast."

Theorem 10 $\mathrm{CBP2d}$ *is the dual of* $\mathrm{CBP2a}$ *and can be solved within* $O(IJ)$ *time.*

$$\begin{aligned} \mathrm{CBP2d}: \quad & \underset{l_{ij}}{Min} \quad l_{I,J} \\ & s.t. \quad l_{ij} \geq l_{ij-1} \; i = 0, \ldots, I; \; j = 1, \ldots, J && (2.15) \\ & \qquad l_{ij} \geq l_{i-1j} + v_{ij} - c_j \; i = 1, \ldots, I; \; j = 0, \ldots, J && (2.16) \\ & \qquad l_{00} = 0. \end{aligned}$$

2.2.3 Comparative Statics

Invoking the sensitivity results from LP, we can infer that the seller profit is concave in c_j and convex in v_{ij} (Theorem 5.3 in Bertsimas et al., 1997). In the following paragraphs, we consider some additional comparative static results.

Consider the cost parameters first. We say that for any two cost vectors c' and c'', the marginal cost of c' is less than that of c'' if for all $j \geq 1$, $c'_j - c'_{j-1} \leq c''_j - c''_{j-1}$. We say the marginal cost is strictly less if the inequality is strict. Although the solution approach of Hitt and Chen (2005) is inadequate, their insight regarding the weak reduction in the size of cardinality bundles with increasing marginal cost still holds.

Corollary 11 *Assume that marginal cost of* c' *is less than that of* c''. *Then, for every optimal allocation* x' *with* c' *there exists an optimal allocation* s *with* c'' *such that each consumer is allocated a bundle of weakly smaller size in* s *than in* x'. *Similarly, for every optimal allocation* x'' *with* c'' *there exists an optimal allocation* t *with* c' *such that each consumer is allocated a bundle of weakly larger size in* t *than in* x''. *If the marginal cost of* c' *is strictly less than that of* c'' *then every optimal allocation* x' *with* c' *allocates a bundle of size no smaller than any optimal allocation* x'' *with* c''.

Next, we study how changes to consumers' WTP affect the solution (e.g., when the seller pursues advertising efforts). Since $\mathrm{CBP2d}$ is convex in v_{ij} and v_{ij} is a linear transformation of w_{ij}, $\mathrm{CBP2d}$ is convex in WTP (Theorem 3.2.2 in Boyd and Vandenberghe, 2004), and therefore, so it is in $\mathrm{CBP2}$. Notice, when WTPs do not satisfy SCP, $\mathrm{CBP2}$ may not be convex in the WTPs.[10]

Increasing WTP (even if it is subject to SCP) does not guarantee an increase in seller profit. From Examples 1 and 2, it should be clear that increasing the consumers' WTP can decrease the profit. However, increasing consumers' WTP (of course, subject to SCP) on the purchased bundles will always increase vendor profits. So, it is important for the sellers to target the WTP increases.

Next, consider the scenario when the seller cannot increase the WTPs but can only shift the WTP from one consumer type to the other (for example, seller pursues homogenization efforts). We first study the profit implications in the context of information goods, where $c_j = 0 \ \forall j$. Let w denote a given $I \times J$ WTP matrix. Define $w' = \mathcal{W}(i_1, i_2, w)$ as a function which

[10] In the following example, we illustrate that when WTPs do not satisfy SCP, $\mathrm{CBP2}$ may not be convex in the WTPs. Consider a scenario with $I = 2$ consumers, $J = 3$ bundle sizes, and costs $c_j = 0$ for all j. Suppose $\mathcal{W}_1$ is one WTP matrix as given in the second and third columns of Table 2.4 and $\mathcal{W}_2$ is another WTP matrix as given in the fourth and fifth columns of Table 2.4. Notice, the WTP of Consumer 1 in $\mathcal{W}_2$ is the same as that of Consumer 2 in $\mathcal{W}_1$ and the WTP of Consumer 2 in $\mathcal{W}_2$ is the same as that of Consumer 1 in $\mathcal{W}_1$. The optimal solutions for both problems are $p_1^* = p_2^* = 12, p_3^* = 34$ and the optimal profits are $\Pi_1^* = \Pi_2^* = 46$. Let $\mathcal{W}_3 = \frac{1}{2}\mathcal{W}_1 + \frac{1}{2}\mathcal{W}_2$, as shown in the last two columns in Table 2.4. The optimal solution is $p_1^* = p_2^* = p_3^* = 26$ and the optimal profit is $\Pi_3^* = 52 > \frac{1}{2}\Pi_1^* + \frac{1}{2}\Pi_2^*$.

Table 2.4: Willingness-to-pay

Bundle	$\mathcal{W}_1$		$\mathcal{W}_2$		$\mathcal{W}_3$	
size	I_1	I_2	I_1'	I_2'	I_1''	I_2''
1	10	16	16	10	13	13
2	12	18	18	12	15	15
3	12	40	40	12	26	26

maps w to another $I \times J$ matrix w', such that, for any j,

$$w'_{ij} = \begin{cases} w_{ij} & \text{if } i < i_1 \text{ or } i > i_2 \\ \frac{1}{i_2 - i_1 + 1} \sum_{i'=i_1}^{i_2} w_{i'j} & \text{if } i_1 \le i \le i_2. \end{cases}$$

That is, consumers indexed between i_1 and i_2 are homogenized so that their individual WTPs in the transformed setting is the average of their original WTPs; whereas the other consumers remain unaffected. Let $\Pi^*_{CBP}(w)$ denote the optimal profit of a CBP problem for a given w WTP matrix.

Proposition 12 *When $c_j = 0$, for each i', $\Pi^*_{CBP}(\mathcal{W}(i', I, w)) \ge \Pi^*_{CBP}(w)$.*

Proposition 12 shows that homogenizing improves the seller profit only if it involves the highest consumer type.[11] A corollary is that homogenizing across all consumer types (*i.e.*, using $\mathcal{W}(1, I, w)$ as WTP) will weakly increase the profit. Notice that, if costs are non-zero, even when the highest consumer type is included for homogenization, the seller profit can decrease.[12]

Apart from the comparative static results, we were also interested in exploring the relationship between v_{ij} and w_{ij} terms. In doing so, we discovered a result that may be tangential to

[11] We illustrate that merging may decrease the seller profit when the highest consumer types are not involved by using the following example.

Table 2.5: Willingness-to-pay

Bundle size	WTP I_1	WTP I_2	WTP I_3
1	2	10	13
2	4	12	20

Consider a scenario with $I = 3$ consumers, $J = 2$ bundle sizes, and costs $c_j = 0$ for all j. Suppose the WTP for the consumers are as given in Table 2.5. Obviously, the optimal solution is $p^*_1 = 10, p^*_2 = 17$ and the optimal profit is 27. If we merge Consumer 1 and 2, then $w'_{11} = w'_{21} = 6, w'_{12} = w'_{22} = 8$, which leads to a new optimal solution of $p'^*_1 = 6, p'^*_2 = 13$ and a lower optimal profit of 25.

[12] Consider a scenario with $I = 2$, $J = 1$, and costs $c_1 = 10$. Suppose $w_{11} = 4$ and $w_{21} = 20$. the optimal solution is $p^*_1 = 20$ and the optimal profit is 10. If we merge Consumer 1 and 2, then $w'_{11} = w'_{21} = 12$, which leads to a new optimal solution of $p'^*_1 = 12$ and a lower optimal profit of 4.

the analysis thusfar but useful (for computational purposes) in quickly generating example values of WTPs satisfying SCP. As mentioned before, given WTPs, $v_{ij} = w_{ij} - (I-i)(w_{i+1j} - w_{ij})$. So,

$$\sum_{i'=i}^{I} v_{i'j} = \sum_{i'=i}^{I}(I-i'-1)w_{ij} - \sum_{i'=i}^{I-1}(I-i')w_{i+1j} = \sum_{i'=i}^{I}(I-i'-1)w_{ij} - \sum_{i'=i+1}^{I}(I-i'-1)w_{ij} = (I-i-1)w_{ij}.$$

Therefore,

$$w_{ij} = \frac{1}{I-i-1}\sum_{i'=i}^{I} v_{i'j}. \tag{2.17}$$

This shows that there is a one-to-one linear transformation relating w to v. Given the relationship, we show next that we may choose v arbitrarily for the first $I-1$ consumers and still find WTPs that satisfy SCP and are increasing in j.

Proposition 13 *Given v_{ij} for $i \in \{1, \ldots, I-1\}$ and $j \in \{1, \ldots, J\}$, there exist w_{ij} for $i \in \{1, \ldots, I\}$ amd $j \in \{0, \ldots, J\}$ that satisfy SCP and are increasing in j.*

2.3 Continuous Case: Model and Analysis

We now investigate a continuous version of the problem treated in Section 3.2. One application of the continuous problem is in quantity discount pricing, which was explored by Spence (1980).[13] The continuous version can also be applied in cardinality bundling, when the goods are not discrete. For example, many restaurants charge based on weight (for e.g., kilos in Brazil) regardless of the kind of food chosen by the consumer on their plate. The main difference is that bundle sizes are not restricted to integer values $1, \ldots, J$ but can take any real value. The problem for the vendor is then to identify the optimal pricing function for all real-valued sizes, which turns out to be significantly more difficult. Nevertheless, we show that the new insights developed in Section 3.2 can be used to approach this problem.

[13]The discrete case analyzed in Hitt and Chen (2005) was heavily inspired by Spence (1980).

2.3.1 Prior Related Work

The model here is similar to that in the previous section except that we use a continuous variable $y \in \mathbb{R}_+$ to represent the bundle sizes, instead of using an index j to denote discrete sizes. Every variable that had an index j before now becomes a function of y instead. In particular: $p(y)$ represents the price of bundle size y; $c(y)$ the cost of Bundle y; $w_i(y)$ the Consumer i's WTP for bundle size y. We also define y_i to denote the bundle size Consumer i purchases and corresponds to $j(i) = \sum_{j=0}^{J} jx_{ij}$ in the discrete case. Spence (1980) also assumes WTPs satisfy SCP and models it as $w_i'(y) < w_{i+1}'(y)$ for all y. We relax these conditions slightly to the weak inequality and generalize them to the non-differentiable case as follows:[14]

$$0 = w_i(0) \leq w_i(y) \leq w_{i+1}(y)\ \forall y \tag{2.18}$$

$$w_i(y+d) - w_i(y) \leq w_{i+1}(y+d) - w_{i+1}(y)\ \forall y\ \forall d \geq 0. \tag{2.19}$$

Assuming $p(0) = 0$, the vendor's decision problem is then as follows:

$$\begin{aligned} \text{CBPc1}: \quad \underset{y_i, p(y)}{\text{Max}} \quad & \sum_{i=1}^{I} (p(y_i) - c(y_i)) \\ \text{s.t.} \quad & w_i(y_i) - p(y_i) \geq w_i(y) - p(y)\ \forall i\ \forall y. \end{aligned} \tag{2.20}$$

We first review the approach suggested in Spence (1980). Assuming that WTPs satisfy SCP conditions with a strict inequality, he shows that every optimal solution must satisfy $y_{i+1} \geq y_i$ for all $i < I - 1$. Then, given y_i, $i = 1, \ldots, I$, he substitutes the optimal prices, obtaining the optimization problem in the space of y variables. Then, the paper ignores the constraints

[14]Since $w_i'(y) \leq w_{i+1}'(y)$ for all y, it follows that $\int_y^d w_i'(y')dy' \leq \int_y^d w_{i+1}'(y')dy'$ for all y and d, which in turn implies that $w_i(y+d) - w_i(y) \leq w_{i+1}(y+d) - w_{i+1}(y)$ for all y and d. On the other hand, $w_{i+1}(y+d) - w_{i+1}(y) \geq w_i(y+d) - w_i(y)$ implies that $\lim_{d\to 0} \frac{w_{i+1}(y+d) - w_{i+1}(y)}{d} \geq \lim_{d \to 0} \frac{w_i(y+d) - w_i(y)}{d}$ or that $w_{i+1}'(y) \geq w_i'(y)$.

$y_{i+1} \geq y_i$ to obtain an unconstrained optimization problem and sets its derivative to zero, yielding the following local optimality condition:

$$(I - i + 1)w_i'(y_i) - (I - i)w_{i+1}'(y_i) = c'(y_i). \tag{2.21}$$

We now interpret the approach of Spence (1980) using our results in Section 3.2. Assume that the optimal bundle sizes the consumers buy are given by y_i^*, $i = 1, \ldots, I$. Then, $\mathrm{CBPc1}$ restricted to these bundle sizes reduces to a discrete problem. Since y_i^*, $i = 1, \ldots, I$ must be optimal to this restricted problem, the results of our previous section still apply. Therefore, with the slightly relaxed SCP conditions (2.18) and (2.19), the results of Spence (1980) still hold. In particular, Proposition 4 shows that there exists an optimal solution with $y_{i+1}^* \geq y_i^*$ for all $i < I$ and Proposition 7 shows that $\mathrm{CBPc1}$ can be rewritten as:

$$\begin{aligned} \mathrm{CBPcy}: \qquad & \underset{y_i}{\text{Max}} \sum_{i=1}^{I} (v_i(y_i) - c(y_i)) \\ & \text{s.t. } \; y_{i+1} \geq y_i \qquad 1 \leq i \leq I - 1, \end{aligned} \tag{2.22}$$

where $v_i(y_i) = w_i(y_i) - (I - i)(w_{i+1}(y) - w_i(y))$ and $w_{I+1}(y)$ is assumed to be $w_I(y)$. Then, Equation (2.21) is the same as setting the derivative of the objective of CBPcy to zero, *i.e.*, $v_i'(y_i) = c'(y_i)$.[15]

Solving (2.21) may not seem hard since each consumer's decision is independent of others. However, this approach only works if Constraints (2.22) are automatically satisfied by the solution. Otherwise, the optimality conditions do not decompose. Once the optimal Lagrangian multipliers are known, the remaining optimality conditions (those of the inner problem of the Lagrangian dual) can still be decomposed. However, for a given i, the Lagrangian multiplier

[15]More generally, when v and c are not necessarily differentiable, then the above optimality condition generalizes to zero belonging to the subdifferential of $v_i(y_i) - c(y_i)$.

of $y_{i+1} \geq y_i$ gets multiplied with the decision of both Consumers i and $i+1$. Therefore, the problem of determining the optimal multipliers links the consumers together.

Besides Constraints (2.22) being ignored in the optimality conditions, there is another subtle issue with Spence's approach. The optimality condition in (2.21 is a local optimality condition, which would be reasonable, if the objective had a unique local maximum (for example if it was strictly concave). However, as shown in the next example, $v_i(\cdot)$ is often nonconvex, and there may be many points where the derivative of the objective of CBPcy is zero.

Example 14 *In* $\mathrm{CBPc1}$, *assume that consumers can choose any bundle size* y, *as long as* $0 \leq y \leq J$, *where* J *is an even number, and let* $c(y)$ *be identically zero. Let* $w_i(y) = 1 + \frac{I}{I-i+1}(\pi y + \log(1+y)) - \cos(\pi y)\ \forall i$. *Each consumer's WTP is increasing in* y *and the WTPs satisfy SCP. It follows that* $v_i(y) = 1 - \cos(\pi y)$. *Therefore, if* y_i *is even, it satisfies (2.21). Since every consumer can be assigned Bundles* $\{0, 2, \ldots, J\}$, $\left(\frac{J}{2}+1\right)^I$ *solutions satisfy Condition (2.21). Moreover, let* $J = 4I - 2$, *and observe that there are exponentially many solutions that satisfy Condition (2.21) and satisfy Constraint (2.22). To see this, consider* 2^I *solutions obtained by allocating bundle sizes in* $\{4(i-1), 4i-2\}$ *to Consumer* i.

Spence (1980) does not mention the fact that there may be many solutions that satisfy Condition (2.21). There is, thus, no guidance available on selecting the best solution among them. If one ignores Constraint (2.22), this situation can be remedied by selecting, for Consumer i, the bundle size y_i that maximizes $v_i(y) - c_i(y)$ by solving a one-dimensional global optimization problem. However, in the presence of Constraint (2.22), the situation is significantly more complex. Thus, the approach based on Condition (2.21) is deficient in that it ignores Constraint (2.22) and does not provide any way of selecting the global optimal solution from many possible local optima.

2.3.2 Reformulation and Approximation

We assume that the vendor only provides bundles of size Y or smaller. This assumption is reasonable since the vendor is typically limited by a production capacity. In other words, we include the constraint $0 \leq y_i \leq Y$ for all i in CBPcy. As illustrated in Section 2.3.1, solving $\mathrm{CBPc1}$ is challenging since it requires the determination of the optimal price function $p(y)$ instead of pricing a discrete set of bundles and has infinitely many incentive compatability constraints of the type (3.33), one for each y. These issues can be somewhat sidestepped by reformulating $\mathrm{CBPc1}$ as CBPcy which has finitely many continuous variables. However, since the resulting functions $v_i(\cdot)$ are in general non-convex, the problem remains challenging to solve, especially in the presence of Constraints (2.22).

First, we remark that it is possible to extend the approach used in formulating $\mathrm{CBP2d}$ to solve the continuous case. In particular, the problem aims to find functions $l_i(\cdot)$, $i = 1, \ldots, I$, such that:

$$
\begin{aligned}
\mathrm{CBPcyd}: \quad & \min_{l_i(y)} \quad l_I(Y) \\
& \text{s.t.} \quad l_i(y) \geq l_{i-1}(y) + v_i(y) - c(y) && i = 1, \ldots, I,\ 0 \leq y \leq Y \quad (2.23) \\
& \qquad l_0(y) = 0 && 0 \leq y \leq Y \quad (2.24) \\
& \qquad l_i(y) \text{ is non-negative and non-decreasing} && i = 1, \ldots, I,\ 0 \leq y \leq Y \quad (2.25)
\end{aligned}
$$

Appendix 5.1.12 shows that CBPcyd is a valid reformulation. The above approach solves the continuous cardinality bundling problem by computing $l_i(y) = \sup\{l_{i-1}(y') + v_i(y') - c(y') \mid y' \leq y\}$ for each i.

We remark that the convex reformulation $\mathrm{CBP2}$ (without the integrality constraints) for the discrete case does not extend easily to the continuous case. Note that, for $\mathrm{CBP2}$, the bundle size that Consumer i buys is $y_i = \sum_{j=0}^{J} j x_{ij}$. At the binary values of x_{ij}, these reduce to $\sum_{j=0}^{J} jx_{i+1j} \geq \sum_{j=0}^{J} jx_{ij}$. However, when x_{ij} take continuous values, Constraints (3.7) are tighter: $\sum_{j=0}^{J} jx_{i+1j} = \sum_{j=1}^{J} \sum_{j'=1}^{j} x_{i+1j} = \sum_{j'=1}^{J} \sum_{j=j'}^{J} x_{i+1j} \geq \sum_{j'=1}^{J} \sum_{j=j'}^{J} x_{ij} = \sum_{j=1}^{J} \sum_{j'=1}^{j} x_{ij} = \sum_{j=0}^{J} jx_{ij}$, where the inequality follows from Constraints (3.7). The converse does not hold for continuous values of x_{ij}.[16] This explains why CBPcy is not convex although $\mathrm{CBP2}$ is a convex program when the superfluous binary restrictions are removed.

For a set, S, let $\mathrm{conv}(S)$ and $\mathrm{proj}_x S$ denote respectively the convex hull of S and the projection of S to the space of x variables. Let $\{k_j\}_{j=0}^{J} \in [0, Y]^{J+1}$, where $0 = k_0 < \cdots < k_J = Y$. Consider $y' \in \mathbb{R}^I$, with $0 \leq y'_i \leq Y$ for all i that satisfies Constraints (2.22) and extend y' to $(y', x') \in \mathbb{R}^I \times \mathbb{R}^{I \times J}$ so that

$$x'_{ij} = \begin{cases} 0 & \text{if } y'_i \leq k_{j-1} \text{ or } y'_i \geq k_{j+1} \\ \frac{y'_i - k_{j-1}}{k_{j-1} - k_j} & \text{if } k_{j-1} < y'_i < k_j \\ \frac{k_{j+1} - y'_i}{k_{j+1} - k_j} & \text{if } k_j \leq y'_i < k_{j+1}, \end{cases} \tag{2.26}$$

where k_{-1} and k_{J+1} are understood to be 0 and $Y + 1$ respectively. Define

$$S = \left\{ (y, x) \;\middle|\; \begin{array}{l} y_i = \sum_{j=0}^{J} k_j x_{ij}, \forall i; \quad \sum_{j=0}^{J} k_j x_{ij} \geq \sum_{j=0}^{J} k_j x_{i+1j}, i = 1, \ldots, I-1 \\ \sum_{j=0}^{J} x_{ij} = 1, \forall i; \quad x_{ij} x_{ij'} = 0, \forall i, j, j' \geq j+2; \quad x_{ij} \geq 0, \forall i, j \end{array} \right\}, \tag{2.27}$$

and observe that (y', x') is the only solution in S that projects to y'. Next, we compute $\mathrm{conv}(S)$.

[16] To see this, let $J = 2$ and define $x_{i0} = x_{i2} = 0.5$, $x_{i1} = x_{i+1,0} = x_{i+1,2} = 0$, and $x_{i+1,1} = 1$. Then, although $\sum_{j=0}^{J} jx_{ij} \leq \sum_{j=0}^{J} jx_{i+1j}$, $x_{iJ} \not\leq x_{i+1J}$.

