Proofs for “An Alternating Move Price-Setting Duopoly Model with Stochastic Costs"

Andrew Eckert *
Department of Economics
University of Alberta
Edmonton Alberta

June 7, 2004

Abstract

This document contains the proofs to Propositions 1 and 2 from "An Alternating Move Price-Setting Duopoly Model with Stochastic Costs."

*Address: 8-14 HM Tory Building, University of Alberta, Edmonton, Alberta T6G 2H4; Phone: 780 492 3959; Fax: 780 439 3300.

1 Proposition 1

Proposition 1: For $c_H - c_L$ sufficiently small and $\alpha = 0.5$, there are no symmetric alternating focal price equilibria supporting p_H^M and p_L^M under high and low costs.

Proof: To prove this proposition, I first prove two lemmas.

Lemma 1: Suppose that $R(p, c_L)$ and $R(p, c_H)$ constitute an alternating focal price equilibrium supporting p_H^M and p_L^M. Then all $p \in R(p_L^M, c_H)$ must be greater than or equal to p_L^M.

Proof: Suppose not. That is, suppose there exists $p < p_L^M$ such that $p \in R(p_L^M, c_H)$. It must be the case that

$$\Pi(p, c_H) + \delta W(p) \geq V(p_L^M, c_L) - \frac{\Pi(p_L^M, c_L)}{2} + \frac{\Pi(p_L^M, c_H)}{2}, \tag{1}$$

to ensure that a firm facing high costs would not strictly prefer to match p_L^M. (Note that since $\alpha = 0.5$, $W(p, c)$ is independent of the marginal costs in the previous period. Therefore, we drop the reference to c, and write the value function as $W(p)$.)

Similarly, to ensure that upon observing p_L^M a firm facing low costs would rather set p_L^M than p, I require that

$$V(p_L^M, c_L) \geq \Pi(p, c_L) + \delta W(p). \tag{2}$$

Conditions (4) and (5) combined imply that

$$(\Pi(p_L^M, c_L) - \Pi(p_L^M, c_H))/2 \geq \Pi(p, c_L) - \Pi(p, c_H), \tag{3}$$

which leads to a contradiction, since $p < p_L^M$ and demand is downward sloping.

Lemma 2: Suppose that $R(p, c_L)$ and $R(p, c_H)$ constitute an alternating focal price equilibrium *supporting p_H^M and p_L^M. Then $R(p_H^M, c_L) = p_L^M$.*

Proof: The proof of Lemma 2 proceeds in 5 steps.

Step 1: First, I show that, for any $p \in (p_L^M, p_H^M)$, there does not exist a $\tilde{p} > p$ such that $\tilde{p} \in R(p,c)$. To see this, suppose not and consider two cases. (a) Suppose $c = c_L$. Then it must be that $\delta W(\tilde{p}) \geq V(p_L^M, c_L) + \Pi(p_L^M, c_L)/2$, which contradicts $R(p_L^M, c_L) = p_L^M$. (b) Suppose $c = c_H$. It must be that $\delta W(\tilde{p}) \geq V(p_L^M, c_L) - \Pi(p_L^M, c_L)/2 + \Pi(p_L^M, c_H)$. This in turn contradicts $R(p_L^M, c_L) = p_L^M$ for $c_H - c_L$ such that $\Pi(p_L^M, c_H) > \Pi(p_L^M, c_L)/2$. Therefore the claim follows.

Step 2: It follows from lemma 1 and the fact that $R(p_L^M, c_L) = p_L^M$ that $R(p,c) \geq p_L^M$ for all $p \in (p_L^M, p_H^M)$. Therefore, $R(p,c) \in [p_L^M, p]$ for all $p \in (p_L^M, p_H^M)$.

Step 3: Next, we show that $R(p,c) = p_L^M$ for all $p \in (p_L^M, p_H^M)$. Suppose not. Define $p*$ as the lowest price $\in (p_L^M, p]$ that a firm is willing to set before dropping to p_L^M, and consider the following cases.

Case(1): Suppose that $R(p*, c_L) = R(p*, c_H) = p_L^M$. For a firm facing marginal cost c to prefer setting $p*$ to p_L^M it must be that

$$\Pi(p*, c) + \delta^2\left(\frac{V(p_L^M, c_L)}{2} + \frac{V(p_L^M, c_H)}{2}\right) \geq \Pi(p_L^M, c) + \frac{\delta}{2}\left(\frac{\Pi(p_L^M, c_L)}{2} + \frac{\Pi(p_L^M, c_H)}{2}\right) + \delta^2\left(\frac{V(p_L^M, c_L)}{2} + \frac{V(p_L^M, c_H)}{2}\right), \quad (4)$$

where the right hand side represents the minimum payoffs from setting p_L^M. This condition can be rewritten as

$$\Pi(p*, c) \geq \Pi(p_L^M, c) + \frac{\delta}{2}\left(\frac{\Pi(p_L^M, c_L)}{2} + \frac{\Pi(p_L^M, c_H)}{2}\right). \quad (5)$$

This condition clearly does not hold for $c_H - c_L$ small.

Case (2): Suppose that $p* \in R(p*, c_L)$. Then it must be that

$$\frac{\Pi(p*, c_L)}{2} + \delta W(p*) \geq \Pi(p_L^M, c_L) + \delta W(p_L^M) \tag{6}$$

which implies

$$\delta W(p*) \geq \Pi(p_L^M, c_L) + \delta W(p_L^M) - \Pi(p*, c_L)/2 > V(p_L^M, c_L), \tag{7}$$

contradicting p_L^M being a best response to p_L^M when costs are low.

Case (3): Suppose that $p* \in R(p*, c_H)$. Noting that $V(p*, c_H) - \frac{\Pi(p*, c_H)}{2} \leq V(p_L^M, c_H)$, such a response can be shown to yield at most

$$\frac{\Pi(p*, c_H)}{2}(1 + \frac{\delta}{2}) + \frac{\delta^2}{4}(\frac{\Pi(p_L^M, c_L)}{2} + \frac{\Pi(p*, c_H)}{2}) + \delta^2(\frac{V(p_L^M, c_L)}{2} + \frac{V(p_L^M, c_H)}{2}). \tag{8}$$

Therefore, for $p*$ to be preferred to p_L^M requires:

$$\frac{\Pi(p*, c_H)}{2}(1 + \frac{\delta}{2}) + \frac{\delta^2}{4}(\frac{\Pi(p_L^M, c_L)}{2} + \frac{\Pi(p*, c_H)}{2}) \geq \Pi(p_L^M, c_H)(1 + \frac{\delta}{4}) + \frac{\delta}{2}\frac{\Pi(p_L^M, c_L)}{2}, \tag{9}$$

which is clearly violated for $c_H - c_L$ not too large.

It follows from the arguments in cases (1)-(3), and from steps 1 and 2, that $R(p, c) = p_L^M$ for all $p \in (p_L^M, p_H^M)$.

