Nonparametric estimation of volatility models with serially dependent innovations

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The "general" model of interest

$$y_t = g(\boldsymbol{x}_t) + \sigma(\boldsymbol{x}_t)\epsilon_t, \qquad t = 1, 2, ..., T. \tag{1}$$

- $y_t \in \mathbb{R},$ and $\boldsymbol{x}_t \in \mathbb{R}^k,$ and $\{(y_t, \boldsymbol{x}_t)\}$ is a strictly stationary process.

- $\mathsf{E}\left(y_t|\boldsymbol{x}_t\right) = g(\boldsymbol{x}_t)$ and $g : \mathbb{R}^k \to \mathbb{R}.$

- $\mathsf{var}\left(y_t|\boldsymbol{x}_t\right) = \sigma(\boldsymbol{x}_t)^2$ and $\sigma : \mathbb{R}^k \to \mathbb{R}_+.$

- ϵ_t is a strictly stationary process with $\mathsf{E}\left(\epsilon_t\right) = 0$ and $\mathsf{var}\left(\epsilon_t\right) = 1.$

1 Applications (short term interest rates)

The ARCH(-M) specification:

- $y_t = \Delta r_t$ and $\boldsymbol{x}_t = (1, \Delta r_{t-1}, \Delta r_{t-2}, ..., \Delta r_{t-k-1})'$.
- $g(\boldsymbol{x}_t) = \boldsymbol{x}_t'\boldsymbol{\beta} \qquad \left(+\sigma(\boldsymbol{x}_t)^2\right)$.
- $\sigma(\boldsymbol{x}_t)^2 = \alpha_0 + \alpha_1 \Delta r_{t-1}^2 + ... + \alpha_p \Delta r_{t-p}^2$.
- $\epsilon_t \sim i.i.d.$

The one-factor diffusion model (cont. time):

- Chan, Karolyi, Longstaff and Saunders (1992) generalized short rate specification

$$dr = (\alpha + \beta r)\, dt + \psi r^{\gamma} dW_t. \qquad (2)$$

- Euler-Maruyama discrete time approximation of (2)

$$\begin{aligned} \Delta r_t &= \alpha + \beta r_{t-1} + \psi r_{t-1}^{\gamma} \epsilon_t, \\ \mathsf{E}\left(\epsilon_t | F_{t-1}\right) &= 0, \qquad \mathsf{E}\left(\epsilon_t^2 | F_{t-1}\right) = 1. \end{aligned}$$

- Brenner, Harjes and Kroner (1996) and Andersen and Lund (1997) suggest

$$\psi_t^2 = \omega + \theta \Delta r_{t-1}^2 + \phi \psi_{t-1}^2.$$

The Markov regime switching model

- Hamilton and Susmel (1994), Cai (1994) and Gray (1996)

$$\begin{aligned} \Delta r_t | F_{t-1} &\sim \begin{cases} \mathsf{N}\left(g_{1t}, \sigma_{1t}^2\right) & \text{w.p. } p_{1t} \\ \mathsf{N}\left(g_{2t}, \sigma_{2t}^2\right) & \text{w.p. } 1 - p_{1t} \end{cases}, \\ p_{1t} &= \mathsf{Pr}\left(S_t = 1 | F_{t-1}\right) \end{aligned}$$

Focus of the paper

- Consider the following process for the time series of interest denoted $y_t \in \mathbb{R}$, $t = 1, 2, ..., T$ where

$$\begin{aligned} y_t &= \sigma_t \epsilon_t, & (3) \\ \epsilon_t &= \phi_0 \epsilon_{t-1} + v_t, & (4) \end{aligned}$$

  - $v_t \sim i.i.d.\,(0,1)$, $\mathsf{E}\left(|v_t|^{l+\gamma}\right) < \infty$ for $l = 1, ..., 4$ and $\gamma > 0$
  - $\phi_0 \in \Theta = (-1; 1)$,
  - $\sigma_t^2 \equiv \sigma(x_t)^2 \in \mathcal{F} = C^2[0,1]$ and $P\left(\sigma_t^2 > 0\right) = 1$
  - ϵ_t is a $\alpha-$mixing sequence with mixing coefficient $-\left(1 + 2/\delta\right)$ for $\delta > 0$.

