Nonparametric estimation of volatility models with serially dependent innovations

by

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The ”general” model of interest

\[ y_t = g(x_t) + \sigma(x_t)\epsilon_t, \quad t = 1, 2, \ldots, T. \]  \hfill (1)

- \( y_t \in \mathbb{R} \), and \( x_t \in \mathbb{R}^k \), and \( \{(y_t, x_t)\} \) is a strictly stationary process.

- \( \mathbb{E} (y_t|x_t) = g(x_t) \) and \( g : \mathbb{R}^k \rightarrow \mathbb{R} \).

- \( \text{var} (y_t|x_t) = \sigma(x_t)^2 \) and \( \sigma : \mathbb{R}^k \rightarrow \mathbb{R}_+ \).

- \( \epsilon_t \) is a strictly stationary process with \( \mathbb{E} (\epsilon_t) = 0 \) and \( \text{var} (\epsilon_t) = 1 \).
1 Applications (short term interest rates)

The ARCH(-M) specification:

• $y_t = \Delta r_t$ and $x_t = (1, \Delta r_{t-1}, \Delta r_{t-2}, \ldots, \Delta r_{t-k-1})'$.

• $g(x_t) = x_t' \beta$ \quad $\left( + \sigma(x_t)^2 \right)$.

• $\sigma(x_t)^2 = \alpha_0 + \alpha_1 \Delta r_{t-1}^2 + \ldots + \alpha_p \Delta r_{t-p}^2$.

• $\epsilon_t \sim i.i.d.$
The one-factor diffusion model (cont. time):

- Chan, Karolyi, Longstaff and Saunders (1992) generalized short rate specification

\[
dr = (\alpha + \beta r) \, dt + \psi r^\gamma \, dW_t.
\]  \hspace{1cm} (2)

- Euler-Maruyama discrete time approximation of (2)

\[
\Delta r_t = \alpha + \beta r_{t-1} + \psi r_{t-1}^\gamma \epsilon_t,
\]

\[
E(\epsilon_t|F_{t-1}) = 0, \quad E(\epsilon_t^2|F_{t-1}) = 1.
\]
\[ \psi_t^2 = \omega + \theta \Delta r_{t-1}^2 + \phi \psi_{t-1}^2. \]

The Markov regime switching model

\[ \Delta r_t | F_{t-1} \sim \begin{cases} 
N \left( g_{1t}, \sigma_{1t}^2 \right) & \text{w.p. } p_{1t} \\
N \left( g_{2t}, \sigma_{2t}^2 \right) & \text{w.p. } 1 - p_{1t} 
\end{cases}, \]
\[ p_{1t} = \Pr \left( S_t = 1 | F_{t-1} \right) \]
Focus of the paper

- Consider the following process for the time series of interest denoted $y_t \in \mathbb{R}$, $t = 1, 2, \ldots, T$ where

$$
\begin{align*}
  y_t &= \sigma_t \epsilon_t, \\
  \epsilon_t &= \phi_0 \epsilon_{t-1} + v_t,
\end{align*}
$$

(3) (4)

- $v_t \sim i.i.d. (0, 1)$, $E \left( |v_t|^{l+\gamma} \right) < \infty$ for $l = 1, \ldots, 4$ and $\gamma > 0$

- $\phi_0 \in \Theta = (-1; 1)$,

- $\sigma_t^2 \equiv \sigma(x_t)^2 \in \mathcal{F} = C^2[0, 1]$ and $P \left( \sigma_t^2 > 0 \right) = 1$

- $\epsilon_t$ is a $\alpha$-mixing sequence with mixing coefficient $-(1 + 2/\delta)$ for $\delta > 0$. 
Econometric/economic implications

• A time varying (functional) coefficient representation of (3)-(4)

\[ y_t = \frac{\sigma_t}{\sigma_{t-1}} \phi y_{t-1} + \sigma_t v_t. \]  (5)

• Change in relative volatility \( \left( \frac{\sigma_t}{\sigma_{t-1}} \right) \) → change in expected return \( (E(y_t|y_{t-1})) \).

• Direction of this effect depends on \( \text{sgn}(\phi) \) (LEVELS) or \( \text{sgn}(\phi y_{t-1}) \).
Figure 1: Modelling the change (w/w) in the 3 month US t-bill ($Dtb3$). Stability of $\rho$ in regression $Dtb3_t = \gamma + \rho Dtb3_{t-1} + \sigma v_t$. 
• Ignoring $\phi \neq 0$ can result in "poor" second-step estimators of $\sigma_t^2$.