Step 4: Next, I show that, if costs are low, in equilibrium a firm does not respond to p_H^M with p_H^M. Suppose instead that $p_H^M \in R(p_H^M, c_L)$. Then it must be that

$$\frac{\Pi(p_H^M, c_L)}{2} + \delta W(p_H^M) \geq \Pi(p_L^M, c_L) + \delta W(p_L^M), \tag{10}$$

or

$$\delta W(p_H^M) \geq \Pi(p_L^M, c_L) - \frac{\Pi(p_L^M, c_L)}{2} + \delta W(p_L^M) > \frac{\Pi(p_L^M, c_L)}{2} + \delta W(p_L^M), \tag{11}$$

which contradicts p_L^M being a best response to p_L^M when costs are low.

Step 5: We can now prove Lemma 2. To see why a firm with low costs does not respond to p_H^M with $p \in (p_L^M, p_H^M)$ note that by the preceding steps, this would yield at most

$$\Pi(p, c_L) + \delta^2 \frac{V(p_L^M, c_L)}{2} + \delta^2 \frac{V(p_L^M, c_H)}{2}, \tag{12}$$

compared with at least

$$\Pi(p_L^M, c_L)(1 + \frac{\delta}{4}) + \delta \frac{\Pi(p_L^M, c_H)}{4} + \delta^2 \frac{V(p_L^M, c_L)}{2} + \delta^2 \frac{V(p_L^M, c_H)}{2}. \tag{13}$$

Lemma 2 follows.

I can now use the above lemmas to prove Proposition 1. Suppose that $p = p_H^M$ and $c = c_H$. Responding with p_H^M can be shown to yield

$$\frac{\Pi(p_H^M, c_H)}{2}(1+\frac{\delta}{2})+\frac{\delta^2}{4}V(p_H^M, c_H)+\frac{\delta^2}{4}V(p_L^M, c_L)+\frac{\delta^2}{4}V(p_L^M, c_H)+\frac{\delta^2}{4}[V(p_L^M, c_L)+\frac{\Pi(p_L^M, c_L)}{2}]. \tag{14}$$

Alternatively, setting p_L^M yields at least

$$\Pi(p_L^M, c_H)(1 + \frac{\delta}{4}) + \frac{\delta}{2}\frac{\Pi(p_L^M, c_L)}{4} + \delta^2(\frac{V(p_L^M, c_L)}{2} + \frac{V(p_H^M, c_H)}{2}). \tag{15}$$

Recalling that $V(p_H^M, c_H) \geq V(p_L^M, c_H)$, a necessary condition for this deviation not to be optimal is therefore

$$\frac{\Pi(p_H^M, c_H)}{2}(1 + \frac{\delta}{2}) + \frac{\delta^2}{4}(\frac{\Pi(p_L^M, c_L)}{2}) \geq \Pi(p_L^M, c_H)(1 + \frac{\delta}{4}) + \delta\frac{\Pi(p_L^M, c_L)}{4}, \tag{16}$$

which is clearly violated for $c_H - c_L$ sufficiently small. QED.

2 Proposition 2

To prove Proposition 2, I first prove a simplified version, given as Proposition 2(a), in which the focal price is restricted to be the low cost monopoly price. I then prove that these strategies are also equilibria for prices on either side of the low cost monopoly price.

Proposition 2(a): Suppose that $p^f = p_L^M$, and that c_H and c_L satisfy

(*i*) $\quad \Pi(c_H, c_L) < \frac{\Pi(p^f, c_H)}{2}$

(*ii*) $\quad \Pi(p^f, c_H) + \frac{1}{2} E\Pi(p^f, c) > \Pi(p_H^M, c_H)$

and

(*iii*) $\quad \Pi(p^f, c_H) > \frac{E\Pi(p^f, c)}{2}$.

Then for δ sufficiently near 1 and k sufficiently near 0, the following strategies constitute an equilibrium:

$$R(p, c_L) = \begin{cases} p^f & \text{for } p > p^f, \\ p^f & \text{for } p = p^f, \\ \underline{p}_H & \text{for } p \in (\underline{p}_H, p^f), \\ p - k & \text{for } p \in (\underline{\underline{p}}_L, \underline{p}_H), \\ \underline{p}_L & \text{for } \underline{\underline{p}}_L \geq p > \underline{p}_L, \\ p^f & \text{for } \underline{p}_L \geq p. \end{cases}$$

$$R(p, c_H) = \begin{cases} p^f & \text{for } p > p^f, \\ p^f & \text{for } p = p^f, \\ p - k & \text{for } p \in (\underline{\underline{p}}_H, p^f), \\ \underline{p}_H & \text{for } p \in (\underline{p}_H, \underline{\underline{p}}_H], \\ p^f & \text{for } \underline{p}_H \geq p, \end{cases}$$

where $\underline{\underline{p}}_H, \underline{p}_H, \underline{\underline{p}}_L$, and $\underline{p}_L$ are defined such that $\underline{p}_L < \underline{\underline{p}}_L < \underline{p}_H < \underline{\underline{p}}_H < p^f$.

Proof: The proof that for low k, high δ and c_L and c_H statisfying (i), (ii) and (iii), the specified strategies are an MPE proceeds in stages. In Stage 1, I define the parameters $\underline{\underline{p}}_H, \underline{p}_H, \underline{\underline{p}}_L$, and $\underline{p}_L$. In Stage 2, given that these parameters exist I demonstrate that the specified strategies are an MPE under (i), (ii) and (iii), and for small k and high δ. In the third stage, I verify that the parameters $\underline{\underline{p}}_H, \underline{p}_H, \underline{\underline{p}}_L$, and $\underline{p}_L$ are well defined for k small, δ high, and $c_H - c_L$ satisfying (i), (ii) and (iii). As well, In the third stage I demonstrate that $\Pi(\underline{p}_H, c_H) > 0$ and that as δ goes to zero, $\underline{p}_H$ does not approach p^f.

Stage 1: $\underline{p}_L$ is defined as the lowest price a firm would be willing to set before setting the focal price p^f, when costs are c_L, the rival's price is less than p^f, and given that the firm's rival will respond by setting p^f, regardless of marginal costs.

If the firm sets p^f, its expected discounted future payoffs are equal to

$$\frac{\delta}{2(1-\delta)} \frac{\Pi(p^f, c_L) + \Pi(p^f, c_H)}{2}. \tag{17}$$

If the firm sets a price p such that its rival restores the focal price next period with certainty, its expected future payoffs are

$$\Pi(p, c_L)(1 + \frac{\delta}{2}) + \frac{\delta}{2}\Pi(p, c_H) + \frac{1}{2}\frac{\delta^2}{1-\delta}\frac{\Pi(p^f, c_L) + \Pi(p^f, c_H)}{2}. \tag{18}$$

Writing $E\Pi(p^f, c) = \Pi(p^f, c_L)/2 + \Pi(p^f, c_H)/2$, $\underline{p}_L$ satisfies

$$\Pi(\underline{p}_L, c_L)(1 + \frac{\delta}{2}) + \frac{\delta}{2}\Pi(\underline{p}_L, c_H) \geq \frac{\delta}{2}E\Pi(p^f, c) > \Pi(\underline{p}_L - k, c_L)(1 + \frac{\delta}{2}) + \frac{\delta}{2}\Pi(\underline{p}_L - k, c_H). \tag{19}$$