Econometric/economic implications

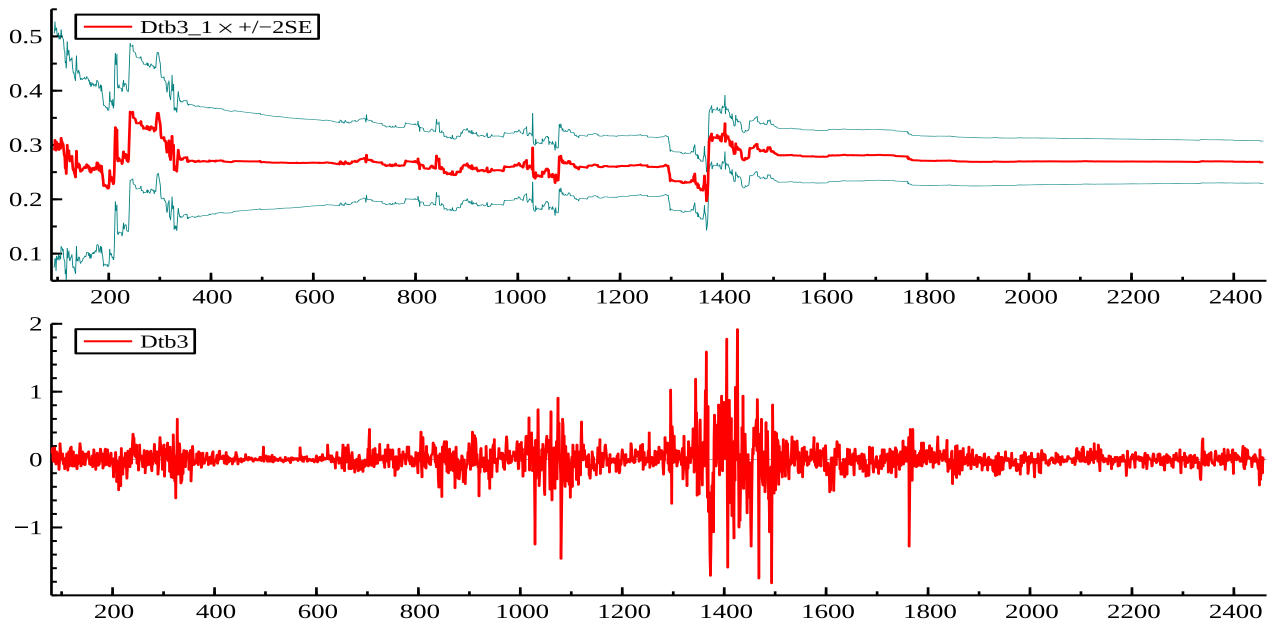
- A time varying (functional) coefficient representation of (3)-(4)

$$y_t = \frac{\sigma_t}{\sigma_{t-1}} \phi y_{t-1} + \sigma_t v_t. \tag{5}$$

- Change in relative volatility $\left(\frac{\sigma_t}{\sigma_{t-1}}\right) \rightarrow$ change in expected return $(\mathsf{E}\left(y_t | y_{t-1}\right))$.

- Direction of this effect depends on sgn(ϕ) (LEVELS) or sgn(ϕy_{t-1})

Figure 1: Modelling the change (w/w) in the 3 month US t-bill ($Dtb3$). Stability of ρ in regression $Dtb3_t = \gamma + \rho Dtb3_{t-1} + \sigma_t v_t$.



- Ignoring $\phi \neq 0$ can result in "poor" second-step estimators of σ_t^2.
  - Estimation properties of σ_t^2 relies on consistency of $\widehat{g}\,(x_t)$.
  - Halunga and Orme (2004): Discuss the asymptotic sensitivity of (G)ARCH estimation in the case of misspecification of $g(x_t)$.
  - Hillebrand (2005): Neglecting parameter changes in (G)ARCH model implies that "the sum of the estimated autoregressive parameters is heavily biased towards unity in finite samples."

- $\left|\sqrt{\frac{\sigma_t^2}{\sigma_{t-1}^2}}\phi\right| > 1$ for some $t \rightarrow$ occasional explosive behavior of y_t.
  - Granger (2005) and Yoon (2003): Models with a similar property (STochastic Unit Roots, Granger and Swanson (1997) and Leybourne et al. (1996a,b)) can generate return series that resemble the behavior of returns (r_t) surprisingly well.
    * r_t have little or no serial correlation
    * The autocorrelations of r_t^2 and $|r_t|^d$ decline slowly with slowest decline for $d = 1$ (Taylor effect).
    * Autocorrelations of sgn(r_t) are all small, insignificant.
    * If one fits a GARCH(1,1) model to the series, then "$\alpha + \beta \approx 1$"

Two Cases of Interest

- Case 1: x_t has a fixed design on the unit interval.

- Case 2: x_t is a strictly stationary process with an $\alpha-$mixing base.

  - Case 2 encompasses the situation where $x_t = y_{t-1}$, hence allows for ARCH/LEVEL effects.

- Main simplifying assumptions

  - Simple AR(1) innovation term structure.

  - The conditional mean function is ignored, i.e., $g(x_t) \equiv 0$.

Remarks

- The model of interest can be viewed as a nonparametric regression with correlated innovation terms.

- Some existing results (e.g., Fan and Masry (1993), Tong (1990), Fan, Yao and Tong(1996)) concerning the estimation of the conditional mean, i.e., $g(x_t) = \mathsf{E}(y_t|x_t)$.

  - Assume that the process satisfies certain mixing/memory conditions. Then

    * The local linear/polynomial estimator $\hat{g}(x_t)$ is asymptotically consistent.

* It is asymptotically normal.

* The rate of convergence in mean squared is $O\left(\frac{1}{Th_T}\right)$, for any bandwidth $h_T = O\left(\frac{1}{T^{2p+3}}\right)$.

- Similar results on the estimation of the conditional variance, $\sigma_t^2 = \text{var}(y_t|x_t)$, is unknown to us!

  – Most general results available assume the innovation process to be a martingale difference sequence (series of recent working papers by O. Linton and coauthors) .

Goals

- Estimate variance-covariance structure of the model

- Provide asymptotic characterization of resulting estimators: all parametric as well as nonparametric estimators of the model.

Characterization of $\widehat{\phi}$ (MINPIN) and its asymptotics

- Let $\widehat{\sigma}_t^2$ denote a (preliminary nonparametric) estimator of σ_t^2.

