# Appendix

**Proof of Proposition 1:** From (8) and (9), the first-order conditions of the downstream firms are as follows:

$$\begin{aligned}\frac{\partial \pi_{d1}}{\partial l_1} &= \frac{(4-4h+2(l_1+3l_2)h-(l_1+l_2)(6-3l_1-l_2))\tau-(1-l_1-l_2)(1+3l_1+l_2)t}{6t(1-l_1-l_2)^2}\\ &\quad\times\frac{(3+l_1-l_2)(1-l_1-l_2)t+(l_1-l_2)(2-l_1-l_2-2h)\tau}{3}, \qquad (20)\\ \frac{\partial \pi_{d2}}{\partial l_2} &= \frac{(4-4h+2(3l_1+l_2)h-(l_1+l_2)(6-l_1-3l_2))\tau-(1-l_1-l_2)(1+l_1+3l_2)t}{6t(1-l_1-l_2)^2}\\ &\quad\times\frac{(3-l_1+l_2)(1-l_1-l_2)t+(l_2-l_1)(2-l_1-l_2-2h)\tau}{3}. \qquad (21)\end{aligned}$$

The second fraction of $\partial \pi_{d1}/\partial l_1$ in (20) and that of $\partial \pi_{d2}/\partial l_2$ in (21) are positive because each of them is equal to the price-cost margin of each firm. Therefore, it is sufficient for us to consider the signs of the first fractions of $\partial \pi_{d1}/\partial l_1$ and $\partial \pi_{d2}/\partial l_2$ in (20) and (21). We solve the following simultaneous equations:

$$\frac{(4-4h+2(l_1+3l_2)h-(l_1+l_2)(6-3l_1-l_2))\tau-(1-l_1-l_2)(1+3l_1+l_2)t}{6t(1-l_1-l_2)^2}=0,$$
$$\frac{(4-4h+2(3l_1+l_2)h-(l_1+l_2)(6-l_1-3l_2))\tau-(1-l_1-l_2)(1+l_1+3l_2)t}{6t(1-l_1-l_2)^2}=0.$$

Solving the following simultaneous equations: $\partial \pi_{d1}/\partial l_1 = 0$ and $\partial \pi_{d2}/\partial l_2 = 0$, we have:

$$(l_1, l_2) = \left(\frac{1}{2}, \frac{1}{2}\right), \quad (l_1, l_2) = \left(\frac{4(1-h)\tau - t}{4(t+\tau)}, \frac{4(1-h)\tau - t}{4(t+\tau)}\right). \qquad (22)$$

Substituting $l_2 = 1/2$ into the numerator of the first fraction in (20), we have:

$$\frac{(1-2l_1)((5-4h)\tau - 3t - 6(t+\tau)l_1)}{4}. \qquad (23)$$

If $(5-4h)\tau - 3t < 0$, for any $l_1 \leq 1/2$, the value in (23) is negative and the best response of $D_1$ is $l_1 = 0$. If $(5-4h)\tau - 3t \geq 0$, the best response of $D_1$ is

$$l_1 = \frac{(5-4h)\tau - 3t}{6(t+\tau)}. \qquad (24)$$

If this is smaller than 1/2, the first pair in (22) is not an equilibrium outcome. The value in (24) is increasing with respect to $\tau$. If $\tau = t$, this is equal to $2(1-2h)/12 < 1/2$. We have assumed $\tau \leq t$. Therefore, this is smaller than $1/2$ and the first pair in (22) is not an equilibrium outcome.

Now, whether or not the second pair in (22) is an equilibrium outcome is checked.

First, we consider the case in which $\tau \leq t/4(1-h)$. In the case, the second pair violates the boundary condition. The optimal location may be $l_1 = l_2 = 0$. We now check whether the location pattern is an equilibrium outcome when $\tau \leq t/4(1-h)$. Given that $D_2$ chooses $l_2 = 0$, the numerator of the first fraction in (20) is:

$$3(t+\tau)l_1^2 - 2(t+(3-h)\tau)l_1 - (t-4(1-h)\tau). \tag{25}$$

When $D_1$ chooses $l_1 = 0$, the value in (25) is negative. When $D_1$ chooses $l_1 = 1/2$, the value in (20) is

$$\frac{-5t+(7-12h)\tau}{4}\left(\leq \frac{-5t+(7-12h)t/4(1-h)}{4} = \frac{-(13-8h)t}{16(1-h)} < 0\right).$$

(25) is convex with respect to $l_1$. Therefore, for any $l_1 \leq 1/2$, the value in (25) is negative, and we find that the best response of $D_1$ is $l_1 = 0$.

Next, we consider the case in which $\tau > t/4(1-h)$. Substituting $l_2 = \frac{4(1-h)\tau-t}{4(t+\tau)}$ into the first fraction of $\partial\pi_{d1}/\partial l_1$ in (20), we have:

$$\frac{[4(1-h)\tau - t - 4(t+\tau)l_1][15t - 4(1-5h)\tau - 12(t+\tau)l_1]}{16(t+\tau)}. \tag{26}$$

When $\tau < 9t/(10-20h)$, if $D_1$ chooses $l_1 < \frac{4(1-h)\tau-t}{4(t+\tau)}$, the value in (26) is positive; otherwise, it is negative. Therefore, if $\tau < 9t/(10-20h)$, the best response of firm 1 is $l_1 = \frac{4(1-h)\tau-t}{4(t+\tau)}$.