  – Estimation properties of $\sigma_t^2$ relies on consistency of $\hat{g}(x_t)$.

  – Halunga and Orme (2004): Discuss the asymptotic sensitivity of (G)ARCH estimation in the case of misspecification of $g(x_t)$.

  – Hillebrand (2005): Neglecting parameter changes in (G)ARCH model implies that "the sum of the estimated autoregressive parameters is heavily biased towards unity in finite samples."
\[
\sqrt{\frac{\sigma^2_t}{\sigma^2_{t-1}}} \phi > 1 \text{ for some } t \rightarrow \text{ occasional explosive behavior of } y_t.
\]

- Granger (2005) and Yoon (2003): Models with a similar property (STochastic Unit Roots, Granger and Swanson (1997) and Leybourne et al. (1996a,b)) can generate return series that resemble the behavior of returns \(r_t\) surprisingly well.

  * \(r_t\) have little or no serial correlation

  * The autocorrelations of \(r_t^2\) and \(|r_t|^d\) decline slowly with slowest decline for \(d = 1\) (Taylor effect).

  * Autocorrelations of \(\text{sgn}(r_t)\) are all small, insignificant.

  * If one fits a GARCH(1,1) model to the series, then "\(\alpha + \beta \approx 1\)"
Two Cases of Interest

- Case 1: $x_t$ has a fixed design on the unit interval.

- Case 2: $x_t$ is a strictly stationary process with an $\alpha-$mixing base.
  - Case 2 encompasses the situation where $x_t = y_{t-1}$, hence allows for ARCH/LEVEL effects.

- Main simplifying assumptions
  - Simple AR(1) innovation term structure.
  - The conditional mean function is ignored, i.e., $g(x_t) \equiv 0$. 
Remarks

• The model of interest can be viewed as a nonparametric regression with correlated innovation terms.

• Some existing results (e.g., Fan and Masry (1993), Tong (1990), Fan, Yao and Tong (1996)) concerning the estimation of the conditional mean, i.e., \( g(x_t) = E(y_t|x_t) \).

  – Assume that the process satisfies certain mixing/memory conditions. Then

  * The local linear/polynomial estimator \( \hat{g}(x_t) \) is asymptotically consistent.
• It is asymptotically normal.

• The rate of convergence in mean squared is $O \left( \frac{1}{Th_T} \right)$, for any bandwidth $h_T = O \left( \frac{1}{T^{2p+3}} \right)$.

• Similar results on the estimation of the conditional variance, $\sigma_t^2 = \text{var}(y_t|x_t)$, is unknown to us!
  
  – Most general results available assume the innovation process to be a martingale difference sequence (series of recent working papers by O. Linton and coauthors).
Goals

- Estimate variance-covariance structure of the model

- Provide asymptotic characterization of resulting estimators: all parametric as well as nonparametric estimators of the model.
Characterization of $\hat{\phi}$ (MINPIN) and its asymptotics

- Let $\hat{\sigma}_t^2$ denote a (preliminary nonparametric) estimator of $\sigma_t^2$.

- Then we can define

$$\hat{\epsilon}_t = \hat{\sigma}_t^{-1} y_t$$

and an estimator of $\phi_0$ could (perhaps) be obtained by applying OLS to

$$\hat{\epsilon}_t = \phi \hat{\epsilon}_{t-1} + \tilde{v}_t$$
• More formally we define the (2-step) estimator of $\phi_0$ as

$$
\hat{\phi} = \arg \min_{\phi \in \Theta} (2T)^{-1} \sum_{t=1}^{T} d(\hat{\sigma}_t, \hat{\sigma}_{t-1}; \phi)
$$

$$
= \left( \frac{1}{T} \sum_{t=1}^{T} \hat{\sigma}_{t-1}^2 y_t^2 \right)^{-1} \left( \frac{1}{T} \sum_{t=1}^{T} \hat{\sigma}_{t-1} \hat{\sigma}_{t-1}^{-1} y_t y_{t-1} \right).
$$

• Note: If $\hat{\sigma}_t$ was a parametric estimator, the asymptotics could be established using the "nice" theory on two-step M-estimators, see, e.g., Section 12.4 in Wooldridge (2002).

• However, as $\hat{\sigma}_t$ is an infinite dimensional estimator a mean value expansion of, say, $d(\hat{\sigma}_t, \hat{\sigma}_{t-1}; \phi)$ about $\hat{\sigma}_t$ is not feasible.
Andrews (1994) provides solutions/tools: MINPIN estimators (estimators that MINimize as criterion function that may depend on a Preliminary Infinite dimensional Nuisance parameter estimator.)