- Then we can define

$$\widehat{\epsilon}_t = \widehat{\sigma}_t^{-1} y_t$$

and an estimator of ϕ_0 could (perhaps) be obtained by applying OLS to

$$\widehat{\epsilon}_t = \phi \widehat{\epsilon}_{t-1} + \widetilde{v}_t$$

- More formally we define the (2-step) estimator of ϕ_0 as

$$\begin{aligned}\widehat{\phi} &= \underset{\phi \in \Theta}{\arg\min} (2T)^{-1} \sum_{t=1}^{T} d\left(\widehat{\sigma}_t, \widehat{\sigma}_{t-1}; \phi\right) \\ &= \left((1/T) \sum_{t=1}^{T} \widehat{\sigma}_{t-1}^{-2} y_{t-1}^2 \right)^{-1} \left((1/T) \sum_{t=1}^{T} \widehat{\sigma}_t^{-1} \widehat{\sigma}_{t-1}^{-1} y_t y_{t-1} \right).\end{aligned}$$

- Note: If $\widehat{\sigma}_t$ was a parametric estimator, the asymptotics could be established using the "nice" theory on two-step M-estimators, see, e.g., Section 12.4 in Wooldridge (2002).

- However, as $\widehat{\sigma}_t$ is an infinite dimensional estimator a mean value expansion of, say, $d\left(\widehat{\sigma}_t, \widehat{\sigma}_{t-1}; \phi\right)$ about $\widehat{\sigma}_t$ is not feasible.

- Andrews (1994) provides solutions/tools: MINPIN estimators (estimators that MINimize as criterion function that may depend on a Preliminary Infinite dimensional Nuisance parameter estimator.)

Theorem 1 Let data be generated according to model (3)-(4) under assumptions i)-iv) with x_t defined as in Case 1 or Case 2. Let $\widehat{\sigma}_t^2$ be a nonparametric estimator of σ_t^2 and suppose a) $\sup_{\phi \in \Theta} \left\| \widehat{\sigma}_t^2 - \sigma_t^2 \right\| \xrightarrow{p} 0$ for some $\sigma_t^2 \in \mathcal{F}$ and b) $P(\widehat{\sigma}_t^2 \in \mathcal{F}) \xrightarrow{p} 1$. Then $\hat{\phi} \xrightarrow{p} \phi_0$.

Theorem 2 Let the Assumptions of Theorem 1 hold. Then under Case 1 and 2, $\sqrt{T}\left(\widehat{\phi} - \phi_0\right) \xrightarrow{d} \mathrm{N}\left(0, 1 - \phi_0^2\right)$.

Proof of Theorem 2 (sketch)

- Define $d\left(\sigma_t, \sigma_{t-1}; \phi\right) = m\left(\sigma_t, \sigma_{t-1}; \phi\right)^2$, where

$$\begin{aligned} m\left(\sigma_t, \sigma_{t-1}; \phi\right) &= \epsilon_{t-1}\left(\epsilon_t - \phi\epsilon_{t-1}\right) \\ &= \sigma_t^{-1}\sigma_{t-1}^{-1}y_t y_{t-1} - \sigma_{t-1}^{-2}\phi y_{t-1}^2 \end{aligned}$$

- Define $\overline{m}_T(\sigma; \phi) = (1/T)\sum_{t=1}^{T} m\left(\sigma_t, \sigma_{t-1}; \phi\right)$.

- Consider mean value expansion of $\sqrt{T}\overline{m}_T(\widehat{\sigma}; \widehat{\phi})$ about ϕ_0 given as

$$\sqrt{T}\overline{m}_T(\widehat{\sigma}; \widehat{\phi}) = \sqrt{T}\overline{m}_T(\widehat{\sigma}; \phi_0) + \frac{\partial}{\partial\phi}\overline{m}_T(\widehat{\sigma}; \phi^*)\sqrt{T}\left(\widehat{\phi} - \phi_0\right), \quad (6)$$

where ϕ^* lies between $\widehat{\phi}$ and ϕ_0.

- If a) $\lim_{T\to\infty}(\partial/\partial\phi)\overline{m}_T(\widehat{\sigma};\phi^*) \xrightarrow{p} M$, where M is nonsingular then it holds that

$$\sqrt{T}\left(\widehat{\phi}-\phi_0\right) = -M^{-1}\left(o_p(1)+\sqrt{T}\overline{m}_T(\widehat{\sigma};\phi_0)\right), \qquad (7)$$

since $\sqrt{T}\overline{m}_T(\widehat{\sigma};\widehat{\phi}) = o_p(1)$ as $\widehat{\phi}$ solves the first order condition $\overline{m}_T(\widehat{\sigma};\widehat{\phi}) = 0$ and $\widehat{\phi}$ belongs to the interior of Θ wp$\to$1.

- Asymptotic normality of $\sqrt{T}\left(\widehat{\phi}-\phi_0\right)$ follows if
  - b) $\sqrt{T}\overline{m}_T(\sigma;\phi_0)$ is asymptotically normally distributed
  - c) $\sqrt{T}(\overline{m}_T(\widehat{\sigma};\phi_0) - \overline{m}_T(\sigma;\phi_0)) \xrightarrow{p} 0$.

- Let $\overline{m}_T^*(\sigma;\phi) = (1/T)\sum_{t=1}^T \mathsf{E}\left(m\left(\sigma_t, \sigma_{t-1};\phi\right)\right)$ and

$$\upsilon_T(\sigma) = \sqrt{T}((1/T)\sum_{t=1}^{T} m\left(\sigma_t, \sigma_{t-1};\phi\right) - \overline{m}_T^*(\sigma;\phi))$$

such that

$$\sqrt{T}\left(\overline{m}_T(\widehat{\sigma};\phi_0) - \overline{m}_T(\sigma;\phi_0)\right) = \upsilon_T(\widehat{\sigma}) - \upsilon_T(\sigma) - \sqrt{T}\overline{m}_T^*(\widehat{\sigma};\phi).$$

- Condition c) is true if $\upsilon_T(\widehat{\sigma})$ is stochastic equicontinuous at σ, i.e.,

$$\lim_{T\to\infty} P\left(\sup_{\rho_{\mathcal{F}}(\widehat{\sigma},\sigma)<\delta'} |\upsilon_T(\widehat{\sigma}) - \upsilon_T(\sigma)| > \eta\right) \to 0$$

and $\sqrt{T}\overline{m}_T^*(\sigma;\phi) \xrightarrow{p} 0$.