Lemma 15 *The convex hull of S is given by:*

$$S' = \left\{ (y,x) \;\middle|\; \begin{array}{l} y_i = \sum_{j=0}^{J} k_j x_{ij}, \forall i; \quad \sum_{j'=j}^{J} x_{ij'} \leq \sum_{j'=j}^{J} x_{i+1j'}, \forall j, i = 1, \ldots, I-1; \\ \sum_{j=0}^{J} x_{ij} = 1, \forall i; \quad x_{ij} \geq 0, \forall i, j \end{array} \right\}. \quad (2.28)$$

Let $A = \{y \mid (2.22), 0 \leq y_i \leq Y, \forall i\}$*. Then,* $\operatorname{proj}_y S' = \operatorname{proj}_y S = A$*. Further,* $\operatorname{conv}(\operatorname{proj}_x S) = \operatorname{proj}_x S'$*.*

By Lemma 15, the continuous cardinality bundling problem can be written as:

$$\mathrm{CBPcx} : \mathrm{Max}_{x_{ij}} \left\{ \sum_{i=1}^{I} \left(v_i \left(\sum_{j=0}^{J} k_j x_{ij} \right) - c \left(\sum_{j=0}^{J} k_j x_{ij} \right) \right) \;\middle|\; x \in \operatorname{proj}_x S. \right\}.$$ [17]

We now show that when $w_i(\cdot)$ and $c(\cdot)$ are piecewise linear functions whose breakpoints form a subset of $\{k_1, \ldots, k_J\}$, then CBPcx can be solved quickly. First, observe that $k_j \leq y \leq k_{j+1}$,

$$w_i(y) = \frac{k_{j+1} - y}{k_{j+1} - k_j} w_i(k_j) + \frac{y - k_j}{k_{j+1} - k_j} w_i(k_{j+1}) = x_{ij} w_i(k_j) + x_{ij+1} w_i(k_{j+1}) = \sum_{j'=0}^{J} x_{ij'} w_i(k_{j'}),$$

where the second equality is from (2.26), and the third equality is because it follows from (2.26) that $x_{ij'} = 0$ for all $j' \notin \{j, j+1\}$. Similarly $c(y) = \sum_{j=0}^{J} x_{ij} c(k_j)$. We define $w_{ij} = w_i(k_j)$, $c_j = c(k_j)$, and $v_{ij} = w_{ij} - (I - i)(w_{i+1j} - w_{ij})$, where w_{I+1j} is understood to be w_{Ij}. Then, CBPcx can be rewritten as:

$$\mathrm{CBPcxL} : \mathrm{Max}_{x_{ij}} \left\{ \sum_{i=1}^{I} \sum_{j=0}^{J} (v_{ij} - c_j) x_{ij} \;\middle|\; x \in \operatorname{proj}_x S. \right\}$$

[17] $\mathrm{Max}_{y_i} \left\{ \sum_{i=1}^{I} \big(v(y_i) - c(y_i)\big) \mid y \in A \right\}$ reformulates to $\mathrm{Max}_{x_{ij}} \left\{ \sum_{i=1}^{I} \Big(v\big(\sum_{j=0}^{J} k_j x_{ij}\big) - c\big(\sum_{j=0}^{J} k_j x_{ij}\big) \Big) \;\Big|\; (y,x) \in S \right\}$, which reduces to CBPcx since the objective only depends on x.

Now, since the objective is linear, by Lemma 15, we replace $\operatorname{proj}_x S$ with $\operatorname{proj}_x S'$ and rewrite CBPcxL as:

$$\text{CBPcxL2} : \text{Max}_{x_{ij}} \left\{ \sum_{i=1}^{I} \sum_{j=0}^{J} (v_{ij} - c_j) x_{ij} \;\middle|\; x \in \operatorname{proj}_x S' \right\}.$$

Thus, we have shown the following result.

Theorem 16 *When $w_i(\cdot)$ and $c(\cdot)$ are piecewise linear functions, whose breakpoints form a subset of $\{k_1, \ldots, k_J\}$, the continuous cardinality bundling problem can be solved as* CBPcxL2.

Observe that CBPcxL2 is identical to the discrete cardinality bundling problem CBP2 for which we developed an $O(IJ)$ algorithm in Section 2.2.2. Therefore, it follows from Theorem 16 that the continuous cardinality bundling problem with piecewise-linear functions can be solved in $O(IJ)$ time.

Corollary 17 *When $w_i(\cdot)$ and $c(\cdot)$ are piecewise linear functions, whose breakpoints form a subset of $\{k_1, \ldots, k_J\}$, there exists an optimal solution where every consumer purchases a bundle in $\{k_1, \ldots, k_J\}$, i.e., $y_i \in \{k_1, \ldots, k_J\}$ for all i.*

Now, we relax the assumption that w_i and c are piecewise linear functions and consider the more general case of Lipschitz continuous functions. Recall that a function $f(x)$ is said to be Lipschitz continuous with Lipschitz constant L_f on an interval $[a, b]$, if there is a non-negative constant L_f such that $|f(x_1) - f(x_2)| \leq L_f |x_1 - x_2|$ for all x_1, x_2 that belong to $[a, b]$. We assume that $w_i(y)$ and $c(y)$ are Lipschitz continuous with Lipschitz constant β. We will construct piecewise linear approximation for $w_i(y)$ (resp. $c(y)$). Say, we wish to approximate the solution within ϵ. Then, we choose $k = \frac{\epsilon}{I(2I+1)\beta}$ and $J = \lceil \frac{Y}{k} \rceil$. We let $k_j = jk$ for

$j \in 0, \ldots, J-1$ and $k_J = Y$. Then, for $k_j \leq y \leq k_{j+1}$, we define

$$w_i^k(y) = \frac{k_{j+1} - y}{k_{j+1} - k_j} w_i(k_j) + \frac{y - k_j}{k_{j+1} - k_j} w_i(k_{j+1}) \text{ and } c^k(y) = \frac{k_{j+1} - y}{k_{j+1} - k_j} c(k_j) + \frac{y - k_j}{k_{j+1} - k_j} c(k_{j+1}).$$

Observe that $w_i^k(\cdot)$ and $c^k(\cdot)$ are piecewise linear functions. Let Π^c be the optimal value of CBPcx and Π^k denote the optimal profit when $w_i^k(y)$ and $c^k(y)$ are the WTP for Consumer i and the cost for producing y.

Theorem 18 *For a given ϵ, define $k = \frac{\epsilon}{I(2I+1)\beta}$. Then, $\Pi^k \leq \Pi^c \leq \Pi^k + \epsilon$. Further, Π^k can be computed in* O $\left(\frac{I^2(I+2)\beta Y}{\epsilon} + I\right)$ *time.*

2.4 Conclusion

Pricing of cardinality bundles has not been widely studied in literature although this bundling scheme is increasingly being adopted in industry. Our paper provides a comprehensive analysis of the problem when the consumer's willingness to pay satisfies Spence-Mirrlees condition and consumers are restricted to buy only one bundle. In this paper, we first study the cardinality bundling problem in the context of discrete bundle sizes, the problem first considered in Hitt and Chen (2005). We provide a solution approach that can solve the problem efficiently. Then we use the underline structures from the discrete problem to revisit the quantity discount problem proposed in Spence (1980) and derive insights and solution approaches.

Chapter 3

Cardinality Bundles with Complex Costs

3.1 Introduction

Cardinality bundling, or, in short, CB, is a kind of bundling strategy where bundles of equal cardinality or size are sold at the same price. One example of how firms adopt CB is the way Disney World sells themepark tickets. Consumers can purchase multi-day (2, 3, 4 or 5 day) passes from Disney World. A consumer who purchases a 2-day pass can choose any two themeparks and enter each one for one day. Similarly, a consumer who purchases a 3-day pass can choose any three themeparks and enter each one for one day, and so on so forth. The key characteristic of CB is that the seller only prices for the cardinality of its goods and let consumers choose the combination of goods they want under the cardinality.

In fact, CB has been adopted by a variety of firms in practice. With the emergence and rapid growth of low-cost reproduction and distribution technologies for information goods, these goods providers are more and more attracted to CB. For example, Eastlink, a cable TV service provider in Canada, sells bundles of either 12 channels or 20 channels and let consumers pick which channels they would like to include in the proposed bundles. Similarly, Netflix, the online DVD rental firm, prices subscription options based on the number of DVDs a consumers rents each time.

Actually several other types of bundling have been adopted in practice. The first one is mixed bundling, wherein every possible combination of goods is sold at a possibly different price. Mixed bundling is the most profitable bundling strategy. However, Hanson and Martin (1990) shows that the pricing problem for mixed bundling is only tractable when the number of goods is small. Other two types of bundling considered are: component bundling and pure bundling. In component bundling, individual components, rather than the bundles, are priced. In pure bundling, only a bundle with all possible products is sold. Pricing for these two types of bundling is relatively easy. But in most cases, these two strategies do not guarantee optimal profits. Chu et al. (2011) shows that in many cases CB is close to the profitability of mixed bundling and more profitable than component pricing and pure bundling.

The models presented in this paper extend the CB models in Kannan et al. (2014b). Before we go into the details of Kannan et al. (2014b), we first review the literature on CB. Hitt and Chen (2005) is the first analytical modeling paper which studies the pricing of CB. They build the basic CB model assuming that consumers' reservation price satisfy Spence-Mirrlees Single Crossing Property (SCP). Their basic model and the SCP assumption are also used in Kannan et al. (2014b) and this paper. They also explore the properties of the optimal solution for the CB problem. Wu et al. (2008) relax the SCP assumption and propose a nonlinear mixed-integer programming approach to analyze the CB problem. Chu et al. (2011) use computational and empirical results to show that in many cases, CB is as profitable as mixed bundling.

Kannan et al. (2014b) solve the basic models of CB problem. They first consider the model presented in Hitt and Chen (2005) and show that it can be solved as a linear programming (LP) problem within polynomial time. They also consider a continuous version of the CB problem where the seller can price bundle sizes at continuous values rather than being restricted to integer values, same as the quantity discount problem that is explored by Spence (1980).

This paper extends Kannan et al. (2014b) in three ways. First, we show that the basic CB model in Kannan et al. (2014b) can be reformulated as a shortest-path problem. The network structure underlying the shortest-path formulation provides many insights into cardinality bundling . Second, we modify the model to include a fixed costs for the seller to add an additional bundle and show that with the fixed costs, the problem can still be solved as a shortest-path problem. We notice that Wu et al. (2008) has a similar setting of including fixed costs. The third extension is to analyze the CB problem without additively separable cost structure. We prove that even without additively separable cost structure, if the production cost is submodular, the above two kinds of problems are still solvable in strong polynomial time. Solutions we developed here can be implemented to a wide scope of industries where economies of scale exists in production.

3.2 Basic Model and a Shortest-Path Reformulation

In this section we first review the model and some important results from Kannan et al. (2014b).[1] Then we show that the problem can be reformulated as a shortest-path problem. The shortest-path structure we develop here not only reveals a simple structure for the CB problem, but also paves a way to solve more complicated problems in the following sections.

3.2.1 Basic Model

The model is developed from a seller's perspective who sells J products to I consumers in the market. In the following, we denote the bundle of size j as Bundle j. WLOG, all bundles, $1, \dots, J$ are assumed to be offered in the market. We denote the price of Bundle j as p_j and the cost of Bundle j as c_j. The seller's objective is to maximize the profit, which is calculated as the sum of all prices for all the bundles sold minus the sum of all the costs for the corresponding

[1] Please refer to Kannan et al. (2014b) for a complete version of the mode setup and analyses.

bundles. We use Bundle 0 to represent consumers' choice of not purchasing and set $p_0 = 0$ and $c_0 = 0$.

We denote the willingness-to-pay (WTP) of Consumer i for Bundle j as w_{ij}. For each consumer i, WTP is assumed to be non-decreasing with the bundle size and w_{i0} is assumed to be zero. Each consumer i is assumed to only purchase one bundle, $j \in \{0, 1, 2, \dots, J\}$, that maximizes her surplus calculated as $w_{ij} - p_j$. Let x_{ij} be 1 if Consumer $i \in \{1, 2, \dots, I\}$ buys Bundle $j \in \{0, 1, 2, \dots, J\}$ and 0 otherwise. Then, CBP can be formulated as follows (see Hitt and Chen, 2005):

Let x_{ij} be 1 if Consumer $i \in \{1, 2, \dots, I\}$ buys Bundle $j \in \{0, 1, 2, \dots, J\}$ and 0 otherwise. Then, the problem can be formulated as follows (see Kannan et al., 2014b):

$$\text{CBP1}: \quad \underset{x_{ij}, p_j}{\text{Max}} \quad \sum_{i=1}^{I} \sum_{j=0}^{J} x_{ij}(p_j - c_j)$$

$$\text{s.t.} \quad \sum_{j'=0}^{J} (w_{ij'} - p_{j'}) x_{ij'} \geq w_{ij} - p_j \ \forall i, \ \forall j \tag{3.1}$$

$$\sum_{j=0}^{J} x_{ij} = 1 \ \forall i \tag{3.2}$$

$$p_0 = 0 \tag{3.3}$$

$$x_{ij} \in \{0, 1\} \ \forall i, \ \forall j. \tag{3.4}$$

Constraints (3.2) represents the assumption that each consumer purchases one bundle. Constraints (3.1) requires that consumer surplus from the purchased bundle to be no less than that from any other alternatives.

Following Kannan et al. (2014b), we assume that consumer valuations satisfy the Spence-Mirrlees Single Crossing Property (SCP) (see Spence, 1980), stating that consumers can be ordered by types. Higher type consumers (with higher indexes) are willing to pay more on each bundle size than lower type consumers (with lower indexes) do, and more for each addition unit. Thus, we impose the following ordering on the consumers' WTP for the bundles:

$$w_{ij} \geq w_{i'j} \ \forall i > i', \tag{3.5}$$

$$w_{ij} - w_{ij'} \geq w_{i'j} - w_{i'j'} \ \forall i > i', \ \forall j > j'. \tag{3.6}$$

In Kannan et al. (2014b), the authors explore a few properties of the optimal solution of $\mathrm{CBP1}$. Next, we review one of those properties that is especially useful in the extended models which will be discussed later on. Proposition 4 in Kannan et al. (2014b) states that consumers with higher indexes always purchase bundle sizes larger than consumers with lower indexes. Thus, we can add the following redundant yet useful constraints into the model:

$$\sum_{j'=j}^{J} x_{i+1j'} \geq \sum_{j'=j}^{J} x_{i,j'} \ i = 1, \ldots, I-1, \ \forall j. \tag{3.7}$$

Notice, the proof of this proposition still holds even if there are fixed costs in the model (discussed in Section 3.3) or the costs have a general submodular form (discussed in Section 3.4).

3.2.2 A Shortest-Path Reformulation

Kannan et al. (2014b) demonstrates that a solution to $\mathrm{CBP1}$ can be obtained by solving a linear programming problem. Next, we show that $\mathrm{CBP1}$ can also be solved as a shortest path problem.

Following Kannan et al. (2014b), we use the following formulation to linearly transform the WTP matrix w_{ij} to another matrix v_{ij}:

$$v_{ij} = w_{i,j} + (I - i)(w_{i,j} - w_{i+1,j}) \tag{3.8}$$

The way we transform w_{ij} to v_{ij} plays an important role in developing the solution approach for $\mathrm{CBP1}$. A detailed discussion on the definition of v_{ij} and its implications is provided in Kannan et al. (2014b). In short, v_{ij} captures that when the seller allocates Consumer i to purchase Bundle j, how his revenue will change. It combines the gain from the bundle sold and the loss from seller's decremental ability to extract surplus from consumers other than i. by transforming w_{ij} to v_{ij}, we are able to reformulate $\mathrm{CBP1}$ to a shortest-path formulation.

Theorem 19 $\mathrm{CBP1}$ *is equivalent to the following shortest path problem on a graph which has* $2I(J+1)+2$ *nodes and* $(I+2)(J+1)+(I-1)(J+1)(J+2)/2$ *edges:*

$$
\begin{aligned}
\mathrm{CBP3}: \quad \mathrm{Min}_{x_{ij},\chi_{ijj'}} \quad & -\sum_{i=1}^{I}\sum_{j=0}^{J}(v_{ij}-c_j)x_{ij} \\
\textit{s.t.} \quad & \sum_{j=0}^{J}\chi_{00j}=1 && (3.9)\\
& \chi_{00j}=x_{1j} && (3.10)\\
& \sum_{j=0}^{J}\chi_{IjJ}=1 && (3.11)\\
& \chi_{IjJ}=x_{Ij} && (3.12)\\
& x_{ij}=\sum_{j'=j}^{J}\chi_{ijj'}\ \forall i\forall j && (3.13)\\
& \sum_{j'=0}^{j}\chi_{i-1,j',j}=x_{ij}\ \forall i\ \forall j && (3.14)\\
& \chi_{ijj'}\in\{0,1\}\ \forall i\ \forall j\ \forall j'\geq j. && (3.15)
\end{aligned}
$$

The problem formulation in Theorem 19 can be seen as a shortest-path problem on a suitable acyclic network. Therefore, one can use a combinatorial algorithm that traverses the vertices in the order generated by a topological sort and finds the shortest path in linear time.

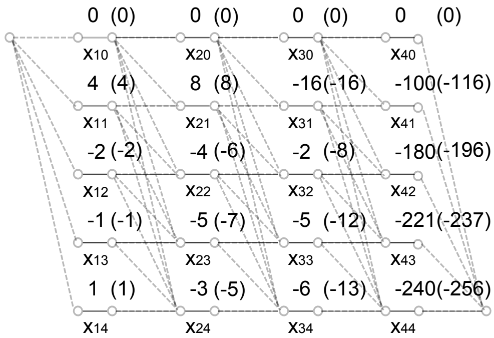
Example 20 *Consider a scenario with* $I=4$ *consumers,* $J=4$ *bundle sizes, and costs* $c_j=0$ *for all* j*. Suppose the WTP for the consumers are as given in Table 3.1. It can be verified easily*

Table 3.1: WTP and v_{ij} values for Example 20

Bundle size	Consumers' WTP I_1	I_2	I_3	I_4
0	0	0	0	0
1	26	36	58	100
2	47	62	91	180
3	58	77	113	221
4	62	83	123	240

Bundle size	v_{ij} I_1	I_2	I_3	I_4
0	0	0	0	0
1	-4	-8	16	100
2	2	4	2	180
3	1	5	5	221
4	-1	3	6	240

Figure 3.1: The shortest path problem formulation for Example 20



that they satisfy SCP. The corresponding v_{ij} values are also shown in Table 3.1.

We illustrate the shortest-path structure of CBP3 on Example 20.Figure 3.1 illustrates the associated network where the source node is the top-left node and the sink node is the bottom-right node. One unit of flow starts from the source node, travels through the network, and finally arrives at the sink node. Each edge in the network is directed from left to right. Observe that, the edges in the network point sideways or downwards, but not upwards. It is consistent with Proposition 4 in Kannan et al. (2014b), stating that the consumers with higher indexes purchase weakly larger sized bundles than consumers with lower indexes. The network has a multipartite structure, each partition corresponding to a consumer. Apart from the source node and the sink node, there are $2I(J + 1)$ nodes, with two nodes for each pair of consumer and bundle size. There are two types of edges in the network, solid and dotted. The solid edges connect two nodes that correspond to Consumer i purchasing Bundle j and the dotted edges connect the different partitions of the network. The variable x_{ij} indicates the flow on the solid edge, and

$\chi_{ijj'}$ is the flow on the dotted edge that connects the end node of the edge with x_{ij} flow with the start node of the edge with $x_{i+1,j'}$ flow. The per unit cost of flow through the solid edge is $-v_{ij} + c_j$ whereas there is no cost for the flow through the dotted edge. In Figure 3.1, there are two numbers shown above each solid edge. The number outside the parenthesis is the cost per unit flow on the edge. The number inside the parenthesis is the shortest distance to this node from the source node. The actual assignment of consumers to bundles is obtained from the nodes that the shortest path visits.Observe that, the shortest path can be computed by keeping track of the predecessor of each node.

We remarked earlier that the algorithm for solving the shortest path problem on an acyclic network takes linear time. Since this network has $O(IJ^2)$ edges, a straightforward implementation takes this much time. However, we can exploit the network structure to make the algorithm faster. We traverse the nodes, one consumer at a time, from left to right. For each consumer, we visit the nodes from smallest bundle size to largest bundle size. For each pair of consumer and bundle sizes, (i, j), there are two nodes, namely the start and end node of the edge with x_{ij} flow. We denote the longest path to the start node of $(i + 1, j)$ as l_{ij}. Then, define $l_{ij} = \max\{l_{ij-1}, l_{i-1j} + v_{ij} - c_j\}$, where l_{0j} is understood to be zero. These computations take $O(IJ)$ time and solve $\mathrm{CBP3}$. The formulation of this method is exactly the same as that of $\mathrm{CBP2b}$ given in Kannan et al. (2014b).

3.3 Fixed Costs

In this section, we discuss the CB problem with fixed costs in setting up each bundle size. Such an extension is particularly important to sellers, such as on-line music stores, who could provide a huge amount of bundle sizes in the market. In fact, sellers in many industries will limit the number of bundles they provide to the consumers. For example, Eastlink only sells bundles of 1 channel, 12 channels, and 20 channels in the market. One important reason is that setting up and managing each additional bundle size is costly to the seller. We notice that the nonlinear mix-integer model developed in Wu et al. (2008) has a constant fixed cost to add each additional bundle size. Next, we modify $\mathrm{CBP1}$ to handle the situation when the seller incurs a fixed cost f_j if at least one consumer is allocated to Bundle j. All the other settings and assumptions in $\mathrm{CBP1}$ are not changed.

Let o_j be 1 if Bundle $j \in \{0, 1, \ldots, J\}$ is available in the market and 0 otherwise. Then, the CB problem with fixed costs can be formulated as follows:

$$\begin{aligned} \mathrm{CBPf}: \quad \underset{x_{ij}, p_j, o_j}{\text{Max}} \quad & \sum_{i=1}^{I} \sum_{j=0}^{J} x_{ij}(p_j - c_j) - \sum_{j=0}^{J} f_j o_j \\ \text{s.t.} \quad & (3.1), (3.2), (4.2), (3.4) \\ & x_{ij} \leq o_j \ \forall i; \forall j \qquad (3.16) \\ & o_j \in \{0, 1\} \ \forall i, \ \forall j. \qquad (3.17) \end{aligned}$$

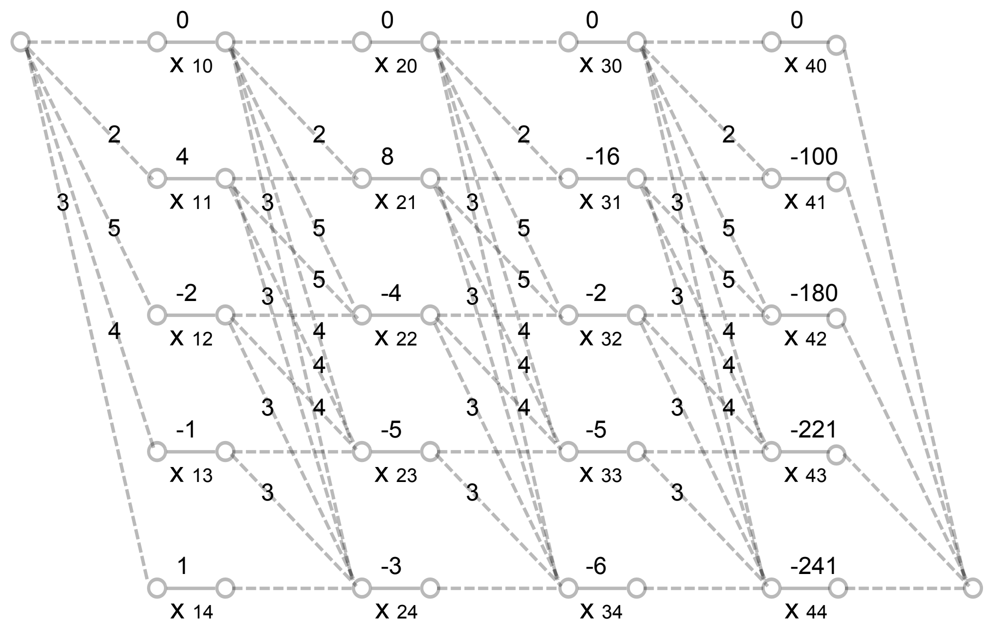
Constraints (3.16) requires Bundle j to be available on the market if it is purchased by any consumer. The additional item $-\sum_{j=0}^{J} f_j o_j$ in the objective function represents the deduction in profit due to the fixed costs. Obviously, if $f_j = 0 \ \forall j$, then CBPf is same as $\mathrm{CBP1}$. Interestingly, we can modify the shortest-path formation $\mathrm{CBP3}$ to make it be able to solve CBPf.

