Cardinality Bundles for Spence-Mirrlees Reservation Prices

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Abstract

This paper analyzes optimal solution of cardinality bundling (CB) problem in which firms set prices only depending on the size of a purchased bundle. Numerical experiments and empirical analyses from prior work suggest that CB tends to be more profitable than unbundling and tends to closely approximate the profits from mixed bundling (MB). Theoretical works on CB use nonlinear mixed-integer programming (MINLP) approach and provide meaningful insights regarding the optimal solution. However, we notice that the existing analytical framework lacks sub-additivity constraints on bundle pricing, which limits its application in reality. In this study, we solve the CB problem with a specific customer preferences structure - Spence-Mirrlees Single Crossing Property (SCP). We first provide a solution for the CB problem without sub-additivity constraints and prove that it is a shortest-path problem. And then, we analyze the CB problem with sub-additivity constraints. We propose a dynamic programming approach based on unit price decreasing property to provide a lower-bound estimation. We convert the MINLP problem to a mixed-integer programming (MIP) one. Our numerical experiments suggest that the proposed approach provides a good approximation.

1 Introduction

Bundling and its benefits have been extensively studied in prior literature. Bakos and Brynjolfsson (1999) show that synergies among products can lead to more profitable opportunities when products are bundled than when they are sold separately. A number of papers have explored the pricing strategies when all the possible combinations of bundles are available (Stigler, 1963, Adams and Yellen, 1976, McAfee et al., 1989). The main problem with mixed bundling is that it is only computationally tractable for a small number of goods (Hanson and Martin, 1990) because a firm may offer up to $2^J - 1$ possible bundles for J goods.

Practically speaking, pricing every bundle combination is quite difficult, so other forms of bundling were considered. Two types of bundling considered are: component bundling, where individual components are priced and not the bundles, and pure bundling, where only a bundle with all possible products is available. Lately, a new type of bundling, cardinality bundles (CB) has become popular. CB sets the same price for bundles of equal size. That is to say, for a firm selling J goods, CB offers one price for a bundle of one good, a second price for a bundle of any two goods, a third price for a bundle of any three goods and so on. Therefore, compared to mixed bundling, CB only requires J prices for J bundles. CB is implemented in reality. For example, Chu et al. (2011) note that CB is used to sell seasonal theater tickets. With the emergence and rapid growth of low-cost reproduction and distribution technologies for information goods, researchers and information goods providers are more and more attracted to CB.

Literature on CB is limited though. To the best of our knowledge, there have been three other papers that have studied CB. Hitt and Chen (2005) study the problem, assuming that the consumer can buy only one bundle. They explore the conditions under which a mixed bundling problem can be reduced to a CB problem. They also analyze optimal solutions for CB problem with an additional assumption about consumers' reservation price, known as the Spence-Mirrlees Single Crossing Property (SCP). Under the same assumption, i.e., on one bundle per consumer, Wu et al. (2008) explore the properties of the CB problem by using a nonlinear mixed-integer programming approach. They propose a Lagrangian relaxation and a subgradient method for solving the CB problems. The authors develop a heuristic solution strategy and provide an upper bound on the profit. However, there is a residual gap between their best upper and lower bounds at termination. Chu et al. (2011) consider a model where unit prices for bundles are decreasing with size. They computationally study how specifying prices for every bundle in this scenario leads to a profit that is close to offering every combination of bundles.

Hitt and Chen (2005) do not require the price scheme to be sub-additive. However, this constraint appears in earlier work on mixed bundling; see Hanson and Martin (1990). This constraint is introduced to ensure that the consumer incentive compatibility is not violated even if consumers are allowed to purchase more than one bundle of goods. For example, consider a firm that sells products A and B individually and also a bundle of the two products. If the firm does not impose sub-additivity on prices, then the consumer would prefer to purchase both A and B individually rather than purchase the bundle. Similarly, in CB, the price of a bundle of two goods should be no more than twice that of one individual item. We will show later that the optimal prices obtained by using the formulation of Hitt and Chen (2005) and Wu et al. (2008) may not be sub-additive even if the consumers consider the goods as substitutes and, thus have sub-additive reservation prices. In other words, the insights obtained by these works do not necessarily extend to situations when the consumers may purchase more than one bundle. For example, consider an on-line music store that sells any bundle of 10 songs for \$20 and any bundle of 20 songs for \$45. If the transaction costs of purchasing two bundles are not significantly larger than that of purchasing a bundle of 20 songs, then rational consumers would rather purchase two bundles of 10 songs than purchase a bundle of 20 songs.

Instead, Chu et al. (2011) require that bundles of all sizes are offered and the unit price for a larger bundle is no more than that of a smaller bundle. If the unit price does not increase with an increase in the bundle size, it can be shown that the resulting prices are sub-additive. However, the converse is not necessarily true. To illustrate this fact, consider a firm that sells each item at a price of \$7, a bundle of two goods at a price of \$11, and a bundle of three goods at a price of

\$17. Then, although the unit price is not decreasing, a consumer cannot purchase three goods for a price below \$17 by purchasing a bundle of two goods and an individual good. A similar price structure is also seen to occur if the firm does not sell bundles of all sizes.

In this study, we define CB problem with price sub-additivity constraints as the CBS problem, CB problem without price sub-additivity constraints as the Customized Bundle-Pricing (CBP) problem, following its definition in Hitt and Chen (2005), and CB problem with non-increasing unit price constraints as the Bundle-Size Pricing (BSP) problem, following its definition in Chu et al. (2011). We will show later that each of CBP, BSP, and CBS may result in different objective values.

In reality, there exist specific industries where each of the three price scheme types can be implemented. In some situations when consumers are limited to purchase no more than one bundle, CBP is often implemented. For example, if a television cable company provides a package of 10 channels at \$20 per month and another package of 20 channels at \$45 per month, some consumers may be willing to purchase the package of 20 channels. They cannot alternatively purchase two packages of 10 channels to get 20 channels since most homes only have one cable connection. However, in some other industries, such as the on-line music industry, CBP is not implementable. When price sub-additivity is required, some firms follow BSP to set prices for their products while some others apply CBS. Pricing in whole-seller/distributor models usually follows BSP. Purchasing more units from a whole-seller usually guarantees a weakly lower unit price. However, there are other scenarios where CBS could be implemented. A music store can offer a single song for \$4 each, and a bundle size 10 for \$10. If someone wants 11 songs, she needs to pay \$14 to get a bundle and a single song, which has a higher unit price than that of bundle size 10.