Theorem 1  Let data be generated according to model (3)-(4) under assumptions i)-iv) with $x_t$ defined as in Case 1 or Case 2. Let $\hat{\sigma}_t^2$ be a nonparametric estimator of $\sigma_t^2$ and suppose a) $\sup_{\phi \in \Theta} \| \hat{\sigma}_t^2 - \sigma_t^2 \| \xrightarrow{p} 0$ for some $\sigma_t^2 \in \mathcal{F}$ and b) $P(\hat{\sigma}_t^2 \in \mathcal{F}) \xrightarrow{p} 1$. Then $\hat{\phi} \xrightarrow{p} \phi_0$.

Theorem 2  Let the Assumptions of Theorem 1 hold. Then under Case 1 and 2, $\sqrt{T} \left( \hat{\phi} - \phi_0 \right) \xrightarrow{d} N \left( 0, 1 - \phi_0^2 \right)$. 
Proof of Theorem 2 (sketch)

- Define $d(\sigma_t, \sigma_{t-1}; \phi) = m(\sigma_t, \sigma_{t-1}; \phi)^2$, where
  
  $$
  m(\sigma_t, \sigma_{t-1}; \phi) = \epsilon_{t-1} (\epsilon_t - \phi \epsilon_{t-1}) = \sigma_{t-1}^{\frac{1}{2}} \sigma_{t-1}^{\frac{1}{2}} y_t y_{t-1} - \sigma_{t-1}^{-2} \phi y_{t-1}^2
  $$

- Define $\overline{m}_T(\sigma; \phi) = (1/T) \sum_{t=1}^{T} m(\sigma_t, \sigma_{t-1}; \phi)$.

- Consider mean value expansion of $\sqrt{T} \overline{m}_T(\hat{\sigma}; \hat{\phi})$ about $\phi_0$ given as

  $$
  \sqrt{T} \overline{m}_T(\hat{\sigma}; \hat{\phi}) = \sqrt{T} \overline{m}_T(\hat{\sigma}; \phi_0) + \frac{\partial}{\partial \phi} \overline{m}_T(\hat{\sigma}; \phi^*) \sqrt{T} \left( \hat{\phi} - \phi_0 \right), \quad (6)
  $$

  where $\phi^*$ lies between $\hat{\phi}$ and $\phi_0$. 
• If $\lim_{T \to \infty} (\partial / \partial \phi) m_T(\hat{\sigma}; \phi^*) \xrightarrow{p} M$, where $M$ is nonsingular then it holds that

$$
\sqrt{T} \left( \hat{\phi} - \phi_0 \right) = -M^{-1} \left( o_p(1) + \sqrt{T} \overline{m}_T(\hat{\sigma}; \phi_0) \right),
$$

(7)

since $\sqrt{T} \overline{m}_T(\hat{\sigma}; \hat{\phi}) = o_p(1)$ as $\hat{\phi}$ solves the first order condition $\overline{m}_T(\hat{\sigma}; \hat{\phi}) = 0$ and $\hat{\phi}$ belongs to the interior of $\Theta$ wp$\to 1$.

• Asymptotic normality of $\sqrt{T} \left( \hat{\phi} - \phi_0 \right)$ follows if

- b) $\sqrt{T} \overline{m}_T(\sigma; \phi_0)$ is asymptotically normally distributed

- c) $\sqrt{T} (\overline{m}_T(\hat{\sigma}; \phi_0) - \overline{m}_T(\sigma; \phi_0)) \xrightarrow{p} 0$. 

• Let \( \overline{m}_T^* (\sigma; \phi) = (1/T) \sum_{t=1}^{T} E( m(\sigma_t, \sigma_{t-1}; \phi) ) \) and
\[
\nu_T(\sigma) = \sqrt{T} ((1/T) \sum_{t=1}^{T} m(\sigma_t, \sigma_{t-1}; \phi) - \overline{m}_T^* (\sigma; \phi))
\]
such that
\[
\sqrt{T} (\overline{m}_T(\hat{\sigma}; \phi_0) - \overline{m}_T(\sigma; \phi_0)) = \nu_T(\hat{\sigma}) - \nu_T(\sigma) - \sqrt{T} \overline{m}_T^* (\hat{\sigma}; \phi).
\]