Theorem 21 CBPf *is equivalent to the following shortest path problem on a graph which has* $2I(J+1)+2$ *nodes and* $(I+2)(J+1)+(I-1)(J+1)(J+2)/2$ *edges:*

$$\begin{aligned} \mathrm{CBP3f}: \quad \mathrm{Min}_{x_{ij}, \chi_{ijj'}} \quad & -\sum_{i=1}^{I} \sum_{j=0}^{J} (v_{ij} - c_j) x_{ij} + \sum_{i=1}^{I} \sum_{j=0}^{J} \sum_{j'=j+1}^{J} \chi_{ijj'} f_{j'} \\ \textit{s.t.} \quad & (3.9), (3.10), (3.11), (3.12), (3.13), (3.14), (3.15). \end{aligned}$$

Observe that the only difference between $\mathrm{CBP3}$ and $\mathrm{CBP3f}$ is that there is an additional item

Figure 3.2: The shortest path problem formulation for Example 22



in the objective function of CBP3f: $+\sum_{i=1}^{I}\sum_{j=0}^{J}\sum_{j'=j+1}^{J}\chi_{ijj'}f_j$. The interpretation of this change is actually straightforward. In any solution of CBP3f, when $\chi_{ijj'}=1$ for some i, j, and $j'>j$, it captures that conditionally on Consumer i is allocated to Bundle j, Consumer $i+1$ is allocated to a larger size Bundle j' larger than Bundle j. As a result, the seller incurs a fixed cost $f_{j'}$ to make Bundle j' available and thus reduces his profit by the same amount.

We now add fixed costs to Example 20 to illustrate the underlining shortest-path structure of CBP3f.

Example 22 *In the setting of Example 20, add $f_1=2$, $f_2=5$, $f_3=4$, and $f_4=3$ as fixed costs for setting each bundle size accordingly.*

Figure 3.2 illustrates the associated network of CBP3f. It is quite similar as the network of CBP3 shown in Figure 3.1. The only difference is that in Figure 3.1, there is no cost for any dotted edges, but in Figure 3.2, each downwards dotted edge has a cost for the flow to go through. Each downwards edge represents that the next consumer purchases a different bundle size than the one purchased by the the previous consumer, and therefore, the seller need to incur a corresponding fixed cost to make this bundle size available. Notice that the all there is no cost for any sidewards dotted edge, representing that if the next consumer makes the same purchase decision as the previous consumer, the seller will not incur any new fixed cost. Same as $\mathrm{CBP3}$, the shortest path on this graph represents the optimal solution of $\mathrm{CBP3f}$.

We next show that there exists a fast algorithm to solve $\mathrm{CBP3}$. We next show that $\mathrm{CBP3f}$ can be reformulated to the following $\mathrm{CBP3a}$, and then show that there exists a fast algorithm to solve the dual of it.

Proposition 23 $\forall i \in \{1, 2, \ldots, I\},\ \forall j \in \{0, 1, \ldots, J\}$, *Let* $a_{ij} = x_{ij}, b_{ij} = \sum_{j'=0}^{j-1} x_{ij'} - \sum_{j'=0}^{j-1} x_{i+1,j'}, c_{ij} = \sum_{j'=0}^{j-1} \chi_{i-1,j'j}$, *and* $d_{ij} = x_{i-1,j}$, *where* $x_{0j},\ \forall j$ *and* $\chi_{0jj'},\ \forall j,\ \forall j'$ *are understood to be* 0. *Let* $a_{I+1,j} = b_{I+1,j} = c_{I+1,j} = d_{I+1,j} = 0,\ \forall j\{0, 1, \ldots, J+1\}$, *and* $a_{i,J+1} = b_{i,J+1} = c_{i,J+1} = d_{i,J+1} = 0,\ \forall i\{1, 2, \ldots, I+1\}$. *Then,* $\mathrm{CBP3f}$ *is equivalent to the following* $\mathrm{CBP3a}$*:*

$$\mathrm{CBP3a}: \quad \underset{a_{ij}, b_{ij}, c_{ij}, d_{ij}}{Max} \quad \sum_{i=1}^{I}\sum_{j=0}^{J}(v_{ij} - c_j - f_j)c_{ij} + \sum_{i=1}^{I}\sum_{j=0}^{J}(v_{ij} - c_j)d_{ij}$$

$$\text{s.t.} \quad a_{ij} + b_{ij} - b_{i,j+1} - c_{i+1,j} = 0 \ \forall i,\ \forall j \tag{3.18}$$

$$-a_{i,j+1} + c_{ij} + d_{ij} - d_{i+1,j} = 0 \ \forall i,\ \forall j \tag{3.19}$$

$$a_{IJ} + b_{IJ} + c_{IJ} + d_{IJ} = 1 \ \forall i,\ \forall j \tag{3.20}$$

$$a_{ij} \geq 0; b_{ij} \geq 0; c_{ij} \geq 0; d_{ij} \geq 0 \ \forall i,\ \forall j. \tag{3.21}$$

Theorem 24 CBP3d *is the dual of* CBP3a *and can be solved within* $O(IJ)$ *time.*

$$\text{CBP3d}: \underset{l^1_{ij}, l^2_{ij}}{\textit{Min}} \quad z$$

$$\textit{s.t.} \quad l^1_{ij} \geq l^1_{ij-1} \; i = 0, \ldots, I; \; j = 1, \ldots, J \tag{3.22}$$

$$l^1_{ij} \geq l^2_{ij-1} \; i = 0, \ldots, I; \; j = 1, \ldots, J \tag{3.23}$$

$$l^2_{ij} \geq l^1_{i-1j} + v_{ij} - c_j - f_j \; i = 1, \ldots, I; \; j = 0, \ldots, J \tag{3.24}$$

$$l^2_{ij} \geq l^2_{i-1j} + v_{ij} - c_j \; i = 1, \ldots, I; \; j = 0, \ldots, J \tag{3.25}$$

$$z \geq l^1_{IJ} \tag{3.26}$$

$$z \geq l^2_{IJ} \tag{3.27}$$

$$l^1_{00} = l^2_{00} = 0.$$

3.4 Submodular Cost Function

In this section, we go back to assume there are no fixed costs and focus on another extension on bundle costs. In Kannan et al. (2014b), the authors solve CB problems with both the discrete bundle sizes and the continuous bundle sizes. For both cases, the authors assume a separable cost structure, *i.e.*, the cost of goods sold to a consumer depends only on the bundle size she buys. As mentioned before, this cost structure was studied earlier by Hitt and Chen (2005) and Spence (1980). Observe also that the shortest-path algorithm applies even if the costs are consumer-specific, *i.e.*, c_j are replaced with c_{ij}. Nevertheless, cost structures, such as scale economies, cannot be accommodated even with consumer-specific costs. For example, these models cannot capture a cost-component that is concave in the sum of the bundle sizes sold to the consumers. This is because the cost of selling an additional unit to a consumer depends on what other consumers purchase. In this section, we extend our analyses to a more general cost structure. We remark that although we express these changes in terms of costs, they can also be used to model additional value generated for the vendor by sale of extra goods to consumers. For example, these ideas can be used to model a convex value function that depends on the total sales, which may capture benefits due to externalities or larger market presence.

3.4.1 Discrete Case

In this section, we consider the discrete case where Bundles $0, \ldots, J$ are offered by the vendor, and a cost function $C'(j_1, \ldots, j_I)$ – where, for each i, j_i represents the bundle size allocated to Consumer i – is submodular in $\{j_1, \ldots, j_I\}$. We define $z_{ij} = \sum_{j'=j}^{J} x_{ij}$. Then, observe that $x_{ij} = z_{ij} - z_{ij+1}$, where z_{iJ+1} is understood to be zero. Also, observe that the bundle j_i that Consumer i purchases is:

$$j_i = \sum_{j=0}^{J} j x_{ij} = \sum_{j=1}^{J} \sum_{j'=1}^{j} x_{ij} = \sum_{j'=1}^{J} \sum_{j=j'}^{J} x_{ij} = \sum_{j'=1}^{J} z_{ij'}. \tag{3.28}$$

If we define $C(z) = C'\left(\sum_{j=1}^{J} z_{ij}, \ldots, \sum_{j=1}^{J} z_{Ij}\right)$, then the vendor's decision problem is:

$$\text{CBPg}: \quad \text{Max}_{z_{ij}} \sum_{i=1}^{I} \sum_{j=0}^{J} v_{ij}(z_{ij} - z_{ij+1}) - C(z)$$

$$\text{s.t.} \quad z_{i0} = 1 \qquad \forall i \tag{3.29}$$

$$z_{ij} \geq z_{ij+1} \qquad \forall i;\ \forall j \leq J-1 \tag{3.30}$$

$$z_{ij} \leq z_{i+1j} \qquad \forall i \leq I-1;\ \forall j \tag{3.31}$$

$$z_{ij} \in \{0, 1\} \qquad \forall i;\ \forall j. \tag{3.32}$$

Given the definition of z_{ij}, Constraint (3.29), Constraint (3.31), and Constraint (3.30) are equivalent to Constraint (3.2), Constraint (3.7), and the non-negativity of x_{ij} variables respectively. Therefore:

Proposition 25 *The feasible region of* CBPg *(resp., relaxation of* CBPg *with no integrality constraints on z) is obtained via a one-to-one linear transformation $z_{ij} = \sum_{j'=j}^{J} x_{ij}$ from the feasible region of* CBP2 *(resp., relaxation of* CBPg *with no integrality constraints on x).*

We next show that the feasible region of CBPg forms a lattice family (Schrijver, 2003). A family $\mathcal{C}$ of sets is called a lattice family if for all A, $B \in \mathcal{C}$, it holds that $A \cup B$, $A \cap B \in \mathcal{C}$. A set A can be mapped into binary values using its incidence vector, *i.e.*, a vector χ^A whose entries are labeled with the elements of the universal set and $\chi_i^A = 1 \Leftrightarrow i \in A$, otherwise $\chi_i^A = 0$. We claim that the binary solutions feasible to CBPg are incidence vectors of a lattice family. In order to show this, consider two feasible solutions z^1 and z^2 and construct z' (resp., z'') such that $z'_{ij} = \max\{z^1_{ij}, z^2_{ij}\}$ (resp., $z''_{ij} = \min\{z^1_{ij}, z^2_{ij}\}$). We verify that z' and z'' are feasible to CBPg. First consider z'. Constraints (3.29) and (3.32) are obviously satisfied. The following shows that z' satisfies Constraint (3.30): $z'_{ij} = \max\{z^1_{ij}, z^2_{ij}\} \geq \max\{z^1_{ij+1}, z^2_{ij}\} \geq \max\{z^1_{ij+1}, z^2_{ij+1}\} = z'_{ij+1}$, where the inequalities follow because z^1 and z^2 satisfy Constraint (3.30). Similarly, it follows that z' satisfies Constraint (3.31). The arguments for showing feasibility of z'' are similar. We refer to z' (respectively, z'') as the join (respectively, the meet) of z^1 and z^2 and denote it as $z^1 \vee z^2$ (respectively, $z^1 \wedge z^2$).

Proposition 26 *For any z in the feasible region of* CBPg *and $i \in \{1, \ldots, I\}$, let j_i be as given in (3.28). Then, $C(z)$ is submodular over the feasible region of* CBPg.

We remark that Proposition 26 only shows the submodularity of $C(z)$ for points feasible to CBPg.[2] We now show that CBPg can be solved in strongly polynomial time, which thereby yields an efficient algorithm for the vendor to price the bundles.

[2]Consider, for example, allocations x^1 and x^2 such that, for some i, $x^1_{i1} = 0.5$, $x^1_{i3} = 0.5$, and $x^2_{i2} = 1$. Let z^1 and z^2 be the corresponding solutions in the z-space and define $z' = z^1 \vee z^2$ and $z'' = z^1 \wedge z^2$. Then, it follows that $z^1_i = (1, 1, 0.5, 0.5)$ and $z^2_i = (1, 1, 1, 0)$. Using (3.28), the first solution corresponds to $j^1_i = 2$, and the second solution also corresponds to $j^2_i = 2$. But, $z'_i = (1, 1, 1, 0.5)$ and $z''_i = (1, 1, 0.5, 0)$ and the corresponding $j'_i = 2.5$ whereas $j''_i = 1.5$. Therefore, for general z, the submodularity does not follow.

Theorem 27 *If $C(z)$ is submodular over the feasible region of* CBPg*, and for a given z, $C(z)$ can be evaluated in strongly polynomial time, then* CBPg *can be solved in strongly polynomial time.*

We remark that the algorithm that is used in Theorem 27 to solve CBPg in polynomial time is based on extending $C(z)$ from the lattice family to a submodular function over $[0,1]^{I\times(J+1)}$, which still attains the same maximum. Then, the new function can be maximized in polynomial time using the algorithm of Schrijver (2000) or Iwata et al. (2001). Exposing the structure of this problem brings many tools from supermodular optimization that can be used to bear on the cardinality bundling problem. For example, one can readily say that the optimal solutions of CBPg forms a non-empty subcomplete lattice of its feasible set (Corollary 2.7.1 in Topkis, 1998).

We now consider a more interesting application. Assume that $C(z) = C''(m,z)$ where m are some parameters of the cost function. Assume $C''(m,z)$ is submodular in $(-m,z)$ space. Then, it follows from Theorem 2.7.6 in Topkis (1998) that the optimal solution is a supermodular function of m. In the setting of Section 3.2, this can be interpreted by letting m denote the marginal cost vector, *i.e.*, $m_j = c_j - c_{j-1}$. Assume now two settings, with marginal costs m^1 and m^2 and assume $m^1 \geq m^2$. Then, the reduction in profit for an increase in m^1 by some Δ is no more than the reduction in profit for an increase in m^2 by the same Δ.

In this setting, we can also extend the result of Corollary 11 in Kannan et al. (2014b). According to this corrollary, if the marginal cost decreases, then every consumer will shift to purchase weakly larger sized bundles. First, we provide a standard definition of set ordering. Given two sets of allocations X' and X'' we say that a set $X' \sqsubseteq X''$, or that X' is smaller than X'', if for every allocation $z' \in X'$ and $z'' \in X''$, it holds that $z' \wedge z'' \in X'$ and $z' \vee z'' \in X''$.

Corollary 28 *If $C''(m,z)$ is submodular in (m,z) then the set of optimal solutions of* CBPg *are increasing in m, i.e., consumers buy non-increasing bundle sizes with increase in m. Let $m^1 > m^2$ and assume that $C''(m,z)$ has strictly increasing differences in (m,z). If z' (resp. z'') is an optimal solution to* CBPg *with m^1 (resp. m^2) then $z' \geq z''$.*

3.4.2 Continuous Case

In this section, we explore the extension of the continuous case to allow for submodular cost functions. The continuous case with separable costs are discussed in Kannan et al. (2014b). Here we first briefly review the model. The model is similar to CBP1 except that we use a continuous variable $y \in \mathbb{R}_+$ to represent the bundle sizes, instead of using an index j to denote discrete sizes. Then all the other variables x_{ij} and p_j, and parameters w_{ij} that are previously indexed with j ,now become a function of y: $x_i(y), p(y),$ and $w_i(y)$. Thus the continuous problem can be modeled as:

$$\begin{aligned} \mathrm{CBPc1}: \quad & \operatorname*{Max}_{y_i, p(y)} \quad \sum_{i=1}^{I} (p(y_i) - c(y_i)) \\ & \text{s.t.} \quad w_i(y_i) - p(y_i) \geq w_i(y) - p(y) \; \forall i \; \forall y. \qquad (3.33) \end{aligned}$$

Next, based on CBPc1, we discuss the continuous case with submodular cost function. As before, we define $v_i(y) = w_i(y) - (I-i)(w_{i+1}(y) - w_i(y))$ for all i, where $w_{I+1}(y)$ is assumed to be $w_I(y)$. Then, , we reformulate the cardinality bundling problem as:

$$\begin{aligned} \mathrm{CBPgcy}: \quad & \mathrm{Max}_{y_i} \quad \sum_{i=1}^{I} v_i(y_i) - C'(y_1, \ldots, y_I) \\ & \text{s.t.} \quad y_{i+1} \geq y_i \qquad 1 \leq i \leq I-1 \qquad (3.34) \\ & \qquad 0 \leq y_i \leq Y \quad \forall i, \qquad (3.35) \end{aligned}$$

where we assume that $C'(y)$ is submodular in y. Let $k(\cdot) : \{0, \ldots, J\} \to [0, Y]$ be such that $0 = k(0) < \cdots < k(J) = Y$. We assume that for all i, $w_i(y)$ are piecewise-linear with breakpoints that belong to $\{k(0), \ldots, k(J)\}$. In Kannan et al. (2014b), we also assumed piecewise-linearity for the cost. We extend this assumption to the current setting. Observe that this requires some detail since $C'(\cdot)$ is now a multi-dimensional function.

Consider a $y \in \mathbb{R}^I$ that is feasible to CBPgcy. For each i, let $a(y_i) = \arg\min_j \{y_i - k(j) \mid k(j) \le y_i\}$. If $k(a(y_i)) = Y$ define $g(y_i) = 0$, otherwise define $g(y_i) = \frac{y_i - k(a(y_i))}{k(a(y_i)+1) - k(a(y_i))}$. Assume that $\pi = (\pi(1), \ldots, \pi(I))$ is a permutation of $\{1, \ldots, I\}$ that sorts $g(y_i)$ such that $g(y_{\pi(1)}) \ge \cdots \ge g(y_{\pi(I)})$. If $g(y_i) = g(y_{i'})$ for some $i < i'$, we assume that $\pi^{-1}(i') < \pi^{-1}(i)$. Let e^i be a unit vector such that $e^i_i = 1$ and $e^i_{i'} = 0$ for $i \ne i'$. For $r = 0, \ldots, I$, define $a^r = (a(y_1), \ldots, a(y_I)) + \sum_{i=1}^r e^{\pi(i)}$. Let y^r be defined so that $y^r_i = k(a^r_i)$ if $a^r_i \le J$ and $y^r_i = Y$ otherwise. It is clear that, for all i, $y^r_i \in \{k_0, \ldots, k_J\}$. Therefore, y^r satisfies Constraint (3.35). We now argue that $a^r_i \le a^r_{i+1}$ for $i \le I - 1$. Observe that this implies that y^r satisfies Constraint (3.34) and is feasible to CBPgcy because $k(j) \le k(j')$ for $j < j'$.

Lemma 29 $a^r_i \le a^r_{i+1}$ *for* $i \le I - 1$. *Consequently,* y^r *is feasible to* CBPgcy.

For notational convenience, define $\pi(0) = 0$, $\pi(I+1) = I+1$, $g(y_0) = 1$, and $g(y_{I+1}) = 0$. We now assume that $C'(y) \ge \sum_{r=0}^{I} \left(g(y_{\pi(r)}) - g(y_{\pi(r+1)})\right) C'(y^r)$. For example, we discuss later that this property is satisfied by piecewise linear or concave cost functions. Observe that, for all r, $g(y_{\pi(r)}) - g(y_{\pi(r+1)}) \ge 0$ and $\sum_{r=0}^{I} \left(g(y_{\pi(r)}) - g(y_{\pi(r+1)})\right) = g(y_0) - g(y_{I+1}) = 1$.

Furthermore

$$
\begin{aligned}
&\sum_{r=0}^{I} \big(g(y_{\pi(r)}) - g(y_{\pi(r+1)})\big)\, y_i^r \\
&= k(a(y_i)) \sum_{r=0}^{\pi^{-1}(i)-1} \big(g(y_{\pi(r)}) - g(y_{\pi(r+1)})\big) + k(a(y_i)+1) \sum_{r=\pi^{-1}(i)}^{I} \big(g(y_{\pi(r)}) - g(y_{\pi(r+1)})\big) \\
&= k(a(y_i))(1 - g(y_i)) + k(a(y_i)+1)g(y_i) \\
&= y_i.
\end{aligned}
\tag{3.36}
$$

Therefore, y can be expressed as a convex combination of feasible points y^r, $r = 0, \ldots, I$ and we have assumed that the convex combination underestimates the cost. This hypothesis is sufficient to show that the solution of the continuous cardinality bundling problem can be restricted to lie on the breakpoints.

Theorem 30 *Assume $w_i(\cdot)$ are piecewise linear with breakpoints in $\{k(0), \ldots, k(J)\}$. Further, assume that $C'(y)$ is submodular and $C'(y) \geq \sum_{r=0}^{I} \big(g(y_{\pi(r)}) - g(y_{\pi(r+1)})\big)\, C'(y^r)$. Then, there exists an optimal solution y^* to* CBPgcy *that is such that, for all i, $y_i^* \in \{k(0), \ldots, k(J)\}$. Let z^* be optimal to*

$$
\mathrm{CBPgcz} : \mathrm{Max}_{z_{ij}} \left\{ \sum_{i=1}^{I} \sum_{j=0}^{J} v_{ij}(z_{ij} - z_{ij+1}) - C(z) \mid (3.29), (3.30), (3.31), (3.32) \right\},
$$

where, $C(z) = C'\left(\sum_{j=1}^{J}\big(k(j) - k(j-1)\big) z_{1j}, \ldots, \sum_{j=1}^{J}\big(k(j) - k(j-1)\big) z_{Ij}\right)$. Then, y^* may be chosen such that $y_i^* = \sum_{j=1}^{J}\big(k(j) - k(j-1)\big) z_{ij}^*$.