In this study, we look at and compare all three types of CB: CBP, BSP, and CBS. In order to get tractable and meaningful results, we additionally assume Spence-Mirrlees Single Crossing Property (SCP) on consumers' reservation price, following a common practice in economics and bundling related literature (Spence, 1974, Tirole, 2003). We consider CBS as the baseline for comparing various models of CB. Our paper establishes that the CBP problem, the problem with one bundle restriction, is a shortest path problem. The CBP optimal profit gives an upper bound on the profit obtained from CBS. BSP is obtained when relaxing the restriction that each consumer purchases one bundle and imposing that unit prices are non-increasing. We propose a dynamic programming algorithm which generates the optimal profit for BSP. The BSP optimal profit provides a lower bound on CBS. We demonstrate the upper and lower bound results on a set of randomly generated examples. Thus, we reconcile the differences in the optimal solutions obtained via different formulations of cardinality bundling in the literature.

The remaining part of this paper is organized as following. In Section 2, we build the CBP model following Hitt and Chen (2005), which is without price sub-additivity constraints, and solve it as a shortest path problem. In Section 3, we propose a pseudo-polynomial algorithm for the BSP problem. In Section 4, we present an MIP approach for the CBS problem and numerically compare it with the CBP problem and the BSP problem. Concluding remarks are presented in Section 5.

2 Solving Customized Bundle-Pricing problem

In this section, we first follow Hitt and Chen (2005) and Wu et al. (2008) to formulate an MINLP problem for the CBP problem. Then, we convert it to a 0-1 integer-programing problem. Finally, we show that the 0-1 IP problem can be re-formulated as a shortest-path problem, which is a well-known linear programming (LP) problem.

2.1 Model

Our main model follows the settings of Hitt and Chen (2005). We form the CBP problem for an information goods provider that distributes J goods to I consumers. The model is developed from the seller's perspective, subject to a set of consumer participation and incentive compatibility constraints. The problem for the seller is to decide p_j, the price at which he sells a bundle of j goods, for $j \in \{1, 2, \ldots, J\}$, so as to maximize his profit. We assume fixed marginal production cost c for the seller. Therefore, $c \cdot j$ is the production cost for bundle j.

In this model, each consumer will buy at most one bundle. Let $w_{ij} \geq 0$ denote consumer i's willingness to pay for bundle j. Each consumer makes a purchase decision maximizing her surplus from purchasing a specific bundle size: $w_{ij} - p_j$. In additon, we assume that when getting same surplus from two bundles, a consumer will purchase the larger size bundle, which is a common assumption in the bundling literature. Without losing generality, we artificially add one bundle with zero size and zero price. If a consumer gets negative surplus from all the available bundles, she will purchase this zero size bundle and get zero surplus.

In order to get tractable and interesting results, we follow Hitt and Chen (2005) to make an additional assumption about consumer valuations known as the Spence-Mirrlees Single Crossing Property. This assumption is used in most models of nonlinear pricing problems. SCP requires that there exists an ordering of consumers such that

$$w_{ij} \geq w_{i'j} \; \forall i > i',$$

$$w_{ij} - w_{ij'} \geq w_{i'j} - w_{i'j'} \; \forall i > i', \; \forall j > j'$$

SCP imposes an ordering of consumer demand over bundles. Consumer indexed with i has greater value for any given customized bundle than any other consumer indexed with a value smaller than i. In addition, these differences are weakly increasing in bundle size. For all subsequent discussion, assume that consumer types are ordered to satisfy this condition.

Let $x_{ij} = 1$ denote consumer $i \in \{1, 2, \ldots, I\}$ buys bundle $j \in \{0, 1, 2, \ldots, J\}$ and $x_{ij} = 0$ otherwise. (Notice that $x_{i0} = 1$ denotes consumer purchase bundle 0.) The CBP problem, which is without price sub-additivity constraints, can be formulated as the following MINLP problem:

$$
\begin{aligned}
CBP_1: \quad \max \quad & \sum_{i=1}^{I}\sum_{j=0}^{J} x_{ij}(p_j - c\cdot j) \\
\text{s.t.} \quad & (w_{ij}-p_j)x_{ij} \geq 0 \; \forall i,\ \forall j && (1)\\
& \sum_{j=0}^{J}(w_{ij}-p_j)x_{ij} \geq w_{ij}-p_j \; \forall i,\ \forall j && (2)\\
& \sum_{j=0}^{J} x_{ij} = 1 \; \forall i && (3)
\end{aligned}
$$

The objective function aggregates seller's profit from each consumer. The term in the parentheses is the profit of selling bundle size j. Constraints (1), known as individual rationality (IR), guarantee that if a consumer chooses to purchase a bundle, it provides nonnegative surplus. Constraints (2), known as incentive compatibility (IC), guarantee that a consumer receives at least as much surplus for purchasing the bundle intended for her as she would from choosing any other bundle. Constraints (3) ensure that each consumer purchases only one bundle. If one consumer i gets negative surplus from any available bundle, she will purchase bundle 0 so that $x_{i0} = 1$ and $x_{ij} = 0, j \geq 1$. A consequence of Constraints (3) is that a consumer can only purchase only one bundle. Therefore, this implies that if the optimal prices are not sub-additive, some consumers may still be forced to purchase larger bundle size, even if they can get the same units by purchasing smaller bundle sizes with lower total price.

We first derive some useful properties for the CBP problem. Then, we convert the MINLP problem CBP_1 to an MIP.

Lemma 1 *In an optimal solution, the bundle size that the first consumer purchases is priced at her willingness-to-pay.*

See Appendix A for proof.

Lemma 2 *For any given pricing scheme, if consumer $i-1$ buys bundle j in an optimal solution, then consumer i will also buy some bundle in that optimal solution, say j'. Furthermore, j' is greater than or equal to j. Therefore, the following constraints are satisfied:*

$$
\sum_{j'=j}^{J} x_{ij'} \geq \sum_{j'=j}^{J} x_{i-1,j'} \; \forall i \geq 2,\ \forall j \tag{4}
$$

See Appendix B for proof.