Properties of the MINPIN estimator:

- $\widehat{\phi}$ converges at $\sqrt{T}$-rate

- $\widehat{\phi}$ is asymptotically efficient

- $\widehat{\phi}$ is adaptive

Difference based estimator of σ_t^2 (Case 1)

- Consider (for unknown $g(\cdot)$ and $e_t \sim i.i.d.$(0,1))

$$y_t = g(x_t) + \sigma e_t, \tag{8}$$

- Hall et al. (1990) suggest estimating σ^2 by

$$\hat{\sigma}_T^2 = (T - (m_1 + m_2))^{-1} \sum_{t=m_1+1}^{T-m_2} \Delta_{t,r}^2$$

$$\Delta_{t,r}^2 = \left(\sum_{j=0}^{r} d_j y_{j+t} \right)^2 \text{ where } \sum_{j=0}^{r} d_j = 0, \text{ and } \sum_{j=0}^{r} d_j^2 = 1.$$

- For $r = 1$: $\Delta_{t,1}^2 = \frac{(y_t - y_{t-1})^2}{2}$

- $\hat{\sigma}_T^2$ enjoys the parametric rate of convergence.

- Consider next (for unknown $g(\cdot)$, $\sigma(\cdot)^2$ and $e_t \sim$i.i.d.(0,1))

$$y_t = g(x_t) + \sigma(x_t)e_t,$$

- Levine (2003) suggests applying a local kernel smoother to all $\Delta_{t,1}^2 = \frac{(y_t - y_{t-1})^2}{2}$ in order to produce the estimator

$$\widehat{\sigma}_t^2 = \frac{\sum_{t=1}^{T-r} \Delta_{t,1}^2 K\left(\frac{x - x_t}{h}\right)}{\sum_{t=1}^{T-r} K\left(\frac{x - x_t}{h}\right)}. \tag{9}$$

- $\widehat{\sigma}_t^2$ has the optimal rate of convergence $T^{-\frac{2p}{2p+1}}$, if $\sigma_t^2 \in C^p[0,1]$ and $g(x) \in C^{p-1}[0,1]$ for $p > 1$.

- Crucial question: Can we obtain a consistent estimator of σ_t^2 in (2)-(3) using the estimator (9)? No! In this case $\widehat{\sigma}_t^2 \stackrel{p}{\longrightarrow} \sigma_t^2/(1+\phi)$.

- Idea: If ϵ_t follows an AR(1) process then $\text{var}(\epsilon_t) - \text{cov}(\epsilon_t, \epsilon_{t-2}) = 1$

- Define $\eta_t = (1/2)^{1/2}\,(y_t - y_{t-2})$ and consider the local linear estimator $\widehat{\sigma}_t^2 = \widehat{\sigma}\,(x_t)^2$ given as $\hat{a}_t$ that solves the problem

$$(\hat{a}_t, \hat{b}_t) = \arg\min_{a_t, b_t} \sum_{s=2}^{T} \left(\eta_s^2 - a_t - (x_s - x_t) b_t\right)^2 K_h\,(x_s - x_t) \qquad (10)$$

Theorem 3 (Consistency of $\widehat{\sigma}_t^2$) Let data be generated according to the model (3)-(4) under Case 1. Suppose that $K(u)$ is a second order non-negative kernel function satisfying: $K(u) \geq 0$ for any $u \in [-1, 1]$, $\mu_1 = \int K(u)\, du = 0$, $\sigma_K^2 \equiv \mu_2 = \int u^2 K(u)\, du \neq 0$ and $R_K = \int K(u)^2\, du$. Then the estimator given by (10) is consistent in mean square with convergence rate $O(T^{-4/5})$. Furthermore, the optimal (in the sense of Parzen (1962) and Rosenblatt (1956)) bandwidth is $h = O(T^{-1/5})$.

- Why will this idea work? Consider the Taylor series expansions (about x)

$$\begin{aligned}\eta_t^2 &= \frac{1}{2}\left(\sigma_t^2\epsilon_t^2 + \sigma_{t-2}^2\epsilon_{t-2}^2 - 2\sqrt{\sigma_t^2\sigma_{t-2}^2}\epsilon_t\epsilon_{t-1}\right)\\ \sigma_t^2 &= \sigma^2 - \mathsf{D}\sigma^2(x-x_t) + \mathsf{D}^2\sigma^2(x-x_t)^2/2 + o(h^2)\\ \sqrt{\sigma_t^2\sigma_{t-1}^2} &= \sigma^2 + \text{higher order expansion terms}\end{aligned}$$

using $\sqrt{1+x} = 1 + x/2 + o(x)$.