Next, $\underline{\underline{p}}_L$ is defined as the lowest price that a firm with low cost would set before dropping its price down to $\underline{p}_L$, if $R(\underline{\underline{p}}_L, c_H) = p^f$, $R(\underline{\underline{p}}_L, c_L) = \underline{p}_L$, amd $R(\underline{p}_L, c) = p^f$ for both cost levels. Setting $\underline{\underline{p}}_L$ when $c = c_L$ yields

$$\Pi(\underline{\underline{p}}_L, c_L) + \frac{\delta}{2}\Pi(\underline{\underline{p}}_L, c_H) + \frac{\delta}{2}\frac{E\Pi(p^f, c)}{2} + \frac{\delta^3}{1-\delta}\frac{E\Pi(p^f, c)}{2}. \tag{20}$$

Therefore, after some algebra, $\underline{\underline{p}}_L$ is defined according to

$$\Pi(\underline{\underline{p}}_L, c_L) + \frac{\delta}{2}\Pi(\underline{\underline{p}}_L, c_H)$$
$$\geq \Pi(\underline{p}_L, c_L)(1 + \frac{\delta}{2}) + \frac{\delta}{2}\Pi(\underline{p}_L, c_H) + \frac{\delta^2}{2}\frac{E\Pi(p^f, c)}{2} > \Pi(\underline{\underline{p}}_L - k, c_L) + \frac{\delta}{2}\Pi(\underline{\underline{p}}_L - k, c_H). \tag{21}$$

Define $\underline{p}_H > \underline{\underline{p}}_L$ as the lowest price a firm facing high costs would set before restoring p^f, if it knows that in all subsequent periods low cost firms will respond to all prices less than or equal to $\underline{p}_H$ according to the reaction function specified above and high cost firms will respond with p^f. Formally, $\underline{p}_H$ is defined as the lowest price such that

$$(1 + \frac{\delta}{2})\Pi(\underline{p}_H, c_H) + \frac{1}{2}\frac{\delta^2}{1-\delta}\frac{E\Pi(p^f, c)}{2} + \frac{1}{4}\delta^2 V(\underline{p}_H - k, c_L) + \frac{1}{4}\delta^2 V(\underline{p}_H - k, c_H) \geq \frac{\delta}{1-\delta}\frac{E\Pi(p^f, c)}{2}. \tag{22}$$

Finally, $\underline{\underline{p}}_H$ is defined as the lowest price a firm facing high costs would be willing to set before dropping down to $\underline{p}_H$, if it believes that all subsequent behavior is determined by the above dynamic reaction functions. Formally, $\underline{\underline{p}}_H$ is defined as the lowest price such that

$$\Pi(\underline{\underline{p}}_H, c_H) + \frac{\delta^2}{2}V(\underline{p}_H, c_H) + \frac{\delta^2}{2}V(\underline{p}_H, c_L)$$
$$\geq (1 + \frac{\delta}{2})\Pi(\underline{p}_H, c_H) + \frac{1}{2}\frac{\delta^2}{1-\delta}\frac{E\Pi(p^f, c)}{2} + \frac{1}{4}\delta^2 V(\underline{p}_H - k, c_L) + \frac{1}{4}\delta^2 V(\underline{p}_H - k, c_H). \tag{23}$$

Stage 2: In this stage we presume that $\underline{p}_L, \underline{\underline{p}}_L, \underline{p}_H, \underline{\underline{p}}_H$ are well defined and that $\underline{p}_H > c_H$. For each possible price set in the previous period and for each marginal cost level, it is sufficient to verify that a firm has no one-period deviation that yields a higher payoff than the strategies specified above. For simplification, the possible prices set by the firm's rival can be divided into different cases.

Case 1: $p \leq \underline{p}_L$.

Suppose first that $c = c_L$. A firm's possible deviations can be divided into several ranges.

(i) Responding with a price $\leq \underline{p}_L$ is not preferred by the definition of $\underline{p}_L$.

(ii) Responding with a price above p^f yields $\delta^2 E\Pi(p^f, c)/2$, which is strictly less than $\delta E\Pi(p^f, c)/2$, the payoff from raising price directly to p^f.

(iii) Since $\Pi(\underline{p}_H, c_H) > 0$ by assumption, and by the definition of $\underline{p}_H$, responding with a price $\in (\underline{p}_L, \underline{p}_H]$ would earn the firm strictly less than restoring p^f.

(iv) To see why responding with a price $\in (\underline{p}_H, \underline{\underline{p}}_H)$ is not preferred, note first that by the definition of $\underline{p}_H$,

$$V(\underline{p}_H, c_L) \leq \frac{\delta}{1-\delta}\frac{E\Pi(p^f, c)}{2} + \Pi(\underline{p}_H - k, c_L) - \Pi(\underline{p}_H - k, c_H), \tag{24}$$

since otherwise a firm with high costs would rather set $\underline{p}_H - k$ than restore the focal price, contradicting the definition of $\underline{p}_H$. Therefore, jumping above a rival's price to a price $\in (\underline{p}_H, \underline{\underline{p}}_H]$ yields at most

$$\frac{\delta^2}{2}(\Pi(\underline{p}_H - k, c_L) - \Pi(\underline{p}_H - k, c_H)) + \frac{\delta^3}{1-\delta}\frac{E\Pi(p^f, c)}{2}, \tag{25}$$

since the response of the rival will be $\underline{p}_H$. For this deviation not to be preferred it is sufficient to show that a firm would prefer to simply restore the focal price. This condition can be written as

$$\frac{\delta^2}{2}(\Pi(\underline{p}_H - k, c_L) - \Pi(\underline{p}_H - k, c_H)) + \frac{\delta^3}{1-\delta}\frac{E\Pi(p^f, c)}{2} \leq \frac{\delta}{1-\delta}\frac{E\Pi(p^f, c)}{2}, \tag{26}$$

or

$$\frac{\delta^2}{2}(\Pi(\underline{p}_H - k, c_L) - \Pi(\underline{p}_H - k, c_H)) \leq (\delta + \delta^2)\frac{E\Pi(p^f, c)}{2}, \tag{27}$$

which follows for δ near 1 by (i) and since $\Pi(\underline{p}_H - k, c_L) - \Pi(\underline{p}_H - k, c_H) \leq \Pi(c_H, c_L)$.

(v) Finally, that responding with a price $\in (\underline{\underline{p}}_H, p^f)$ is not preferred will follow when it is demonstrated that in response to p^f, a firm facing high costs will not deviate to $p^f - k$. This argument is deferred until Case 5.

Suppose next that $c = c_H$. Since $V(p, c_L) = V(p, c_H)$ for all $p \leq \underline{p}_L$ and the payoffs to all deviations above p are the same for high cost and low cost firms, it follows that a firm facing high costs would not deviate.

Case 2: $p \in (\underline{p}_L, \underline{\underline{p}}_L]$.

Suppose first that $c = c_L$. Deviating to a price $\in (\underline{p}_L, p]$ is ruled out by the definition of $\underline{\underline{p}}_L$. All other deviations are ruled out by arguments employed in Case 1.

Next suppose $c = c_H$. That a firm has no preferred deviation follows from the arguments employed in Case 1.

Case 3: $p \in (\underline{\underline{p}}_L, \underline{p}_H]$.