• Condition c) is true if \( \nu_T(\hat{\sigma}) \) is stochastic equicontinuous at \( \sigma \), i.e.,
\[
\lim_{T \to \infty} P \left( \sup_{\rho_F(\hat{\sigma}, \sigma) < \delta'} |\nu_T(\hat{\sigma}) - \nu_T(\sigma)| > \eta \right) \to 0
\]
and \( \sqrt{T} \overline{m}_T^* (\sigma; \phi) \overset{p}{\to} 0. \)
Properties of the MINPIN estimator:

- $\hat{\phi}$ converges at $\sqrt{T}$-rate
- $\hat{\phi}$ is asymptotically efficient
- $\hat{\phi}$ is adaptive
Difference based estimator of $\sigma_t^2$ (Case 1)

- Consider (for unknown $g(\cdot)$ and $e_t \sim i.i.d.(0,1)$)
  \[ y_t = g(x_t) + \sigma e_t, \quad (8) \]

- Hall et al. (1990) suggest estimating $\sigma^2$ by
  \[
  \hat{\sigma}_T^2 = (T - (m_1 + m_2))^{-1} \sum_{t=m_1+1}^{T-m_2} \Delta_{t,r}^2
  \]
  \[
  \Delta_{t,r}^2 = \left( \sum_{j=0}^{r} d_{j}y_{j+t} \right)^2 \quad \text{where} \quad \sum_{j=0}^{r} d_{j} = 0, \quad \text{and} \quad \sum_{j=0}^{r} d_{j}^2 = 1.
  \]
- For $r = 1$: $\Delta^2_{t,1} = \frac{(y_t - y_{t-1})^2}{2}$

- $\hat{\sigma}^2_T$ enjoys the parametric rate of convergence.

- Consider next (for unknown $g(\cdot)$, $\sigma(\cdot)^2$ and $e_t \sim \text{i.i.d.}(0,1)$)

$$y_t = g(x_t) + \sigma(x_t)e_t,$$

- Levine (2003) suggests applying a local kernel smoother to all $\Delta^2_{t,1} = \frac{(y_t - y_{t-1})^2}{2}$ in order to produce the estimator

$$\hat{\sigma}^2_t = \frac{\sum_{t=1}^{T-r} \Delta^2_{t,1} K \left( \frac{x - x_t}{h} \right)}{\sum_{t=1}^{T-r} K \left( \frac{x - x_t}{h} \right)}.$$  \hspace{1cm} (9)
• $\hat{\sigma}_t^2$ has the optimal rate of convergence $T^{-\frac{2p}{2p+1}}$, if $\sigma_t^2 \in C^p[0, 1]$ and $g(x) \in C^{p-1}[0, 1]$ for $p > 1$.

• Crucial question: Can we obtain a consistent estimator of $\sigma_t^2$ in (2)-(3) using the estimator (9)? No! In this case $\hat{\sigma}_t^2 \xrightarrow{p} \sigma_t^2/(1 + \phi)$.

• Idea: If $\varepsilon_t$ follows an AR(1) process then $\text{var}(\varepsilon_t) - \text{cov}(\varepsilon_t, \varepsilon_{t-2}) = 1$

• Define $\eta_t = (1/2)^{1/2} (y_t - y_{t-2})$ and consider the local linear estimator $\hat{\sigma}_t^2 = \hat{\sigma} (x_t)^2$ given as $\hat{a}_t$ that solves the problem

$$ (\hat{a}_t, \hat{b}_t) = \arg\min_{a_t, b_t} \sum_{s=2}^{T} \left( \eta_s^2 - a_t - (x_s - x_t)b_t \right)^2 K_h(x_s - x_t) $$

(10)
Theorem 3 (Consistency of $\hat{\sigma}_t^2$) Let data be generated according to the model (3)-(4) under Case 1. Suppose that $K(u)$ is a second order non-negative kernel function satisfying: $K(u) \geq 0$ for any $u \in [-1, 1]$, $\mu_1 = \int K(u) du = 0$, $\sigma_K^2 \equiv \mu_2 = \int u^2 K(u) du \neq 0$ and $R_K = \int K(u)^2 du$. Then the estimator given by (10) is consistent in mean square with convergence rate $O(T^{-4/5})$. Furthermore, the optimal (in the sense of Parzen (1962) and Rosenblatt (1956)) bandwidth is $h = O(T^{-1/5})$. 
• Why will this idea work? Consider the Taylor series expansions (about $x$)

$$
\eta_t^2 = \frac{1}{2} \left( \sigma_t^2 \epsilon_t^2 + \sigma_{t-2}^2 \epsilon_{t-2}^2 - 2 \sqrt{\sigma_t^2 \sigma_{t-2}^2} \epsilon_t \epsilon_{t-1} \right)
$$

$$
\sigma_t^2 = \sigma^2 - D \sigma^2 (x - x_t) + D^2 \sigma^2 (x - x_t)^2 / 2 + o(h^2)
$$

$$
\sqrt{\sigma_t^2 \sigma_{t-1}^2} = \sigma^2 + \text{higher order expansion terms}
$$

using $\sqrt{1 + x} = 1 + x/2 + o(x)$.