- Then

$$\begin{aligned}\mathsf{E}\left(\eta_t^2\right) &= \frac{1}{2}\left(\sigma_t^2 + \sigma_{t-2}^2\right)\mathsf{var}(\epsilon_t) - \sqrt{\sigma_t^2\sigma_{t-2}^2}\,\mathsf{cov}(\epsilon_t, \epsilon_{t-2})\\ &= \left(\mathsf{var}(\epsilon_t) - \mathsf{cov}(\epsilon_t, \epsilon_{t-2})\right)\sigma^2 + \text{higher order expansion terms}\\ &= \sigma^2 + \text{ higher order expansion terms}\end{aligned}$$

as $\text{var}(\epsilon_t) - \text{cov}(\epsilon_t, \epsilon_{t-2}) = (\gamma_0 - \gamma_2) = 1.$

Theorem 4 (Asymptotic Normality of $\widehat{\sigma}_t^2$)   Let the Assumptions of Theorem 1 hold. Then,

$$\widehat{\sigma}_t^2 \xrightarrow{d} \mathsf{N}\left(\sigma_t^2 + Bias(\widehat{\sigma}_t^2),\ R_K C(\phi_0)\sigma^4\,(Th)^{-1} + o((Th)^{-1})\right).$$

as $T \to \infty$, $h \to 0$ and $Th \to \infty$, where

$$Bias(\widehat{\sigma}_t^2) \;=\; \left[\mathsf{D}^2\sigma^2/4 - \gamma_2(\mathsf{D}\sigma^2)^2/\sigma^2\right] h^2\sigma_K^2/2 + o(h^2).$$

Fan and Yao's estimator of σ_t^2 (Case 2)

- Fan and Yao prove consistency and asymptotic normality of the estimators of $\mathsf{E}(y_t|\boldsymbol{x}_t)$ and $\mathsf{var}(y_t|\boldsymbol{x}_t)$ denoted $\widehat{a}_t$ and $\widehat{\alpha}_t$ respectively and given as

$$
\begin{aligned}
(\hat{a}_t, \widehat{\boldsymbol{b}_t}) &= \arg\min_{a_t, \boldsymbol{b}_t} \sum_{s=3}^{T} (y_s - a_t - (\boldsymbol{x}_s - \boldsymbol{x}_t)\boldsymbol{b}_t)^2 K_H(\boldsymbol{x}_s - \boldsymbol{x}_t), \\
(\widehat{\alpha}_t, \widehat{\beta}_t) &= \arg\min_{\alpha_t, \beta_t} \sum_{s=3}^{T} \left(\widehat{r}_s^2 - \alpha_t - (x_s - x_t)\beta_t\right)^2 K_h(x_s - x_t),
\end{aligned}
$$

where $\widehat{r}_t = y_t - \hat{a}_t$, and $\boldsymbol{x}_t = (x_t, x_{t-1}, y_{t-1})$.

Simulation results

Table 1: Alternative data generating processes. $\varphi(\cdot)$ is the standard normal c.d.f.

	Specifications
Model 1	$y_t = \sqrt{0.1 + 0.5x_t^2}\epsilon_t$
Model 2	$y_t = \sqrt{0.4\exp(-2x_t^2) + 0.2}\epsilon_t$
Model 3	$y_t = \sqrt{\varphi(x_t + 1.2) + 1.5\varphi(x_t - 1.2)}\epsilon_t$

Figure 2: MSE from the Difference based variance function estimator (solid line) and the Fan-Yao estimator (dotted line) under alternative variance function specifications and alternative values of ϕ_0. $T = 1000$ and the number of Monte Carlo replications equals 1000.