When $\tau \geq 9t/(10-20h)$, if $D_1$ chooses $l_1 < \frac{4(1-h)\tau-t}{4(t+\tau)}$ or $\frac{15t-4(1-5h)\tau}{12(t+\tau)} < l_1 (\leq 1/2)$, the value in (26) is positive; otherwise, it is negative. The best response of $D_1$ may be $l_1 = 1/2$. However, given that $D_2$ chooses $l_2 = \frac{4(1-h)\tau-t}{4(t+\tau)}$, if $D_1$ chooses $l_1 = 1/2$, the wholesale prices set by the upstream firms may not be the rival firms' transport costs. That is, an upstream firm may set its price lower than the rival firm's transport cost. Because of the asymmetric location, the quantity supplied by $D_2$ is not large enough. Setting a lower wholesale price, $U_B$ tries to expand the quantity supplied by $D_2$. As a result, $D_2$'s procurement cost decreases, and the price competition between the downstream firms is tougher. In the following proof, the upstream firm's pricing strategy is not considered. Therefore, $D_1$'s incentive to deviate from choosing $l_1 = \frac{4(1-h)\tau - t}{4(t+\tau)}$ is evaluated excessively.

We now show the *sufficient* condition that the best response of $D_1$ is $l_1 = \frac{4(1-h)\tau - t}{4(t+\tau)}$. Given that $D_2$ chooses $l_2 = \frac{4(1-h)\tau - t}{4(t+\tau)}$, when $D_1$ chooses $l_1 = 1/2$, the profit of $D_1$ is

$$\frac{(15t + 2(1-2h)\tau)^2(3t - 2(1-2h)\tau)}{1152t(t+\tau)}. \tag{27}$$

The difference between the profit in which $D_1$ chooses $l_1 = \frac{4(1-h)\tau - t}{4(t+\tau)}$ and that in which it chooses $l_1 = 1/2$ is:

$$\begin{aligned} & \frac{t(3t - 2(1-2h)\tau)}{4(t+\tau)} - \frac{(15t + 2(1-2h)\tau)^2(3t - 2(1-2h)\tau)}{1152t(t+\tau)} \\ = \; & \frac{[3t - 2(1-2h)\tau][63t^2 - 60(1-2h)t\tau - 4(1-2h)^2\tau^2]}{1152t(t+\tau)}. \end{aligned} \tag{28}$$

The first bracket is positive. If the second bracket is positive, the value in (28) is positive. The condition in which the second bracket is positive is:

$$63t^2 - 60(1-2h)t\tau - 4(1-2h)^2\tau^2 > 0 \quad \Rightarrow \quad \tau < \frac{3(4\sqrt{2} - 5)t}{2(1-2h)}. \tag{29}$$

If the inequality holds, the second pair in (22) is an equilibrium outcome. Q.E.D.

**Proof of Proposition 2:** We now show the location pattern is an equilibrium outcome. As shown by Ziss (1993), when $l_1 \leq (1 - l_2)$ and $c_1 < c_2$, firm 1 supplies all consumers if and only if

$$c_2 - c_1 \geq t(1 - l_2 - l_1)(3 - l_1 + l_2). \tag{30}$$

In this case, the profit of firm 1 is $c_2 - c_1 - t(1 - l_2 - l_1)(1 - l_1 + l_2)$. When $(1 - l_2) \leq l_1$ and $c_1 < c_2$, firm 1 supplies all consumers if and only if

$$c_2 - c_1 \geq t(l_1 - 1 + l_2)(3 + l_1 - l_2). \tag{31}$$

In this case, the profit of firm 1 is $c_2 - c_1 - t(l_1 - 1 + l_2)(l_1 + 1 - l_2)$.

First, we consider the case in which $t/4 < \tau < t/2$. If the following inequalities hold (see (30) and (31)), firm 1 has all of the market demand:

$$\begin{aligned}
&\tau\left(1-\frac{4\tau-t}{4(t+\tau)}\right)^2-\tau\left(1-l_1\right)^2 \geq t\left(1-\frac{4\tau-t}{4(t+\tau)}-l_1\right)\left(3-l_1+\frac{4\tau-t}{4(t+\tau)}\right), \\
&\text{if } \ l_1 \leq 1-l_2, \\
&\tau\left(1-\frac{4\tau-t}{4(t+\tau)}\right)^2-\tau\left(1-l_1\right)^2 \geq t\left(l_1-1+\frac{4\tau-t}{4(t+\tau)}\right)\left(3+l_1-\frac{4\tau-t}{4(t+\tau)}\right), \\
&\text{if } \ l_1 \geq 1-l_2.
\end{aligned}$$

That is,

$$(H_a \equiv)\frac{4(2t+\tau)-\sqrt{3t(3t+16\tau)}}{4(t+\tau)} \leq l_1 \leq \frac{-4(t-\tau)+\sqrt{3t(27t-16\tau)}}{4(t+\tau)}(\equiv H_b).$$

In the range $[H_a, H_b]$, the optimal location of firm 1 is $l_1 = 1-l_2$, that is, the same place as that of firm 2. We now show it.