• Then

$$
E \left( \eta_t^2 \right) = \frac{1}{2} \left( \sigma_t^2 + \sigma_{t-2}^2 \right) \text{var}(\epsilon_t) - \sqrt{\sigma_t^2 \sigma_{t-2}^2} \text{cov}(\epsilon_t, \epsilon_{t-2})
$$

$$
= (\text{var}(\epsilon_t) - \text{cov}(\epsilon_t, \epsilon_{t-2})) \sigma^2 + \text{higher order expansion terms}
$$

$$
= \sigma^2 + \text{higher order expansion terms}
$$
as $\text{var}(\epsilon_t) - \text{cov}(\epsilon_t, \epsilon_{t-2}) = (\gamma_0 - \gamma_2) = 1$.

Theorem 4 (Asymptotic Normality of $\hat{\sigma}_t^2$) Let the Assumptions of Theorem 1 hold. Then,

$$\hat{\sigma}_t^2 \xrightarrow{d} N \left( \sigma_t^2 + \text{Bias}(\hat{\sigma}_t^2), R_K C(\phi_0) \sigma^4 (Th)^{-1} + o((Th)^{-1}) \right).$$

as $T \to \infty$, $h \to 0$ and $Th \to \infty$, where

$$\text{Bias}(\hat{\sigma}_t^2) = \left[ D^2 \sigma^2 / 4 - \gamma_2 (D \sigma^2)^2 / \sigma^2 \right] h^2 \sigma_K^2 / 2 + o(h^2).$$
Fan and Yao’s estimator of $\sigma^2_t$ (Case 2)

- Fan and Yao prove consistency and asymptotic normality of the estimators of $E(y_t|x_t)$ and $\text{var}(y_t|x_t)$ denoted $\hat{a}_t$ and $\hat{\alpha}_t$ respectively and given as

$$
(\hat{a}_t, \hat{b}_t) = \arg \min_{a_t, b_t} \sum_{s=3}^{T} \left( y_s - a_t - (x_s - x_t)b_t \right)^2 K_H (x_s - x_t),
$$

$$
(\hat{\alpha}_t, \hat{\beta}_t) = \arg \min_{\alpha_t, \beta_t} \sum_{s=3}^{T} \left( \hat{r}_s^2 - \alpha_t - (x_s - x_t)\beta_t \right)^2 K_h (x_s - x_t),
$$

where $\hat{r}_t = y_t - \hat{a}_t$, and $x_t = (x_t, x_{t-1}, y_{t-1})$. 
Simulation results

Table 1: Alternative data generating processes. $\varphi(\cdot)$ is the standard normal c.d.f.

<table>
<thead>
<tr>
<th>Specifications</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model 1 $y_t = \sqrt{0.1 + 0.5x_t^2}\epsilon_t$</td>
</tr>
<tr>
<td>Model 2 $y_t = \sqrt{0.4 \exp(-2x_t^2) + 0.2\epsilon_t}$</td>
</tr>
<tr>
<td>Model 3 $y_t = \varphi(x_t + 1.2) + 1.5\varphi(x_t - 1.2)\epsilon_t$</td>
</tr>
</tbody>
</table>
Figure 2: MSE from the Difference based variance function estimator (solid line) and the Fan-Yao estimator (dotted line) under alternative variance function specifications and alternative values of $\phi_0$. $T = 1000$ and the number of Monte Carlo replications equals 1000.
Figure 3: Finite sample (simulated) densities and the asymptotic density of 
\( \sqrt{T} \left( \hat{\phi} - \phi_0 \right) / \sqrt{1 - \phi_0^2} \) under alternative variance function specifications for \( T = 1000 \) and \( \phi_0 = 0.5 \). Solid line: N(0,1). Dashed Line: Fan-Yao. Dotted line: Difference based. The number of Monte Carlo replications equals 1000.
Empirical illustration (3m t-bill)