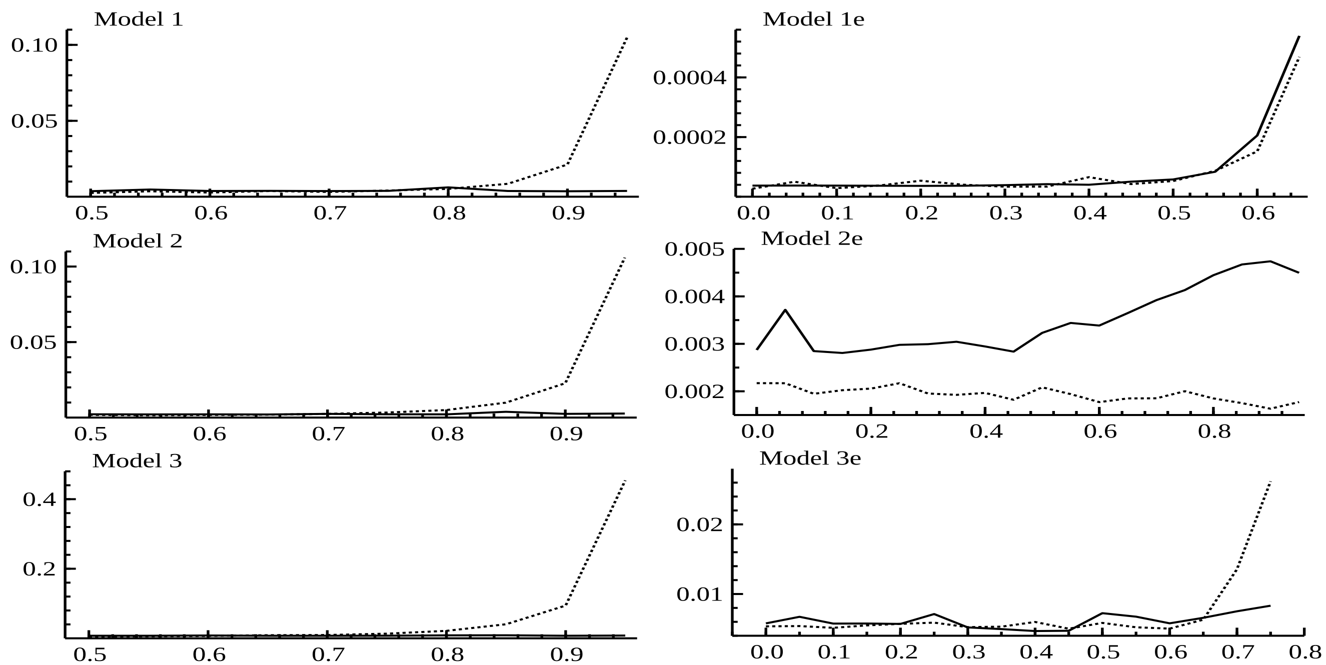
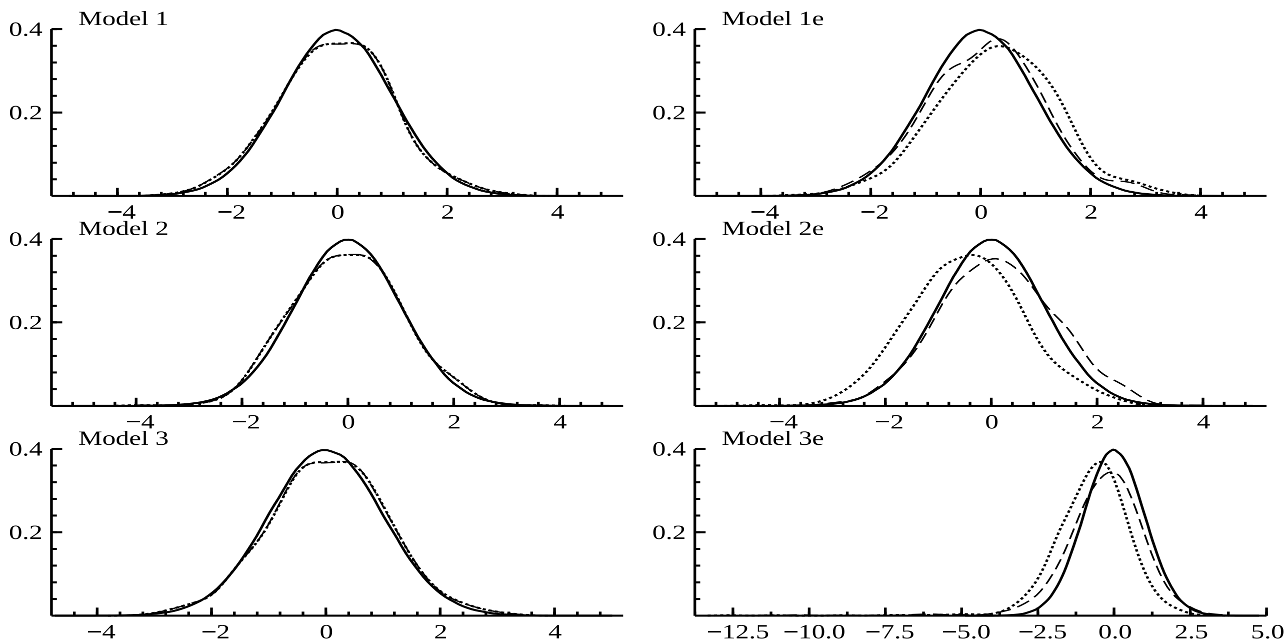


Figure 3: Finite sample (simulated) densities and the asymptotic density of $\sqrt{T}\left(\widehat{\phi}-\phi_0\right)/\sqrt{1-\phi_0^2}$ under alternative variance function specifications for T $= 1000$ and $\phi_0 = 0.5$. Solid line: N$(0,1)$. Dashed Line: Fan-Yao. Dotted line: Difference based. The number of Monte Carlo replications equals 1000.



Empirical illustration (3m t-bill)

- GARCH-M

$$
\begin{aligned}
\Delta r_t &= \gamma + \rho r_{t-1} + \delta\sigma_t + \sigma_t v_t, \qquad v_t \sim i.i.d(0,1), \\
\sigma_t^2 &= \alpha_0 + \alpha_1 \Delta r_{t-1}^2 + \beta\sigma_{t-1}^2.
\end{aligned}
$$

- GARCH with AR(1) innovations

$$
\begin{aligned}
\Delta r_t &= \gamma + \rho r_{t-1} + \phi\left(\frac{\sigma_t}{\sigma_{t-1}}\right)\Delta r_{t-1} + \kappa_1\left(\frac{\sigma_t}{\sigma_{t-1}}\right) r_{t-2} + \kappa_2\left(\frac{\sigma_t}{\sigma_{t-1}}\right) + \sigma_t v_t, \\
\sigma_t^2 &= \alpha_0 + \alpha_1 \Delta r_{t-1}^2 + \beta\sigma_{t-1}^2, \\
\kappa_1 &= -\phi\rho, \\
\kappa_2 &= -\phi\gamma.
\end{aligned}
$$

- Nonparametric model (NonP a.)

$$r_t = g(r_{t-1}, r_{t-2}; \phi) + \sigma_t v_t,$$

- Nonparametric model (NonP b.)

$$\begin{aligned} \Delta r_t &= \gamma + \rho r_{t-1} + \phi \frac{\widehat{\sigma}_t^{np}}{\widehat{\sigma}_{t-1}^{np}} \Delta r_{t-1} + \kappa_1 \frac{\widehat{\sigma}_t^{np}}{\widehat{\sigma}_{t-1}^{np}} r_{t-2} + \kappa_2 \frac{\widehat{\sigma}_t^{np}}{\widehat{\sigma}_{t-1}^{np}} + \sigma_t v_t. \\ \sigma_t^2 &= \alpha_0 + \alpha_1 \Delta r_{t-1}^2 + \beta \sigma_{t-1}^2. \end{aligned}$$

Table 2: Estimation results. Sample period: 01/08/1954-12/31/1999.