Since, by Theorem 27, CBPgcz can be solved in strongly polynomial time, we have the following:

Corollary 31 *Assume* $w_i(\cdot)$ *are piecewise linear with breakpoints in* $\{k(0), \ldots, k(J)\}$. *Further, assume that* $C'(y)$ *is submodular and* $C'(y) \geq \sum_{r=0}^{I} \left(g(y_{\pi(r)}) - g(y_{\pi(r+1)})\right) C'(y^r)$. *Then,* CBPgcy *can be solved in strongly polynomial time (assuming* $k(0), \ldots, k(J)$ *are part of the input).*

We remark that Theorem 30 generalizes Theorem 16 and Corollary 17 in Kannan et al. (2014b). This is because the cost functions treated in Kannan et al. (2014b) are additively separable, *i.e.*, sum of one-dimensional functions, which are always submodular. Further, piecewise-linearity assumed in Kannan et al. (2014b) is a special case of the requirement $C'(y) \geq \sum_{r=0}^{I} \left(g(y_{\pi(r)}) - g(y_{\pi(r+1)})\right) C'(y^r)$ in Theorem 30. In fact, in the case of one-dimensional functions, the right-hand side is precisely the piecewise linear function with breakpoints at $(k(0), \ldots, k(J))$. Since the inequality holds trivially, the generalization follows. Similarly, the following result can be easily obtained.

Corollary 32 *Assume* $w_i(\cdot)$ *are piecewise linear with breakpoints in* $\{k(0), \ldots, k(J)\}$. *Further, assume that* $C'(y)$ *is concave and submodular. Then,* CBPgcy *can be solved using* CBPgcz *in strongly polynomial time (assuming* $k(0), \ldots, k(J)$ *are part of the input).*

Now, we consider the general case, where the WTP and cost functions are not necessarily piecewise-linear. We assume that $w_i(\cdot)$ and $C'(\cdot)$ are Liptschitz continuous with Lipschitz constant β, *i.e.*, for all i, $|w_i(y_i) - w_i(y_i')| \leq \beta|y_i - y_i'|$ and $|C'(y) - C'(y')| \leq \beta\|y - y'\|$. We show that Theorem 30 gives an approach to approximate the solution of this more general problem. We construct piecewise-linear approximations of $w_i(y)$ and $C'(\cdot)$. Assume we choose $k = \frac{\epsilon}{2\beta(I^2+\sqrt{I})\beta}$ and $J = \lceil \frac{Y}{k} \rceil$. We let $k(j) = jk$ for $j = 0, \ldots, J-1$ and $k(J) = Y$. Then, we define: $w_i^k(y) = \frac{k_{j+1}-y}{k_{j+1}-k_j} w_i(k(j)) + \frac{y-k_j}{k_{j+1}-k_j} w_i(k(j+1))$ and ${C'}^k(y) = \sum_{r=0}^{I} \left(g(y_{\pi(r)}) - g(y_{\pi(r+1)})\right) C'(y^r)$. Observe that $w_i^k(\cdot)$ and ${C'}^k(\cdot)$ satisfy the hypotheses of Theorem 30. Let Π^c be the optimal value of CBPgcy and Π^k denote the optimal profit when

$w_i^k(\cdot)$ and ${C'}^k(\cdot)$ are the WTP for Consumer i and the cost function respectively.

Theorem 33 *For a given ϵ, define $k = \frac{\epsilon}{2\beta(I^2+\sqrt{I})\beta}$. Then, $\Pi^k \leq \Pi^c \leq \Pi^k + \epsilon$. Further, Π^k can be computed in time that is polynomial in I, Y, β, $\frac{1}{\epsilon}$, and the time taken by the oracle call to compute $C'(y)$.*

3.5 Conclusion

In this paper, we first extend the existing CB model to allow fixed costs in adding additional bundles. We show that CB problem with fixed costs can be solved as a shortest-path problem. We then extend the CB model in another way to solve CB problem with submodular cost structure. Such an analysis is especially useful when there exists economies of scale in production.

Chapter 4

Cardinality bundles with Constrained Prices

4.1 Introduction

This paper studies a bundling schemed called cardinality bundling (CB). in CB, sellers price for the number of goods and let consumers choose with specific products they what. Pricing for toppings of pizza is a simple example of CB. In many pizza stores, consumers are priced for the number of toppings regardless of the specific topping types. Similarly, Disney World uses CB to sell theme park tickets. Instead of selling tickets for each park separately, Disney World prices consumers for the number of visits to all its theme parks. More generally, information goods providers such as Netflix and Blockbuster, telecommunication service providers such as AT&T, and cable TV providers such as Eastlink, are also implementing CB in selling their products or services.

The current literature on CB is relatively sparse and we review it briefly here. Most relevant to the current paper is Hitt and Chen (2005), where they study the pricing of cardinality bundles assuming that each consumer is restricted to buy at most one bundle. They explore conditions under which CB can attain the same profit as mixed bundling. Further assuming that consumers' reservation price satisfy Spence-Mirrlees Single Crossing Property (SCP), they propose and analyze a readily Wu et al. (2008) also restrict the consumer to purchase at most one bundle and seek to solve the CB pricing problem as a nonlinear mixed-integer program. They develop a heuristic algorithm based on Lagrangian relaxation and subgradient ascent to solve the problem, and provide a lower-bound on the profit. Even at the termination of their algorithm, they report a significant gap between the lower- and upper-bounds on the profit. Chu et al. (2011) consider a CB model where unit prices for bundles decrease with increasing size. They use computations and real data to argue that profit from their CB model is almost the same as that from mixed bundling. Lahiri et al. (2013) study cardinality bundling in the context of pricing of wireless services. They analytically compare CB with another pricing regime in which each wireless service is charged separately and show that both regimes may perform better than the other under different conditions. computable pricing strategy.

Kannan et al. (2014b) analytically studies the optimal pricing strategies for CB problems with SCP consumer valuations. They show that the optimal prices to the problem can be obtained, in strongly polynomial time, by solving a shortest-path problem. Based on the network structure underlying the shortest path formulation, they develop an algorithm to solve the quantity-discount problem proposed by Spence (1980). Lastly, they also study the characteristics of the underlying problem that lead to similar strongly polynomial time solution approaches.

The models in Hitt and Chen (2005) and Kannan et al. (2014b) assume that each consumer can only purchase no more than one bundle. This assumption is valid in some industries. For example, each home usually has no more than one cable TV connection and therefore is only able to purchase at most one cable TV bundle. Other examples include toppings of pizza and cellular data pricing plans. However, in some other industries, consumers are not restricted to only purchase one bundle. For example, consumers can easily purchase multiple bundles of songs at on-line music stores. As a result, the insights obtained by these works do not necessarily extend to situations when the consumers may purchase more than one bundle. In this paper, we relax the one bundle per consumer assumption. We introduce sub-additive constraints on bundle prices to ensure that the consumer incentive compatibility is not violated even if consumers are allowed to purchase more than one bundle of goods.

In reality, three main types of sub-additive price schemes are used in different industries. (1) Marginal decreasing prices (MDP) where the marginal price of each additional unit is weakly decreasing, which is also known as multiple-part tariff pricing (Wilson, 1993). (2) Unit decreasing prices where the unit price of each bundle is weakly decreasing. Since this type of price scheme is first introduced by Chu et al. (2011) as bundle-size pricing (BSP), we also call it BSP in this paper. (3) General form of sub-additive prices (CBSP) where the price of any bundle is no less than the total price of any two other bundles which can together form the previous one. In this paper, we study these various kinds of CB problems with different constraints on bundle prices. In order to get tractable and meaningful results, we additionally assume Spence-Mirrlees Single Crossing Property (SCP) on consumers' reservation price. We first develop a shortest-path solution approach for MDP. Second, we propose a dynamic programming algorithm to solve BSP. Third, we analyze the CB problem with sub-additive prices and convert its MINLP formulation to a mixed-integer programming (MIP) one. Finally, we provide analytical and numerical analysis on the gaps between different CB models.

4.2 Marginal Decreasing Prices (MDP)

4.2.1 Model

In this section, we consider the cardinality bundling problem with marginal decreasing prices. The model is built upon that in Kannan et al. (2014b) and we review it here for the sake of completeness. Consider a vendor who sells J products and assume that there are I consumers in the market. In the following, we denote the bundle of size j as Bundle j. We assume WLOG that all bundles, $1, \ldots, J$ are offered in the market and the vendor decides their prices. We denote the price of Bundle j as p_j. Obviously, the consumer does not pay anything for Bundle 0, whose price is therefore fixed at 0. We assume that the cost of the Bundle j for vendor is c_j and that the total cost to the vendor is the sum of the costs for all the bundles sold. Clearly, c_0 is 0. The model makes a reasonable assumption that a consumer's willingness-to-pay (WTP) is non-decreasing with the bundle size, which would be trivially true if extra units can be freely disposed.[1] The model further assumes WTP for each consumer to be concave in j.

Let $w_{ij} \geq 0$ denote the WTP of Consumer i for Bundle j. For every i, we set w_{i0} to zero to denote that consumers, who do not purchase anything, do not derive any value out of the vendor's products. Since WTPs are non-decreasing with bundle size, $w_{ij} \geq w_{ij'}$ for $j \geq j'$. Since the choice of the bundle rests with the consumer, if Consumer i purchases Bundle j_i, this bundle must maximize her consumer surplus, *i.e.*, $j_i \in \arg\max_j \{w_{ij} - p_j\}$. Let J_i be the set of bundles Consumer i prefers with price vector p. If $|J_i| > 1$, we assume that Consumer i purchases a Bundle j_i that belongs to $\arg\max_j \{p_j - c_j \mid j \in J_i\}$, *i.e.*, the surplus-maximizing bundle that yields the most profit to the vendor. This assumption is typical in the literature and is without loss of generality.[2]

Kannan et al. (2014b) assume that each consumer can only purchase no more than one bundle. We relax this assumption in this paper and allow consumers to purchase more than one bundle. As a result, such a relaxation leads to enforce the seller to always set the bundle prices that satisfies sub-additive constraints. In this section, we impose the first kind of sub-additive prices: marginal decreasing prices, or MDP. It is straightforward that if the marginal price for

[1] Hitt and Chen (2005) imposes WTP for each consumer to be concave in j, which we relax in our model.
[2] see Kannan et al. (2014b) for more details about this assumption.

each additional unit is weakly decreasing, than the price of any bundle will always be weakly less than the total price of any other two smaller-sized bundles which can form the previous one. As a result, any rational consumer will never purchase more than one bundle.

Let x_{ij} be 1 if Consumer $i \in \{1, 2, \ldots, I\}$ buys Bundle $j \in \{0, 1, 2, \ldots, J\}$ and 0 otherwise. Then, MDP can be formulated as follows:

$$\text{MDP1}: \quad \underset{x_{ij}, p_j}{\text{Max}} \quad \sum_{i=1}^{I} \sum_{j=0}^{J} x_{ij}(p_j - c_j)$$

$$\text{s.t.} \quad \sum_{j'=0}^{J} (w_{ij'} - p_{j'}) x_{ij'} \geq w_{ij} - p_j \ \forall i, \ \forall j \tag{4.1}$$

$$p_0 = 0 \tag{4.2}$$

$$p_j - p_{j-1} \leq p_{j-1} - p_{j-2} \ \forall j \geq 2 \tag{4.3}$$

$$\sum_{j=0}^{J} x_{ij} = 1 \ \forall i \tag{4.4}$$

$$x_{ij} \in \{0, 1\} \ \forall i, \ \forall j. \tag{4.5}$$

Let (x^*, p^*) be a solution that generates the maximum profit for the vendor. Constraints (4.1) enforce incentive compatability (IC) and individual rationality (IR) for Consumer i. The left hand side models the consumer surplus from the purchase decision and the right hand side models the consumer surplus from the purchase of alternate bundles. The case with $j = 0$ ensures that consumer only purchases bundles with non-negative surplus. Constraints (4.3) enforce the marginal price of each bundle to be weakly decreasing. Constraints (4.4) enforce that each consumer purchases only one bundle.

Like in other nonlinear pricing problems, Kannan et al. (2014b) assume that consumer valuations satisfy the Spence-Mirrlees Single Crossing Property (SCP) (see Spence, 1980). We also make the same assumption, which imposes the following ordering on the consumers' WTP for the bundles:

$$w_{ij} \geq w_{i'j} \; \forall i > i', \tag{4.6}$$

$$w_{ij} - w_{ij'} \geq w_{i'j} - w_{i'j'} \; \forall i > i', \; \forall j > j'. \tag{4.7}$$

The interpretation of these conditions is straightforward. A consumer with a higher index has a (weakly) higher WTP for any bundle. Also, the WTP exhibits increasing differences, *i.e.*, as bundle size increases, the WTP for a higher-indexed consumer increases more rapidly than the WTP for a lower-indexed consumer. Essentially, this assumption states that consumers can be ordered by types, with higher type consumers valuing the products and marginal changes in bundle sizes more than the lower type ones.

4.2.2 Properties of the Optimal Solution

First, we identify some properties of the optimal solution.[3] Let $w_{ij}^m = w_{ij} - w_{i,j-1} \; \forall i \; \forall j \geq 1$ and $w_{i0}^m = 0$ be the marginal WTP of each consumer i for each additional unit of goods j. Let p_j^m be the marginal price for each unit of goods j. Similarly, let c_j^m be the marginal cost for each unit of goods j.

Proposition 34 *There exists an optimal solution to* MDP1 *that satisfies:*

$$\sum_{j'=j}^{J} x_{i+1j'} \geq \sum_{j'=j}^{J} x_{i,j'} \; i = 1, \ldots, I-1, \; \forall j. \tag{4.8}$$

That is, there exists an optimal solution where the mapping from consumer types to bundle sizes is non-decreasing, i.e., for any $i < I$, if Consumer i buys Bundle j, then Consumer $i+1$ buys a Bundle j' such that $j' \geq j$. Further, for any given price vector, there exists a feasible allocation of bundle sizes to consumer types that is non-decreasing.

[3] All the proofs are omitted.

Proposition 35 *There exists an optimal pricing scheme such that if two bundle sizes j and j' are bought by some consumers and $j' > j$ then $p_{j'} - c_{j'} > p_j - c_j$.*

Lemma 36 *Among the consumers purchasing a non-zero bundle size, the lowest indexed one is charged at her WTP in every optimal solution.*

Proposition 34, 35, and Lemma 36 are proved in Kannan et al. (2014b) when there is no constraints on bundle prices. In this paper, we show that in CB models such as MDP, BSP, or CBSP, where the bundle prices are constrained with various kinds of conditions, Proposition 34, 35, and Lemma 36 are still valid and therefore can be useful to derive solution approaches for these problems.

In $\mathrm{MDP1}$, we have an additional property that plays a critical roll to solve the problem.

Proposition 37 *In the optimal solution, p_j^{m*}, the optimal marginal price for any unit j, satisfies the following condition:*

$$p_1^{m*} \in \{w_{11}^m, \cdots, w_{I1}^m\},\ p_j^{m*} \in \{w_{1j}^m, \cdots, w_{i'j}^m, p_{j-1}^{m*}\} \forall j \geq 2,$$

where $i' = \arg\mathrm{Max}_i\{w_{ij}^m \leq p_{j-1}^{m*}\}$. That is, the optimal marginal price for any unit j is priced at the same marginal price as Bundle $j-1$, or at some consumer's marginal WTP on Bundle j that is no more than p_{j-1}^{m*}.

By Proposition 37, we know that $p_1^{m*} \in \{w_{11}^m, \cdots, w_{i'1}^m\}$, or, the marginal price of the first unit is priced at one consumer's marginal WTP for the first unit. Next, we can easily get $p_2^{m*} \in \{w_{11}^m, \cdots, w_{I1}^m,$
$w_{12}^m, \cdots, w_{I2}^m\}$, or, the marginal price of the first unit is priced at one consumer's marginal WTP for the first two units. Recursively, we have $p_j^{m*} \in \{w_{11}^m, \cdots, w_{I1}^m, w_{12}^m, \cdots, w_{I2}^m, \cdots, w_{1j}^m, \cdots, w_{Ij}^m\}$.

Proposition 38 *Let* $v_{jij'} = (I - i' + 1)(w^m_{ij'} - c^m_j)$ *where* $i' = \arg\text{Min}_{i''}\{w^m_{i''j} \geq w^m_{ij}\}$. $v_{jij'}$ *capture how the total profit will change if the marginal price of Bundle* j *is priced at the marginal WTP of Consumer* i *for Bundle* j'.

Next, we convert MDP1 to a shortest path problem. Let $x_{jij'}, j' \leq j$ be a binary variable to indicate whether the marginal price of Bundle j is priced at the marginal WTP of Consumer i for Bundle j'. Let $chi^{\tilde{i}\tilde{j}'}_{jij'}, j' \leq j, \tilde{j}' \leq j - 1$ be a binary variable to indicate whether the marginal price of Bundle j is priced at $w^m_{ij'}$ while that of Bundle $j - 1$ is priced at $w^m_{\tilde{i}\tilde{j}'}$. Let $v^{\tilde{i}\tilde{j}'}_{jij'}, j' \leq j, \tilde{j}' \leq j - 1$ captures how the total profit will change if $\chi^{\tilde{i}\tilde{j}'}_{jij'} = 1$.

$$v^{\tilde{i}\tilde{j}'}_{jij'} = \begin{cases} 0 & \text{if } w^m_{ij'} = w^m_{\tilde{i}\tilde{j}'} \\ 0 & \text{if } w^m_{ij'} \leq w^m_{\tilde{i}\tilde{j}'} \text{ and } j' = j \\ -\sum_{i=1}^{I} w_{iJ} & \text{otherwise.} \end{cases} \tag{4.9}$$

Then we can reformulate MDP1 to the following shortest-path problem:

Theorem 39 MDP1 *is equivalent to the following shortest path problem on a graph which has* $(I + 1)I(J + 1) + 2$ *nodes and no more than* $I^3(J + 1)^2/2$ *edges:*

$$\text{MDP2}: \quad \mathop{\textit{Min}}_{x_{jij'},\chi^{\tilde{i}\tilde{j}'}_{jij'}} \quad -\sum_{j=0}^{J}\sum_{i=1}^{I}\sum_{j'=1}^{j} v_{jij'}x_{jij'} - \sum_{j=0}^{J}\sum_{\tilde{i}=1}^{I}\sum_{\tilde{j}'=1}^{j-1}\sum_{i=1}^{I}\sum_{j'=1}^{j} v^{\tilde{i}\tilde{j}'}_{jij'}\chi^{\tilde{i}\tilde{j}'}_{jij'}$$

$$\textit{s.t.} \quad \sum_{i=1}^{I} x_{1i1} = 1 \tag{4.10}$$

$$\sum_{j'=1}^{J}\sum_{i=1}^{I} x_{Jij'} = 1 \tag{4.11}$$

$$\sum_{\tilde{i}=1}^{I}\sum_{\tilde{j}'=1}^{j-1} \chi^{\tilde{i}\tilde{j}'}_{jij'} = x_{jij'}, \ \forall i \ \forall j \ \forall j' \leq j \tag{4.12}$$

$$x_{j\tilde{i}\tilde{j}'} = \sum_{i=1}^{I}\sum_{j'=1}^{j} \chi^{\tilde{i}\tilde{j}'}_{jij'}, \ \forall \tilde{i} \ \forall j \ \forall \tilde{j}' \leq j-1 \tag{4.13}$$

$$\chi^{\tilde{i}\tilde{j}'}_{jij'} \in \{0,1\} \ \forall i \ \forall j \ \forall j' \leq j \ \forall \tilde{i} \ \forall \tilde{j}' \leq j-1. \tag{4.14}$$

4.3 Unit Decreasing Prices (BSP)

Instead of imposing the marginal decreasing prices, another way to solving the problem without the single bundle restriction on the consumer is to impose a non-increasing unit price constraint on the prices set by the vendor. In such a case, naturally, no consumer will have an incentive to buy more than one bundle to form the bundle of her desired size. Chu et al. (2011) considers such a restriction, $\frac{p_j}{j} \geq \frac{p_{j+1}}{(j+1)} \ \forall j \leq J-1$, in their cardinality bundling formulation and refer to it as the Bundle-Size Pricing (BSP). The vendor's decision problem is then

$$\text{BSP1}: \quad \mathop{\text{Max}}_{x_{ij},p_j} \quad \sum_{i=1}^{I}\sum_{j=0}^{J} x_{ij}(p_j - c_j)$$

$$\text{s.t.} \quad (4.1), (4.2), (4.4), (4.5),$$

$$p_j/j \geq p_{j+1}/(j+1) \ \forall j \leq J-1 \tag{4.15}$$

The non-increasing unit price constraint is specified in Equation (4.15). Because of this constraint, the problem does not retain the structure of the shortest-path problem for $\mathrm{MDP1}$. Notice, Proposition 34, 35, and Lemma 36 are still valid under $\mathrm{BSP1}$ but Proposition 37 is not valid anymore. We next develop some new properties for $\mathrm{BSP1}$. From now on, we relax the concavity assumption on consumers' WTP.

Proposition 40 *For a given price scheme, assume that Bundle j is purchased by some consumer(s). Also assume that $p_{j+1}, p_{j+2}, \cdots, p_J$ are all high enough so that no consumer purchase any bundle size greater than j. If we reduce p_{j+1} to a certain level such that some consumer change to purchase p_{j+1}, then this consumer purchases Bundle j before p_{j+1} is changed.*

Next, we develop a unit-price based dynamic programming algorithm for solving BSP_1 when the costs are separable in bundle sizes. In this algorithm, the unit prices can only take discrete values. The feasible set of unit prices correspond to a grid of length ϵ. There are total K points on the grid. K is determined by $K = W_{IJ}/\epsilon$, where W_{IJ} is last consumer's willingness-to-pay for the largest bundle size. We use the variable $k = 1$ to denote the individual grid points and u_k as the corresponding unit price. largest bundle size. We use $k = 0, 1, \ldots, K$ for grid step index and u_k for unit price on grid step k.

According to the definition of BSP, all bundle sizes are available in the market and the unit price of each bundle is no more than a smaller-sized bundle. Our algorithm start with finding out the maximum total profit when the unit price of Bundle 1 is priced at u_k and the unit price of any other larger-sized bundle is also priced at u_k. This situation is same as providing all bundles with the same unit price u_k. For each grid index k, we can easily find out which consumer i is the lowest type consumer starting to purchase and how many units she want to purchase according to her WTP. Similarly, we can also find out how many units each other higher type consumer purchases and then get the total profit by the vendor. We denote this profit value as Π_{i1k}. More generally, let Π_{ijk} be the maximum total profit if bundle size j is the first one to be provided at unit price u_k (i.e., the unit price of any smaller-sized bundle is greater than u_k) and consumer i is the first one to start purchasing this bundle.

We have already show how to calculate Π_{i1k} $\forall i,\ \forall k$. We can then calculate Π_{i2k} $\forall i,\ \forall k$, based on Π_{i1k} results. We use a function $\Delta(i, 2, k, i', 1, k')$ to calculate the change in total profit for a reducing in unit price. It basically calculates which consumers will switch from purchasing Bundle 1 to Bundle 2 because of the availability of Bundle 2 and how many units each of these consumers purchase with the new unit price. Therefore, we have $\Pi_{i2k} = \max \Pi_{i'1k'} + \Delta(i, 2, k, i', 1, k')$. By using the same recursive logic, we can continue to calculate Π_{ijk} for any larger bundle size j as well and can finally find the optimal solution for the BSP problem.