First suppose $c = c_H$. That there is no preferred deviation follows from previous arguments and the definition of $\underline{p}_H$.

Next, suppose that $c = c_L$. To verify that a firm has no preferred one-shot deviation, I proceed iteratively. First suppose that $p = \underline{\underline{p}}_L + k$. By the definition of $\underline{\underline{p}}_L$ and previous arguments, all deviations are ruled out except matching $\underline{\underline{p}}_L + k$. Such a deviation would yield

$$\frac{\Pi(\underline{\underline{p}}_L + k, c_L)}{2} + \frac{\delta}{2}\Pi(\underline{\underline{p}}_L + k, c_H) + \frac{\delta^2}{2}\frac{E\Pi(p^f, c)}{2(1-\delta)} + \frac{\delta^3}{4}\frac{E\Pi(p^f, c)}{2(1-\delta)} + \frac{\delta^2}{4}V(\underline{\underline{p}}_L, c_L). \tag{28}$$

The payoff from setting $\underline{\underline{p}}_L$ would be

$$\pi(\underline{\underline{p}}_L, c_L) + \frac{\delta}{2}\Pi(\underline{\underline{p}}_L, c_H) + \frac{\delta^2}{2}\frac{E\Pi(p^f, c)}{2(1-\delta)} + \frac{\delta^3}{4}\frac{E\Pi(p^f, c)}{2(1-\delta)} + \frac{\delta^2}{4}V(\underline{p}_L, c_L). \tag{29}$$

As k becomes small, $\Pi(\underline{\underline{p}}_L + k, c_L)$ approaches $\Pi(\underline{\underline{p}}_L, c_L)$ and $V(\underline{\underline{p}}_L, c_L)$ approaches $V(\underline{p}_L, c_L)$. Therefore, for k small, the payoff from matching $\underline{\underline{p}}_L + k$ is less than the payoff from setting $\underline{\underline{p}}_L$.

Next, suppose $p = \underline{\underline{p}}_L + 2k$. Following the specified strategies yields an expected present value payoff of

$$\Pi(\underline{\underline{p}}_L + k, c_L) + \frac{\delta}{2}\Pi(\underline{\underline{p}}_L + k, c_H) + \frac{\delta^2}{2}\frac{E\Pi(p^f, c)}{2} + \frac{\delta^2}{4}V(\underline{\underline{p}}_L, c_L) + \frac{\delta^2}{4}V(\underline{\underline{p}}_L, c_H). \tag{30}$$

All deviations except matching and setting $p = \underline{\underline{p}}_L$ are ruled out by arguments given in Cases 1 and 2. That matching is not optimal for small k follows from the same reasoning as in the previous paragraph. Setting $\underline{\underline{p}}_L$ yields expected present value payoffs of

$$\Pi(\underline{\underline{p}}_L, c_L) + \frac{\delta}{2}\Pi(\underline{\underline{p}}_L, c_H) + \frac{\delta^2}{2}\frac{E\Pi(p^f, c)}{2} + \frac{\delta^2}{4}V(\underline{p}_L, c_L) + \frac{\delta^2}{4}V(\underline{p}_L, c_H), \tag{31}$$

That this deviation is not optimal follows from the definitions of $\underline{\underline{p}}_L$ and $\underline{p}_L$ and that $\Pi(\underline{\underline{p}}_L + k, c) > \Pi(\underline{\underline{p}}_L, c)$ for each c.

Next, consider $p = \underline{\underline{p}}_L + 3k$. Using similar arguments, that a firm facing c_L would not undercut by more than a grid size follows since p is less than p_L^M. After some manipulation, it can be shown that the firm will prefer $\underline{\underline{p}}_L + 2k$ to matching $\underline{\underline{p}}_L + 3k$ only if

$$\Pi(\underline{\underline{p}}_L + 2k, c_L) - \frac{\Pi(\underline{\underline{p}}_L + 3k, c_L)}{2} + \frac{\delta}{2}(\Pi(\underline{\underline{p}}_L + 2k, c_H) - \Pi(\underline{\underline{p}}_L + 3k, c_H)) \geq \frac{\delta^2}{4}(V(\underline{\underline{p}}_L + 2k, c_L) - V(\underline{\underline{p}}_L + k, c_L)). \tag{32}$$

From above we know that in response to $\underline{\underline{p}}_L + k$ and $c = c_L$, matching is not preferred to setting $\underline{\underline{p}}_L$. Therefore,

$$V(\underline{\underline{p}}_L + 2k, c_L) - V(\underline{\underline{p}}_L + k, c_L) \leq \frac{\Pi(\underline{\underline{p}}_L + k, c_L)}{2}. \tag{33}$$

A sufficient condition for the firm not to match $\underline{\underline{p}}_L + 3k$ can then be written

$$\Pi(\underline{\underline{p}}_L + 2k, c_L) - \frac{\Pi(\underline{\underline{p}}_L + 3k, c_L)}{2} + \frac{\delta}{2}(\Pi(\underline{\underline{p}}_L + 2k, c_H) - \Pi(\underline{\underline{p}}_L + 3k, c_H)) \geq \frac{\delta^2}{4}\frac{\Pi(\underline{\underline{p}}_L + k, c_L)}{2}. \tag{34}$$

Clearly, this is satisfied for k small.

By iterating upwards and using similar arguments we can show that a firm facing low costs and any $p \in (\underline{\underline{p}}_L + 3k, \underline{p}_H]$ would not choose to match or undercut by more than a grid size.

Case 4: $p \in (\underline{p}_H, \underline{\underline{p}}_H]$

Consider first $c = c_H$. Based on above arguments the only deviations left to consider are to prices $\in (\underline{p}_H, p]$. However, these are ruled out by the definition of $\underline{\underline{p}}_H$.

Next consider $c = c_L$. Again by previous arguments the firm will not deviate to prices below $\underline{p}_H$ or greater than p, so we may focus on deviations to prices in $(\underline{p}_H, p]$. If the firm sets $\tilde{p} > \underline{p}_H$ it earns at most

$$\Pi(\tilde{p}, c_L) + \delta W(\tilde{p}) = (\Pi(\tilde{p}, c_L) - \Pi(\tilde{p}, c_H)) + \Pi(\tilde{p}, c_H) + \delta(\tilde{p}). \tag{35}$$

If it sets $\underline{p}_H$ it earns

$$\Pi(\underline{p}_H, c_L) - \Pi(\underline{p}_H, c_H) + \Pi(\underline{p}_H, c_H) + \delta W(\underline{p}_H). \tag{36}$$

By the definition of $\underline{\underline{p}}_H$ the last two terms of the latter are greater than the last two terms of the former. As well, by the fact that demand is downward sloping, we know that

$$\Pi(\underline{p}_H, c_L) - \Pi(\underline{p}_H, c_H) > \Pi(\tilde{p}, c_L) - \Pi(\tilde{p}, c_H). \tag{37}$$

Therefore, this deviation cannot be preferred. By similar reasoning we can show that the low cost firm would not prefer to match $\underline{\underline{p}}_H$.