Lemma 1 describes how to price the first bundle that consumers start to purchase, which is used as a start point to determine prices for all the bundles in optimal. Hitt and Chen (2005) also observe the same property for CBP. Lemma 2 provides important constraints to CBP_1. Actually, we will show that by imposing these constraints we can eventually convert the original MINLP

Table 1: MIP example 1 with no marginal cost

	Consumers' WTP			
Bundle	I_1	I_2	I_3	I_4
J_1	26	36	58	100
J_2	47	62	91	180
J_3	58	77	113	221
J_4	62	83	123	241
	v_{ij}			
Bundle	I_1	I_2	I_3	I_4
J_1	-4	-8	16	100
J_2	2	4	2	180
J_3	1	5	5	221
J_4	-1	3	6	241

problem to an LP problem. Furthermore, it is easy to show that the proofs of both Lemmas 1 and 2 go through even with price sub-additivity constraints.

2.2 Converting CBP_1 to an MIP problem CBP_2

We use Lemmas 1 and 2 to reformulate MINLP problem CBP_1 to the following 0-1 integer programming problem.

Proposition 3 *Define $v_{ij} = (w_{i,j} - jc) + (I - i)(w_{i,j} - w_{i+1,j})$. The MINLP problem CBP_1 can be reformulated as the following 0-1 integer linear problem CBP_2:*

$$CBP_2: \quad \max \quad \sum_{i=1}^{I}\sum_{j=0}^{J} v_{ij}x_{ij}$$

$$s.t. \quad v_{ij} = (w_{ij} - jc) + (I - i)(w_{ij} - w_{i+1,j}) \tag{5}$$

$$\sum_{j=0}^{J} x_{ij} = 1 \ \forall i \tag{6}$$

$$\sum_{j'=j}^{J} x_{ij'} \geq \sum_{j'=j}^{J} x_{i-1,j'} \ \forall i \geq 2, \ \forall j \tag{7}$$

See Appendix C for proof.

The example in Table 1 demonstrates the algorithm in Proposition 3. In the top part, we list four consumers' willingness-to-pay for four bundle sizes respectively. We maximize seller's profit by making an optimal pricing scheme for the four bundle sizes.

The basic incentive for Proposition 3 is to calculate the total profit by looking at the incremental value each bundle provides to the profit. To do this, we define v_{ij} as equation (5) to captures the incremental change of profit if consumer i buys bundle j. There are two parts in v_{ij}. The first part $(w_{ij} - jc)$ captures the incremental gain of offering bundle j to consumer i.

By making consumer i purchase bundle size j, we can get additional revenue w_{ij} with additional marginal cost jc. The second part $(I-i)(w_{ij}-w_{i+1,j})$ captures the incremental loss of offering that bundle. Since now every consumer $i+1, i+2, \ldots, I$ must get a surplus of at least $(w_{ij}-w_{i+1,j})$, $(I-i)(w_{ij}-w_{i+1,j})$ is the lost revenue.

In the above example, where $I=4$ and $J=4$, we now calculate each v_{ij}. For example, $v_{11}=(w_{11}-c)-3(w_{21}-w_{11})=-4$. The first part $w_{11}-c=26$ represents how much the seller will gain if consumer 1 buys bundle 1, which is her WTP for that bundle. The second part $-3(w_{21}-w_{11})=-30$ represents how much the seller will lose if consumer 1 buys bundle 1, which is a reduction from the highest price he can charge for all the other consumers. In order to push consumer 2 to buy any bundle other than bundle 1, the seller needs to give her a surplus of at least $(w_{21}-w_{11})=10$. Since there are three other consumers after consumer 1, the total lost revenue is 30. Similarly, $v_{22}=(w_{22}-2c)-2(w_{32}-w_{22})=4$. Notice that the price p_2 is $w_{22}-(w_{21}-w_{11})=52$ rather than $w_{22}=62$. However, $-(w_{21}-w_{11})=10$ has already be evaluated in v_{11}. Therefore, the seller incrementally gains $w_{22}-2c=62$ from providing consumer 2 with bundle 2 and loses $2(w_{32}-w_{22})=58$ from all the other consumers (consumers 3 and 4 in this case). By using the same method, we can generate all v_{ij} values which are shown in the bottom part of Table 1.

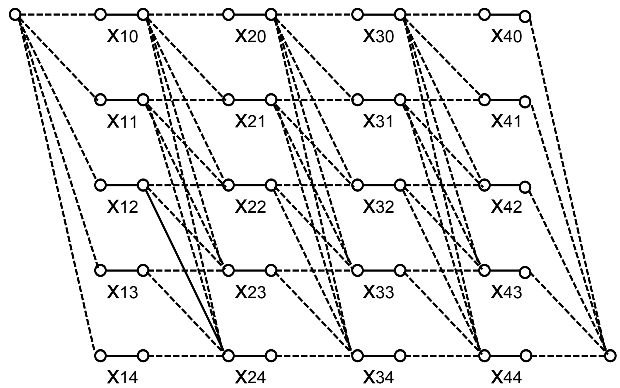
Given all the v_{ij} value, we can easily find the seller's total profit for a given x_{ij} that satisfies Constraints (6) and (7). For example, if a seller tries to serve consumer 1 with bundle size 1, consumer 2 with bundle size 2, consumer 3 with bundle size 3, and consumer 4 with bundle size 4, then the maximum total profit is $v_{11}+v_{22}+v_{33}+v_{44}=247$. Therefore, the primary problem is converted to a 0-1 IP problem which tries to find x_{ij} subject to Constraints (6) and (7) and maximizes $\sum_{i=1}^{I}\sum_{j=0}^{J} v_{ij}x_{ij}$.

2.3 A shortest-path solution for CBP_2

With Proposition 3, we reformulate CBP_1 to an MIP problem CBP_2. The incentive of the new objective function is to maximize the total incremental impact of each consumer's purchase choice on the total profit, subject to the constraints that each consumer can buy no more than one bundle (Constraints (6)) and each consumer i cannot purchase a smaller-sized bundle than $i-1$ does (Constraints (7)). Although a general 0-1 IP problem is NP-hard, we will use Constraints (4) to prove that CBP_2 is a shortest-path problem, which is polynomially solvable.