- GARCH-M

\[
\Delta r_t = \gamma + \rho r_{t-1} + \delta \sigma_t + \sigma_t v_t, \quad v_t \sim i.i.d(0, 1), \\
\sigma_t^2 = \alpha_0 + \alpha_1 \Delta r_{t-1}^2 + \beta \sigma_{t-1}^2.
\]

- GARCH with AR(1) innovations

\[
\Delta r_t = \gamma + \rho r_{t-1} + \phi \left( \frac{\sigma_t}{\sigma_{t-1}} \right) \Delta r_{t-1} + \kappa_1 \left( \frac{\sigma_t}{\sigma_{t-1}} \right) r_{t-2} + \kappa_2 \left( \frac{\sigma_t}{\sigma_{t-1}} \right) + \sigma_t v_t, \\
\sigma_t^2 = \alpha_0 + \alpha_1 \Delta r_{t-1}^2 + \beta \sigma_{t-1}^2, \\
\kappa_1 = -\phi \rho, \\
\kappa_2 = -\phi \gamma.
\]
• Nonparametric model (NonP a.)

\[ r_t = g(r_{t-1}, r_{t-2}; \phi) + \sigma_t v_t, \]

• Nonparametric model (NonP b.)

\[
\begin{align*}
\Delta r_t &= \gamma + \rho r_{t-1} + \phi \frac{\hat{\sigma}^\text{np}_t}{\hat{\sigma}^\text{np}_{t-1}} \Delta r_{t-1} + \kappa_1 \frac{\hat{\sigma}^\text{np}_t}{\hat{\sigma}^\text{np}_{t-1}} r_{t-2} + \kappa_2 \frac{\hat{\sigma}^\text{np}_t}{\hat{\sigma}^\text{np}_{t-1}} + \sigma_t v_t, \\
\sigma_t^2 &= \alpha_0 + \alpha_1 \Delta r_{t-1}^2 + \beta \sigma_{t-1}^2.
\end{align*}
\]
<table>
<thead>
<tr>
<th></th>
<th>GARCH-M</th>
<th>GARCH-AR(1)</th>
<th>NonP (a)</th>
<th>NonP (b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma$</td>
<td>0.016 (0.005)</td>
<td>-0.044 (0.045)</td>
<td>·</td>
<td>-0.398 (0.329)</td>
</tr>
<tr>
<td>$\rho$</td>
<td>-0.003 (0.001)</td>
<td>0.011 (0.009)</td>
<td>·</td>
<td>0.111 (0.084)</td>
</tr>
<tr>
<td>$\delta$</td>
<td>0.007 (0.045)</td>
<td>·</td>
<td>·</td>
<td>·</td>
</tr>
<tr>
<td>$\alpha_0$</td>
<td>0.000 (0.000)</td>
<td>0.001 (0.000)</td>
<td>·</td>
<td>0.001 (0.000)</td>
</tr>
<tr>
<td>$\alpha_1$</td>
<td>0.169</td>
<td>0.156</td>
<td>·</td>
<td>0.154</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.830 (0.036)</td>
<td>0.844 (0.018)</td>
<td>·</td>
<td>0.845 (0.018)</td>
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<tr>
<td>$\phi$</td>
<td>·</td>
<td>0.177 (0.021)</td>
<td>0.252 (0.020)</td>
<td>0.212 (0.037)</td>
</tr>
<tr>
<td>$\kappa_1$</td>
<td>·</td>
<td>-0.014 (0.009)</td>
<td>·</td>
<td>-0.115 (0.084)</td>
</tr>
<tr>
<td>$\kappa_2$</td>
<td>·</td>
<td>0.060 (0.045)</td>
<td>·</td>
<td>0.415 (0.329)</td>
</tr>
</tbody>
</table>

Table 2: Estimation results. Sample period: 01/08/1954-12/31/1999.
Directions for ongoing/future research

- Include $g(x_t)$ in (3)-(4): Estimation parametrically/non-parametrically?

- Generalization of the innovations process to AR($p$).

- Capturing GARCH effects (nonparametrically)

- Evaluate the predictive ability of the model based on simulated/actual data

- Evaluate the significance of $\phi$ (failure of the martingale difference hypothesis)
Include \( g(x_t) \)

- Case 1: Under weak regularity conditions the effect of \( g(x_t) \) will be removed by the "difference" operation.