The pseudo-code for the algorithm is shown as follows:

for $i, j; i <= I, j <= J$ **do**
  $u_0 = w_{ij}/j$;
  $\pi_{ijK} = \Pi(i, j)$;
  for $i_1, j_1, k_1; i \leq i_1 \leq I, j \leq j_1 \leq J, k_1 \leq K$ **do**
    for $i_2, j_2, k_2; i_2 \leq i_1, j_2 \leq j_1, k_2 \leq k_1$ **do**
      $\Pi_{temp} = \Pi_{i_2 j_2 k_2} + \Delta(i_2, j_2, k_2, i_1, j_1, k_1)$
      if $\Pi_{temp} \geq \Pi_{i_1 j_1 k_1}$ **then**
        $\Pi_{i_1 j_1 k_1} = \Pi_{temp}$
      end if
    end for
  end for
  if $\max_k\{\Pi_{IJk}\} > \Pi_{max}$ **then**
    $\Pi_{max} = \max_k\{\Pi_{IJk}\}$;
  end if
end for

Theorem 41 *When the costs are separable in bundle sizes, for any given total error ϵ_t, let the grid step length parameter be $\epsilon = 2\epsilon_t/(J+1)JI$. Then the proposed algorithm guarantees that the gap between the optimal profit and the solution generated by the algorithm is no more than ϵ_t. Moreover, the computation complexity is $O(I^3 J^4 K^2)$, where $K = W_{IJ}/\epsilon$.*

4.4 Sub-Additive Price (CBSP)

In the previous section, we remove consumers' incentives to purchase more than one bundle by imposing the non-decreasing unit price constraint. However, in some cases it may be a more strict constraint than necessary. The following example illustrates that the non-decreasing unit price constraint may reduce vendor's profits.

Example 42 *A music store can offer a single song for $4 each, and a bundle size 10 for $10. If someone wants 11 songs, she needs to pay $14 to get a bundle and a single song, which has a higher unit price than that of bundle size 10. Imposing the non-decreasing unit price constraint in this scenario will reduce vendor's profits.*

To overcome this issue, we propose a CBSP model, cardinality bundling problems with sub-additive prices, in this section. Formulating the CBSP problem is similar to $\mathrm{BSP1}$, except replacing the non-increasing unit price Constraints (4.15) with the following price sub-additivity constraints:

$$p_j \leq p_{j'} + p_{j-j'} \ \forall j \ \forall j' < \frac{1}{2}(j+1)$$

Proposition 43 *Solutions to CBP and BSP are respectively the lower and upper bounds for CBSP.*

CBP is the CB problem without any constraints on bundle prices. It is easy to understand the rationale behind this result. On one hand, CBP is the same problem as CBSP except that the price sub-additivity constraints are relaxed. One the other hand, price constraints in BSP are stricter constraints than sub-additivity constraints in CBSP, leading to an underestimation of CBSP.

When the costs are separable, it is possible to create an MIP formulation. Notice that the nonlinearity of the objective function in CBSP comes from $x_{ij}p_j$. Therefore, we introduce

$q_{ij} = x_{ij}p_j$ to replace all the nonlinear items. By adding Constraints (4.20) - (4.23), we can reformulate CBSP as an MIP:

$$\text{CBSP1}: \quad \underset{x_{ij}, q_{ij}}{\text{Max}} \quad \sum_{i=1}^{I}\sum_{j=0}^{J} q_{ij} - x_{ij}c_j$$

$$\text{s.t.} \quad \sum_{j=0}^{J} x_{ij} = 1 \ \forall i \tag{4.16}$$

$$\sum_{j'=0}^{J} (w_{ij'}x_{ij'} - q_{ij'}) \geq w_{ij} - p_j \ \forall i, \ \forall j \tag{4.17}$$

$$p_j \leq p_{j'} + p_{j-j'} \ \forall j \ \forall j' < \frac{1}{2}(j+1) \tag{4.18}$$

$$p_j \leq p_{j+1} \ \forall j \leq J-1 \tag{4.19}$$

$$q_{ij} \geq x_{ij}p_j^L \ \forall j \tag{4.20}$$

$$q_{ij} \leq x_{ij}p_j^U \ \forall j \tag{4.21}$$

$$q_{ij} \geq x_{ij}p_j^U + p_j - p_j^U \ \forall j \tag{4.22}$$

$$q_{ij} \leq x_{ij}p_j^L + p_j - p_j^L \ \forall j. \tag{4.23}$$

Here, p_j^L and p_j^U are upper and lower bound for each p_j. Constraints (4.20) - (4.23) ensure that if $x_{ij} = 0$, then $q_{ij} = 0$, and if $x_{ij} = 1$, then $q_{ij} = p_j$. Therefore, MIP formulation $CBSP_1$ always has the same solution as the MINLP CBSP problem.

4.5 Gap Analyses

We also numerically evaluated how well the three mechanisms compare when the costs are zero. Table 4.1 shows five numerical examples with 20 consumers and 20 bundles sizes. All consumers' WTP is randomly generated according to SCP. In Column three to five, optimal profits for CBP, CBSP, and BSP are shown. We can see that for all the problems, CBP optimal value is (weakly) greater than that of CBSP which is (weakly) greater than that of BSP. We

Table 4.1: Comparison of CBP, CBSP, and BSP

		Optimal profit			Gap	
Problem No.	Problem size(I,J)	CBP	CBSP	BSP	CBP	BSP
1	20,20	152.384	149.884	148.679	1.67%	-0.80%
2	20,20	0.8	0.78	0.775	2.56%	-0.64%
3	20,20	16.199	16.123	16.119	0.47%	-0.02%
4	20,20	22.536	22.536	22.504	0.00%	-0.14%
5	20,20	39.435	39.014	38.902	1.08%	-0.29%
Average		46.271	45.667	45.396	1.16%	-0.38%

observe that the gaps can be large when using CBP compared to BSP. To investigate this issue further, we have also theoretically analyzed the gap between CBP and CBSP, and that between BSP and CBSP when the costs are separable in bundle sizes. Let Π^*_{CBP}, Π^*_{BSP}, and Π^*_{CBSP} be the optimal profits if the seller implements CBP, BSP, or CBSP respectively.

Proposition 44 *When the costs are separable in bundle sizes:*

- *The gap between the optimal profits of CBP and CBSP can be infinity.*

$$\max\left\{\frac{\Pi^*_{CBP}}{\Pi^*_{CBSP}}\right\} = \infty.$$

- *The gap between the optimal profits of CBSP and BSP is smaller than a factor of* 8.

$$\max\left\{\frac{\Pi^*_{CBSP}}{\Pi^*_{BSP}}\right\} \leq 8.$$

4.6 Conclusion

In this study, we first study the CB problem with marginal decreasing prices and prove that it is a shortest-path problem. Second, we propose a dynamic programming algorithm to solve the CB problem with unit decreasing prices. Third, we analyze the CB problem with sub-additive prices and convert its MINLP formulation to a mixed-integer programming (MIP) one.

Finally, we provide analytical and numerical analysis on the gaps between different CB models. We reconcile the differences in the optimal solutions obtained via different formulations of cardinality bundling in the literature.

There are several ways to extend the current study. First, there is still room to improve the performance of proposed dynamic programming algorithm for the BSP problem by combining it with LP cuttings. Second, CBSP problem has only been converted to an MIP, which is still N-P hard. Third, the gap analysis between MDP to BSP is still missing. Last but not least, analyzing cardinality bundling problems without Spence-Mirrlees condition can provide a wider application of these pricing schemes in reality.

Chapter 5

Appendix

5.1 Proofs

5.1.1 Proof of Proposition 3

Proof. Assume p' is an optimal price vector that is not non-decreasing and k, the smallest index for which $p'_k > p'_{k+1}$, is the largest among all optimal price vectors. We claim that for every feasible solution to CBP1 and for all i, $x_{ik} = 0$. Otherwise, Constraint (3.2) implies that $x_{ik'} = 0$ for all $k' \neq k$. Since $w_{ik} \leq w_{ik+1}$, $w_{ik} - p'_k < w_{ik+1} - p'_{k+1}$ which violates Constraint (3.1). Therefore, $x_{ik} = 0$. Consider now a price vector p such that $p_j = p'_j$ for all $j \neq k$ and $p_k = p_{k+1}$. Let (x, p') be feasible to CBP1. Since $x_{ik} = 0$, the objective value for x is the same for both p' and p. We claim that (x, p) is also feasible to CBP1 and therefore the optimal value with price p does not decrease. This is because $\sum_{j'=0}^{J}(w_{ij'} - p_{j'})x_{ij'} \geq \sum_{j'=0}^{J}(w_{ij'} - p'_{j'})x_{ij'} \geq w_{ik+1} - p'_{k+1} \geq w_{ik} - p_k$, where the first inequality follows since $p' \geq p$, the second because (x, p') is feasible, and the last because $w_{ik+1} \geq w_{ik}$ and $p_k = p'_{k+1}$. Further, existence of $k' > k$ such that $p_{k'} > p_{k'+1}$ contradicts the choice of p'. Therefore, p must be non-decreasing. ■

5.1.2 Proof of Proposition 4

Proof. We show the result for a fixed price vector. Then, the first part follows by applying the argument to an optimal price vector. Observe that there are finitely many solutions to $\mathrm{CBP1}$ in the x-space for a given p. We consider the allocations that yield the most profit and order them arbitrarily. Let $j_k(i')$ denote the bundle Consumer i' buys in the k^{th} such solution to $\mathrm{CBP1}$. Then, let $k' = \arg\max_k \min_{i''}\{i'' \mid j_k(i'') > j_k(i''+1)\}$. This means that k' is the optimal solution where the first consumer that buys a larger sized bundle than her immediate successor is of the highest type. Let $i \in \arg\min_{i''}\{i'' \mid j_{k'}(i'') > j_{k'}(i''+1)\}$. Now, construct the solution $j(\cdot)$ where $j(i') = j_{k'}(i')$ when $i' \neq i+1$ and $j(i+1) = j_{k'}(i)$. We show that $j(\cdot)$ is a feasible assignment of bundles to consumers which achieves at least the same objective function value, thus deriving a contradiction to the choice of k'. Since we do not change the assignment for any $i' \neq i+1$, we only need to verify that $j(\cdot)$ satisfies $w_{i+1j(i+1)} - p_{j(i+1)} \geq w_{i+1j} - p_j$ for all j. Now, consider the following chain of inequalities:

$$
\begin{aligned}
0 &\geq w_{i+1j_{k'}(i)} - p_{j_{k'}(i)} - w_{i+1j_{k'}(i+1)} + p_{j_{k'}(i+1)} \\
&\geq w_{ij_{k'}(i)} - p_{j_{k'}(i)} - w_{ij_{k'}(i+1)} + p_{j_{k'}(i+1)} \\
&\geq 0,
\end{aligned}
$$

where the first inequality follows because $i+1$ chooses $j_{k'}(i+1)$, the second inequality because $j_{k'}(i) > j_{k'}(i+1)$ implies by SCP that $w_{i+1j_{k'}(i)} - w_{i+1j_{k'}(i+1)} \geq w_{ij_{k'}(i)} - w_{ij_{k'}(i+1)}$ and the last inequality because i chooses $j_{k'}(i)$. Therefore, equality holds throughout. Then, for any j, it follows that

$$
w_{i+1j(i+1)} - p_{j(i+1)} = w_{i+1j_{k'}(i)} - p_{j_{k'}(i)} = w_{i+1j_{k'}(i+1)} - p_{j_{k'}(i+1)} \geq w_{i+1j} - p_j,
$$

where the first equality follows because $j(i+1) = j_{k'}(i)$, the second equality follows from the argument above, and the first inequality because $i+1$ chooses $j_{k'}(i+1)$ under the feasible solution $j_{k'}(\cdot)$. Therefore, we have shown that $j(\cdot)$ is a feasible assignment of bundles to consumers. Now, we show that the corresponding objective value does not decrease. This follows since

$$\sum_{i'} \left(p_{j(i')} - c_{j(i')}\right) = \sum_{i' \neq i+1} \left(p_{j_{k'}(i')} - c_{j_{k'}(i')}\right) + p_{j_{k'}(i)} - c_{j_{k'}(i)} \geq \sum_{i'} \left(p_{j_{k'}(i')} - c_{j_{k'}(i')}\right),$$

where the first equality follows by the definition of $j(\cdot)$. The first inequality follows because $p_{j_{k'}(i)} - c_{j_{k'}(i)} \geq p_{j_{k'}(i+1)} - c_{j_{k'}(i+1)}$ is implied by $w_{ij_{k'}(i)} - p_{j_{k'}(i)} - w_{ij_{k'}(i+1)} + p_{j_{k'}(i+1)} = 0$ and optimality of $j_{k'}(\cdot)$ for p. Otherwise, $j_{k'}(i+1)$ yields the same surplus to i' as $j_{k'}(i)$, which means $j'(i') = j_{k'}(i')$ for $i' \neq i$ and $j'(i) = j_{k'}(i+1)$ is feasible, yielding a strictly higher objective value than $j_{k'}(\cdot)$. ■

5.1.3 Proof of Proposition 5

Proof. Consider an optimal solution such that no other optimal solution allocates a subset of the bundle sizes to the consumers. Assume that the bundle sizes sold are $\{j_k, \ldots, j_1\}$ where $j_k < \cdots < j_1$ and the corresponding price vector is p'. If $J \notin \{j_k, \ldots, j_1\}$, we assume without loss of generality that $p'_J = w_{IJ} + \epsilon$ for some $\epsilon > 0$. Similarly, we assume that for $j \notin \{j_k, \ldots, j_1\} \cup \{J\}$, the price is $\min\{p_{j'} \mid \exists j' \geq j, j' \in \{j_k, \ldots, j_1\} \cup \{J\}\}$. So, by optimality of j', no consumer purchases any bundles not in $\{j_k, \ldots, j_1\}$.

We assume that $k \geq 2$ since there is nothing to show otherwise. We show by induction on r that $p'_{j_{r+1}} - c_{j_{r+1}} < p'_{j_r} - c_{j_r}$ for all $r < k$. Consider $r = 1$. By Proposition 4, Consumers $i, \ldots, I$ purchase Bundle j_1 for some $i \leq I$. Construct a price vector p'' where $p''_j = p'_j$ for $j < j_1$ and $p''_{j_1} = w_{Ij_1} + \epsilon$. Any consumer that does not purchase j_1 does not alter her decision since the surplus of non-preferred bundles only decreased with p''. Since Consumer $i-1$ continues to buy Bundle j_2, by Proposition 4, Consumers $i, \ldots, I$ only consider bundles j_2 or higher. Since j_1 does not offer any surplus, all these consumers will purchase Bundle j_2. Observe that $p'_{j_2} - c_{j_2} \leq p'_{j_1} - c_{j_1}$. Otherwise, the optimal solution with p'' attains a strictly higher profit. If $p'_{j_2} - c_{j_2} = p'_{j_1} - c_{j_1}$, the optimal profit attained with p'' is the same as that with p'. However, this contradicts the selection of the optimal solution with minimal number of bundles allocated to consumers. Therefore, $p'_{j_2} - c_{j_2} < p'_{j_1} - c_{j_1}$. Now, for the induction step, we assume that $p'_{j_r} - c_{j_r} < p'_{j_{r-1}} - c_{j_{r-1}}$ and show that $p'_{j_{r+1}} - c_{j_{r+1}} < p'_{j_r} - c_{j_r}$. Let $\{i_1, \ldots, i_t\}$ be the consumers that purchase Bundle j_r. Then, consider the price vector p'' such that $p''_j = p'_j$ for $j \neq j_r$ and $p''_{j_r} = p'_{j_{r-1}}$. Observe that any consumer who does not purchase j_r does not change their decision since, by Proposition 3, the surplus of non-preferred items only reduced with the price change. It follows from Proposition 4 that any consumer in $\{i_1, \ldots, i_t\}$ now purchases one of the bundles $\{j_{r+1}, j_r, j_{r-1}\}$. We first show that with p'', no consumer strictly prefers j_r. Let $i \in \{i_1, \ldots, i_t\}$. Then, $w_{ij_r} - p''_{j_r} = w_{ij_r} - p'_{j_{r-1}} \leq w_{ij_{r-1}} - p'_{j_{r-1}} = w_{ij_{r-1}} - p''_{j_{r-1}}$. Therefore, Consumer i weakly prefers Bundle j_{r-1} over j_r under price p''. Since we assumed that consumers purchase bundle sizes that offer most profit to the vendor (among the sizes that offer maximum surplus), it follows from the induction hypothesis that each consumer prefers Bundle j_{r-1} over j_r. Now, assume that $p'_{j_{r+1}} - c_{j_{r+1}} \geq p'_{j_r} - c_{j_r}$, *i.e.*, Bundle j_{r+1} offers more profit to the vendor as compared to j_r. Since all the consumers in $\{i_1, \ldots, i_t\}$ now purchase either Bundle j_{r-1} or j_{r+1}, both of which offer either same or more profit to the vendor compared to $p'_{j_r} - c_{j_r}$, the profit under p'' must be optimal, and thus contradicts the minimality of the bundles allocated to consumers. Therefore, $p'_{j_{r+1}} - c_{j_{r+1}} < p'_{j_r} - c_{j_r}$. ■

5.1.4 Proof of Proposition 6

Proof. Let i_1 be the lowest indexed consumer who purchases a bundle of non-zero size, say

$j_1 > 0$. Let p^* be the optimal price vector. Clearly, $p^*_{j_1} \leq w_{i_1 j_1}$. Now, assume that $p^*_{j_1} < w_{i_1 j_1}$. Consider $p' = p^* + \Delta$, where $\Delta = w_{i_1 j_1} - p^*_{j_1} > 0$ and a consumer i' that purchased a bundle, $j' > 0$. Then, $w_{i'j'} - p^*_{j'} \geq w_{i'j_1} - p^*_{j_1} \geq w_{ij_1} - p^*_{j_1} = \Delta$, where the first inequality is because i' prefers j' over j_1 and the second inequality follows from SCP and $i' > i_1$. Therefore, $w_{i'j'} - p'_{j'} = w_{i'j'} - p^*_{j'} - \Delta \geq 0$. This shows that any consumer that purchases j' with p^* still prefers j' to not purchasing anything. For any consumer, the relative preference between bundles of non-zero size does not change. Therefore, all consumers that purchased any product still purchase the same product. The consumers that did not purchase a product with p^* do not have incentive to purchase a product with p' because the surpluses have reduced. Therefore, the consumer purchasing decisions do not change. If I' is the set of consumers that purchase a bundle of non-zero size, the vendor makes an additional $|I'|\Delta$ profit due to the increase in price. Since $i_1 \in I'$, it follows that $|I'| \geq 1$. However, this yields a contradiction to the optimality of p^* since p' yields a strictly higher profit. ■

5.1.5 Proof of Proposition 7

Proof. For a given x that satisfies (3.2), (3.4), and (3.7), we obtain the optimal prices. Let J' be the set of bundles of non-zero size that some consumers buy. We will derive the prices for the bundles in J' by solving an optimization model. Given the prices of the bundles in J', we show how to price the remaining bundles. If $J \notin J'$, the price for Bundle J is assigned to be $w_{IJ} + \epsilon$. The price of Bundle 0 is fixed at 0. Now consider a remaining bundle, $j \in \{1, \ldots, J\} \backslash (J' \cup \{J\})$. The vendor does not want any consumer to purchase this bundle. Therefore, he may price j at the price of Bundle $j' = \min\{j'' \mid j'' \geq j, j'' \in J' \cup \{J\}\}$. Since $j \leq J$, it follows that the minimum in the definition of j' is attained.

Now, we compute prices for the bundles in J' by solving $\mathrm{CBP1}$ with x variables fixed to the values given. To emphasize that optimization is in the space of the p variables, we refer to this formulation as CBP$_p$. We will show that CBP$_p$ can be reformulated into a model that is much simpler. We replace the consumers that do not purchase any bundle with the highest type consumer that does not purchase any bundle. (Clearly, if this consumer does not have an incentive to purchase a bundle, the lower-type consumers will not either.) If every consumer purchases some bundle, we create a consumer whose WTP for all bundles is 0 and therefore does not buy any bundle. Then, we reindex the consumers to $1, \ldots, I'$. We denote the reindexed WTP as w' and $w'_{I'+1j} = w'_{I'j}$. We denote by $j(i)$ the bundle that is assigned to Consumer i. We reformulate CBP$_p$ as:

$$\mathrm{CBP1a}: \quad \underset{p_{j(i)}}{\text{Max}} \quad \sum_{i=1}^{I'} (p_{j(i)} - c_{j(i)}) \tag{5.1}$$

$$\text{s.t.} \quad w'_{ij(i)} - p_{j(i)} \geq w'_{ij(i')} - p_{j(i')} \;\; 1 \leq i, i' \leq I' \tag{5.2}$$

$$p_0 = 0 \tag{5.3}$$

It can be verified easily that $\mathrm{CBP1a}$ and CBP$_p$ are equivalent. We assume without loss of generality, by re-indexing the bundles, that the bundles sizes are $\{0, 1, \ldots, |J'|\}$.

Let $\{i_0, i_1, \ldots, i_{J'}\}$ be the lowest-type consumers who buy Bundle j, where by definition, $i_0 = 1$. Now, we rewrite Constraint (5.2) as $w'_{ij(i)} - p_{j(i)} \geq w'_{ij} - p_j$ for all i and $j \in \{0, \ldots, |J'|\}$. Since the constraint for $j = j(i)$ holds trivially, we decompose this constraint for a Consumer i as follows:

$$w'_{ij(i)} - p_{j(i)} \geq w'_{ij} - p_j \quad \forall j < j(i) \tag{5.4}$$

$$w'_{ij(i)} - p_{j(i)} \geq w'_{ij} - p_j \quad \forall j > j(i). \tag{5.5}$$

We show that all constraints in (5.4) are redundant except those corresponding to some $i \in \{i_1, \ldots, i_{|J'|}\}$ and $j = j(i) - 1$. Note that there is no constraint of the type (5.4) for $i = 1$.