Proposition 4 *CBP_2 is equivalent to the following shortest path problem on a graph which has*

Figure 1. Shortest-path problem structure



$2IJ+2I+2$ nodes and $(I+2)(J+1)+(I-1)(j+1)J/2$ edges:

$$
\begin{aligned}
CBP_3: \quad \min \quad & -\sum_{i=1}^{I}\sum_{j=0}^{J} v_{ij}x_{ij} \\
s.t. \quad & v_{ij} = (w_{ij} - jc) + (I-i)(w_{ij} - w_{i+1,j}) && (8)\\
& \sum_{j=0}^{J} y_{01j} = 1 && (9)\\
& \sum_{j=0}^{J} y_{IJj} = 1 && (10)\\
& x_{ij} = \sum_{j'=j}^{J} y_{ijj'} \ \forall i \forall j && (11)\\
& \sum_{j'=0}^{j} y_{i-1,j',j} = x_{ij} \ \forall i \forall j && (12)\\
& y_{ijj'} \in 0,1 \ \forall i, \ \forall j \ \forall j' >= j && (13)
\end{aligned}
$$

See Appendix D for proof.

Figure 1 demonstrates a shortest-path problem structure for a four-consumers and four-bundle-sizes problem. In this graph, one unit of flow starts from the top-left node, travels through the network, and finally arrives at the bottom-right node. If it passes any solid lines, then the corresponding x_{ij} is equal to 1 and otherwise x_{ij} is equal to 0. Passing through each solid line costs $-v_{ij}$. The above graph is multipartite with alternative partitions marked with dotted and solid lines. The flow must alternate between these types of edges to travel to the destinatation. $y_{ijj'}$ is used to count the flow on the dotted edges. If $x_{ij} = x_{i+1,j'} = 1$, then $y_{ijj'} = 1$. The cost of passing dash lines is set to zero.

Since the shortest-path problem is an LP problem, we can relax 0-1 integer constraints on x_{ij} to the following constraints to convert CBP_2 to an LP problem: $0 \leq x_{ij} \leq 1 \ \forall i, \ \forall j$.

Solving Example 1 with the LP model gives an optimal profit that equals 256, with $p_1^* = 58$ and $p_4^* = 198$. In the optimal solution, consumer 3 buys bundle size 1 and consumer 4 buys bundle size 4, whereas consumers 1 and 2 do not buy anything. Following Result 3 in Hitt and Chen (2005), we would obtain the prices as $p_2^* = 47, p_3^* = 62$, and $p_4^* = 72$. Under such a pricing scheme, consumer 1 buys bundle size 2, 2 buys size 3, and both 3 and 4 buy size 4. The corresponding profit is 253 which is lower than the profit generated by our algorithm.

3 Solving the CBP problem with price sub-additivity constraints

In this section, we look at BSP, cardinality bundling with the following non-increasing unit price constraints:

$$p_j/j \geq p_{j+1}/(j+1)\ \forall j \leq J-1$$

Following is a MINLP model for the BSP problem.

$$BSP_1: \quad \max \quad \sum_{i=1}^{I}\sum_{j=0}^{J} x_{ij}(p_j - c \cdot j) \tag{14}$$

$$\text{s.t.} \quad \sum_{j=0}^{J} x_{ij} = 1\ \forall i \tag{15}$$

$$\sum_{j=0}^{J} (w_{ij} - p_j)x_{ij} \geq w_{ij} - p_j\ \forall i,\ \forall j \tag{16}$$

$$(w_{ij} - p_j)x_{ij} \geq 0\ \forall i,\ \forall j \tag{17}$$

$$p_j/j \geq p_{j+1}/(j+1)\ \forall j \leq J-1 \tag{18}$$

Now we propose a unit-price based dynamic programming algorithm for the BSP problem. This algorithm can provide not only solutions which can be used for BSP problems, but also lower-bound estimation for the CBP problem since it has stricter constraints than the CBP problem does.

New parameters and functions: ϵ = grid step length.

K = total step number.

$k = 1, \ldots, K$ is step index.

u_k as unit price on grid step k.

pro_{ijk} = maximum total profit if bundle size j is the first one to be provided at price u_k and consumer i is the first one to start purchasing this bundle.

Let $\Pi(i, j)$ be a function that calculates the total profit obtained by the seller if he starts to serve consumer i with bundle size j and it is the only available bundle in the market. In this situation, unit price u_0 is equal to w_{ij}/j for all bundles starting from size j. We can easily calculate how many units each consumer buys and then calculate $\Pi(i, j)$ as unit price times total units sold.

let $\Delta(i, j, k, i', j', k')$ be a function that calculates the change in total profit for a reducing in unit price. Suppose currently j is the largest available bundle size at unit price u_k and consumer i is the first consumer starting to purchase this bundle. Now we add one larger bundle size j' with a lower unit price $u_{k'}$ and assume a higher-end consumer i' is the first one starting to switch from buying several units $j", j \leq j" < j'$ with unit price u_k to buying j' units with a lower unit price $u_{k'}$. Adding bundle j' motivates consumers after i' to purchase more units with a lower unit price. We use $\Delta(i, j, k, i', j', k')$ to capture the total change in profit from pro_{ijk} to $pro_{i'j'k'}$.

Pseudo-code for unit-price based algorithm:

for $i, j; i <= I, j <= J$ **do**
  $u_0 = w_{ij}/j$;
  $pro_{jiK} = \Pi(i, j)$;
  for $i_1, j1, k1; i \leq i_1 \leq I, j \leq j_1 \leq J, k_1 \leq K$ **do**
    for $i_2, j2, k2; i_2 \leq i_1, j_2 \leq j_1, k_2 \leq k_1$ **do**
      $temp\_pro = pro_{i_2 j_2 k_2} + \Delta(i_2, j_2, k_2, i_1, j_1, k_1)$
      if $temp\_pro \geq pro_{i_1 j_1 k_1}$ **then**
        $pro_{i_1 j_1 k_1} = temp\_pro$
      end if
    end for
  end for
  if $\max k, pro_{IJk} > opt\_pro$ **then**
    $opt\_pro = \max k, pro_{IJk}$;
  end if
end for

Proposition 5 *For any given total error ϵ_t, let the grid step length parameter be $\epsilon = 2\epsilon_t/(J+1)JI$. Then the unit-price based dynamic programming algorithm guarantees that the gap between the optimal profit and the solution generated by the algorithm is no more than ϵ_t. Moreover, the computation complexity is $O(I^3J^4K^2)$, where $K = W_{IJ}/\epsilon$.*

See Appendix E for proof.