- Case 2: As long as \( k \) is not too large \( g(x_t) \) is consistently estimated in the first step of the Fan and Yao (1998) estimation procedure. If additional assumptions on \( g(\cdot) \) is imposed, such as additivity, \( k \) can be large without affecting the asymptotics.
Generalization of the innovations process: AR($p$) - structure

- Consider now the following model

\[
\begin{align*}
    y_t &= \sigma_t \epsilon_t \quad \text{(11)} \\
    \epsilon_t &= \sum_{i=1}^{p} \phi_i \epsilon_{t-i} + \nu_t \quad \text{(12)}
\end{align*}
\]
Case 1:

- Define $\tilde{\eta}_t = (1/2)^{1/2} (y_t - y_{t-1})$ and $\phi = (\phi_1, ..., \phi_p)'$. By using the same expansion-techniques as above we have

$$E(\tilde{\eta}_t^2) = \frac{1}{2} (\sigma_t^2 + \sigma_{t-1}^2) \text{var}(\epsilon_t) - \sqrt{\sigma_t^2 \sigma_{t-1}^2} \text{cov}(\epsilon_t, \epsilon_{t-2})$$

$$= (\gamma_0(\phi) - \gamma_1(\phi)) \sigma^2 + \text{higher order expansion terms}$$

and consequently one can show that

$$\hat{\sigma}_t^2 \xrightarrow{p} (\gamma_0(\phi) - \gamma_1(\phi)) \sigma^2$$

where $\hat{\sigma}_t^2$ is obtained using local linear regression to $\tilde{\eta}_t^2$.

- Consider the following estimator of $\phi$ given as

$$\hat{\phi} = A (\hat{\sigma}_{t-1}, ..., \hat{\sigma}_{t-p})^{-1} B (\hat{\sigma}_t, ..., \hat{\sigma}_{t-p})$$
where

\[ A \left( \hat{\sigma}_{t-1}, \ldots, \hat{\sigma}_{t-p} \right) = \begin{bmatrix}
\frac{1}{T} \sum_{t=p+1}^{T} \frac{y_{t-1}^2}{\hat{\sigma}_{t-1}^2} & \cdots & \frac{1}{T} \sum_{t=p+1}^{T} \frac{y_{t-1}y_{t-p}}{\hat{\sigma}_{t-1}\hat{\sigma}_{t-p}} \\
\frac{1}{T} \sum_{t=p+1}^{T} \frac{y_{t-2}y_{t-1}}{\hat{\sigma}_{t-2}\hat{\sigma}_{t-1}} & \cdots & \frac{1}{T} \sum_{t=p+1}^{T} \frac{y_{t-2}y_{t-p}}{\hat{\sigma}_{t-2}\hat{\sigma}_{t-p}} \\
\vdots & \ddots & \vdots \\
\frac{1}{T} \sum_{t=p+1}^{T} \frac{y_{t-p}y_{t-1}}{\hat{\sigma}_{t-p}\hat{\sigma}_{t-1}} & \cdots & \frac{1}{T} \sum_{t=p+1}^{T} \frac{y_{t-p}^2}{\hat{\sigma}_{t-p}^2}
\end{bmatrix} \]

and

\[ B \left( \hat{\sigma}_{t}, \ldots, \hat{\sigma}_{t-p} \right) = \begin{bmatrix}
\frac{1}{T} \sum_{t=p+1}^{T} \frac{y_{t-1}y_{t}}{\hat{\sigma}_{t-1}\hat{\sigma}_{t}} \\
\frac{1}{T} \sum_{t=p+1}^{T} \frac{y_{t-2}y_{t}}{\hat{\sigma}_{t-2}\hat{\sigma}_{t}} \\
\vdots \\
\frac{1}{T} \sum_{t=p+1}^{T} \frac{y_{t-p}y_{t}}{\hat{\sigma}_{t-p}\hat{\sigma}_{t}}
\end{bmatrix} \]
Proposition E1  Let data be generated according to the model given by (3)-(4). Then
\[ A \left( \hat{\sigma}_{t-1}, \ldots, \hat{\sigma}_{t-p} \right) \xrightarrow{p} \left( \gamma_0(\phi) - \gamma_1(\phi) \right)^{-1} A \left( \sigma_{t-1}, \ldots, \sigma_{t-p} \right) \]

Proposition E2  Let data be generated according to the model given by (3)-(4). Then
\[ B \left( \hat{\sigma}_t, \ldots, \hat{\sigma}_{t-p} \right) \xrightarrow{p} \left( \gamma_0(\phi) - \gamma_1(\phi) \right)^{-1} B \left( \sigma_t, \ldots, \sigma_{t-p} \right) \]

Theorem E1  Let data be generated according to the model given by (3)-(4). Then
\[ \hat{\phi} \xrightarrow{p} \phi. \]
• A consistent estimator of $\sigma^2$ can then be obtained as

$$\left(\gamma_0(\hat{\phi}) - \gamma_1(\hat{\phi})\right)^{-1} \hat{\sigma}_t^2$$

where $\hat{\sigma}_t^2$ denotes the preliminary (and inconsistent) nonparametric estimator.