Case 5: $p \in (\underline{\underline{p}}_H, p^f)$

First, suppose that $p \in (\underline{\underline{p}}_H, p^f)$ and $c = c_H$. All deviations can be ruled out immediately, except matching and undercutting to a price in $[\underline{\underline{p}}_H, p - 2k]$. I begin by proving that a firm would not prefer to undercut to a price in $[\underline{\underline{p}}_H, p - 2k]$.

Suppose first that $p = \underline{\underline{p}}_H + 2k$. Setting $\underline{\underline{p}}_H + k$ yields

$$\Pi(\underline{\underline{p}}_H + k, c_H) + \frac{\delta^2}{4} V(\underline{\underline{p}}_H, c_H) + \frac{\delta^2}{4} V(\underline{\underline{p}}_H, c_L) + \frac{\delta^2}{4} V(\underline{p}_H, c_H) + \frac{\delta^2}{4} V(\underline{p}_H, c_L), \tag{38}$$

whereas setting $\underline{\underline{p}}_H$ yields

$$\Pi(\underline{\underline{p}}_H, c_H) + \frac{\delta^2}{2} V(\underline{p}_H, c_H) + \frac{\delta^2}{2} V(\underline{p}_H, c_L). \tag{39}$$

Recall that from previous arguments and the definition of $\underline{p}_H$, $V(\underline{\underline{p}}_H, c_H) \geq V(\underline{p}_H, c_H)$ and $V(\underline{\underline{p}}_H, c_L) \geq V(\underline{p}_H, c_L)$. As well, recall that $\Pi(\underline{\underline{p}}_H + k, c_H) > \Pi(\underline{\underline{p}}_H, c_H)$. It follows that the payoffs from setting $\underline{\underline{p}}_H$ when costs are high are less than the payoffs from setting $\underline{\underline{p}}_H + k$.

Next, suppose $p = \underline{\underline{p}}_H + 3k$. By the previous argument we know that setting $\underline{\underline{p}}_H + k$ is preferred to setting $\underline{\underline{p}}_H$ when costs are high. The payoffs from setting $\underline{\underline{p}}_H + k$ are given above. Setting $\underline{\underline{p}}_H + 2k$ yields

$$\Pi(\underline{\underline{p}}_H + 2k, c_H) + \frac{\delta^2}{4} V(\underline{\underline{p}}_H + k, c_L) + \frac{\delta^2}{4} V(\underline{\underline{p}}_H + k, c_L) + \frac{\delta^2}{4} V(\underline{p}_H, c_H) + \frac{\delta^2}{4} V(\underline{p}_H, c_L). \tag{40}$$

Again we know that $V(\underline{\underline{p}}_H + k, c_H) \geq V(\underline{\underline{p}}_H, c_H)$ and $V(\underline{\underline{p}}_H + k, c_L) = V(\underline{\underline{p}}_H, c_L)$. It follows that a firm facing high costs would not deviate by setting $\underline{\underline{p}}_H + k$ or $\underline{\underline{p}}_H$. Similar arguments can be used to show that a firm would not deviate in response to high costs and a price $p \in (\underline{\underline{p}}_H, p^f)$ with any price in $[\underline{\underline{p}}_H, p - 2k]$.

I now verify that a firm facing high costs would not prefer to match the observed price rather than undercut. First, suppose $p = \underline{\underline{p}}_H + k$. Matching yields

$$\frac{\Pi(\underline{\underline{p}}_H + k, c_H)}{2} + \frac{\delta^2}{4} V(\underline{\underline{p}}_H, c_H) + \frac{\delta^2}{4} V(\underline{\underline{p}}_H, c_L) + \frac{\delta^2}{4} V(\underline{p}_H, c_H) + \frac{\delta^2}{4} V(\underline{p}_H, c_L), \tag{41}$$

whereas setting $\underline{\underline{p}}_H + k$ yields

$$\Pi(\underline{\underline{p}}_H, c_H) + \frac{\delta^2}{2} V(\underline{p}_H, c_H) + \frac{\delta^2}{2} V(\underline{p}_H, c_L). \tag{42}$$

Recall that as k approaches zero, $V(\underline{\underline{p}}_H, c_H)$ approaches $V(\underline{p}_H, c_H)$ and $V(\underline{\underline{p}}_H, c_L)$ approaches $V(\underline{p}_H, c_L)$. Therefore for k small matching is not optimal.

Next, consider any $p \in (\underline{\underline{p}}_H, p^f)$ such that the number of prices on the grid between p and $\underline{\underline{p}}_H$ is even (the odd case follows almost identical reasoning). We need to demonstrate that

$$V(p, c_H) \geq \frac{\Pi(p, c_H)}{2} + \frac{\delta^2}{4} V(\underline{p}_H, c_H) + \frac{\delta^2}{4} V(\underline{p}_H, c_L) + \frac{\delta^2}{4} V(p-k, c_H) + \frac{\delta^2}{4} V(p-k, c_L), \tag{43}$$

which can be rewritten

$$\Pi(p-k, c_H) - \frac{\Pi(p, c_H)}{2} \geq \frac{\delta^2}{4} (V(p-k, c_H) - V(p-2k, c_H)). \tag{44}$$

The right hand side can be written as

$$\begin{aligned} &\frac{\delta^2}{4} (\Pi(p-2k, c_H) - \Pi(p-3k, c_H)) + \frac{\delta^4}{4^2} (\Pi(p-4k, c_H) - \Pi(p-5k, c_H)) + \\ &\quad ... + \frac{\delta^{2t}}{4^t} (\Pi(\underline{\underline{p}}_H + k, c_H) - \Pi(\underline{\underline{p}}_H, c_H)) + \frac{\delta^{2t+2}}{4^{t+1}} (V(\underline{\underline{p}}_H, c_H) - V(\underline{p}_H, c_H)). \end{aligned} \tag{45}$$

Therefore, a sufficient condition for a firm facing high costs not to deviate by matching $p \in (\underline{\underline{p}}_H, p^f)$ is

$$\Pi(p-k, c_H) - \frac{\Pi(p, c_H)}{2} \geq \frac{\delta^2}{4} \Pi(p-2k, c_H) + \frac{\delta^{2t+2}}{4^{t+1}} (V(\underline{\underline{p}}_H, c_H) - V(\underline{p}_H, c_H)), \tag{46}$$

which holds for k sufficiently small.

Next I consider $c = c_L$. The only deviations not ruled out by previous arguments are matching and undercutting to a price greater than or equal to $\underline{\underline{p}}_H$. I will demonstrate first that for all $p \in (\underline{\underline{p}}_H, p^f)$, no price $\in [\underline{\underline{p}}_H, p)$ can be a preferred deviation. It then follows immediately that a firm facing low costs will not prefer to match p.

Suppose first that $p = \underline{\underline{p}}_H + k$. Setting $\underline{p}_H$ yields $V(\underline{\underline{p}}_H, c_H) + (\Pi(\underline{p}_H, c_L) - \Pi(\underline{p}_H, c_H))$, which for k small is approximately $\frac{\delta E\Pi(p^f, c)}{2(1-\delta)} + (\Pi(\underline{p}_H, c_L) - \Pi(\underline{p}_H, c_H))$. Similarly, as k approaches zero, the payoff from setting $\underline{\underline{p}}_H$ approaches $\frac{\delta E\Pi(p^f, c)}{2(1-\delta)} + (\Pi(\underline{\underline{p}}_H, c_L) - \Pi(\underline{\underline{p}}_H, c_H))$. Therefore, since demand is downward sloping, deviating by setting $\underline{\underline{p}}_H$ is clearly not preferred for k near zero.