Observe that:

$$w'_{ij(i)} - w'_{ij} = \sum_{j'=j}^{j(i)-1} (w'_{ij'+1} - w'_{ij}) \geq \sum_{j'=j}^{j(i)-1} (w'_{i_{j'+1}j'+1} - w'_{i_{j'+1}j'}) \geq \sum_{j'=j}^{j(i)-1} (p_{j'+1} - p_{j'}) = p_{j(i)} - p_j,$$

where the first inequality is because of SCP and $i_{j'} \leq i$ for all $j' \leq j(i)$, and the second inequality uses (5.4) for some $i \in \{i_1, \ldots, i_{|J'|}\}$ and where $j = j(i) - 1$. Therefore, we replace (5.4) by the following:

$$w'_{i_j j} - p_j \geq w'_{i_j j-1} - p_{j-1} \quad 1 \leq j \leq |J'|. \tag{5.6}$$

We will now show that in every optimal solution the inequalities in (5.6) are binding. We consider a feasible p to $\mathrm{CBP1a}$ where at least one of the (5.6) is not binding. Then, we show that p is not optimal by constructing p' which is feasible, has at least one more (5.6) binding, and has a higher objective function value than p. Let $j' = \arg\min\{j \mid w'_{i_j j} - p_j > w'_{i_j j-1} - p_{j-1}\}$, the index of the first inequality that is not binding, and $\Delta = w'_{i_{j'} j'} - p_{j'} > w'_{i_{j'} j'-1} - p_{j'-1}$. Then, consider the price vector p', where $p'_j = p_j$ for $j < j'$ and $p'_j = p_j + \Delta$ otherwise. Then, it is easy to see that for $j \neq j'$, the left hand side of (5.6) changes by the same amount as the right hand side. Therefore, if the inequality was binding for p then it remains binding for p'. Further, the adjustment of $p'_{j'}$ guarantees that (5.6) is binding for $j = j'$. Now, we show that p' is also feasible to (5.5). If $j(i) < j'$ the inequality follows since the surplus of bundles that the consumer does not buy only increases. If $j(i) \geq j'$, then both sides of the inequality decrease by the same amount. Therefore, the constraint holds. Now,

$$\sum_{i=1}^{I'} p_{j(i)} - c_{j(i)} < \sum_{i=1}^{i_{j'}-1} (p'_{j(i)} - c'_{j(i)}) + \sum_{i=i_{j'}}^{I'} (p'_{j(i)} - c'_{j(i)}),$$

where the inequality follows because $i_{j'} \leq I'$ and $p'_i > p_i$ for $i \geq i_{j'}$.

Constraint (5.5) is redundant since:

$$\begin{aligned} w'_{ij} - w'_{ij(i)} &= \sum_{j'=j(i)+1}^{j} (w'_{ij'} - w'_{ij'-1}) \\ &\leq \sum_{j'=j(i)+1}^{j} (w'_{i_{j'}j'} - w'_{i_{j'}j'-1}) \\ &= \sum_{j'=j(i)+1}^{j} (p_{j'} - p_{j'-1}) \\ &= p_j - p_{j(i)}. \end{aligned}$$

Here, the first inequality follows from SCP and that $i_{j'} \geq i$ whenever $j' \geq j(i) + 1$, the second equality because (5.6) is tight at an optimal solution.

For any $j \in J'$, we give a closed-form formula for p_j. Since $p_{j(1)} = 0$ and Constraint (5.6) is binding,

$$p_j = \sum_{r=1}^{j} \left(w'_{i_r j(i_r)} - w'_{i_r j(i_{r-1})} \right).$$

It is easy to verify that the above formula is equivalent to (2.10). We define $i_{|J'|+1} = I' + 1$ and let $w'_{I'+1j} = w'_{I'j}$. We now evaluate the objective function value for $\mathrm{CBP1a}$.

$$
\begin{aligned}
\sum_{i=1}^{I'} \left(p_{j(i)} - c_{j(i)}\right) &= \sum_{j=1}^{|J'|} \sum_{i=i_j}^{i_{j+1}-1} p_j - \sum_{i=1}^{I} \sum_{j=1}^{J} c_{ij} x_{ij} \\
&= \sum_{j=1}^{|J'|} \sum_{i=i_j}^{i_{j+1}-1} \sum_{r=1}^{j} \left(w'_{i_r j(i_r)} - w'_{i_r j(i_{r-1})}\right) - \sum_{i=1}^{I} \sum_{j=1}^{J} c_{ij} x_{ij} \\
&= \sum_{r=1}^{|J'|} \sum_{j=r}^{|J'|} \sum_{i=i_j}^{i_{j+1}-1} \left(w'_{i_r j(i_r)} - w'_{i_r j(i_{r-1})}\right) - \sum_{i=1}^{I} \sum_{j=1}^{J} c_{ij} x_{ij} \\
&= \sum_{r=1}^{|J'|} \left((I' - i_r + 1) w'_{i_r j(i_r)} - (I' - i_{r+1} + 1) w'_{i_{r+1} j(i_r)}\right) - \sum_{i=1}^{I} \sum_{j=1}^{J} c_{ij} x_{ij} \\
&= \sum_{r=1}^{|J'|} \sum_{i=i_r}^{i_{r+1}-1} \left((I' - i + 1) w'_{i_r j(i_r)} - (I' - i) w'_{i_r+1 j(i_r)}\right) - \sum_{i=1}^{I} \sum_{j=1}^{J} c_{ij} x_{ij} \\
&= \sum_{i=1}^{I} \sum_{j=1}^{J} \left(v_{ij} - c_{ij}\right) x_{ij}.
\end{aligned}
$$

Here, the forth equality uses $w'_{10} = 0$ and $w'_{I'+1j} = w'_{I'j}$. By Proposition 4, every consumer i', who purchases a bundle of non-zero size, is reindexed to some consumer in $\{1, \ldots, I'\}$. Let this index be i and observe that $I - i' = I' - i$. Therefore, the last equality follows. ■

5.1.6 Proof of Proposition 8

Proof. Consider any subset, T of the allocation variables, x_{ij}. By Theorem III.1.2.7 in Nemhauser and Wolsey (1988), the constraint matrix of $\mathrm{CBP2}$ is totally unimodular if and only if T can be partitioned into two subsets T_1 and T_2 such that for every constraint, $\sum_{i=1}^{I} \sum_{j=1}^{J} d_{ij} x_{ij} \leq d_0$, in $\mathrm{CBP2}$:

$$
\left| \sum_{(i,j) \in T_1} d_{ij} - \sum_{(i,j) \in T_2} d_{ij} \right| \leq 1. \tag{5.7}
$$

We construct such a partition. For Consumer i, let T contain $\{x_{ij_1} \ldots, x_{ij_{k_i}}\}$. If k_i is odd, we include $\{x_{ij_1}, x_{ij_3}, \ldots, x_{ij_{k_i}}\}$ in T_1. Otherwise, we include $\{x_{ij_2}, x_{ij_4}, \ldots, x_{ij_{k_i}}\}$. The remaining variables are in T_2. We do the same for every consumer. Now, consider the variables in T that have a non-zero coefficient in Constraint (3.7). Among these, let the number of variables for Consumer i that belong to T_1 (resp. T_2) be a_i (resp. b_i). Clearly, $b_i \in \{a_i - 1, a_i\}$ and the same conclusion holds for Consumer $i+1$'s allocation. Then, for Constraint (3.7), the sum of the coefficients for variables in T_1 minus the sum of coefficients for variables in T_2 equals $a_i - b_i - a_{i+1} + b_{i+1}$. Then,

$$-1 \leq -a_{i+1} + b_{i+1} \leq a_i - b_i - a_{i+1} + b_{i+1} \leq a_i - b_i \leq 1.$$

We have thus verified (5.7) for Constraint (3.7). Verification for Constraint (3.2) is easy since $\lceil \frac{k_i}{2} \rceil - \lfloor \frac{k_i}{2} \rfloor \leq 1$. Further, (5.7) holds trivially for bound constraints since they have only one variable with non-zero coefficient. ■

5.1.7 Proof of Proposition 9

Proof. We first show that $\mathrm{CBP2}$ can be reformulated to the following problem:

$$\begin{aligned} \mathrm{CBP2b}: \quad \underset{x_{ij}}{\text{Max}} \quad & \sum_{i=1}^{I} \sum_{j=0}^{J} (v_{ij} - c_j) x_{ij} \\ \text{s.t.} \quad & (3.7) \\ & \sum_{j=0}^{J} x_{Ij} \leq 1 && (5.8) \\ & x_{ij} \geq 0 \ \forall i, \ \forall j. && (5.9) \end{aligned}$$

First, observe that $x_{ij} \leq \sum_{j'=0}^{J} x_{ij} = 1$. Therefore, $x_{ij} \leq 1$ can be dropped from $\mathrm{CBP2}$. Obviously, (5.8) is implied by (3.2). Therefore, $\mathrm{CBP2b}$ is a relaxation of $\mathrm{CBP2}$. We now show the reverse inclusion. We next prove that (3.2) is also implied by (5.8) and (3.7). Clearly, for any i,

$$\sum_{j=0}^{J} x_{ij} \leq \sum_{j=0}^{J} x_{Ij} \leq 1.$$

The first inequality follows from (3.7) and the second from (5.8). Since x_{i0} does not appears in the objective function of $\mathrm{CBP2}$, if $\sum_{j=0}^{J} x_{ij} < 1$ for some i, we can set $x_{i0} = 1 - \sum_{j=0}^{J} x_{ij}$ and make $\sum_{j=0}^{J} x_{ij} = 1$ without affecting the objective value or any other constraints. Therefore, $\mathrm{CBP2b}$ is equivalent to $\mathrm{CBP2}$.

Next, we prove that $\mathrm{CBP2a}$ is equivalent to $\mathrm{CBP2b}$. Given a solution x feasible to $\mathrm{CBP2b}$. We show that we can construct (a, x) that is feasible to $\mathrm{CBP2a}$ that has the same objective. Let $a_{ij} = \sum_{j'=j}^{J} x_{i+1,j'} - \sum_{j'=j}^{J} x_{ij'}$. Here, $x_{I+1,J}$ is assumed to be 1 and $x_{I+1,j}, \forall j \neq J$ is 0. Observe that (2.11) is satisfied by definition:

$$\begin{aligned}
& a_{ij} - a_{i,j+1} + x_{ij} - x_{i+1,j} \\
= \;& \sum_{j'=j}^{J} x_{i+1,j'} - \sum_{j'=j}^{J} x_{ij'} - \Big(\sum_{j'=j+1}^{J} x_{i+1,j'} - \sum_{j'=j+1}^{J} x_{ij'}\Big) + x_{ij} - x_{i+1,j} \\
= \;& x_{i+1,j} - x_{ij} + x_{ij} - x_{i+1,j} \\
= \;& 0.
\end{aligned}$$

Further, (2.12) is satisfied because of $a_{IJ} + x_{IJ} = 1 - x_{IJ} + x_{IJ} = 1$. (2.14) follows from (3.7) and (5.9).

Let (a, x) be a feasible solution to $\mathrm{CBP2a}$. Observe that $0 \leq a_{ij} = a_{ij} - a_{iJ} = \sum_{j'=j}^{J} x_{i+1,j'} - \sum_{j'=j}^{J} x_{ij'}$. The first inequality is by (2.14), first equality is by (2.13), second equality is by summing (2.11) for j' from j to J. Therefore, x satisfies (3.7). Now, consider

$$0 \leq a_{I0} = a_{I0} - a_{IJ} + a_{IJ} = 0 - \sum_{j'=0}^{J-1} x_{Ij'} + 1 - x_{IJ},$$

where the first inequality is by (2.14), second equality is by summing (2.11) for j' from j to $J-1$ with (2.12). Therefore, x satisfies (5.8). Clearly, $x_{ij} \geq 0$ by (2.14). Since the objective depends only on x, we have shown that x is feasible to $\mathrm{CBP2b}$ with the same objective value as (a, x) in CBP2a. Therefore, we have shown the converse. ■

5.1.8 Proof of Theorem 10

Proof. Let $l_{ij}, (i, j) \neq (IJ)$ be the multiplier for each constraint in (2.11) with corresponding (i, j) and l_{IJ} for (2.12). Then we can easily get that CBP2d is the dual of CBP2a.

We next show that there exists an optimal solution to CBP2d such that for each (i, j), $l_{ij} = \max\{l_{ij-1}, l_{i-1j} + v_{ij} - c_j\}$, where l_{0j} is understood to be zero. Clearly, in any feasible solution, we have $l_{ij} \geq \max\{l_{ij-1}, l_{i-1j} + v_{ij} - c_j\}$ $\forall i;\ \forall j$ because otherwise either (2.15) or (2.16) cannot hold. Assume l^* is an optimal solution containing some l^*_{ij} values such that $l^*_{ij} > \max\{l^*_{ij-1}, l^*_{i-1j} + v_{ij} - c_j\}$. Let $\hat{j} = \arg\min\{j \mid l^*_{ij} > \max\{l^*_{ij-1}, l^*_{i-1j} + v_{ij} - c_j\}\}$ and $\hat{i} = \arg\min\{i \mid l^*_{i\hat{j}} > \max\{l^*_{i\hat{j}-1}, l^*_{i-1\hat{j}} + v_{i\hat{j}} - c_{\hat{j}}\}\}$. Clearly, $(\hat{i}, \hat{j}) \neq (I, J)$ because otherwise we can get a better solution $l^{*'}_{IJ} = \max\{l^*_{IJ-1}, l^*_{I-1J} + v_{IJ} - c_J\} < l^*_{IJ}$ without violating any of the constraints. If we create a new solution $\hat{l}$ such that $\hat{l}_{ij} = l^*_{ij}$ $\forall (i, j) \neq (\hat{i}, \hat{j})$ and $\hat{l}_{\hat{i}\hat{j}} = \max\{\hat{l}_{\hat{i}\hat{j}-1}, \hat{l}_{\hat{i}-1\hat{j}} + v_{\hat{i}\hat{j}} - c_{\hat{j}}\} = \max\{l^*_{\hat{i}\hat{j}-1}, l^*_{\hat{i}-1\hat{j}} + v_{\hat{i}\hat{j}} - c_{\hat{j}}\}$, we argue that $\hat{l}$ is also feasible. Changing $l^*_{\hat{i}\hat{j}}$ to $\hat{l}_{\hat{i}\hat{j}}$ only affects four constraints: (1) $\hat{l}_{\hat{i}\hat{j}} \geq l^*_{\hat{i}\hat{j}-1}$; (2) $\hat{l}_{\hat{i}\hat{j}} \geq l^*_{\hat{i}-1\hat{j}} + v_{\hat{i}\hat{j}} - c_{\hat{j}}$; (3) $l^*_{\hat{i}\hat{j}+1} \geq \hat{l}_{\hat{i}\hat{j}}$; and (4) $l^*_{\hat{i}+1\hat{j}} \geq \hat{l}_{\hat{i}\hat{j}} + v_{\hat{i}+1\hat{j}} - c_{\hat{j}}$. Clearly, the first two inequalities still hold because of the definition of $\hat{l}_{\hat{i}\hat{j}}$. The last two inequalities also hold because the left-hand-side values of both are not changed and the right-hand-side values are reduced. Since the objective value l_{IJ} is not affected, $\hat{l}$ is also an optimal solution. By using the same procedure, we can sequentially update all l^*_{ij} values such that $l^*_{ij} > \max\{l^*_{ij-1}, l^*_{i-1j} + v_{ij} - c_j\}$ to $l^{*'}_{ij} = \max\{l^*_{ij-1}, l^*_{i-1j} + v_{ij} - c_j\}$ while maintaining the optimality of the solution. Finally, we can obtain an optimal solution in which for each (i, j), $l_{ij} = \max\{l_{ij-1}, l_{i-1j} + v_{ij} - c_j\}$.

Now we illustrate how to use $l_{ij} = \max\{l_{ij-1}, l_{i-1j} + v_{ij} - c_j\}$ to quickly find out the optimal solution. we start with $i = 0$: $l_{0j} = \max\{l_{0j-1}, 0\} = \max\{\max\{l_{0j-2}, 0\}, 0\} = \ldots = 0$. Next, we have $l_{10} = \max\{0, l_{00} + v_{10} - c_0\} = 0$. If l_{1j-1} is known, then we can calculate

$l_{1j} = \max\{l_{1j-1}, l_{0j}+v_{1j}-c_j\}$. Thus, we can calculate l_{1j} $\forall j$ in $J+1$ step. Sequentially, we can calculate $l_{2j}, \ldots, l_{Ij}$ and finally reach l_{IJ} in $(I+1)(J+1)$ steps. Therefore, the computational complexity for CBP2d is $O(IJ)$. ■

5.1.9 Proof of Corollary 11

Proof. Let c' and c'' be two (non-decreasing) cost vectors such that, for any $j \geq 1$, $c'_j - c'_{j-1} \leq c''_j - c''_{j-1}$. Let x' be optimal with c'. For c'', we construct an optimal allocation, s, where consumers purchase bundles of weakly decreasing size compared to x'. Assume, x'' is optimal with cost c'' and some consumer purchases a bundle smaller than in x'. Since $c_0 = 0$,

$$\sum_{i=1}^{I}\sum_{j=1}^{J} c_j x'_{ij} = \sum_{i=1}^{I}\sum_{j=1}^{J}(c_j - c_0)x'_{ij} = \sum_{i=1}^{I}\sum_{j=1}^{J} x'_{ij}\sum_{j'=1}^{j}(c_{j'} - c_{j'-1}) = \sum_{i=1}^{I}\sum_{j'=1}^{J}(c_{j'} - c_{j'-1})\sum_{j=j'}^{J} x'_{ij}. \tag{5.10}$$

Now, consider

$$s_{ij} = \min\left\{\sum_{j'=j}^{J} x'_{ij'}, \sum_{j'=j}^{J} x''_{ij'}\right\} - \min\left\{\sum_{j'=j+1}^{J} x'_{ij'}, \sum_{j'=j+1}^{J} x''_{ij'}\right\}$$

and let

$$t_{ij} = \max\left\{\sum_{j'=j}^{J} x'_{ij'}, \sum_{j'=j}^{J} x''_{ij'}\right\} - \max\left\{\sum_{j'=j+1}^{J} x'_{ij'}, \sum_{j'=j+1}^{J} x''_{ij'}\right\}.$$

Both s_{ij} and t_{ij} are feasible and $s_{ij} + t_{ij} = x'_{ij} + x''_{ij}$. Then:

$$\begin{aligned}
&\sum_{i=1}^{I}\sum_{j=1}^{J} v_{ij}(s_{ij}+t_{ij}) - \sum_{i=1}^{I}\sum_{j=1}^{J}\left[(c'_j - c'_{j-1})\sum_{j'=j}^{J} t_{ij'} + (c''_j - c''_{j-1})\sum_{j'=j}^{J} s_{ij'}\right] \\
&\quad\leq \sum_{i=1}^{I}\sum_{j=1}^{J} v_{ij}(x'_{ij}+x''_{ij}) - \sum_{i=1}^{I}\sum_{j=1}^{J}\left[(c'_j - c'_{j-1})\sum_{j'=j}^{J} x'_{ij'} + (c''_j - c''_{j-1})\sum_{j'=j}^{J} x''_{ij'}\right] \qquad (5.11) \\
&\quad\leq \sum_{i=1}^{I}\sum_{j=1}^{J} v_{ij}(s_{ij}+t_{ij}) - \sum_{i=1}^{I}\sum_{j=1}^{J}\left[(c'_j - c'_{j-1})\sum_{j'=j}^{J} t_{ij'} + (c''_j - c''_{j-1})\sum_{j'=j}^{J} s_{ij'}\right],
\end{aligned}$$

where the first inequality is by optimality of x' with cost c' and the optimality of x'' with cost c'' and the second inequality follows $s_{ij} + t_{ij} = x'_{ij} + x''_{ij}$ and rearrangement inequality because $c'_j - c'_{j-1} \leq c''_j - c''_{j-1}$, $\sum_{j'=j}^{J} s_{ij'} = \min\left\{\sum_{j'=j}^{J} x'_{ij'}, \sum_{j'=j}^{J} x''_{ij'}\right\}$, and $\sum_{j'=j}^{J} t_{ij'} = \max\left\{\sum_{j'=j}^{J} x'_{ij'}, \sum_{j'=j}^{J} x''_{ij'}\right\}$. Therefore, equality holds throughout. Since x' and x'' are optimal with c' and c'' respectively, s is a feasible allocation which yields optimal profit to the vendor when the cost is c''. Since

$$\sum_{j'=0}^{j} s_{ij'} = 1 - \min\left\{\sum_{j'=j+1}^{J} x'_{ij'}, \sum_{j'=j+1}^{J} x''_{ij'}\right\} \geq 1 - \sum_{j'=j+1}^{J} x'_{ij'} = \sum_{j'=0}^{j} x'_{ij'},$$

it follows that s allocates smaller bundle sizes to all consumers compared to x'. Similarly, for every optimal allocation x'' with c'', there exists an optimal allocation t with c' where each consumer buys a bundle of size at least as large as in x''.