Suppose that $l_1 \in [H_a, 1-l_2]$. The profit of firm 1 and the first-order condition are:

$$\begin{aligned}
\pi_1 &= \tau\left(1-\frac{4\tau-t}{4(t+\tau)}\right)^2-\tau\left(1-l_1\right)^2-t\left(1-\frac{4\tau-t}{4(t+\tau)}-l_1\right)\left(1-l_1+\frac{4\tau-t}{4(t+\tau)}\right), \\
\frac{\partial \pi_1}{\partial l_1} &= 2(1-l_1)(t+\tau) > 0.
\end{aligned}$$

On $[H_a, 1-l_2]$, $l_1 = 1-l_2$ is the best response of firm 1.

Suppose that $l_1 \in [1-l_2, H_b]$. The profit of firm 1 and the first-order condition are:

$$\begin{aligned}
\pi_1 &= \tau\left(1-\frac{4\tau-t}{4(t+\tau)}\right)^2-\tau\left(1-l_1\right)^2-t\left(l_1-1+\frac{4\tau-t}{4(t+\tau)}\right)\left(l_1+1-\frac{4\tau-t}{4(t+\tau)}\right), \\
\frac{\partial \pi_1}{\partial l_1} &= 2(\tau-(t+\tau)l_1) < 0.
\end{aligned}$$

On $[1-l_2, H_b]$, $l_1 = 1-l_2$ is the best response of firm 1. The profit is

$$\pi_1 = \frac{\tau(3t-2\tau)}{2(t+\tau)}. \tag{32}$$

We now suppose that $l_1 \in [0, H_a]$. The profit of firm 1 and the first-order condition are:

$$\begin{aligned}
\pi_1 &= \frac{\left(\tau\left(1-\frac{4\tau-t}{4(t+\tau)}\right)^2-\tau\left(1-l_1\right)^2+t\left(1-\frac{4\tau-t}{4(t+\tau)}-l_1\right)\left(3+l_1-\frac{4\tau-t}{4(t+\tau)}\right)\right)^2}{18t\left(1-\frac{4\tau-t}{4(t+\tau)}-l_1\right)}, \\
\frac{\partial \pi_1}{\partial l_1} &= (4\tau-t-4(t+\tau)l_1)(15t-4\tau-12(t+\tau)l_1) \\
&\quad \times\frac{65t^2-16t\tau-16\tau^2-32(t^2-\tau^2)l_1-16(t+\tau)^2l_1^2}{288t(5t-4(t+\tau)l_1)^2}.
\end{aligned}$$

The best response of firm 1 is $l_1 = \frac{4\tau - t}{4(t+\tau)}$ and the profit of it is

$$\pi_1 = \frac{t(3t-2\tau)}{4(t+\tau)}. \tag{33}$$

We now suppose that $l_1 \in [H_b, 1]$. The profit of firm 1 and the first-order condition are:

$$\begin{aligned}
\pi_1 &= \frac{\left(\tau\left(1-\frac{4\tau-t}{4(t+\tau)}\right)^2 - \tau(1-l_1)^2 + t\left(l_1 - 1 + \frac{4\tau-t}{4(t+\tau)}\right)\left(3 - l_1 + \frac{4\tau-t}{4(t+\tau)}\right)\right)^2}{18t\left(l_1 - 1 + \frac{4\tau-t}{4(t+\tau)}\right)}, \\
\frac{\partial \pi_1}{\partial l_1} &= (5t + 4\tau - 4(t+\tau)l_1)(21t - 4\tau - 12(t+\tau)l_1) \\
&\quad \times \frac{55t^2 + 16t\tau + 16\tau^2 - 32(2t+\tau)(t+\tau)l_1 + 16(t+\tau)^2 l_1^2}{288t(5t - 4(t+\tau)l_1)^2} < 0.
\end{aligned}$$

On $[H_b, 1]$, the best response of firm 1 is $l_1 = H_b$. As mentioned earlier, however, $l_1 = H_b$ is dominated by $l_1 = 1 - l_2$.

We now have two candidates of the best response: $l_1 = \frac{4\tau - t}{4(t+\tau)}$ and $l_1 = 1 - l_2$. From (32) and (33), if the following inequality holds, the best response of firm 1 is $l_1 = \frac{4\tau-t}{4(t+\tau)}$:

$$\frac{t(3t-2\tau)}{4(t+\tau)} - \frac{\tau(3t-2\tau)}{2(t+\tau)} = \frac{(t-2\tau)(3t-2\tau)}{4(t+\tau)} > 0, \quad (\tau < t/2).$$

Second, we consider the case in which $\tau < t/4$. If the following inequalities hold (see (30)), firm 1 has all of the market demand:

$$\tau - \tau(1-l_1)^2 \geq t(1-l_1)(3-l_1).$$

That is,

$$(H_c \equiv)\frac{2t + \tau - \sqrt{t^2 + t\tau + \tau^2}}{t+\tau} \leq l_1.$$

In the range $[H_c, 1]$, the optimal location of firm 1 is $l_1 = 1$, that is, the same place as that of firm 2. We now show it.

Suppose that $l_1 \in [H_c, 1]$. The profit of firm 1 and the first-order condition are:

$$\begin{aligned}
\pi_1 &= \tau - \tau(1-l_1)^2 - t(1-l_1)(1-l_1), \\
\frac{\partial \pi_1}{\partial l_1} &= 2(1-l_1)(t+\tau) > 0.
\end{aligned}$$

On $[H_c, 1]$, $l_1 = 1$ is the best response of firm 1.