3.1 Computational results on CBS

In this section, we present a MIP linearization of the CBS problem. We use numerical experiments to show that our proposed estimations can provide a good bound for the MINLP problem. In addition we show that Constraints (4) from Lemma 2 can improve MIP performance.

Formulating the CBS problem is similar to BSP_1, except replacing the non-increasing unit price Constraints (18) to the following price sub-additivity constraints:

$$p_j \leq p_{j'} + p_{j-j'} \ \forall j \ \forall j' < \frac{1}{2}(j+1)$$

$$p_(j) \leq p_{j+1} \ \forall j \leq J-1$$

The nonlinearity of the objective function in BSP_1 comes from $x_{ij}p_j$. Therefore, we introduce $q_{ij} = x_{ij}p_j$ to replace all the nonlinear items. By adding Constraints (24) - (27), we can reformulate the CBS problem to an MIP problem CBS_1 as following:

$$\begin{aligned}
CBS_1: \quad \max \quad & \sum_{i=1}^{I}\sum_{j=0}^{J} q_{ij} - x_{ij}\cdot c \cdot j \\
\text{s.t.} \quad & \sum_{j=0}^{J} x_{ij} = 1 \ \forall i && (19)\\
& \sum_{j=0}^{J} (w_{ij}x_{ij} - q_{ij}) \geq w_{ij} - p_j \ \forall i, \ \forall j && (20)\\
& w_{ij}x_{ij} - q_{ij} \geq 0 \ \forall i, \ \forall j && (21)\\
& p_j \leq p_{j'} + p_{j-j'} \ \forall j \ \forall j' < \frac{1}{2}(j+1) && (22)\\
& p_(j) \leq p_{j+1} \ \forall j \leq J-1 && (23)\\
& q_{ij} \geq x_{ij}p_j^L \ \forall j && (24)\\
& q_{ij} \leq x_{ij}p_j^U \ \forall j && (25)\\
& q_{ij} \geq x_{ij}p_j^U + p_j - p_j^U \ \forall j && (26)\\
& q_{ij} \leq x_{ij}p_j^L + p_j - p_j^L \ \forall j && (27)
\end{aligned}$$

Here p_j^L and p_j^U are upper and lower bound for each p_j. Constraints (24) - (27) ensure that if $x_{ij} = 0$, then $q_{ij} = 0$, and if $x_{ij} = 1$, then $q_{ij} = p_j$. Therefore, MIP problem CBS_1 always has the same solution as the MINLP CBS problem.

Table 2 shows twenty numerical examples with 20 consumers and 20 bundles sizes. All consumers' WTP is randomly generated according to SCP. In Column three to five, optimal profits for CBP, CBS, and BSP are shown. We can see that for all the problems, CBP optimal value is (weakly) greater than that of CBS which is (weakly) greater than that of BSP. This result is consistent to our expectation because on one hand, CBP is the same problem as CBS except that the price sub-additivity constraints are relaxed. One the other hand, price constraints in BSP are stricter constraints than sub-additivity constraints in CBS, leading to an underestimation of CBS.

Therefore, solutions in CBP and BSP can provide lower and upper bounds for CBS. In column six in Table 2, we show the gap between CBP and CBS, and that between BSP and CBS. CBP solutions can provide upper-bound estimation for CBS with an average gap of 0.39% and BSP solution can provide lower-bound estimation with an average gap of 0.22%.

Table 2: Comparison of CBP, CBS, and BSP

		Optimal profit			Gap	
Problem No.	Problem size(I,J)	CBP	CBS	BSP	CBP	BSP
1	20,20	152.384	149.884	148.679	1.67%	-0.80%
2	20,20	137.001	136.731	136.354	0.20%	-0.28%
3	20,20	165.083	164.234	163.944	0.52%	-0.18%
4	20,20	200.971	200.525	199.596	0.22%	-0.46%
5	20,20	215.764	215.764	215.764	0.00%	0.00%
6	20,20	168.497	167.739	166.911	0.45%	-0.49%
7	20,20	182.589	182.109	181.755	0.26%	-0.19%
8	20,20	210.917	210.917	210.917	0.00%	0.00%
9	20,20	204.398	204.137	203.134	0.13%	-0.49%
10	20,20	0.8	0.78	0.775	2.56%	-0.64%
11	20,20	6.755	6.755	6.755	0.00%	0.00%
12	20,20	10.782	10.781	10.781	0.01%	0.00%
13	20,20	16.199	16.123	16.119	0.47%	-0.02%
14	20,20	22.536	22.536	22.504	0.00%	-0.14%
15	20,20	33.838	33.838	33.822	0.00%	-0.05%
16	20,20	36.514	36.514	36.514	0.00%	0.00%
17	20,20	39.435	39.014	38.902	1.08%	-0.29%
18	20,20	60.817	60.817	60.817	0.00%	0.00%
19	20,20	46.294	46.281	46.116	0.03%	-0.36%
20	20,20	47.662	47.543	47.543	0.25%	0.00%
Average		97.9618	97.6511	97.3851	0.39%	-0.22%

4 Conclusion

In this study, we look at three types of cardinality bundling problems, all of which involve setting a different price for each different-sized bundle: 1) CBS with price sub-additivity constraints; 2) CBP without price sub-additivity constraints; 3) BSP with non-increasing unit price constraints. In order to get tractable and interesting results, we impose a Spence-Mirrlees reservation price on consumers' willingness-to-pay and discuss all three problems under this condition. We show a LP algorithm for the CBP problem and a dynamic programming algorithm for the BSP problem. Finally we numerically analyze the CBS problem and show that CBP and BSP can provide good bound estimation for CBS.