Moreover, if $c'_j - c'_{j-1} < c''_j - c''_{j-1}$ and there is a consumer i such that $\sum_{j'=j}^{J} x'_{ij} = 0$ and $\sum_{j'=j}^{J} x''_{ij'} = 1$, then the second inequality in (5.11) is strict and yields a contradiction. Therefore, if the marginal cost of selling an additional unit (from $j-1$ to j) with c' is strictly smaller than that with c'', then no consumer purchases a bundle size less than j with c' but at least j with c''. If, for all $j \geq 1$, $c'_j - c'_{j-1} < c''_j - c''_{j-1}$, then with c' no consumer purchases a bundle size smaller than with c''. Or, in every optimal solution with c' consumers purchase a bundle size smaller than in any optimal solution with c''. ■

5.1.10 Proof of Proposition 12

Proof. Let w be an arbitrary set of WTPs satisfying SCP and v be a set of the corresponding v_{ij} values. Consider $w' = \mathcal{W}(i', I, w)$, wherein WTPs of consumers indexed i' through I are homogenized. Then, the corresponding v'_{ij} values of w' can be written as:

$$v'_{ij} = \begin{cases} v_{ij} & \text{if } i \leq i' - 2 \\ (I - i' + 2)w_{i'-1j} - \sum_{i''=i'}^{I} w_{i''j} & \text{if } i = i' - 1 \\ \frac{1}{I-i'+1} \sum_{i''=i'}^{I} w_{i''j} & \text{if } i \geq i'. \end{cases}$$

Call the CBP problem with WTPs w as $CBP(w)$ and that with w' as $CBP(w')$. Let $j(i)$ denote the bundle size that Consumer i purchases in an optimal solution of $CBP(w)$. Consider an allocation $j'(i)$ such that $j'(i) = j(i)$ when $i < i'$ and $j'(i) = J$ when $i \geq i'$. Obviously, $j'(i)$ is a feasible bundle allocation for $CBP(w')$. We next show that $j'(i)$ in $CBP(w')$ leads to a profit $\sum_{i=1}^{I} v'_{ij'(i)}$ that is weakly higher than $\sum_{i=1}^{I} v_{ij(i)}$, the optimal profit of $CBP(w)$.

$$\begin{aligned} \sum_{i=1}^{I} v'_{ij'(i)} &= \sum_{i=1}^{i'-2} v'_{ij'(i)} + +v'_{i'-1,j(i'-1)} + \sum_{i=i'}^{I} v'_{ij'(i)} \\ &= \sum_{i=1}^{i'-2} v_{ij(i)} + (I - i' + 2)w_{i'-1,j(i'-1)} + \sum_{i=i'}^{I} (w_{iJ} - w_{ij(i'-1)}) \\ &\geq \sum_{i=1}^{i'-2} v_{ij(i)} + (I - i' + 2)w_{i'-1,j(i'-1)} + \sum_{i=i'}^{I} (w_{ij(i)} - w_{ij(i'-1)}) \\ &= \sum_{i=1}^{i'-2} v_{ij(i)} + (I - i' + 2)w_{i'-1,j(i'-1)} + \sum_{i=i'}^{I} \sum_{i''=i'}^{i} (w_{ij(i'')} - w_{i,j(i''-1)}) \\ &\geq \sum_{i=1}^{i'-2} v_{ij(i)} + (I - i' + 2)w_{i'-1,j(i'-1)} + \sum_{i=i'}^{I} \sum_{i''=i'}^{i} (w_{i''j(i'')} - w_{i''j(i''-1)}) \\ &= \sum_{i=1}^{i'-2} v_{ij(i)} + (I - i' + 2)w_{i'-1,j(i'-1)} + \sum_{i=i'}^{I} (I - i + 1)(w_{ij(i)} - w_{i-1j(i)}) \\ &= \sum_{i=1}^{I} v_{ij(i)}, \end{aligned}$$

wherein the equalities are because of either reorganization or by invoking the definitions; the first inequality is because $w_{iJ} \geq w_{ij(i)}$ $\forall i$; and the second inequality is due to SCP. ■

5.1.11 Proof of Proposition 13

Proof. From Equation (2.17), $w_{ij} = \sum_{i'=i}^{I} v_{i'j}$. We only need to fix v_{Ij} for all j to define the WTPs, w_{ij} for all i and j. We will show that for large enough v_{Ij}, the corresponding WTPs are non-decreasing in j and satisfy SCP. Observe that:

$$w_{ij+1} - w_{ij} = \frac{1}{I-i+1} \sum_{i'=i}^{I} (v_{i'j+1} - v_{i'j}).$$

Therefore, $w_{1j+1} - w_{1j} \geq 0$ is equivalent to $v_{Ij+1} - v_{Ij} \geq -\sum_{i'=1}^{I-1}(v_{i'j+1} - v_{i'j})$. We define $\bar{v}_{j+1} = \sum_{i'=1}^{I-1}(v_{i'j+1} - v_{i'j})$. Further, in order that w satisfy SCP, we require that $(w_{i+1j+1} - w_{i+1j}) - (w_{ij+1} - w_{ij}) \geq 0$. This simplifies to:

$$0 \leq (I-i+1) \sum_{i'=i+1}^{I} (v_{i'j+1} - v_{i'j}) - (I-i) \sum_{i'=i}^{I} (v_{i'j+1} - v_{i'j}) = \sum_{i'=i+1}^{I} (v_{i'j+1} - v_{i'j}) - (I-i)(v_{ij+1} - v_{ij}).$$

In other words,

$$v_{Ij+1} - v_{Ij} \geq (I-i)(v_{ij+1} - v_{ij}) - \sum_{i'=i+1}^{I-1} (v_{i'j+1} - v_{i'j}).$$

We define $v'_{j+1} = \max_i \left\{ (I-i)(v_{ij+1} - v_{ij}) - \sum_{i'=i+1}^{I-1}(v_{i'j+1} - v_{i'j}) \right\}$. Then, we may define $v_{Ij} = \sum_{j'=1}^{j} \max\{\bar{v}_j, v'_j\}$ to ensure that WTPs satisfy SCP and are non-decreasing in j. ■

5.1.12 Why CBPcyd is a valid reformulation of CBPcy with $0 \leq y \leq Y$

Let $v_i(y_i) - c(y_i) = f_i(y_i)$ for all i. Assume that $l_i(y_i)$ has a finite upper bound over $[0, Y]$. It follows that CBPcy has a finite upper bound for all i. Assume that y' is feasible to CBPcy. We show by induction that $l_i(y'_i) \geq \sum_{i'=1}^{i} f_{i'}(y'_{i'})$. For $i = 1$, $l_1(y'_1) \geq f_1(y'_1)$ because of

Constraints (2.23) and (2.24). Assume $l_i(y'_i) \geq \sum_{i'=1}^{i} f_{i'}(y'_{i'})$. Then

$$l_{i+1}(y'_{i+1}) \geq l_i(y'_{i+1}) + f_{i+1}(y'_{i+1}) \geq l_i(y'_i) + f_{i+1}(y'_{i+1}) \geq \sum_{i'=1}^{i+1} f_{i'}(y'_{i'}),$$

where the first inequality is by (2.23), second by (2.25), and the third by induction. Since $l_I(y) \geq l_I(y'_I)$, the optimal value of CBPcyd is at least that of CBPcy. Assume now that the optimal value of the former exceeds that of the latter by an $\epsilon \geq 0$. Let $\lambda_i^k(\bar{y})$ be the k^{th} element of a sequence that converges monotonically to the optimal value in $l_i(y) = \sup\{l_{i-1}(y'') + f_i(y'') \mid y'' \leq \bar{y}\}$. Let $\lambda_{i \cdots i'}^k(\bar{y})$ denote $\lambda_i^k \circ \cdots \circ \lambda_{i'}^k(\bar{y})$. Then, let $y^k = (\lambda_{1 \cdots I}^k(Y), \ldots, \lambda_I^k(Y))$, and observe that y^k is feasible to CBPcy. Therefore, $l_I(Y) - \epsilon \geq \sum_{i=1}^{I} f_i(y_{i-1}^k)$. Taking $k \to \infty$,

$$\lim_{k \to \infty} \sum_{i=1}^{I} f_i(y_i^k) \leq l_I(Y) - \epsilon.$$

For any $\delta > 0$ and k, we can find $k'(k)$ nondecreasing in k, such that for all i,

$$l_i(y_i^k) \leq l_{i-1}(y_{i-1}^{k'(k)}) + f_i(y_{i-1}^{k'(k)}) + \delta.$$

Summing for all i, we obtain

$$\sum_{i=1}^{I} l_i(y_i^k) \leq \sum_{i=1}^{I-1} l_i(y_i^{k'(k)}) + \sum_{i=1}^{I} f_i(y_{i-1}^{k'(k)}) + I\delta.$$

Taking the limit as $k \to \infty$,

$$l_I(Y) \leq \lim_{k \to \infty} \sum_{i=1}^{I} f_i(y_{i-1}^{k'(k)}) + I\delta \leq l_I(Y) - \epsilon + I\delta.$$

Since δ was chosen arbitrarily, the above implies $\epsilon \leq 0$, proving that the optimal value of CBPcyd matches that of CBPcy.

5.1.13 Proof of Lemma 15

Proof. To show that $S' = \mathrm{conv}(S)$, we show that $\mathrm{vert}(S') \subseteq S \subseteq S'$, where $\mathrm{vert}(S')$ are the vertices of S'. Then, the result follows because S' is bounded, $\mathrm{conv}(S') = \mathrm{conv}(\mathrm{vert}(S')) \subseteq \mathrm{conv}(S) \subseteq \mathrm{conv}(S') = S'$, and $\mathrm{conv}(S') = S'$. We first show that $S \subseteq S'$. Let $(y', x') \in S$. Then, y' satisfies (2.22) and x' is as defined in (2.26). We show that $(y', x') \in S'$. For that, we need to show that $\sum_{j'=j}^{J} x'_{ij} \leq \sum_{j'=j}^{J} x'_{i+1j}$ for all j. Let j_1 be such that $k_{j_1} \leq y'_i \leq k_{j_1+1}$. Because of (2.26), for $0 \leq j \leq j_1$, $\sum_{j'=j}^{J} x'_{i+1j} = 1 \geq \sum_{j'=j}^{J} x'_{ij}$. For $j > j_1 + 1$, $\sum_{j'=j}^{J} x'_{ij} = 0 \leq \sum_{j'=j}^{J} x'_{i+1j}$. Therefore, we only need to consider $j = j_1 + 1$. Then,

$$
\begin{aligned}
k_{j_1} + (k_{j_1+1} - k_{j_1})x'_{ij_1+1} = \sum_{j=0}^{J} k_j x'_{ij} \leq \sum_{j=0}^{J} k_j x'_{i+1j} = k_{j_1} + \sum_{j=j_1+1}^{J} (k_j - k_{j-1}) \sum_{j'=j}^{J} x_{i+1j'} \\
\leq k_{j_1} + (k_{j_1+1} - k_{j_1}) \sum_{j=j_1+1}^{J} x_{i+1j_1+1}
\end{aligned}
$$

Since $k_{j_1+1} - k_{j_1} > 0$, it follows that

$$
\sum_{j=j_1+1}^{J} x'_{ij} = x'_{ij_1+1} \leq \sum_{j=j_1+1}^{J} x_{i+1j_1+1}.
$$

Now, we show that $\mathrm{vert}(S') \subseteq S$. Let $(y', x') \in \mathrm{vert}(S')$. Obviously, $x' \in \mathrm{vert}(\mathrm{proj}_x(S'))$. However, by Proposition 8, the constraint matrix defining $\mathrm{proj}_x(S')$ is totally unimodular. Therefore, x' is binary-valued. Then, it follows from $\sum_{j=0}^{J} x_{ij} = 1$ that $x_{ij} x_{ij'} = 0$ for all

$j \neq j'$. Finally,

$$\sum_{j=0}^{J} k_j x_{ij} = \sum_{j=1}^{J} \sum_{j'=1}^{j} (k_{j'} - k_{j'-1}) x_{ij} = \sum_{j'=1}^{J} (k_{j'} - k_{j'-1}) \sum_{j=j'}^{J} x_{ij} \leq \sum_{j'=1}^{J} (k_{j'} - k_{j'-1}) \sum_{j=j'}^{J} x_{i+1j}$$
$$= \sum_{j=1}^{J} \sum_{j'=1}^{j} (k_{j'} - k_{j'-1}) x_{i+1j} = \sum_{j=0}^{J} k_j x_{i+1j},$$

where the inequality follows because $k_{j'} \geq k_{j'-1}$ and $\sum_{j=j'}^{J} x_{ij} \leq \sum_{j=j'}^{J} x_{i+1j}$.

We now show that $\text{proj}_y(S') = \text{proj}_y(S) = A$. Towards this end, we prove that $\text{proj}_y(S) \subseteq A$. Let $(y, x) \in S$. It follows that $0 \leq y_i \leq Y$ because $0 = k_0 \sum_{j=0}^{J} x_{ij} \leq \sum_{j=0}^{J} k_j x_{ij} \leq k_J \sum_{j=0}^{J} x_{ij} = Y$. Also, $y_i \leq y_{i+1}$ follows directly from $\sum_{j=0}^{J} k_j x_{ij} \leq \sum_{j=0}^{J} k_j x_{i+1j}$. Since $A \subseteq \text{proj}_y(S)$ follows directly from (2.26), it follows that $\text{proj}_y(S) = A$. Then, $\text{proj}_y(S') = A$ follows from

$$\text{proj}_y(S') = \text{proj}_y(\text{conv}(S)) = \text{conv}(\text{proj}_y(S)) = \text{conv}(A) = A,$$

where the second equality because a linear transformation commutes with convexification, the third equality because $\text{proj}_y(S) = A$ and the last equality because A is convex. The last statement in the lemma follows from $\text{conv}(\text{proj}_x(S)) = \text{proj}_x(\text{conv}(S)) = \text{proj}_x(S')$. ■

5.1.14 Proof of Theorem 18

Proof. We first show that $\Pi^k \leq \Pi^c$. Define $w_{ij} = w_i^k(k_j)$, $c_j = c^k(k_j)$, and $v_{ij} = w_{ij} - (I - i)(w_{i+1j} - w_{ij})$. Then, we solve CBP2 to find Π^k and the optimal solution x_{ij} for all i, j. The prices p_j are assumed to satisfy Proposition 5. Now, for any $y' \in [0, Y]$, define $p'(y') = \min\{p(k_j) \mid k_j \geq y', j = 0, \ldots, J\}$. Observe that since $y \leq Y = k_J$, the minimum in the formula is attained. Let $y_i = \sum_{j=0}^{J} k_j x_{ij}$. We claim that (y, p') is feasible to CBPc1 and has an objective value of Π^k. Consider Constraints (3.33). Let $k_{j'-1} < y \leq k_{j'}$ for some j'.

Then, since (x, p) is feasible to CBP1, it follows that

$$w_i(y_i) - p'(y_i) = \sum_{j=0}^{J}(w_{ij} - p_j)x_{ij} \geq w_{ij'} - p_{j'} = w_i(k_{j'}) - p'(k_{j'}) \geq w_i(y) - p'(y).$$

The objective function value of (y, p') is then

$$\sum_{i=1}^{I}(p'(y_i) - c(y_i)) = \sum_{i=1}^{I}\sum_{j=0}^{J}(p_j - c_j)x_{ij} = \Pi^k.$$

Since (y, p') is feasible to CBPc1 and has an objective value of Π^k, it follows that the optimal value Π^c to CBPc1 is at least Π^k.

Now, we show that $\Pi^c \leq \Pi^k + \epsilon$. Let (y', p') be the optimal assignment and price for CBPc1. Now consider CBPc1 where the $w_i(\cdot)$ and $c(\cdot)$ functions are replaced with $w_i^k(\cdot)$ and $c^k(\cdot)$ and call this problem Q. Since $w_i^k(\cdot)$ and $c^k(\cdot)$ are piecewise-linear with breakpoints in $\{k_1, \ldots, k_J\}$, it follows from Theorem 16 that the optimal value of Q is Π^k. Now, we define $p''(y) = \min\{p(y_i') - i\delta \mid y_i' \geq y, i = 0, \ldots, I+1\}$, where y_0' and y_{I+1}' are assumed to be 0 and Y respectively and δ will be fixed later. Assume $p''(\cdot)$ is the price in Q. We show that there is a feasible solution (y'', p'') to Q, where $y_i'' \in \{y_1', \ldots, y_I'\}$ for each i. Instead, let y_i be an allocation to Consumer i such that $y_{i'-1}' < y_i < y_{i'}'$. However,

$$w_i^k(y_i) - p''(y_i) \leq w_i^k(y_{i'}') - p''(y_{i'}'),$$

where the inequality follows since $y_i < y_{i'}'$ implies that $w_i^k(y_i) \leq w_i^k(y_{i'}')$ and the definition of $p''(\cdot)$ implies that $p''(y_i) = p''(y_{i'}')$. Therefore, the consumer may substitute $y_{i'}'$ for y_i without loss of surplus. Now, observe that the choice set of each consumer is finite, therefore there exists a bundle size that provides maximum surplus to the consumer. Now, we show that, by suitably choosing δ, we can ensure that there exists a feasible solution that satisfies $y_i'' \geq y_i'$ for all i. Assume otherwise and consider a Consumer i who purchases $y_i'' = y_{i'}' < y_i'$. First, observe that Lipschitz continuity of $w_i(\cdot)$ and $c(\cdot)$ guarantees that for any y,

$$\begin{aligned} |w_i(y) - w_i^k(y)| &\leq \max\{w_i^k(k_{j+1}) - w_i(y), w_i(y) - w_i^k(k_j)\} \\ &= \max\{w_i(k_{j+1}) - w_i(y), w_i(y) - w_i(k_j)\} \leq k\beta \end{aligned} \tag{5.12}$$

$$\begin{aligned} |c(y) - c^k(y)| &\leq \max\{|c(y) - c^k(k_j)|, |c^k(k_{j+1}) - c(y)|\} \\ &= \max\{|c(y) - c(k_j)|, |c(k_{j+1}) - c(y)|\} \leq k\beta, \end{aligned} \tag{5.13}$$

where j is chosen such that $k_j \leq y < k_{j+1}$. The first inequality follows since w^k is non-decreasing and the first equality because $w_i(\cdot)$ (resp. $c(\cdot)$) match $w^k(\cdot)$ (resp. $c^k(\cdot)$) at all $y \in \{k_1, \ldots, k_J\}$. Then, choosing $\delta = 2k\beta$ it follows that:

$$\begin{aligned} w_i^k(y_{i'}') - p''(y_{i'}') &= w_i^k(y_{i'}') - (p(y_{i'}') - i'\delta) \leq w_i(y_{i'}') + k\beta - (p(y_{i'}') - i'\delta) \\ &\leq w_i(y_i') + k\beta - (p(y_i') - i'\delta) \leq w_i^k(y_i') + 2k\beta - (p(y_i') - i'\delta) \\ &\leq w_i^k(y_i') + 2k\beta - (i - i')\delta - (p(y_i') - i\delta) \leq w_i^k(y_i') - p''(y_i'). \end{aligned}$$

Therefore, no consumer purchases a smaller sized bundle and so, for any $i' > i$:

$$p''(y_{i'}') - c^k(y_{i'}') \geq p(y_{i'}') - c(y_{i'}') - i'\delta - k\beta \geq p(y_i) - c(y_i) - i'\delta - k\beta \geq p(y_i) - c(y_i) - (2I+1)k\beta,$$

where the first inequality follows from the definition of p'' and (5.13) and the second inequality from Proposition 5 and $i' > i$, and the third inequality because $\delta = 2k\beta$. Therefore,

$$\Pi^k \geq \Pi^c - I(2I+1)k\beta = \Pi^c - \epsilon.$$

Because $J = \lceil \frac{Y}{k} \rceil$ and CBP2 can be solved in $\text{O}(IJ)$ time, CBPc1 can be approximated within ϵ in $\text{O}\left(\frac{I^2(I+2)\beta Y}{\epsilon} + I\right)$ time. ■

5.1.15 Proof of Theorem 19

Proof. According to Proposition 7 in Kannan et al. (2014b), we prove that we can reformulate the MINLP CBP1 into the following 0-1 IP problem CBP2:

$$\text{CBP2 : Max}_{x_{ij}} \left\{ \sum_{i=1}^{I} \sum_{j=0}^{J} (v_{ij} - c_j) x_{ij} \;\middle|\; (3.2), (3.4), (3.7) \right\}.$$

We add the following χ variables to CBP2 and reformulate the problem into CBP3:

$$\chi_{00j} = x_{1j} \forall j, \qquad \chi_{ijj'} = x_{ij} x_{i+1,j'} 1 \le i < I \;\forall j \;\forall j' \ge j, \qquad \text{and} \qquad \chi_{IjJ} = x_{Ij} \forall j. \quad (5.14)$$

We show that constraints of CBP3 are implied by CBP2 and (5.14). Constraints (3.10), (3.12), and (3.15) follow from (5.14) and (3.4). Constraint (3.9) holds since $\sum_{j=0}^{J} \chi_{00j} = \sum_{j=0}^{J} x_{1j} = 1$, where the equalities are due to (5.14) and (3.2) respectively. Constraint (3.11) holds since $\sum_{j=0}^{J} \chi_{IjJ} = \sum_{j=0}^{J} x_{Ij} = 1$, where the equalities are due to (5.14) and (3.2) respectively. Constraint (3.13) and (3.14) hold because:

$$x_{ij} = x_{ij} \sum_{j'=j}^{J} x_{ij'} \le x_{ij} \sum_{j'=j}^{J} x_{i+1j'} = \sum_{j'=j}^{J} \chi_{ijj'} = x_{ij} \sum_{j'=j}^{J} x_{i+1j'} \le x_{ij} \sum_{j'=0}^{J} x_{i+1j'} = x_{ij}$$
$$x_{ij} = x_{ij} \sum_{j'=0}^{j} x_{ij'} \le x_{ij} \sum_{j'=0}^{j} x_{i-1j'} = \sum_{j'=0}^{j} \chi_{i-1j'j} = x_{ij} \sum_{j'=0}^{j} x_{i-1j'} \le x_{ij} \sum_{j'=0}^{J} x_{i-1j'} = x_{ij},$$

imply that equality holds throughout. In both cases, the first equality holds because (3.4) and (3.2) imply that $x_{ij}^2 = x_{ij}$ and $x_{ij} x_{ij'} = 0$ if $j' \ne j$. The first inequality follows from (3.7) (and (3.2) in the second case), the second and third equalities from (5.14), the second inequality from (3.4) and the last equality from (3.2).