We now suppose that $l_1 \in [0, H_c]$. The profit of firm 1 and the first-order condition are:

$$\begin{aligned}
\pi_1 &= \frac{\left(\tau - \tau\,(1-l_1)^2 + t(1-l_1)(3+l_1)\right)^2}{18t(1-l_1)}, \\
\frac{\partial \pi_1}{\partial l_1} &= -\frac{(3t - 2(t-\tau)l_1 + (t+\tau)l_1^2)(t - 4\tau + 2(t+3\tau)l_1 + 3(t+\tau)l_1^2)}{18t(1-l_1)^2}.
\end{aligned}$$

The best response of firm 1 is $l_1 = 0$, and the profit of it is:

$$\pi_1 = \frac{t}{2}.$$

We now have two candidates of the best response: $l_1 = 0$ and $l_1 = 1$. If the following inequality holds, the best response of firm 1 is $l_1 = 0$:

$$\frac{t}{2} - \tau = \frac{t - 2\tau}{2} > 0.$$

Therefore, the former part of proposition 2 holds.

We now show the latter part of proposition 2. The locations in (16) are similar to those in d'Aspremont *et al.* (1979). In the locations, the procurement costs of the downstream firms are zero. As show by d'Aspremont *et al.* (1979), in the locations, the price effect dominates the demand effect. Moreover, if a downstream firm relocate to another point, it incurs a procurement cost. Therefore, the location pattern is an equilibrium outcome.

Q.E.D.

**Proof of Proposition 3:** We now consider two cases: (i) full integration (two mergers occur) and (ii) partial integration (one merger occurs).

First, we consider the full integration case. In the case, the marginal costs of the firms are $\tau l_1^2$ and $\tau l_2^2$, respectively. From (1), (4), and (5), we derive the profits of the integrated firms

$$\pi_{I1} = \frac{[(3 + l_1 - l_2)(1 - l_1 - l_2)t - (l_1 - l_2)(l_1 + l_2)\tau]^2}{18t(1 - l_1 - l_2)}, \tag{34}$$

$$\pi_{I2} = \frac{[(3 - l_1 + l_2)(1 - l_1 - l_2)t - (l_2 - l_1)(l_1 + l_2)\tau]^2}{18t(1 - l_1 - l_2)}. \tag{35}$$

Calculating the first-order conditions, we have:

$$l_1 = l_2 = 0. \tag{36}$$

The profits of each integrated firm are:

$$\pi_{I1} = \pi_{I2} = \frac{t}{2}. \tag{37}$$

Second, we consider a partial integration case. We have assumed that the integrated firm ($U_B$ and $D_2$) cannot commit not to supply units of input to the rival downstream firm ($D_1$). We solve the problem by backward induction. We only consider the case in which $l_1 \leq 1 - l_2$.

In the third and fourth stages, for a consumer living at $x$ in (1), the total cost is the same at either of the two firms. Each downstream firm's profit is given by

$$\begin{aligned} \pi_{d1} &\equiv (p_1 - \min\{w_{A1}, w_{B1}\}) \left( \frac{1 + l_1 - l_2}{2} + \frac{p_2 - p_1}{2t(1 - l_1 - l_2)} \right), & (38) \\ \pi_I &\equiv (p_2 - \tau l_2^2) \left( \frac{1 - l_1 + l_2}{2} + \frac{p_1 - p_2}{2t(1 - l_1 - l_2)} \right). & (39) \end{aligned}$$

Calculating the first-order conditions of downstream firms, we have:

$$\begin{aligned} \pi_{d1} &= \frac{((1 - l_1 - l_2)(3 + l_1 - l_2)t - \min\{w_{A1}, w_{B1}\} + \tau l_2^2\})^2}{18(1 - l_1 - l_2)t}, & (40) \\ \pi_I &= \frac{((1 - l_1 - l_2)(3 - l_1 + l_2)t + \min\{w_{A1}, w_{B1}\} - \tau l_2^2\})^2}{18(1 -_1 - l_2)t}. & (41) \end{aligned}$$

In the second stage, for $D_1$ (non integrated downstream firm), $U_A$ (non integrated upstream firm) and the integrated firm engage in price competition. In this case, $U_A$ supplies to $D_1$ and sets the wholesale price at $w_{A1} = \tau(1 - l_1)^2$.[17]

In the first stage, each downstream firm locates at each point. Given the locations of the downstream firms, the profits of the firms are:

$$\begin{aligned} \pi_{d1} &= \frac{(1 - l_1 - l_2)((3 + l_1 - l_2)t - (1 - l_1 + l_2)\tau)^2}{18t}, & (42) \\ \pi_I &= \frac{(1 - l_1 - l_2)((3 - l_1 + l_2)t + (1 - l_1 + l_2)\tau)^2}{18t}. & (43) \end{aligned}$$

[17] If $\tau$ is sufficiently large, the "monopoly" price set by an upstream firm may be smaller than the rival's wholesale price (which is equal to the rival's transport cost per unit). In this paper, we only consider the case in which $\tau \leq t$. In the case, the problem does not occur.