There are several ways to extend the current study. First, there is still room to improve the performance of proposed dynamic programming algorithm for the BSP problem by combining it with LP cuttings. Second, CBS problem has only been converted to an MIP, which is still N-P hard. Last but not least, analyzing cardinality bundling problems without Spence-Mirrlees condition can provide a wider application of these pricing schemes in reality.

A Proof of lemma 1

Proof. Suppose there is a solution that consumer i_1 is the first one who purchases a bundle size $j_1 > 0$. The current price scheme is $p_{j_1}, p_{j_1+1}, \ldots, p_J$ and we have $p_{j_1} \leq w_{i_1 j_1}$. We can see that if we increase p_{j_1} to make it equal to $w_{i_1 j_1}$ and simultaneously increase the price for all the other bundles by the same amount $\Delta = w_{i_1 j_1} - p_{j_1}$, all the consumers will not change their purchasing decisions. For consumer i_1, her surplus of buying any bundle decreases by the same amount. She originally buys bundle size j_1, and it is assumed now she will still buy bundle size j_1, given that she still get highest non-negative surplus. Similarly, for any other consumer $i' > i_1$ who originally buys bundle size $j' \geq j_1$, she still get positive surplus from buying bundle j', since $w_{i'j'} - (p_{j'} + \Delta) = (w_{i'j'} - p_{j'}) - (w_{i_1 j_1} - p_{j_1}) \geq (w_{i'j_1} - p_{j_1}) - (w_{i_1 j_1} - p_{j_1}) = w_{i'j_1} - w_{i_1 j_1}$. As all the prices are increased by the same amount, her decision will not change. Therefore, increase p_{j_1} to $w_{i_1 j_1}$ will increase the revenue. On the other side, it is obvious that we cannot increase p_{j_1} to be more than $w_{i_1 j_1}$ without changing consumers' purchase decisions, as consumer i_1 cannot continue to purchase bundle j_1. ■

B Proof of lemma 2

Proof. Let s_{ij} denote consumer i's surplus of buying bundle j, $s_{ij} = w_{ij} - p_j$. Suppose for a given i, j, we have$\sum_{j'=j}^{J} x_{ij'} < \sum_{j'=j}^{J} x_{i-1,j'}$.Since we have Constraints 3, we then have $\sum_{j'=j}^{J} x_{ij'} = 0$ and $\sum_{j'=j}^{J} x_{i-1,j'} = 1$, which results in $x_{ij'} = 0, j' = j, j+1, \ldots, J$.

Therefore, we know that $\exists k, j \leq k \leq J$,so that $x_{i-1,k} = 1$, and $x_{i-1,j'} = 0, j' = j, j+1, \ldots, k-1, k+1, \ldots, J$.

Case 1: If $x_{ij'} = 0, j' = 1, 2, \ldots, j-1$, we will have $\sum_{j'=1}^{J} x_{ij'} = 0$, which implies $w_{ik} - p_k \geq w_{i-1,k} - p_k \geq 0$.

Let $k' = \arg\max_{j' \in (1,2\ldots,k-1,k+1,\ldots,J)} s_{ij'}$. If $s_{ik'} > s_{ik}$,then $x_{ik'} = 1$. If $s_{ik'} < s_{ik}$,then $x_{ik} = 1$. If $s_{ik'} = s_{ik}$, then either $x_{ik} = 1$ or $x_{ik'} = 1$, depending on which one has a larger size. In any of these situations, we will have $\sum_{j'=1}^{J} x_{ij'} = 1$ which is contradictory to equation B.

Case 2: If $\exists k', 1 \leq k' \leq j-1$, so that $x_{ik'} = 1$ and $x_{ij'} = 0$ $j' = 1, 2, \ldots, k'-1, k'+1, \ldots, j-1$.

We will have $w_{i,k'} - p_{k'} = (w_{ik'} - w_{i-1,k'}) + (w_{i-1,k'} - p_{k'}) \leq w_{i-1,k'} - p_{k'}$. Since $x_{i-1,k} = 1$, by Constraints 2 we have $s_{i-1,k'} \leq s_{i-1,k}$. Therefore, we have $s_{i,k'} \leq w_{i-1,k} - p_k \leq w_{i,k} - p_k = s_{i,k}$. Thus, $x_{i,k'}$ can never be 1 since this bundle is always dominated by another bundle with a larger size, k, which is contradictory to our assumption that $x_{ik'} = 1$. ■

C Proof of proposition 3

Proof. For a given X_{ij}, which satisfies Constraints (3) and (4), we show how to find an optimal pricing scheme.

Let $w_i^{i'}$ denote consumer i' willingness to pay for the bundle that consumer i' buys in optimal. Let P^i denote the price for the bundle that consumer i buys. Let c^i denote the cost for the bundle that consumer i buys. For a given X_{ij} which satisfies Constraints (3) and (4), the optimal problem to maximize profit is

$$\max \quad \sum_{i=1}^{I} (P^i - c^i) \tag{28}$$

$$\text{s.t.} \quad w_i^i - P^i \geq w_i^{i'} - P^{i'} \ \forall i, i' \tag{29}$$

$$w_1^1 - P^1 \geq 0 \tag{30}$$

Rewrite (29) as the following constraints:

$$w_i^i - P^i \geq w_i^{i'} - P^{i'} \ \forall i > i' \tag{31}$$

$$w_i^i - P^i \geq w_i^{i'} - P^{i'} \ \forall i < i' \tag{32}$$

Introduce the adjacent IC constraints:

$$w_i^i - P^i \geq w_i^{i-1} - P^{i-1} \ \forall i \tag{33}$$

In the following part, we prove that if Constraints (33) hold, then Constraints (31) also hold. We then relax (32), and get the optimal solution P^{i*} for (28) s.t.(33), (30). Finally we show that the optimal solution always satisfies Constraints (32).