Now, we show that Constraints (3.9)-(3.15) imply constraints of $\mathrm{CBP2}$. We use induction on i to show (3.2). The base case follows since $\sum_{j=0}^{J} x_{1j} = \sum_{j=0}^{J} x_{00j} = 1$. For the induction step, we assume $\sum_{j=0}^{J} x_{ij} = 1$ and show $\sum_{j=0}^{J} x_{i+1j} = 1$. Then,

$$\sum_{j=0}^{J} x_{i+1j} = \sum_{j=0}^{J} \sum_{j'=0}^{j} \chi_{ij'j} = \sum_{j'=0}^{J} \sum_{j=j'}^{J} \chi_{ij'j} = \sum_{j'=0}^{J} x_{ij'} = 1,$$

where the first equality follows from (3.14), the third equality by (3.13), and the last equality by induction. Constraint (3.7) follows because $\sum_{j_1=j}^{J} x_{ij_1} = \sum_{j_1=j}^{J} \sum_{j_2=0}^{j_1} \chi_{i-1j_2j_1} \geq \sum_{j_1=j}^{J} \sum_{j_2=j}^{j_1} \chi_{i-1j_2j_1} = \sum_{j_2=j}^{J} \sum_{j_1=j_2}^{J} \chi_{i-1j_2j_1} = \sum_{j_2=j}^{J} x_{i-1j_2}$, where the first equality follows from (3.14), the first inequality since some terms are dropped, and the last equality from (3.13). Constraint (3.4) follows since x_{ij} is non-negative and integer-valued by Constraints (3.15) and (3.13) and it cannot take a value larger than one by (3.2). ■

5.1.16 Proof of Proposition 26

Proof. Define $f_i(z) = \sum_{j'=1}^{J} z_{ij'}$ and $f(z) = (f_1(z), \ldots, f_I(z))$. For z' and z'' feasible to CBPg,

$$\begin{aligned} C(z') + C(z'') &= C'(f(z')) + C'(f(z'')) \\ &\geq C'(\max\{f_1(z'), f_1(z'')\}, \ldots, \max\{f_I(z'), f_I(z'')\}) \\ &\quad + C'(\min\{f_1(z'), f_1(z'')\}, \ldots, \min\{f_I(z'), f_I(z'')\}) \\ &= C(z' \vee z'') + C(z' \wedge z''), \end{aligned} \tag{5.15}$$

since C' is submodular. ■

5.1.17 Proof of Theorem 27

Proof. It follows from (49.25) in Schrijver (2003) that a supermodular function can be maximized over a lattice family in strongly polynomial time if the following two conditions hold. First, there is an oracle to compute the supermodular function in strongly polynomial time. This is true by assumption. Second, the lattice family $\breve{C}$ is described using the smallest set, the largest set and a pre-order $\preceq$ such that if $u \preceq v \Leftrightarrow$ each $U \in \breve{C}$ containing v also contains u. For us, the smallest element corresponds to the case when all consumers purchase nothing, *i.e.*, $z_{ij} = 0$ for all $j \geq 1$. The largest element is when all consumers purchase bundle size J, *i.e.*, $z_{ij} = 1$ for all $j \leq J$. For the pre-order $\preceq$, observe that if $z_{ij} = 1$, then $z_{i'j'} = 1$ for all $i' \geq i$ and $j' \leq j$ and for all i' if $j' = 0$. In other words, $(i', j') \preceq (i, j)$ if and only if either $i' \geq i$ and $j' \leq j$ or $j' = 0$. If $z_{ij} = 1$ then $z_{i'j'} = 1$ using Constraints (3.29), (3.30), and (3.31). Now, consider $(i', j') \npreceq (i, j)$. Then, we show that there is a feasible allocation that sets $z_{i'j'} = 0$ but $z_{ij} = 1$. From the definition of the pre-order, we know that $j' > 0$ and either $i' < i$ or $j' > j$. In either case, the sought solution is one where Consumers $1, \ldots, i-1$ purchase nothing and Consumers $i, \ldots, I$ purchase Bundle j. ■

5.1.18 Proof of Corollary 28

Proof. The two results follow from Lemma 2.8.1 and Theorem 2.8.4 in Topkis (1998) respectively. ■

5.1.19 Proof of Lemma 29

Proof. We prove by induction on r. Consider y^0. Since $y_1 \leq \cdots \leq y_I$, it follows that $a(y_1) \leq \cdots \leq a(y_I)$. Now, assume that y^{r-1} is feasible to CBPgcy. We show that $a_i^r \leq a_{i+1}^r$ for $i \leq I - 1$. The inequality holds by induction whenever $i \notin \{\pi(r), \pi(r) - 1\}$ because $a_{i'}^r = a_{i'}^{r-1}$ for all $i' \neq \pi(r)$. For $i = \pi(r) - 1$, it follows because $a_{\pi(r)-1}^r = a_{\pi(r)-1}^{r-1} \leq a_{\pi(r)}^{r-1} <$

$a^r_{\pi(r)}$, where the first inequality follows from induction. Now, consider $i = \pi(r)$. Assume $\pi^{-1}(i+1) < \pi^{-1}(i) = r$. Then, $a^r_{i+1} = a^0_{i+1} + 1 \geq a^0_i + 1 = a^r_r$, where the inequality follows by the base case. Now, assume that $\pi^{-1}(i+1) > \pi^{-1}(i) = r$. Then, $a^0_i \leq a^0_{i+1}$ because $y_i \leq y_{i+1}$. If $a^0_i < a^0_{i+1}$, the result follows since $a^r_{i+1} = a^0_{i+1}$ and $a^r_i = a^0_i + 1$. We show that a^0_i cannot equal a^0_{i+1}. Otherwise, $a(y_{i+1}) = a(y_i)$ and $y_{i+1} \geq y_i$ yield $g(y_{i+1}) \geq g(y_i)$ and thus a contradiction to $\pi^{-1}(i+1) > \pi^{-1}(i)$ using the definition of π. ■

5.1.20 Proof of Theorem 30

Proof. We describe the architecture of the proof. We construct a relaxation of CBPgcy, which we call $\mathrm{CBPgcyR}$. Then, we relax $\mathrm{CBPgcyR}$ to $\mathrm{CBPgczR}$ which has the same objective value as CBPgcz. This shows that the objective value of CBPgcy is no more than that of CBPgcz. Finally, we show the converse and recover the optimal solution y^* of CBPgcy from the optimal solution z^* of CBPgcz. Let

$$C''(y) = \sum_{r=0}^{I} \left(g(y_{\pi(r)}) - g(y_{\pi(r+1)}) \right) C'(y^r). \tag{5.16}$$

Let $\mathrm{Feas}(Q)$ be the feasible region for any problem Q. Consider the following optimization problem:

$$\mathrm{CBPgcyR} : \mathrm{Max}_{y_i} \left\{ \sum_{i=1}^{I} v_i(y_i) - C''(y) \mid (3.34), (3.35) \right\}.$$

Observe that $\mathrm{CBPgcyR}$ is a relaxation of CBPgcy, because $C''(y) \leq C'(y)$ and, therefore, the objective function value is overestimated. For any $y \in \mathrm{Feas}(\mathrm{CBPgcy})$, define z_{ij} as follows

$$z_{ij} = \begin{cases} 1 & \text{if } j \leq a(y_i) \\ g(y_i) & \text{if } j = a(y_i) + 1 \\ 0 & \text{otherwise,} \end{cases} \tag{5.17}$$

and observe that $\sum_{j=1}^{J} \big(k(j) - k(j-1)\big) z_{ij} = y_i$. Since w_i is piecewise-linear, $w_i(y_i) = w_i(k(a(y_i))) + g(y_i)\big(w_i(k(a(y_i)) + 1) - w_i(k(a(y_i)))\big) = \sum_{j=1}^{J} \big(w_i(k(j)) - w_i(k(j-1))\big) z_{ij}$. For all j, define $w_{ij} = w_i(k(j))$ and, as before, $v_{ij} = w_{ij} - (I - i)(w_{i+1j} - w_{ij})$, where w_{I+1j} is assumed to be w_{Ij}. Then, $v_i(y_i) = \sum_{j=0}^{J} v_{ij}(z_{ij} - z_{ij+1})$.

Now, consider CBPgcz, where we replace (3.32) with $0 \leq z_{ij} \leq 1$, call the resulting feasible region R, and extend $C(z)$ over all of R. Observe that, as in the proof of Proposition 26, $C(\cdot)$ restricted to binary vertices is submodular.

To extend $C(z)$ to R, we construct the convex envelope of $C(z)$ restricted to $\mathrm{Feas}(\mathrm{CBPgcz})$ over R. For any $z \in R$, let $\gamma_z(\cdot, \cdot) : \{1, \ldots, I\} \times \{1, \ldots, J\} \to \{1, \ldots, IJ\}$ be a one-to-one mapping such that $\gamma_z(i, j) \geq \gamma_z(i', j')$ whenever $z_{i,j} \leq z_{i',j'}$. In addition, we require that ties such as $z_{i,j} = z_{i',j'}$ are resolved in the following manner. If $i' > i$ then $\gamma_z(i, j) > \gamma_z(i', j')$. Otherwise, if $i' = i$ and $j' < j$ then $\gamma_z(i, j) > \gamma_z(i', j')$. Observe that this definition guarantees that $\gamma_z(i + 1, j) < \gamma_z(i, j)$ and $\gamma_z(i, j - 1) < \gamma_z(i, j)$. Let z^0 be defined such that $z_{i0} = 1$ for all i, and $z_{ij} = 0$ otherwise. For $r = 1, \ldots, IJ$, define $z^r = z^0 + \sum_{s=1}^{r} e^{\gamma_z^{-1}(s)}$. Observe that, for all r, z^r is feasible to CBPgcz, because of the definition of γ_z. Then, using the insight from

Corollary 2.3 in Tawarmalani et al. (2013), we extend

$$\breve{C}(z) = C(z^0) + \sum_{r=1}^{IJ} \left[z_{\gamma_z^{-1}(r)} \left(C(z^r) - C(z^{r-1}) \right) \right]. \tag{5.18}$$

When z is binary, it is a vertex of R and $\breve{C}(z) = C(z)$. So, $\breve{C}(z)$ is a valid extension for $C(z)$ and

$$\begin{aligned} \text{CBPgczR}: \quad \text{Max}_{z_{ij}} \quad & \sum_{i=1}^{I} \sum_{j=0}^{J} v_{ij}(z_{ij} - z_{ij+1}) - \breve{C}(z) \\ \text{s.t.} \quad & (3.29), (3.30), (3.31) \\ & 0 \le z_{ij} \le 1 \quad \forall i;\ \forall j \end{aligned} \tag{5.19}$$

is a relaxation of CBPgcz. We now show that CBPgczR is also a relaxation of CBPgcyR. We begin by showing that for each $y \in \text{Feas(CBPgcy)}$, if z is defined as in (5.17), then $\breve{C}(z) = C''(y)$. The result is clear if $y \in \{k(0), \ldots, k(J)\}^I$ since in this case, $\breve{C}(z) = C(z) = C'(y) = C'(y^0) = C''(y)$, where the first equality follows since z is a vertex of R, the second equality because of the definition of $C(z)$, and the third equality because $y^0 = y$. Since $g(y_0) = 1$ and $g(y_i) = 0$, for $1 \le i \le I$, the last equality follows. When, $y \notin \{k(0), \ldots, k(J)\}^I$, we define z_{ij} as in (5.17). Let $I' \subseteq \{1, \ldots, I\}$ be the set of consumers for which $g(y_i) \notin \{0, 1\}$, *i.e.*, $y_i \notin \{k(0), \ldots, k(J)\}$. It follows from the definition of γ that for any $i, i' \in I'$, $\gamma_z(i, a(y_i) + 1) \ge \gamma_z(i', a(y_{i'}) + 1)$ if and only if (i) $g(y_i) < g(y_{i'})$, or (ii) $g(y_i) = g(y_{i'})$ and $i < i'$. However, this implies that for $i \in I'$, the relative ordering of $\gamma_z(i, a(y_i) + 1)$ for $1 \le i \le I$ is consistent with $\pi^{-1}(i)$. Let t be the smallest value such that $z_{\gamma^{-1}(t)} \notin \{0, 1\}$. Then, it follows that for $i \in I'$, $\gamma(i, a(y_i) + 1) = \pi^{-1}(i) + t - 1$. Since, for $r = 0, \ldots, |I'|$, $y^r \in \{k(0), \ldots, k(J)\}^I$, it follows from the discussion above that $C(z^{r+t-1}) = C'(y^r)$ for

$r = 0, \ldots, |I'|$. Let $z_{\gamma_z^{-1}(I+1)} = 0$ and observe that

$$\begin{aligned}
\breve{C}(z) &= C(z^{t-1}) + \sum_{r=t}^{t+|I'|-1} z_{\gamma_z^{-1}(r)} \left(C(z^r) - C(z^{r-1}) \right) \\
&= (1 - z_{\gamma_z^{-1}(t)}) C(z^{t-1}) + \sum_{r=t}^{t+|I'|-1} \left(z_{\gamma_z^{-1}(r)} - z_{\gamma_z^{-1}(r+1)} \right) C(z^r) \\
&= (1 - g(y_{\pi(1)}) C'(y^0) + \sum_{r=1}^{|I'|} \left(g(y_{\pi(r)}) - g(y_{\pi(r+1)}) \right) C'(y^r) \\
&= \sum_{r=0}^{|I'|} \left(g(y_{\pi(r)}) - g(y_{\pi(r+1)}) \right) C'(y^r) \\
&= \sum_{r=0}^{I} \left(g(y_{\pi(r)}) - g(y_{\pi(r+1)}) \right) C'(y^r) = C''(y),
\end{aligned}$$

where the first equality is from (5.18) and that $z_{\gamma_z^{-1}(r)} = 0$ for $r \geq t + |I'|$; the second follows by rearranging terms and because the definition of I' implies that $z_{\gamma_z^{-1}(t+|I'|)} = 0$; the third is by realizing that $z_{\gamma_z^{-1}(r)} = g(y_{\pi(r-t+1)})$ for $r = t, \ldots, t + |I'| - 1$ and $C(z^r) = C'(y^{r-t+1})$ for $r = t-1, \ldots, t + |I'| - 1$; the fourth is because $g(y_{\pi(0)}) = 1$ and $g(y_{\pi(I+1)}) = 0$; the fifth is because the terms in the summation with $r > |I'|$ are zero; and the final is by (5.16). Thus, we have shown that, for any $y \in \text{Feas(CBPgcy)}$, if z is defined as in (5.17), then $\breve{C}(z) = C''(y) \leq C'(y)$. Finally, observe that we already showed that piecewise-linearity of $w_i(\cdot)$ implies that $v_i(y_i) = \sum_{j=0}^{J} v_{ij}(z_{ij} - z_{ij+1})$.

We have thus shown CBPgczR is a relaxation of CBPgcy. Now, we show that there exists an optimal solution in CBPgczR where each z_{ij} is binary. Assume that this is not the case. Now, take an optimal solution $\bar{z}$, compute the corresponding $\gamma_{\bar{z}}(i,j)$ mapping and $\bar{z}^0, \ldots, \bar{z}^{IJ}$. We show that $\bar{z}^0_{ij} + \sum_{r=1}^{IJ} \left[\bar{z}_{\gamma_{\bar{z}}^{-1}(r)} \left(\bar{z}^r_{ij} - \bar{z}^{r-1}_{ij} \right) \right] = \bar{z}_{ij}$. First observe that all terms in the summation, except when $r = \gamma_{\bar{z}}(i,j)$, are zero since $\bar{z}^r_{ij} = \bar{z}^{r-1}_{ij}$ otherwise. When $r = \gamma_{\bar{z}}(i,j)$, the term yields $\bar{z}_{ij}$ because $\bar{z}^r_{ij} = 1$ and $\bar{z}^{r-1}_{ij} = 0$. We have shown binary $z^0, \ldots, z^r$ that are feasible to CBPgcz such that $(\bar{z}, \breve{C}(\bar{z})) \in \text{conv}\left((\bar{z}^0, C(\bar{z}^0)), \ldots, (\bar{z}^{IJ}, C(\bar{z}^{IJ})) \right)$. Then, let

$F(z) = \sum_{i=1}^{I} \sum_{j=0}^{J} v_{ij}(z_{ij} - z_{ij+1}) - \breve{C}(z)$ be the objective function of CBPgczR and note that we have shown that there exist $\lambda_0, \ldots, \lambda_{IJ}$, each non-negative, such that $\sum_{r=0}^{IJ} \lambda_{IJ} = 1$ and $(\bar{z}, F(\bar{z})) = \sum_{r=0}^{IJ} \lambda_r(\bar{z}^r, F(\bar{z}^r))$. In particular, $\lambda_0 = (1 - \bar{z}_{\gamma_{\bar{z}}(1)})$; $\lambda_r = (\bar{z}_{\gamma_{\bar{z}}(r)} - \bar{z}_{\gamma_{\bar{z}}(r+1)})$ for $r = 1, \ldots, IJ - 1$; and $\lambda_{IJ} = \bar{z}_{\gamma_{\bar{z}}(IJ)}$. Therefore, $F(\bar{z}) = \sum_{r=0}^{IJ} \lambda_r F(\bar{z}^r) \leq \sum_{r=0}^{IJ} \lambda_r \max_{r'} F(\bar{z}^{r'}) = \max_{r'} F(\bar{z}^{r'})$, *i.e.*, there exists one of $\bar{z}^0, \ldots, \bar{z}^{IJ}$, say z^r, that achieves the same objective function value as $\bar{z}$. Since $\bar{z}^r$ is feasible to CBPgcz with the same objective value as in CBPgczR, it follows that the optimal value of CBPgcz matches that of CBPgczR.

We showed that the optimal value of CBPgcy is no more than that of CBPgcz. We now show the converse. Consider z' feasible to CBPgcz and let $y_i = \sum_{j=1}^{J}(k(j) - k(j-1))z_{ij}$. By definition of $C(\cdot)$ and piecewise-linearity of $w_i(\cdot)$, y has the same objective function value in CBPgcy as does z in CBPgcz. Further, y is feasible to CBPgcy. Observe that, y satisfies (3.34) because $y_i = \sum_{j=1}^{J}(k(j) - k(j-1))z_{ij} \leq \sum_{j=1}^{J}(k(j) - k(j-1))z_{i+1j} = y_{i+1}$, because $k(j) > k(j-1)$ and $z_{i+1j} \geq z_{ij}$. Also, y satisfies (3.35) because $0 \leq \sum_{j=1}^{J}(k(j) - k(j-1))z_{ij} \leq \sum_{j=1}^{J}(k(j) - k(j-1)) = k(J) - k(0) = Y$. ■

5.1.21 Proof of Corollary 32

Proof. If $C'(y) \geq \sum_{r=0}^{I} \left(g(y_{\pi(r)}) - g(y_{\pi(r+1)})\right) C'(y^r)$ then the result follows from Theorem 27 and Corollary 31. Observe that $g(y_{\pi(r)}) \geq g(y_{\pi(r+1)})$ and $\sum_{r=0}^{I} \left(g(y_{\pi(r)}) - g(y_{\pi(r+1)})\right) = 1$. Therefore, it follows from (3.36) that $y \in \text{conv}(y^0, \ldots, y^I)$. By Lemma 29, $y^0, \ldots, y^I$ are feasible to CBPgcy. Then, $C'(y) \geq \sum_{r=0}^{I} \left(g(y_{\pi(r)}) - g(y_{\pi(r+1)})\right) C'(y^r)$, because $C'(\cdot)$ is concave. ■

5.1.22 Proof of Theorem 33

Proof. Assume y' is the optimal solution to CBPgcy. Let $\{i_1, \ldots, i_k\}$ be the lowest-type consumers that purchase a bundle of a certain size. The pricing p then follows from Proposition

7 in Kannan et al. (2014b):

$$p_j = w_{i_0 j(i_0)} + \sum_{r=1}^{r'(j)} \left(w_{i_r j(i_r)} - w_{i_r j(i_{r-1})} \right). \tag{5.20}$$

More formally, for any s such that $0 \leq s \leq Y$, define $r'(s) = \arg\min_r \{ i_r \mid y'_{i_r} \geq y_s \}$, *i.e.*, the lowest-type consumer who purchases a bundle of larger size. If there is no consumer that purchases a bundle of size s or larger, we define its price to be $w_{IY} + \delta$ for some $\delta > 0$. Otherwise, we define $p_s = \sum_{r=1}^{r'(s)} \left((w_{i_r}(y'_{i_r}) - w_{i_r}(y'_{i_{r-1}}) \right)$.

We denote CBPgcy with WTPs replaced with $w^k(\cdot)$ and cost replaced with $C'^k(y)$ as Problem T. We now construct a feasible solution to T by providing a pricing strategy such that, for all i, Consumer i still purchases y'_i. For this we again utilize (5.20). For bundle size s, we set the price to $p^k_s = \sum_{r=1}^{r'(s)} \left((w^k_{i_r}(y'_{i_r}) - w^k_{i_r}(y'_{i_{r-1}}) \right)$. It follows from Proposition 7 in Kannan et al. (2014b) that Consumer i continues to purchase bundle size y'_i in Problem T. For any s satisfying $0 \leq s \leq Y$, let $s_k = \max\{k(j) \mid k(j) \leq s\}$ and $s'_k = \min\{k(j) \mid k(j) \geq s\}$. Then, for any i:

$$\begin{aligned} |w^k_i(s) - w_i(s)| &= |w^k_i(s) - w^k_i(s_k) + w_i(s_k) - w_i(s)| \\ &\leq \max\{w^k_i(s) - w^k_i(s_k), w_i(s) - w_i(s_k)\} \\ &\leq \max\{w^k_i(s'_k) - w^k_i(s_k), w_i(s) - w_i(s_k)\} \\ &= w_i(s'_k) - w_i(s_k) \\ &\leq k\beta, \end{aligned} \tag{5.21}$$

where the first equality follows because $w^k_i(s_k) = w_i(s_k)$; the first inequality because $w^k_i(s) - w^k_i(s_k) \geq 0$ and $w_i(s_k) - w_i(s) \leq 0$; the second inequality because $w^k_i(s'_k) \geq w^k_i(s) \geq w^k_i(s_k)$; the second equality because $w^k_i(s'_k) = w_i(s'_k)$, $w^k_i(s_k) = w_i(s_k)$ and $w_i(s'_k) \geq w_i(s) \geq w_i(s_k)$;

and the last inequality because of Lipschitz continuity of w. Observe that

$$\begin{aligned} p_s - p_s^k &= \sum_{r=1}^{r'(s)} \left(w_{i_r}(y'_{i_r}) - w_{i_r}^k(y'_{i_r}) - w_{i_r}(y'_{i_{r-1}}) + w_{i_r}^k(y'_{i_{r-1}}) \right) \\ &\leq \sum_{r=1}^{r'(s)} \left(|w_{i_r}(y'_{i_r}) - w_{i_r}^k(y'_{i_r})| + |w_{i_r}(y'_{i_{r-1}}) - w_{i_r}^k(y'_{i_{r-1}})| \right) \\ &\leq 2kI\beta, \end{aligned}$$

where the last inequality follows from (5.21). Consider a vector y feasible to CBPgcy and construct $y^0, \ldots y^I$ as described before Lemma 29. Let $t^1 = \arg\min\{C'(t) \mid t \in \{y^0, \ldots, y^I\}\}$ and $t^2 = \arg\max\{C'(t) \mid t \in \{y^0, \ldots, y^I\}\}$. Then:

$$\begin{aligned} |C'^k(y) - C'(y)| &= |C'^k(y) - C'^k(t^1) + C'_i(t^1) - C'(s)| \\ &\leq |C'^k(y) - C'^k(t^1)| + |C'(t^1) - C'(y)| \\ &\leq |C'^k(t^2) - C'^k(t^1)| + |C'(t^1) - C'(y)| \\ &\leq |C'(t^2) - C'(t^1)| + |C'(t^1) - C'(y)| \\ &\leq 2\sqrt{I}k\beta \end{aligned} \tag{5.22}$$

where the second inequality follows from (3.36) and since C'^k is linear in $\mathrm{conv}\{y^0, \ldots, y^I\}$, the second equality follows since $C'^k(\cdot)$ matches $C'(\cdot)$ at each of $\{y^0, \ldots, y^I\}$, and the last inequality follows because $\|t^2 - t^1\| \leq \sqrt{I}k$ and $\|y - t^1\| \leq \sqrt{I}k$. Therefore, $\Pi^k \geq \Pi^c - 2k\beta(I^2 + \sqrt{I})$. Since $k = \frac{\epsilon}{2\beta(I^2+\sqrt{I})}$, $\Pi^k \geq \Pi^c + \epsilon$. Since the optimal solution of T occurs at the breakpoints, it is also feasible to CBPgcy, as long as the price of the intermediate sizes is set high enough (to that of the next breakpoint). Therefore, $\Pi^k \leq \Pi^c$.

Finally, observe that, by Theorem 27, CBPgcz can be solved in time polynomial in I and J and the oracle time to compute $C'(y)$. Then, the algorithm is polynomial in I, Y, β, $\frac{1}{\epsilon}$ because $J = \lceil \frac{Y}{k} \rceil$ implies that $J \leq \frac{2Y\beta(I^2+\sqrt{I})}{\epsilon} + 1$. ■

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