The first-order conditions are:

$$
\begin{aligned}
\frac{\partial \pi_{d1}}{\partial l_1} &= -\frac{((3+l_1-l_2)t-(1-l_1+l_2)\tau)((1+3l_1+l_2)t-(3-3l_1-l_2)\tau)}{18t}, && (44)\\
\frac{\partial \pi_{I}}{\partial l_2} &= -\frac{((3-l_1+l_2)t+(1-l_1+l_2)\tau)((1+l_1+3l_2)t-(1-l_1-3l_2)\tau)}{18t} < 0. && (45)
\end{aligned}
$$

Solving the simultaneous equations: $\frac{\partial \pi_{d1}}{\partial l_1} = 0$ and $l_2 = 0$, we have:

$$
\begin{aligned}
&\text{if } \tau < \frac{t}{3}, \quad l_1 = l_2 = 0, \\
&\text{if } \frac{t}{3} < \tau, \quad l_1 = \frac{3\tau - t}{3(t+\tau)}, \; l_2 = 0,
\end{aligned}
\tag{46}
$$

In equilibrium, the profits of the firms are:

$$
\begin{aligned}
&\text{if } \tau < \frac{t}{3}, \quad \pi_{d1} = \frac{(3t-\tau)^2}{18t}, \; \pi_{uA} = \frac{\tau(3t-\tau)}{6t}, \; \pi_I = \frac{(3t+\tau)^2}{18t}, \\
&\text{if } \frac{t}{3} < \tau, \quad \pi_{d1} = \frac{128t^2}{243(t+\tau)}, \; \pi_{uA} = \frac{4\tau(5t-3\tau)}{27(t+\tau)}, \; \pi_I = \frac{200t^2}{243(t+\tau)}.
\end{aligned}
\tag{47}
$$

To derive Proposition 3, we consider two cases: (i) given that a vertical merger occurs, whether or not the non integrated firms merge; (ii) given that no vertical merger occur, whether or not a pair of upstream and downstream firms merge.

We now consider the first case. If the sum of the profits of the non-integrated downstream firm and the non-integrated upstream one is smaller than the integrated firm's profit in which two mergers occur, another vertical merger occurs. From $\pi_{I1}$ in (37), $\pi_{d1}$ in (47), and $\pi_{uA}$ in (47), we find the difference between them:

$$
\begin{aligned}
&\text{if } \tau < \frac{t}{3}, \quad \pi_{d1} + \pi_{uA} - \pi_{I1} = \frac{\tau(3t-2\tau)}{18t} > 0, \\
&\text{if } \frac{t}{3} < \tau, \quad \pi_{d1} + \pi_{uA} - \pi_{I1} = \frac{13t^2 + 117t\tau - 216\tau^2}{486(t+\tau)}.
\end{aligned}
\tag{48}
$$

In the latter equation in (48), the righthand side is negative, if and only if $\tau > \frac{(39+\sqrt{2769})t}{144} \sim 0.636t$.

Before we discuss the second case, we calculate the profits of upstream firms in which no merger takes place. In equilibrium, $U_A$ supplies to $D_1$. The price for $D_1$ is $\tau(1-l_1)^2$ (see

(6)). The transport cost per unit is $\tau l_1^2$. The quantity demanded by $D_1$ is $1/2$. Therefore, the profit of $U_A$ is:

$$\pi_{uA} = \frac{\tau(1-l_1)^2 - \tau l_1^2}{2} = \frac{\tau(1-2l_1)}{2}. \tag{49}$$

From (10) and (11), the profits of $U_A$ and $U_B$ are:

$$\begin{aligned} &\text{if } \tau < \frac{t}{4}, \quad \pi_{uA} = \pi_{uB} = \frac{\tau}{2}, \\ &\text{if } \frac{t}{4} < \tau, \quad \pi_{uA} = \pi_{uA} = \frac{\tau(3t-2\tau)}{4(t+\tau)}. \end{aligned} \tag{50}$$

To consider the second case, we have to compare the following two profits: (i) the sum of the profit of a downstream firm and an upstream one in the case where no merger takes place, and (ii) the profit of the merged firm in which two mergers take place. As mentioned earlier, given that a vertical merger occurs, the non integrated firms merge if $\tau > 0.636t$. If $\tau > 0.636t$ and the profit of (i) is larger than that of (ii), the first merger does not take place because the first merger induces the second one and then the profit of the first integrated firm is smaller than before. From $\pi_{I1}$ in (37), $\pi_{uA}$ in (50), and $\pi_{d1}$ in (12) and (13), we find the difference between them:

$$\begin{aligned} &\text{if } \tau < \frac{t}{4}, \quad \pi_{uA} + \pi_{d1} - \pi_{I1} = \frac{\tau}{2} > 0, \\ &\text{if } \frac{t}{4} < \tau, \quad \pi_{uA} + \pi_{d1} - \pi_{I1} = \frac{t-2\tau}{4}. \end{aligned} \tag{51}$$

In the latter equation in (51), the righthand side is positive, if and only if $\tau < t/2$. Therefore, the following situation does not exist: no merger occur even though the first merger is profitable.

We now consider the second case. If the sum of the downstream firm's profit and the upstream firm's one is smaller than the integrated firm's profit in which one mergers occurs, the vertical merger occurs. From $\pi_{I1}$ in (47), $\pi_{d1}$ in (12) and (13), and $\pi_{uA}$ in (50), we find the difference between them:

$$\begin{aligned}
&\text{if } \tau < \frac{t}{4}, && \pi_{d1} + \pi_{uA} - \pi_{I1} = \frac{\tau(3t-\tau)}{18t} > 0, \\
&\text{if } \frac{t}{4} < \tau < \frac{t}{3}, && \pi_{d1} + \pi_{uA} - \pi_{I1} = \frac{9t^3 - 21t^2\tau - 32t\tau^2 - 2\tau^3}{36t(t+\tau)}, \\
&\text{if } \frac{t}{3} < \tau, && \pi_{d1} + \pi_{uA} - \pi_{I1} = -\frac{71t^2 - 243t\tau + 486\tau^2}{972(t+\tau)} < 0.
\end{aligned} \tag{52}$$

In the second equation in (52), the righthand side is negative, if and only if $\tau > \frac{3(3\sqrt{3}-5)t}{2}$.