For $\forall i > i'$,

$$w_i^i - w_i^{i'} = \sum_{k=i'}^{i-1}(w_i^{k+1} - w_i^k) \geq \sum_{k=i'}^{i-1}(P^{k+1} - P^k) = P^i - P^{i'}$$

Therefore, we have $w_i^i - P^i \geq w_i^{i'} - P^{i'}$

With Constraints (33) and (30), we have:

$$P^1 \leq w_1^1$$
$$P^i \leq w_i^i - w_i^{i-1} + P^{i-1}$$

In order to maximize (28), we will set all above $\leq$ to $=$ to get the optimal solution for each P^{i*}:

$$P^{1*} = w_1^1; P^{i*} = w_i^i - w_i^{i-1} + P^{i-1} = (w_i^i - w_i^{i-1}) + (w_{i-1}^{i-1} - w_{i-1}^{i-2}) + \ldots + w_1^1 \ \forall i > 1$$

For $\forall i < i'$,

$$\begin{aligned} P^{i'} - P^i = ((w_{i'}^{i'} - w_{i'}^{i'-1}) + (w_{i'-1}^{i'-1} - w_{i'-1}^{i'-2}) + \ldots + w_1^1) \\ -((w_i^i - w_i^{i-1}) + (w_{i-1}^{i-1} - w_{i-1}^{i-2}) + \ldots + w_1^1) \\ = ((w_{i'}^{i'} - w_{i'}^{i'-1}) + (w_{i'-1}^{i'-1} - w_{i'-1}^{i'-2}) + \ldots + (w_{i+1}^{i+1} - w_{i+1}^{i})) \\ \geq ((w_i^{i'} - w_i^{i'-1}) + (w_i^{i'-1} - w_i^{i'-2}) + \ldots + (w_i^{i+1} - w_i^i)) \\ = w_i^{i'} - w_i^i \end{aligned}$$

Therefore, the optimal solution P^{i*} for problem 28 is:

$$P^{1*} = w_1^1; P^{i*} = w_i^i - w_i^{i-1} + P^{i-1} = (w_i^i - w_i^{i-1}) + (w_{i-1}^{i-1} - w_{i-1}^{i-2}) + \ldots + w_1^1 \ \forall i > 1$$

Using the definition v_{ij} in section 2, we can rewrite 28 as following:

$$
\begin{aligned}
\sum_{i=1}^{I}(P^i - c^i) &= (w_I^J - c^I) + (2w_{I-1}^{I-1} - w_I^{I-1} - c^{I-1}) + \dots \\
&\quad +((I-i+1)w_i^i - (I-i)w_{i+1}^i - c^i) + \dots + (Iw_1^1 + (I-1)w_2^1 - c^1) \\
&= \sum_{j=0}^{J} v_{Ij}x_{Ij} + \sum_{j=0}^{J} v_{I-1,j}x_{I-1,j} + \dots \\
&\quad + \sum_{j=0}^{J} v_{ij}x_{ij} + \dots + \sum_{j=0}^{J} v_{1j}x_{1j} \dots \\
&= \sum_{i=1}^{I}\sum_{j=0}^{J} v_{ij}x_{ij} \qquad (34)
\end{aligned}
$$

■

D Proof of proposition 4

Proof. We start from the formulation of CPB_1. First, adding the following non-linear constraints to CPB_1 will not change the problem, since they just introduce new 0-1 variables $y_{ijj'}$ to the problem.

$$
\begin{aligned}
y_{00j} &= x_{1j} \ \forall j \\
y_{ijj'} &= x_{ij}x_{i+1,j'} \ \forall j \ \forall j' \geq j \\
y_{IJj} &= x_{Ij} \ \forall j
\end{aligned}
$$

Then we show that the above three constraints together with Constraints (6) and (7) can always imply Constraints (9) to (13).

$$
\begin{aligned}
\sum_{j=0}^{J} y_{01j} &= \sum_{j=0}^{J} x_{1j} = 1 \\
\sum_{j=0}^{J} y_{IJj} &= \sum_{j=0}^{J} x_{Jj} = 1
\end{aligned}
$$

$\forall i \ \forall j$ if $x_{ij} = 0$, then

$$\sum_{j'=j}^{J} y_{ijj'} = \sum_{j'=j}^{J} x_{ij}x_{i+1,j'} = 0 = x_{ij}$$
$$\sum_{j'=0}^{j} y_{i-1,j',j} = \sum_{j'=0}^{j} x_{i-1,j'}x_{ij} = 0 = x_{ij}$$

If $x_{ij} = 1$, then we have

$$\sum_{j'=j}^{J} x_{i+1,j'} \geq \sum_{j'=j}^{J} x_{i,j'} = 1$$
$$\sum_{j'=j}^{J} x_{i+1,j'} \leq \sum_{j'=0}^{J} x_{i+1,j'} = 1$$
$$\sum_{j'=j}^{J} x_{i+1,j'} = 1$$

Similarly, we also have

$$\sum_{j'=0}^{j} x_{i-1,j'} = 1$$

Therefore, we have

$$\sum_{j'=j}^{J} y_{ijj'} = \sum_{j'=j}^{J} x_{ij}x_{i+1,j'} = \sum_{j'=j}^{J} x_{i+1,j'} = 1 = x_{ij}$$
$$\sum_{j'=0}^{j} y_{i-1,j',j} = \sum_{j'=0}^{j} x_{i-1,j'}x_{ij} = \sum_{j'=0}^{j} x_{i-1,j'} = 1 = x_{ij}$$

Therefore, CPB_2 is equivalent to the following nonlinear mixed-integer problem CPB_{2a}:

$$
\begin{aligned}
CBP_{2a}: \quad \max \quad & \sum_{i=1}^{I}\sum_{j=0}^{J} v_{ij}x_{ij} \\
\text{s.t.} \quad & v_{ij} = (w_{ij} - jc) + (I - i)(w_{ij} - w_{i+1,j}) && (35)\\
& \sum_{j=0}^{J} x_{ij} = 1 \ \forall i && (36)\\
& \sum_{j'=j}^{J} x_{ij'} \geq \sum_{j'=j}^{J} x_{i-1,j'} \ \forall i \geq 2, \ \forall j && (37)\\
& y_{ijj'} = x_{ij}x_{i+1,j'} \ \forall j \ \forall j' \geq j && (38)\\
& y_{00j} = x_{1j} \ \forall j && (39)\\
& y_{IJj} = x_{Ij} \ \forall j && (40)\\
& \sum_{j=0}^{J} y_{01j} = 1 && (41)\\
& \sum_{j=0}^{J} y_{IJj} = 1 && (42)\\
& x_{ij} = \sum_{j'=j}^{J} y_{ijj'} \ \forall i \forall j && (43)\\
& \sum_{j'=0}^{j} y_{i-1,j',j} = x_{ij} \ \forall i \forall j && (44)\\
& y_{ijj'} \in 0,1 \ \forall i, \ \forall j \ \forall j' >= j && (45)
\end{aligned}
$$

Next, we show that Constraints (39) to (45) make Constraints (36) to (38) redundant so that they can be relaxed without changing the problem.