	GARCH-M	GARCH-AR(1)	NonP (a)	NonP (b)
γ	0.016 (0.005)	-0.044 (0.045)	·	-0.398 (0.329)
ρ	-0.003 (0.001)	0.011 (0.009)	·	0.111 (0.084)
δ	0.007 (0.045)	·	·	·
α_0	0.000 (0.000)	0.001 (0.000)	·	0.001 (0.000)
α_1	0.169	0.156	·	0.154
β	0.830 (0.036)	0.844 (0.018)	·	0.845 (0.018)
ϕ	·	0.177 (0.021)	0.252 (0.020)	0.212 (0.037)
κ_1	·	-0.014 (0.009)	·	-0.115 (0.084)
κ_2	·	0.060 (0.045)	·	0.415 (0.329)

Directions for ongoing/future research

- Include $g(x_t)$ in (3)-(4): Estimation parametrically/non-parametrically?

- Generalization of the innovations process to AR(p).

- Capturing GARCH effects (nonparametrically)

- Evaluate the predictive ability of the model based on simulated/actual data

- Evaluate the significance of ϕ (failure of the martingale difference hypothesis)

Include $g(x_t)$

- Case 1: Under weak regularity conditions the effect of $g(x_t)$ will be removed by the "difference" operation.

- Case 2: As long as k is not too large $g(x_t)$ is consistently estimated in the first step of the Fan and Yao (1998) estimation procedure. If additional assumptions on $g(\cdot)$ is imposed, such as additivity, k can be large without affecting the asymptotics.

Generalization of the innovations process: AR(p) - structure

- Consider now the following model

$$
\begin{aligned}
y_t &= \sigma_t \epsilon_t && (11) \\
\epsilon_t &= \sum_{i=1}^{p} \phi_i \epsilon_{t-i} + v_t && (12)
\end{aligned}
$$

Case 1:

- Define $\widetilde{\eta}_t = (1/2)^{1/2}(y_t - y_{t-1})$ and $\boldsymbol{\phi} = (\phi_1, ..., \phi_p)'$. By using the same expansion-techniques as above we have

$$\begin{aligned} \mathsf{E}\left(\widetilde{\eta}_t^2\right) &= \frac{1}{2}\left(\sigma_t^2 + \sigma_{t-1}^2\right)\mathsf{var}(\epsilon_t) - \sqrt{\sigma_t^2\sigma_{t-1}^2}\,\mathsf{cov}(\epsilon_t, \epsilon_{t-2}) \\ &= \left(\gamma_0(\boldsymbol{\phi}) - \gamma_1(\boldsymbol{\phi})\right)\sigma^2 + \text{higher order expansion terms} \end{aligned}$$

and consequently one can show that

$$\widehat{\sigma}_t^2 \xrightarrow{p} \left(\gamma_0(\boldsymbol{\phi}) - \gamma_1(\boldsymbol{\phi})\right)\sigma^2$$

where $\widehat{\sigma}_t^2$ is obtained using local linear regression to $\widetilde{\eta}_t^2$.

- Consider the following estimator of $\boldsymbol{\phi}$ given as

$$\widehat{\boldsymbol{\phi}} = \boldsymbol{A}\left(\widehat{\sigma}_{t-1}, ..., \widehat{\sigma}_{t-p}\right)^{-1}\boldsymbol{B}\left(\widehat{\sigma}_t, ..., \widehat{\sigma}_{t-p}\right)$$

where

$$\boldsymbol{A}\left(\widehat{\sigma}_{t-1},...,\widehat{\sigma}_{t-p}\right)=\begin{bmatrix} \frac{1}{T}\sum_{t=p+1}^{T}\frac{y_{t-1}^2}{\widehat{\sigma}_{t-1}^2} & \cdots & \frac{1}{T}\sum_{t=p+1}^{T}\frac{y_{t-1}y_{t-p}}{\widehat{\sigma}_{t-1}\widehat{\sigma}_{t-p}} \\ \frac{1}{T}\sum_{t=p+1}^{T}\frac{y_{t-2}y_{t-1}}{\widehat{\sigma}_{t-2}\widehat{\sigma}_{t-1}} & \cdots & \frac{1}{T}\sum_{t=p+1}^{T}\frac{y_{t-2}y_{t-p}}{\widehat{\sigma}_{t-2}\widehat{\sigma}_{t-p}} \\ \vdots & \ddots & \vdots \\ \frac{1}{T}\sum_{t=p+1}^{T}\frac{y_{t-p}y_{t-1}}{\widehat{\sigma}_{t-p}\widehat{\sigma}_{t-1}} & \cdots & \frac{1}{T}\sum_{t=p+1}^{T}\frac{y_{t-p}^2}{\widehat{\sigma}_{t-p}^2} \end{bmatrix}$$

and

$$\boldsymbol{B}\left(\widehat{\sigma}_{t},...,\widehat{\sigma}_{t-p}\right)=\begin{bmatrix} \frac{1}{T}\sum_{t=p+1}^{T}\frac{y_{t-1}y_t}{\widehat{\sigma}_{t-1}\widehat{\sigma}_t} \\ \frac{1}{T}\sum_{t=p+1}^{T}\frac{y_{t-2}y_t}{\widehat{\sigma}_{t-2}\widehat{\sigma}_t} \\ \vdots \\ \frac{1}{T}\sum_{t=p+1}^{T}\frac{y_{t-p}y_t}{\widehat{\sigma}_{t-p}\widehat{\sigma}_t} \end{bmatrix}$$