From the discussion, we have Proposition 3. Q.E.D.

**Proof of Proposition 4:** The profits of the upstream firms are:

$$\begin{aligned}
\pi_{uA} &= (w_{A1} - \tau(h_A - l_1)^2)x \\
&= (\tau(1-h_B-l_1)^2 - \tau(h_A-l_1)^2) \\
&\quad \times \frac{(3+l_1-l_2)(1-l_1-l_2)t - \tau(1-h_B-l_1)^2 + \tau(1-h_A-l_2)^2}{6(1-l_1-l_2)t},
\end{aligned} \tag{53}$$

$$\begin{aligned}
\pi_{uB} &= (w_{B2} - \tau(h_B - l_2)^2)(1-x) \\
&= (\tau(1-h_A-l_2)^2 - \tau(h_B-l_2)^2) \\
&\quad \times \frac{(3-l_1+l_2)(1-l_1-l_2)t + \tau(1-h_B-l_1)^2 - \tau(1-h_A-l_2)^2}{6(1-l_1-l_2)t}.
\end{aligned} \tag{54}$$

The first-order conditions are:

$$\begin{aligned}
\frac{\partial \pi_{uA}}{\partial h_A} &= \frac{\tau(-2\tau h_A^3 + 3(1+l_1-l_2)\tau h_A^2 - (3+l_1-l_2)(1-l_1-l_2)th_A)}{3(1-l_1-l_2)t} \\
&+ \frac{(l_1^2 + 4l_1(1-l_2) + (1-l_2)^2 - 2(1-l_1-h_B)^2)\tau^2 h_A}{3(1-l_1-l_2)t} \\
&+ \frac{\tau(l_1(3+l_1-l_2)(1-l_1-l_2)t - (1+l_1-l_2)((1-l_1-h_B)^2 - (1-l_2)l_1)\tau)}{3(1-l_1-l_2)t},
\end{aligned} \tag{55}$$

$$\begin{aligned}
\frac{\partial \pi_{uB}}{\partial h_B} &= \frac{\tau(-2\tau h_B^3 + 3(1-l_1+l_2)\tau h_B^2 - (3-l_1+l_2)(1-l_1-l_2)th_B)}{3(1-l_1-l_2)t} \\
&+ \frac{(l_2^2 + 4l_2(1-l_1) + (1-l_1)^2 - 2(1-l_2-h_A)^2)\tau^2 h_B}{3(1-l_1-l_2)t} \\
&+ \frac{\tau(l_2(3-l_1+l_2)(1-l_1-l_2)t - (1-l_1+l_2)((1-l_2-h_A)^2 - (1-l_1)l_2)\tau)}{3(1-l_1-l_2)t}.
\end{aligned} \tag{56}$$

An equilibrium outcome is now estimated. As shown in Section 3, given that upstream firms locate symmetrically ($h_A = h_B = h$), the best responses of the downstream firms are

$l_1 = l_2 = \frac{4\tau - t - 4\tau h}{4(t+\tau)}$. Substituting them into the first-order conditions of the upstream firms in (55) and (56), we have the same equation:

$$-\frac{\tau(3t - 7\tau + 2(6t + 5\tau)h + 8\tau h^2)}{12(t+\tau)} = 0. \tag{57}$$

Solving the equation, we have a candidate of the best responses:

$$\begin{aligned} &\text{if } \tau > \frac{3t}{7} \quad h_A = h_B = h = \frac{-(6t + 5\tau) + 3\sqrt{4t^2 + 4t\tau + 9\tau^2}}{8\tau}, \\ &\text{if } \tau \leq \frac{3t}{7} \quad h_A = h_B = h = 0. \end{aligned} \tag{58}$$

Substituting them into $l_1 = l_2 = \frac{4\tau - t - 4\tau h}{4(t+\tau)}$, we have:

$$\begin{aligned} &\text{if } \tau > \frac{3t}{7} \quad && l_1 = l_2 = \frac{4t + 13\tau - 3\sqrt{4t^2 + 4t\tau + 9\tau^2}}{8(t+\tau)}, \\ &\text{if } \frac{t}{4}\tau \leq \frac{3t}{7} \quad && l_1 = l_2 = \frac{4\tau - t}{4(t+\tau)}, \\ &\text{if } \tau \leq \frac{t}{4} \quad && l_1 = l_2 = 0. \end{aligned} \tag{59}$$

The locations in (58) and (59) are similar to those in (18).

We now check whether the location patterns in (18) ((58) and (59)) are the best responses of the firms.

From the result in Section 3, each downstream firm does not have incentives to move to the other points. This has been shown in (59).

We now check the locations of the upstream firms.