Next suppose that $p = \underline{\underline{p}}_H + 2k$. Setting $\underline{\underline{p}}_H + k$ yields no more than

$$\Pi(\underline{\underline{p}}_H + k, c_L) + \delta^2 V(\underline{\underline{p}}_H, c_H) + \frac{\delta^2}{2} [\Pi(\underline{p}_H, c_L) - \Pi(\underline{p}_H, c_H)]. \tag{47}$$

Setting $\underline{p}_H$ yields

$$V(\underline{\underline{p}}_H, c_H) + \Pi(\underline{p}_H, c_L) - \Pi(\underline{p}_H, c_H). \tag{48}$$

Therefore, a sufficient condition not to deviate is (after some manipulation)

$$V(\underline{\underline{p}}_H, c_H)(1-\delta^2) + (1-\frac{\delta^2}{2})[\Pi(\underline{p}_H, c_L) - \Pi(\underline{p}_H, c_H)] \geq \Pi(\underline{\underline{p}}_H + k, c_L). \tag{49}$$

Replacing $V(\underline{\underline{p}}_H, c_H)$ with $\frac{\delta}{(1-\delta)}\frac{E\Pi(p^f,c)}{2}$, $\Pi(\underline{\underline{p}}_H + k, c_L)$ with $\Pi(p^f, c_L)$ and $(1-\frac{\delta^2}{2})$ with $1/2$, a new sufficient condition is

$$\frac{1}{2}[\Pi(\underline{p}_H, c_L) - \Pi(\underline{p}_H, c_H)] \geq \Pi(p^f, c_L) - (1+\delta)\delta\frac{E\Pi(p^f, c)}{2}. \tag{50}$$

Since as δ approaches one $\underline{p}_H$ is bounded away from p^f and the right hand side approaches $\frac{1}{2}[\Pi(p^f, c_L) - \Pi(p^f, c_H)]$, this deviation is not preferred.

Finally, suppose $p \in [\underline{\underline{p}}_H + 3k, p^f)$ and $c = c_L$. Setting $\underline{p}_H$ yields

$$V(\underline{\underline{p}}_H, c_H) + (\Pi(\underline{p}_H, c_L) - \Pi(\underline{p}_H, c_H)). \tag{51}$$

Undercutting by a grid size yields at most

$$\Pi(p-k, c_L) + \frac{\delta^2}{4}V(p-2k, c_H) + \frac{3\delta^2}{4}V(\underline{\underline{p}}_H, c_H) + \frac{\delta^2}{2}[\Pi(\underline{p}_H, c_L) - \Pi(\underline{p}_H, c_H)]. \tag{52}$$

Therefore, a sufficient condition for undercutting by a grid size not to be optimal is

$$V(\underline{\underline{p}}_H, c_H)(1-\frac{3\delta^2}{4}) + (1-\frac{\delta^2}{2})[\Pi(\underline{p}_H, c_L) - \Pi(\underline{p}_H, c_H)] \geq \Pi(p-k, c_L) + \frac{\delta^2}{4}V(p-2k, c_H). \tag{53}$$

The following upper bound can be written for $V(p-2k, c_H)$:

$$V(p-2k, c_H) \leq \Pi(p-3k, c_H) + \frac{3\delta^2}{4}V(\underline{\underline{p}}_H, c_H) + \frac{\delta^2}{2}[\Pi(\underline{p}_H, c_L) - \Pi(\underline{p}_H, c_H)] + \frac{\delta^2}{4}V(p-2k, c_H), \tag{54}$$

or

$$V(p-2k, c_H) \leq \frac{1}{1-\delta^2/4}(\Pi(p-3k, c_H) + \frac{3\delta^2}{4}V(\underline{\underline{p}}_H, c_H) + \frac{\delta^2}{2}[\Pi(\underline{p}_H, c_L) - \Pi(\underline{p}_H, c_H)]). \tag{55}$$

Substituting this upper bound for $V(p-2k, c_H)$ into the sufficient condition and simplifying yields the new condition:

$$V(\underline{\underline{p}}_H, c_H)(1-\delta^2)+(1-\frac{3\delta^2}{4})(\Pi(\underline{p}_H, c_L)-\Pi(\underline{p}_H, c_H)) \geq (1-\frac{\delta^2}{4})\Pi(p-k, c_L)+\frac{\delta^2}{4}\Pi(p-3k, c_H). \quad (56)$$

Since $(1-\delta^2)V(\underline{\underline{p}}_H, c_H) \geq \delta(1+\delta)E\Pi(p^f, c)/2$, $\Pi(p^f, c_L) > \Pi(p-k, c_L)$ and $\Pi(p^f, c_H) > \Pi(p-3k, c_H)$, a new sufficient condition is:

$$(1-\frac{3\delta^2}{4})(\Pi(\underline{p}_H, c_L)-\Pi(\underline{p}_H, c_H)) \geq \pi(p^f, c_L)(1-\frac{\delta(1+\delta)}{4}-\frac{\delta^2}{4})+\Pi(p^f, c_H)(\frac{\delta^2}{4}-\frac{\delta(1+\delta)}{4}). \quad (57)$$

For δ near 1, this condition is approximately

$$\pi(\underline{p}_H, c_L)-\Pi(\underline{p}_H, c_H) \geq \Pi(p^f, c_L)-\Pi(p^f, c_H). \quad (58)$$

Therefore, since $\underline{p}_H$ does not approach p^f as δ approaches 1, and since demand is downward sloping, forδ sufficiently near 1 the deviation is not preferred.

Case 6: $p = p^f$

Suppose first that $c = c_L$. The most profitable deviation is to set $\underline{p}_H$, which yields:

$$V(\underline{\underline{p}}_H, c_H)+\Pi(\underline{p}_H, c_L)-\Pi(\underline{p}_H, c_H), \quad (59)$$

which can be rewritten as

$$\frac{\delta}{1-\delta}\frac{E\Pi(p^f, c)}{2}+\Pi(\underline{p}_H, c_L)-\Pi(\underline{p}_H, c_H)+\Omega, \quad (60)$$

where Ω approaches zero as k approaches zero. The payoff from setting p^f is

$$\frac{\Pi(p^f, c_L)}{2}+\frac{\delta}{1-\delta}\frac{E\Pi(p^f, c)}{2}. \quad (61)$$

Therefore,for k small an approximate sufficient condition for this deviation not to be preferred to setting p^f is

$$\frac{\Pi(p^f, c_L)}{2} > \Pi(\underline{p}_H, c_L) - \Pi(\underline{p}_H, c_H), \tag{62}$$

which is satisfied by condition (i).