Fist, we show that Constraints (39) to (45) always imply the non-linear Constraints (38). $\forall i \ \forall j \ \forall j' geqj$, if $x_{ij} = 0$, then $y_{ijj'} \leq \sum_{j'=j}^{J} y_{ijj'} = x_{ij} = 0$. Therefore $y_{ijj'} = 0 = x_{ij}x_{i+1,j'}$.

If $x_{ij} = 1$ and $x_{i+1,j'} = 0$, then $y_{ijj'} \leq \sum_{j=0}^{j'} y_{ijj'} = x_{i+1,j'} = 0$. Therefore $y_{ijj'} = 0 = x_{ij}x_{i+1,j'}$.

If $x_{ij} = 1$ and $x_{i+1,j'} = 1$, and if $y_{ijj'} = 0$, then since $x_{ij} = \sum_{j'=j}^{J} y_{ijj'} = 1$, there exist $j'' <> j', y_{ijj''} = 1$. Therefore,$x_{i+1,j''} = \sum_{j'=0}^{j''} y_{i,j',j''} \geq 1$, which leading to $\sum_{j=0}^{J} x_{i+1,j} \geq x_{i+1,j'} + x_{i+1,j''} \geq 2$. Contradictory to Constraints (36). Therefore, $y_{ijj'} = 1 = x_{ij}x_{i+1,j'}$. Thus, Constraints (38) are redundant.

Second, we show that Constraints (39) to (45) always imply Constraints (36).

$$\begin{aligned}\sum_{j=0}^{J} x_{ij} &= \sum_{j'=0}^{0} y_{i-1,j',0} + \sum_{j'=0}^{1} y_{i-1,j',1} + \ldots + \sum_{j'=0}^{J} y_{i-1,j',J} \\ &= \sum_{j'=0}^{J} y_{i-1,0,j'} + \sum_{j'=1}^{J} y_{i-1,1,j'} + \ldots + \sum_{j'=J}^{J} y_{i-1,J,j'} \\ &= \sum_{j=0}^{J} x_{i-1,j} = \ldots = \sum_{j=0}^{J} x_{1,j} = \sum_{j=0}^{J} y_{00j} = 1\end{aligned}$$

Finally, we show that Constraints (39) to (45) always imply Constraints (37).

$$\begin{aligned}\sum_{j'=j}^{J} x_{ij'} &= \sum_{j'=0}^{j} y_{i-1,j',j} + \sum_{j'=0}^{j+1} y_{i-1,j',j+1} + \ldots + \sum_{j'=0}^{J} y_{i-1,j',J} \\ &= \sum_{j'=j}^{J} y_{i-1,0,j'} + \sum_{j'=j}^{J} y_{i-1,1,j'} + \ldots + \sum_{j'=j}^{J} y_{i-1,J,j'} \\ &\geq \sum_{j'=j}^{J} y_{i-1,j,j'} + \sum_{j'=j}^{J} y_{i-1,j+1,j'} + \ldots + \sum_{j'=j}^{J} y_{i-1,J,j'} \\ &= \sum_{j'=j}^{J} x_{i-1,j'}\end{aligned}$$

As a conclusion, we show that Constraints (39) to (45) make Constraints (36) to (38) redundant so that they can be relaxed without changing the problem. Therefore, after changing the objective function from $\max \sum_{i=1}^{I} \sum_{j=0}^{J} v_{ij} x_{ij}$ to $\min - \sum_{i=1}^{I} \sum_{j=0}^{J} v_{ij} x_{ij}$,$CPB_{2a}$ is equivalent to CPB_3. ■

E Proof of proposition 5

Proof. The error in the total profit is caused by grid errors in each unit price. Suppose u_{j_1} is the unit price for the first bundle that consumers start to purchase. All those smaller-sized bundles will have $u_j = u_{j_1} \forall j < j1$ and have no impact on the total profit. Therefore, we only focus on the unit price for bundle sizes larger than u_{j_1} and check how the grid error ϵ impacts the total profit.

We use function $E()$ to capture the largest possible error, which is the maximum possible gap between the true value in optimal and the value generated by our algorithm- $E(u_{j_1}) = \epsilon$, since $E(u_{j_1})$ is always a constant in our algorithm and the only source for the error comes from the grid rounding, which is no more than ϵ. Therefore, $E(p_{j_1}) = j_1 \epsilon$.

Now let's consider the price for the next bundle size $j_1 + 1$. It can keep the same unit price, which leads to $E_1(p_{j_1+1}) = (j_1+1)\epsilon$. Or if someone switches to purchase bundle size j_1+1, we will have $p_{j_1+1} = p_{j_1} + Con + \epsilon'$, where Con is a constant based on consumers' WTP and $\epsilon' \leq (j_1+1)\epsilon$ is an additional grid error. Therefore in this case $E_2(p_{j_1+1}) = E(u_{j_1}) + \epsilon' = j_1\epsilon + (j_1+1)\epsilon = (2j_1+1)\epsilon$ which is greater than $E_1(p_{j_1+1})$.

Similarly, with the same logic we can show that for any bundle size j', generating $p_{j'}$ by switching a consumer from buying bundle size $j'-1$ always creates a greater $E(p_{j'})$ than leaving j' with the same unit price as bundle size $j'-1$. Thus, for $\forall j' > j_1$, we have $E(p_{j'}) = \sum_{j=j_1}^{j'} j\epsilon = \frac{(}{1})2(j'+j_1)(j'-j_1+1)\epsilon \leq \frac{(}{1})2(j'+1)j'\epsilon \leq \frac{(}{1})2(J+1)J\epsilon$. Therefore, we can get an error from each consumer which is no more than $\frac{(}{1})2(J+1)J\epsilon$. The maximum total error is $\frac{(}{1})2(J+1)JI\epsilon$. ■

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