First, we consider the case in which $3t/7 < \tau$. Given the other three firms' locations in (18), the first-order condition of $U_A$ in (55) is:

$$\begin{aligned} \frac{\partial \pi_{uA}}{\partial h_A} \quad = \quad & -64\tau^2(t^2 + 2t\tau - \tau^2)h_A^3 - 48\tau^2\left((t+\tau)\sqrt{4t^2 + 4t\tau + 9\tau^2} - (5t+\tau)\tau\right)h_A^2 \\ & -6\left((6t^3 + 27t^2\tau + 22t\tau^2 + \tau^3)\sqrt{4t^2 + 4t\tau + 9\tau^2}\right)h_A \\ & \quad +(12t^4 + 36t^3\tau + 65t^2\tau^2 + 72t\tau^3 + 11\tau^4))h_A \\ & 3(t+\tau)(6t^2 + 21t\tau + 5\tau^2)\sqrt{4t^2 + 4t\tau + 9\tau^2} \\ & \quad -(36t^4 + 216t^3\tau + 187t^2\tau^2 + 260t\tau^3 + 53\tau^4). \end{aligned} \tag{60}$$

For any $\tau < t$, the coefficients of $h_A$, $h_A^2$ and $h_A^3$ are negative. Therefore, $\partial\pi_{uA}/\partial h_A$ in (60) is decreasing with respect to $h_A$. The constant term in (60) is positive if $3t/7 < \tau < t$. When $3t/7 < \tau < t$, substituting $h_A = \frac{-(6t+5\tau)+3\sqrt{4t^2+4t\tau+9\tau^2}}{8\tau}$ into $\frac{\partial\pi_{uA}}{\partial h_A}$ in (60), we have $\frac{\partial\pi_{uA}}{\partial h_A} = 0$. Therefore, $h_A = \frac{-(6t+5\tau)+3\sqrt{4t^2+4t\tau+9\tau^2}}{8\tau}$ is the best response of $U_A$. By symmetry, this is also the best response of $U_B$.

Second, we consider the case in which $t/4 < \tau < 3t/7$. Given the other three firms' locations in (18), the first-order condition of $U_A$ in (55) is:

$$\begin{aligned}\frac{\partial\pi_{uA}}{\partial h_A} \quad = \quad & -\frac{16\tau(t+\tau)^2h_A^3 - 24\tau(t+\tau)^2h_A^2}{12t(3t-2\tau)(t+\tau)} \\ & -\frac{2(18t^3-5t^2\tau+6t\tau^2+4\tau^3)h_A + t(3t-7\tau)(3t-2\tau)}{12t(3t-2\tau)(t+\tau)}.\end{aligned} \tag{61}$$

Differentiating $\frac{\partial\pi_{uA}}{\partial h_A}$ in (61) with respect to $h_A$, we have:

$$\frac{d}{dh_A}\left(\frac{\partial\pi_{uA}}{\partial h_A}\right) \quad = \quad -\frac{48\tau(t+\tau)^2h_A^2 - 48\tau(t+\tau)^2h_A^2 + 2(18t^3-5t^2\tau+6t\tau^2+4\tau^3)}{12t(3t-2\tau)(t+\tau)}. \tag{62}$$

For any $h_A$ and $\tau$, which satisfies $t/4 < \tau < 3t/7$, this is negative. If $t/4 < \tau < 3t/7$, the constant term in (61) is negative, and then $\frac{\partial\pi_{uA}}{\partial h_A}$ in (61) is negative. Therefore, choosing $h_A = 0$ is the best response of $U_A$. By symmetry, this is also the best response of $U_B$.

Third, we consider the case in which $\tau \leq t/4$. Given the locations of the other three firms in (18), the first-order condition of $U_A$ in (55) is:

$$\frac{\partial\pi_{uA}}{\partial h_A} \quad = \quad -\frac{\tau(\tau(1+h_A)+3(t-\tau h_A)h_A+2\tau h_A^3)}{3t} < 0. \tag{63}$$

Therefore, choosing $h_A = 0$ is the best response of $U_A$. By symmetry, this is also the best response of $U_B$. Q.E.D.

**Proof of the first paragraph in Section 3.3:** We now show that the following location pattern does not occur as an equilibrium outcome: $l_1 < 1/2$ and $l_2 > 1/2$ (or $l_1 > l/2$ and $l_2 < 1/2$).

By symmetry, it is sufficient to consider the case in which $l_1 \leq 1/2$ and $l_2 > 1/2$. In this case, both downstream firms procure their inputs from $U_A$.

Using (2) and (3), we derive the prices of the downstream firms:

$$p_1 = \frac{(1-l_1-l_2)(3+l_1-l_2)t + 2\min\{w_{A1}, w_{B1}\} + \min\{w_{A2}, w_{B2}\}}{3}, \tag{64}$$

$$p_2 = \frac{(1-l_1-l_2)(3-l_1+l_2)t + \min\{w_{A1}, w_{B1}\} + 2\min\{w_{A2}, w_{B2}\}}{3}, \tag{65}$$

From (1), (64), and (65), the quantities supplied by $D_1$ and $D_2$ are:

$$\begin{aligned} x &= \frac{w_{A2} - w_{A1} + (1-l_1-l_2)(3+l_1-l_2)t}{6(1-l_1-l_2)t}, \\ 1-x &= \frac{w_{A1} - w_{A2} + (1-l_1-l_2)(3-l_1+l_2)t}{6(1-l_1-l_2)t}, \end{aligned}$$

where $w_{A1}$ and $w_{A2}$ are the wholesale prices set by $U_A$. Given the location of the downstream firms, $U_A$'s maximization problem is:

$$\begin{aligned} \pi_{uA} &= (w_{A1} - \tau(h-l_1)^2)\frac{w_{A2} - w_{A1} + (1-l_1-l_2)(3+l_1-l_2)t}{6(1-l_1-l_2)t} \\ &\quad + (w_{A2} - \tau(1-h-l_2)^2)\frac{w_{A1} - w_{A2} + (1-l_1-l_2)(3-l_1+l_2)t}{6(1-l_1-l_2)t}, \\ &\quad s.t.\ \ \tau(1-h-l_1)^2 \geq w_{A1},\ \ \tau(l_2-h)^2 \geq w_{A2}. \end{aligned} \tag{66}$$