Secondly, suppose that $c = c_H$. In this case, the best deviation is to set $p^f - k$. This yields at most

$$\Pi(p^f - k, c_H) + \frac{\delta^2}{4} V(p^f - 2k, c_H) + \frac{3\delta^2}{4} V(\underline{\underline{p}}_H, c_H) + \frac{\delta^2}{2}[\Pi(\underline{p}_H, c_L) - \Pi(\underline{p}_H, c_H)], \tag{63}$$

Whereas matching yields

$$\frac{\Pi(p^f, c_H)}{2} + \frac{\delta}{1-\delta}\frac{E\Pi(p^f, c)}{2}. \tag{64}$$

A sufficient condition for deviating not to be optimal is

$$\frac{\Pi(p^f, c_H)}{2} + \frac{\delta}{1-\delta}\frac{E\Pi(p^f, c)}{2} \geq \Pi(p^f - k, c_H) + \frac{\delta^2}{4} V(p^f - 2k, c_H) + \frac{3}{4}\delta^2 V(\underline{\underline{p}}_H, c_H) + \frac{1}{2}\delta^2[\Pi(\underline{p}_H, c_L) - \Pi(\underline{p}_H, c_H)]. \tag{65}$$

From Case (5) above we know that

$$V(p^f - 2k, c_H) + \Pi(p^f - 3k, c_L) - \Pi(p^f - 3k, c_H) \leq V(\underline{\underline{p}}, c_H) + [\Pi(\underline{p}_H, c_L) - \Pi(\underline{p}_H, c_H)], \tag{66}$$

since a low cost firm will not respond to $p^f - 2k$ with $p^f - 3k$.

Therefore a new sufficient condition is

$$\frac{\Pi(p^f, c_H)}{2} + \frac{\delta}{1-\delta}\frac{\Pi(p^f, c)}{2} \geq \Pi(p^f - k, c_H) + \delta^2 V(\underline{\underline{p}}_H, c_H) + \frac{3\delta^2}{4}[\Pi(\underline{p}_H, c_L) - \Pi(\underline{p}_H, c_H)]. \tag{67}$$

Using the fact that, by (i) and downward sloping demand , $[\Pi(\underline{p}_H, c_L) - \Pi(\underline{p}_H, c_H)] < \Pi(p^f, c_H)/2$, we can write the sufficient condition

$$\frac{\Pi(p^f, c_H)}{2} + \frac{\delta + \delta^2}{2} E\Pi(p^f, c) \geq \Pi(p^f - k, c_H) + \frac{3}{4}\delta^2 \frac{\Pi(p^f, c_H)}{2} + \delta^2[V(\underline{\underline{p}}_H, c_H) - \frac{\delta}{1-\delta}\frac{E\Pi(p^f, c)}{2}], \tag{68}$$

which is satisfied for δ near one and k near zero.

Case 7: $p > p^f$

First suppose $c = c_L$. The only deviation that is not ruled out by previous arguments is setting a price $> p^f$. Such a deviation earns at most

$$\Pi(p, c_L) + \frac{\delta^2}{1-\delta} \frac{E\Pi(p^f, c)}{2}, \tag{69}$$

whereas setting p^f yields

$$\Pi(p^f, c_L) + \delta \frac{E\Pi(p^f, c)}{2} + \frac{\delta^2}{1-\delta} \frac{E\Pi(p^f, c)}{2}. \tag{70}$$

Therefore the result follows since $\Pi(p^f, c_L) > \Pi(p, c_L)$.

Secondly, suppose that $c = c_H$. Setting p^f earns

$$\Pi(p^f, c_H) + \delta \frac{E\Pi(p^f, c)}{2} + \frac{\delta^2}{1-\delta} \frac{E\Pi(p^f, c)}{2}. \tag{71}$$

The most profitable $p > p^f$ under high costs is p_H^M. Setting this price yields at most

$$\Pi(p_H^M, c_H) + \frac{\delta^2}{1-\delta} \frac{E\Pi(p^f, c)}{2}. \tag{72}$$

Therefore, I require that $\Pi(p^f, c_H) + \delta \frac{E\Pi(p^f,c)}{2} > \Pi(p_H^M, c_H)$, which for δ sufficiently near one is satisfied by condition (ii).

Stage 3: Finally, I need to verify that the parameters of the strategies given above are well defined. As well, I demonstrate that $\Pi(\underline{p}_H, c_H) > 0$ and that as (δ, k) approaches $(1, 0)$ $\underline{p}_H$ does not approach p^f.

$\underline{p}_L$: For $\underline{p}_L$ to be well defined requires that there exists $p < p^f$ such that

$$\Pi(p, c_L)(1 + \frac{\delta}{2}) + \delta \frac{\Pi(p, c_H)}{2} \geq \frac{\delta}{2} \frac{E\Pi(p^f, c)}{2}, \tag{73}$$

which is clearly the case.

$\underline{\underline{p}}_L$: For $\underline{\underline{p}}_L$ to be well defined requires that there exists $p \in (\underline{p}_L, p^f)$ such that

$$\Pi(p, c_L) + \frac{\delta}{2}\Pi(p, c_H) \geq \Pi(\underline{p}_L, c_L)(1 + \frac{\delta}{2}) + \frac{\delta}{2}\Pi(\underline{p}_L, c_H) + \frac{\delta^2}{2}\frac{E\Pi(p^f, c)}{2}. \tag{74}$$

Recall that as k decreases the payoffs from setting $\underline{p}_L$ approach the payoffs of restoring the focal price, implying that

$$\Pi(\underline{p}_L, c_L)(1 + \frac{\delta}{2}) + \frac{\delta}{2}\Pi(\underline{p}_L, c_H) = \frac{\delta}{2}E\Pi(p, c) + \Omega \tag{75}$$

where Ω is positive and approaches zero as k decreases. Therefore, I require that there exists $p \in (\underline{p}_L, p^f)$ such that

$$\Pi(p, c_L) + \frac{\delta}{2}\Pi(p, c_H) \geq (\frac{\delta}{2} + \frac{\delta^2}{4})E\Pi(p^f, c) + \Omega. \tag{76}$$

Clearly, such a price exists for k sufficiently near zero.

$\underline{p}_H$: For $\underline{p}_H$ to be well defined I require that there exists $p \in (\underline{\underline{p}}_L, p^f)$ such that (after some manipulation)

$$\Pi(p, c_H) + \frac{\delta}{2}\Pi(p, c_H) + \frac{\delta^2}{2}\frac{E\Pi(p^f, c)}{2} + \frac{3\delta^3}{4(1-\delta)}\frac{E\Pi(p^f, c_L)}{2} + \frac{\delta^2}{4}V(R(p, c_L), c_L) \geq \frac{\delta}{1-\delta}\frac{E\Pi(p^f, c)}{2}, \tag{77}$$

where $R(p, c_L) = p - k$ if $p > \underline{\underline{p}}_L$ and $R(p, c_L) = \underline{p}_L$ if $p = \underline{\underline{p}}_L$. Note that for k sufficiently small this cannot be satisfied at $\underline{p}_H = \underline{\underline{p}}_L$, by the definition of $\underline{\underline{p}}_L$. It is sufficient therefore to show that there exists $p \in (\underline{\underline{p}}_L, p^f)$ such that

$$\Pi(p, c_H)(1 + \frac{\delta}{2}) \geq (\delta + \frac{\delta^2}{2})\frac{E\Pi(p^f, c)}{2}, \tag{78}$$

which is satisfied by condition (iii).