Case 2:

• Rewrite the model as

$$y_t = g(y_{t-1}, \ldots, y_{t-p-1}) + \sigma_t v_t$$

where

$$g(y_{t-1}, \ldots, y_{t-p-1}) = \phi_1 \frac{\sigma_t}{\sigma_{t-1}} y_{t-1} + \phi_2 \frac{\sigma_t}{\sigma_{t-2}} y_{t-2} + \ldots + \phi_p \frac{\sigma_t}{\sigma_{t-p}} y_{t-p}$$
• Although convergence will be slow an estimator of \( g(y_{t-1}, \ldots, y_{t-p-1}) \)
can be obtained by the Fan and Yao (1998) approach.

• Based on simulations the following "iterated" Fan and Yao estimator seems promising:

1. Obtain a preliminary consistent estimates of \( g(y_{t-1}, \ldots, y_{t-p-1}) \)
   and \( \sigma_t^2 \) using Fan-Yao.

2. Obtain \( \hat{\phi} = A \left( \hat{\sigma}_{t-1}, \ldots, \hat{\sigma}_{t-p} \right)^{-1} B \left( \hat{\sigma}_t, \ldots, \hat{\sigma}_{t-p} \right) \)

3. Form \( \hat{r}_t = y_t - \hat{\phi}_1 \frac{\hat{\sigma}_t}{\sigma_{t-1}} y_{t-1} - \hat{\phi}_2 \frac{\hat{\sigma}_t}{\sigma_{t-2}} y_{t-2} - \ldots - \hat{\phi}_p \frac{\hat{\sigma}_t}{\sigma_{t-p}} y_{t-p} \) and
update the estimator of $\sigma^2_t$ (given as $\hat{\alpha}_t$) from

$$(\hat{\alpha}_t, \hat{\beta}_t) = \arg \min_{\alpha_t, \beta_t} \sum_{s=3}^{T} \left( \hat{r}_s^2 - \alpha_t - (y_{s-1} - y_{t-1})\beta_t \right)^2 K_h(x_s - x_t)$$

4. Repeat steps 2. and 3. until convergence.

• Alternatively to the "iterated" Fan-Yao estimator one could apply the estimation algorithms proposed by Fan, Yao and Cai (2003) for so-called Varying Coefficient Linear Models.
Capturing GARCH effects

- Consider the following semi-parametric GARCH specification

\[ \sigma_t^2 = m(y_{t-1}) + \beta \sigma_{t-1}^2 \]

or equivalently (provided that \(|\beta| < 1\))

\[ \sigma_t^2 = \sum_{j=1}^{\infty} \beta^{j-1} m(y_{t-j}) \]
Next, consider the approximation (after truncation and replacing $\sigma_t$ with $\hat{r}_t = y_t - \hat{a}_t$)

\[
\hat{r}_t^2 = \sum_{j=1}^{J} \beta_j^{-1} m(y_{t-j}) + \omega_t
\]

(13)

or

\[
\hat{r}_t^2 = \sum_{j=1}^{J} m_j(y_{t-j}) + \omega_t
\]

(14)

Note that (14) is an ordinary additive model where the component functions $m_j(\cdot)$ are linked by $\beta$, i.e., $m_j(x) = \beta_j^{-1} m_1(x)$. 
• Carroll, Haerdle and Mammen (2002) suggest estimating \( m(\cdot) \) and \( \beta \) by the following procedure

1. Estimate \( m_j(\cdot) \) for \( j = 1, \ldots, J \), based on (14) and note that \( m_1(x) = m(x) \)

2. Obtain the estimator of \( \beta \) as

\[
\hat{\beta} = \arg \min_{\beta} \sum_{t=J+1}^{T} \sum_{j=2}^{J} w_j(y_{t-j}) \left[ \hat{m}_j(y_{t-j}) - \beta^{j-1} \hat{m}_1(y_{t-1}) \right]^2
\]