The constraints are induced by the price competition between $U_A$ and $U_B$. If the constraints are satisfied, the quantity supplied by $U_A$ is 1 (constant). Therefore, at least one constraint is binding.

$$\mathcal{L} = \pi_{uA} + \lambda(\tau(1-h-l_1)^2 - w_{A1}) + \Lambda(\tau(l_2-h)^2 - w_{A2}), \tag{67}$$

$$\frac{\partial \mathcal{L}}{\partial l_1} = \frac{2(w_{A2} - w_{A1}) + (1-l_1-l_2)((3+l_1-l_2-6\lambda)t - (1+l_1-l_2-2h)\tau)}{6(1-l_1-l_2)t}, \tag{68}$$

$$\frac{\partial \mathcal{L}}{\partial l_1} = \frac{2(w_{A1} - w_{A2}) + (1-l_1-l_2)((3-l_1+l_2-6\Lambda)t + (1+l_1-l_2-2h)\tau)}{6(1-l_1-l_2)t}. \tag{69}$$

First, we consider the case in which the first constraint in (66) is binding. In the case, if the second constraint is also binding, $w_{A1} = \tau(1-h-l_1)^2$ and $w_{A2} = \tau(l_2-h)^2$. The value of $\Lambda$ is

$$\Lambda = \frac{(3-l_1+l_2)t + (3-l_1+l_2-6h)\tau}{6t} > 0. \tag{70}$$

For any $h$ and $\tau < t$, $\Lambda$ is positive. If the first constraint is binding, then the second constraint is also binding. In this case, the profits of the downstream firms are:

$$\pi_{d1} = \frac{(1-l_1-l_2)[(3+l_1-l_2)t - (1-l_1+l_2-2h)\tau]^2}{18t}, \tag{71}$$

$$\pi_{d2} = \frac{(1-l_1-l_2)[(3-l_1+l_2)t+(1-l_1+l_2-2h)\tau]^2}{18t}. \tag{72}$$

The first-order conditions are as follows:

$$\frac{\partial \pi_{d1}}{\partial l_1} = \frac{[(3-3l_1-l_2-2h)\tau-(1+3l_1+l_2)t][(3+l_1-l_2)t-(1-l_1+l_2-2h)\tau]}{18t}, \tag{73}$$

$$\frac{\partial \pi_{d2}}{\partial l_2} = \frac{[(1-l_1-3l_2+2h)\tau-(1+l_1+3l_2)t][(3-l_1+l_2)t+(1-l_1+l_2-2h)\tau]}{18t}. \tag{74}$$

The second brackets in (73) and (74) are positive because these are similar to price-cost margins. We now show that the first bracket in (74) is negative for any $l_1 \leq 1/2$ and $l_2 > 1/2$. The first bracket in (74) is decreasing with respect to $l_2$. Substituting $l_2 = 1/2$ into the first bracket in (74), we have

$$-\frac{5t-4h\tau+\tau+2(t+\tau)l_1}{2}.$$

The value is negative because $0 < \tau < t$ and $0 \leq l_1 \leq 1/2$. Firm 2's behavior in which it chooses $l_2 > 1/2$ is dominated by that in which it chooses $l_2 = 1/2$.

Second, we consider the case in which the second constraint in (66) is binding. In the case, if the first constraint is not binding, $w_{A2} = \tau(l_2-h)^2$ and $\lambda = 0$. $w_{A1}$ is

$$\frac{(3+l_1-l_2)(1-l_1-l_2)t-(1-l_1^2-2l_2-l_2^2-2h+2l_1h+6l_2h-2h^2)\tau}{2}.$$

In this case, the profits of the downstream firms are:

$$\pi_{d1} = \frac{(1-l_1-l_2)[(3+l_1-l_2)t+(1+l_1-l_2-2h)\tau]^2}{72t}, \tag{75}$$

$$\pi_{d2} = \frac{(1-l_1-l_2)[(9-l_1+l_2)t-(1+l_1-l_2-2h)\tau]^2}{72t}. \tag{76}$$

The first-order conditions are as follows:

$$\frac{\partial \pi_{d1}}{\partial l_1} = \frac{[(1-3l_1-l_2+2h)\tau-(1+3l_1+l_2)t][(3+l_1-l_2)t+(1+l_1-l_2-2h)\tau]}{72t}, \tag{77}$$

$$\frac{\partial \pi_{d2}}{\partial l_2} = -\frac{[(7+l_1+3l_2)t-(3-l_1-3l_2-2h)\tau][(9-l_1+l_2)t-(1+l_1-l_2-2h)\tau]}{18t} \tag{78}$$

The second bracket in (78) is positive because this is similar to price-cost margins. The first bracket in (78) is also positive because $\tau < t$. Therefore, $\partial \pi_{d2}/\partial l_2$ in (78) is negative. Firm 2's behavior in which it chooses $l_2 > 1/2$ is dominated by that in which it chooses $l_2 = 1/2$. Therefore, the first paragraph in Section 3.3 is true. Q.E.D.