$\underline{\underline{p}}_H$: Finally, I need to show that $\underline{\underline{p}}_H$ is well defined. Recall that the payoff from setting $\underline{p}_H$ is equal to $\frac{\delta}{1-\delta}\frac{E\Pi(p^f,c)}{2} + \Lambda_1$, where Λ_1 approaches zero as k approaches zero. Therefore, we need to show that there exists $p \in (\underline{p}_H, p^f)$ such that

$$\Pi(p, c_H) + \frac{\delta^2}{2}V(\underline{p}_H, c_L) + \frac{\delta^2}{2}V(\underline{p}_H, c_H) \geq \frac{\delta}{1-\delta}\frac{E\Pi(p^f, c)}{2} + \Lambda_1 \tag{79}$$

Using the definition of $\underline{p}_H$, we can write

$$V(\underline{p}_H, c_L) = \frac{\delta}{1-\delta}\frac{E\Pi(p^f, c)}{2} + \Pi(\underline{p}_H - k, c_L) - \Pi(\underline{p}_H - k, c_H) + \Lambda_2. \tag{80}$$

Therefore we need to show that there exists $p \in (\underline{p}_H, p^f)$ such that

$$\delta^2(\Pi(\underline{p}_H - k, c_L) - \Pi(\underline{p}_H - k, c_H)) \geq (\delta + \delta^2)\frac{E\Pi(p^f, c)}{2} - \Pi(p, c_H) + (\Lambda_1 - \frac{\delta^2}{2}\Lambda_2), \tag{81}$$

which is true for (δ, k) sufficiently near $(1, 0)$, since demand is downward sloping.

Next we demonstrate that $\Pi(\underline{p}_H, c_H) > 0$. First note that after some manipulation, $\underline{p}_H$ can be defined as the lowest price p such that

$$(1 + \frac{\delta}{2})\Pi(\underline{p}_H, c_H) \geq \frac{\delta^2}{4}[\frac{E\Pi(p^f, c)}{2(1-\delta)} - V(\underline{p}_H - k, c_L)] + (\delta + \frac{\delta^2}{4})\frac{E\Pi(p^f, c)}{2}. \tag{82}$$

Therefore, to demonstrate that $\underline{p}_H > c_H$ it is sufficient to show that $\frac{E\Pi(p^f,c)}{2} - V(\underline{p}_H - k, c_L)$ is positive. Now,

$$V(\underline{p}_H - k, c_L) - \Pi(\underline{p}_H - 2k, c_L) + \Pi(\underline{p}_H - 2k, c_H) \leq \frac{\delta}{(1-\delta)}\frac{E\Pi(p^f, c)}{2}, \tag{83}$$

since otherwise a firm facing $\underline{p}_H - k$ and high costs would prefer to set $\underline{p}_H - 2k$ than p^f, contradicting the definition of $\underline{p}_H$. After some manipulation this can be rewritten as

$$\frac{E\Pi(p^f, c)}{2} - [\Pi(\underline{p}_H - 2k, c_L) - \Pi(\underline{p}_H - 2k, c_H)] \leq \frac{1}{1-\delta}\frac{E\Pi(p^f, c)}{2} - V(\underline{p}_H - k, c_L). \tag{84}$$

But by (i) and the fact that demand is downward sloping, the left hand side of this inequality is positive, yielding our result.

Finally, we demonstrate that as (δ, k) approach $(1, 0)$, $\underline{p}_H$ is bounded away from p^f. From previous results we know that $\underline{p}_H$ is the lowest price p satisfying

$$\Pi(p, c_H)(1 + \frac{\delta}{2}) \geq (\delta + \frac{\delta^2}{2})\frac{E\Pi(p^f, c)}{2} + \frac{\delta^2}{4}[\frac{\delta}{1-\delta}\frac{E\Pi(p^f, c)}{2} - V(R(p, c_L), c_L)]. \tag{85}$$

Therefore, $\underline{p}_H$ is below the lowest price satisfying

$$\Pi(p, c_H) \geq \frac{\delta E\Pi(p^f, c)}{2}, \tag{86}$$

which for k small is strictly below p^f by (iii).

Focal Prices Other Than p_L^M: I now explain how these strategies can be used to support other prices near p_L^M, by identifying the deviations that must be considered differently than in the previous proof. For convenience I divide the possible prices into two cases: $p^f > p_L^M$ and $p^f < p_L^M$. I will also continue to suppose $p^f < p_H^M$.

$p^f < p_L^M$: The only condition that has to be rechecked in this case is that, upon observing a price above p^f, a firm facing low costs would set p^f rather than a price above p^f. The most tempting deviation is to set p_L^M. This yields $\Pi(p_L^M, c_L) + \frac{\delta^2}{1-\delta}\frac{E\Pi(p^f,c)}{2}$. Setting p^f yields $\Pi(p^f, c_L) + \delta\frac{E\Pi(p^f,c)}{2} + \frac{\delta^2}{1-\delta}\frac{E\Pi(p^f,c)}{2}$. Therefore, a new sufficient condition is that $\Pi(p_L^M, c_L) \leq (1+\frac{\delta}{2})\Pi(p^f, c_L)$. This is satisfied for p^f sufficiently close to p_L^M.

$p^f > p_L^M$. Suppose that $p^f - p_L^M$ is small enough to ensure that the parameters $\underline{p}_H$, $\underline{p}_L$, $\underline{\underline{p}}_L$ and $\underline{\underline{p}}_H$ are less than p_L^M. In this case, it again turns out that there is only one deviation that needs to be rechecked. Consider prices $\in (p_L^M, p^f)$. I need to identify conditions under which a firm facing low costs does not prefer to undercut instead of setting $\underline{p}_H$. Suppose first that $p = p_L^M + k < p^f$ (the other cases will follow from similar reasoning and will not be discussed here).

Sufficient condition (40) can be rewritten in this case as

$$V(\underline{\underline{p}}_H, c_H)(1-\delta^2) + (1-\frac{3\delta^2}{4})(\Pi(\underline{p}_H, c_L) - \Pi(\underline{p}_H, c_H)) \geq \Pi(p_L^M, c_L) + \frac{\delta^2}{4}\Pi(p_L^M, c_H), \tag{87}$$

or

$$\begin{aligned}(1-\frac{3\delta^2}{4})(\Pi(\underline{p}_H, c_L) - \Pi(\underline{p}_H, c_H)) \geq \pi(p^f, c_L)(1 - \frac{\delta(1+\delta)}{4} - \frac{\delta^2}{4}) \\ +\Pi(p^f, c_H)(\frac{\delta^2}{4} + \frac{\delta(1+\delta)}{4}) + (\Pi(p_L^M, c_L) - \Pi(p^f, c_L)).\end{aligned} \tag{88}$$

For δ near 1 this condition is approximately

$$\Pi(\underline{p}_H, c_L) - \Pi(\underline{p}_H, c_H) \geq (\Pi(p^f, c_L) - \Pi(p^f, c_H)) + (\Pi(p_L^M, c_L) - \Pi(p^f, c_L)). \tag{89}$$

Therefore, since $\underline{p}_H$ does not approach p^f as δ approaches 1 and since demand is downward sloping, for p^f sufficiently close to p_L^M and δ sufficiently close to 1,the deviation is not preferred.