Proposition E1 Let data be generated according to the model given by (3)-(4). Then

$$\boldsymbol{A}\left(\widehat{\sigma}_{t-1}, ..., \widehat{\sigma}_{t-p}\right) \xrightarrow{p} \left(\gamma_0(\boldsymbol{\phi}) - \gamma_1(\boldsymbol{\phi})\right)^{-1} \boldsymbol{A}\left(\sigma_{t-1}, ..., \sigma_{t-p}\right)$$

Proposition E2 Let data be generated according to the model given by (3)-(4). Then

$$\boldsymbol{B}\left(\widehat{\sigma}_{t}, ..., \widehat{\sigma}_{t-p}\right) \xrightarrow{p} \left(\gamma_0(\boldsymbol{\phi}) - \gamma_1(\boldsymbol{\phi})\right)^{-1} \boldsymbol{B}\left(\sigma_{t}, ..., \sigma_{t-p}\right)$$

Theorem E1 Let data be generated according to the model given by (3)-(4). Then

$$\widehat{\boldsymbol{\phi}} \xrightarrow{p} \boldsymbol{\phi}.$$

- A consistent estimator of σ^2 can then be obtained as

$$\left(\gamma_0(\widehat{\phi}) - \gamma_1(\widehat{\phi})\right)^{-1} \widehat{\sigma}_t^2$$

where $\widehat{\sigma}_t^2$ denotes the preliminary (and inconsistent) nonparametric estimator

Case 2:

- Rewrite the model as

$$y_t = g(y_{t-1}, ..., y_{t-p-1}) + \sigma_t v_t$$

where

$$g(y_{t-1}, ..., y_{t-p-1}) = \phi_1 \frac{\sigma_t}{\sigma_{t-1}} y_{t-1} + \phi_2 \frac{\sigma_t}{\sigma_{t-2}} y_{t-2} + ... + \phi_p \frac{\sigma_t}{\sigma_{t-p}} y_{t-p}$$

- Although convergence will be slow an estimator of $g(y_{t-1}, ..., y_{t-p-1})$ can be obtained by the Fan and Yao (1998) approach.

- Based on simulations the following "iterated" Fan and Yao estimator seems promising:

1. Obtain a preliminary consistent estimates of $g(y_{t-1}, ..., y_{t-p-1})$ and σ_t^2 using Fan-Yao.

2. Obtain $\widehat{\boldsymbol{\phi}} = \boldsymbol{A}\left(\widehat{\sigma}_{t-1}, ..., \widehat{\sigma}_{t-p}\right)^{-1} \boldsymbol{B}\left(\widehat{\sigma}_{t}, ..., \widehat{\sigma}_{t-p}\right)$

3. Form $\widehat{r}_t = y_t - \widehat{\phi}_1 \frac{\widehat{\sigma}_t}{\widehat{\sigma}_{t-1}} y_{t-1} - \widehat{\phi}_2 \frac{\widehat{\sigma}_t}{\widehat{\sigma}_{t-2}} y_{t-2} - ... - \widehat{\phi}_p \frac{\widehat{\sigma}_t}{\widehat{\sigma}_{t-p}} y_{t-p}$ and

update the estimator of σ_t^2 (given as $\widehat{\alpha}_t$) from

$$(\widehat{\alpha}_t, \widehat{\beta}_t) = \arg\min_{\alpha_t, \beta_t} \sum_{s=3}^{T} \left(\widehat{r}_s^2 - \alpha_t - (y_{s-1} - y_{t-1})\beta_t \right)^2 K_h \left(x_s - x_t \right)$$

4. Repeat steps 2. and 3. until convergence.

- Alternatively to the "iterated" Fan-Yao estimator one could apply the estimation algorithms proposed by Fan, Yao and Cai (2003) for so-called Varying Coefficent Linear Models.

Capturing GARCH effects

- Consider the following semi-parametric GARCH specification

$$\sigma_t^2 = m(y_{t-1}) + \beta\sigma_{t-1}^2$$

or equivalently (provided that $|\beta| < 1$)

$$\sigma_t^2 = \sum_{j=1}^{\infty} \beta^{j-1} m(y_{t-j})$$

- Next, consider the approximation (after truncation and replacing σ_t with $\widehat{r}_t = y_t - \hat{a}_t$)

$$\widehat{r}_t^2 = \sum_{j=1}^{J} \beta^{j-1} m(y_{t-j}) + \omega_t \tag{13}$$

or

$$\widehat{r}_t^2 = \sum_{j=1}^{J} m_j(y_{t-j}) + \omega_t \tag{14}$$

- Note that (14) is an ordinary additive model where the component functions $m_j(\cdot)$ are linked by β, i.e., $m_j(x) = \beta^{j-1} m_1(x)$.

- Carroll, Haerdle and Mammen (2002) suggest estimating $m(\cdot)$ and β by the following procedure

1. Estimate $m_j(\cdot)$ for $j = 1, ..., J$, based on (14) and note that $m_1(x) = m(x)$

2. Obtain the estimator of β as

$$\widehat{\beta} = \arg\min_{\beta} \sum_{t=J+1}^{T} \sum_{j=2}^{J} w_j(y_{t-j}) \left[\widehat{m}_j(y_{t-j}) - \beta^{j-1} \widehat{m}_1(y_{t-1}) \right]^2$$