Economics 251
Homework #4
Answer key
20 points

1. (3 points) For the duopoly game with 13 different output levels, find the profit function for Firm 1. To do this, use the inverse demand function

$$\begin{aligned} p &= D^{-1}(q_1, q_2) \\ &= 30 - (q_1 + q_2) \end{aligned}$$

to form the Firm 1 revenue function

$$R_1(q_1, q_2) = D^{-1}(q_1, q_2)\, q_1$$

and use the cost function $C_1(q_1) = 6\, q_1$ to form the profit function $\pi_1(q_1, q_2) = R_1(q_1, q_2) - C_1(q_1)$.

Solution The profit function is

$$\begin{aligned} \pi_1(q_1, q_2) &= R_1(q_1, q_2) - C_1(q_1) \\ &= D^{-1}(q_1, q_2)\, q_1 - 6\, q_1 \\ &= (30 - q_1 - q_2)\, q_1 - 6\, q_1 \\ &= (24 - q_1 - q_2)\, q_1. \end{aligned}$$

2. (1 point) Calculate the profit for Firm 1 for outputs between 3 and 5, assuming that Firm 2 produces 3 units. Compare your answers to the numbers from the table. (The payoff table is posted to the web site right below the problem set.) Why do the profits of firm 1 increase when their output increases and then eventually decline?

Solution With $q_2 = 3$, the profit function (from problem 1) is

$$\begin{aligned} \pi_1(q_1, 3) &= (24 - q_1 - 3)\, q_1 \\ &= (21 - q_1)\, q_1. \end{aligned}$$

The profits for q_1 between $q_1 = 3$ and $q_1 = 5$ are $\pi_1(3, 3) = 18 \cdot 3 = 54$, $\pi_1(4, 3) = 17 \cdot 4 = 68$, and $\pi_1(5, 3) = 16 \cdot 5 = 80$.

When the output level is low, adding one more unit of output constitutes a large percentage increase in output, but only a small percentage decrease to the profit margin. Since profit is the product of the output quantity q_1 times the profit margin $(24 - q_1 - q_2)$, the net result is an increase in profit when the output is low.

When the output level is high, this result is turned around. Adding one more unit of output constitutes a small percentage increase in output, but a large percentage decrease in profit margin. Since profit is the product of the output quantity q_1 times the profit margin $(24 - q_1 - q_2)$, the net result is a decrease in profit when the output is high.

Two examples should clarify this. When q_1 increase from $q_1 = 3$ to $q_1 = 4$, the quantity increases by over 33%, but the profit margin falls from $24 - q_1 - q_2 = 24 - 3 - 3 = 18$ to $24 - 4 - 3 = 17$, which is a decline of less than 6%. The result is an increase in profit of over 25%. When q_1 increase from $q_1 = 14$ to $q_1 = 15$, the quantity increases by just over 7%, but the profit margin falls from $24 - q_1 - q_2 = 24 - 14 - 3 = 7$ to $24 - 15 - 3 = 6$, which is a decline of over 14%. The result is a decrease in profit of over 8%.

3. (1 point) Calculate the profit for Firm 1 for output 3, assuming that Firm 2 produces 4 units and then assuming that Firm 2 produces 5 units. Compare your answers to the numbers from the table. Why do the profits of firm 1 decline when the output of firm 2 increases?

Solution With $q_1 = 3$, the profit function (from problem 1) is

$$\begin{aligned} \pi_1(3, q_2) &= (24 - 3 - q_2)\, q_1 \\ &= (21 - q_2)\, q_1. \end{aligned}$$

The profits for q_2 between $q_2 = 3$ and $q_2 = 5$ are $\pi_1(3,3) = 18 \cdot 3 = 54$, $\pi_1(3,4) = 17 \cdot 3 = 51$, and $\pi_1(3,5) = 16 \cdot 3 = 48$.

For a fixed value of q_1, the profit of firm 1 declines as the output of firm 2 increases, because the profit margin $(24 - q_1 - q_2)$ declines each time that the quantity of firm 2 increases.

4. (3 points) Using the payoff table, trace out the best-response dynamic starting from the output levels $q_1 = 15$ and $q_2 = 3$.

Solution Output levels in period 1 are $(q_1^1, q_2^1) = (15, 3)$. If firm 1 expects firm 2 to produce the same amount in period 2 that firm 2 produced in period 1, then firm 1 will choose output $q_1^2 = 11$, because that would result in the payoff $\pi_1 = 110$, and all of the other payoffs in column 1 (with $q_2 = 3$) are below that.

If firm 2 expects firm 1 to produce the same amount in period 2 that firm 1 produced in period 1, then firm 2 will choose output $q_2^2 = 5$, because that would result in the payoff $\pi_2 = 20$, and all of the other payoffs in row 13 (with $q_1 = 15$) are below that.

So the period 2 outputs are $(q_1^2, q_2^2) = (11, 5)$.

By the same type of argument, the outputs in the next 3 periods are $(q_1^3, q_2^3) = (11, 5)$, $(q_1^4, q_2^4) = (8, 7)$, and $(q_1^5, q_2^5) = (8, 8)$.

The response to $(q_1^5, q_2^5) = (8, 8)$ is $(q_1^6, q_2^6) = (8, 8)$, so from that time on, play will be at $(q_1, q_2) = (8, 8)$ in all successive periods.

Problems 5 through 8 are based on the lecture notes on monopoly and duopoly.

5. (3 point) For the monopoly market with

$$\begin{aligned} p &= D^{-1}(q) \\ &= 45 - q \end{aligned}$$

and $MC(q) = 3+q$ we found the monopoly price and quantity in the lecture notes. Use this information and the definitions of consumers' and producer's surplus for monopoly to find their surplus.

Solution The monopoly output is $q^* = 31$ and the monopoly price is $q^* = 14$. The consumers' surplus is

$$\begin{aligned} CS &= \frac{1}{2} q^* \left(D^{-1}(0) - p^*\right) \\ &= \frac{1}{2} 14 \,(45 - 31) \\ &= 98. \end{aligned}$$

The producer's surplus is

$$\begin{aligned} PS &= q^* \left(p^* - MC(q^*)\right) + \frac{1}{2} q^* \left(MC(q^*) - MC(0)\right) \\ &= 14\,(31 - 17) + \frac{1}{2} 14\,(17 - 3) \\ &= 196 + 98 \\ &= 294. \end{aligned}$$

Note: If you use the marginal revenue formula with $\delta = 1$ then the marginal revenue function is $MR(q) = 46 - 2\,q$. The monopoly quantity is the same when it is an integer, and the monopoly price is also the same. There is a slight difference in the calculation for the producer's surplus though depending on whether you use the formula $PS = q^*\,(p^* - MC(q^*)) + \frac{1}{2}\,q^*\,(MC(q^*) - MC(0))$ or the formula $PS = q^*\,(p^* - MR(q^*)) + \frac{1}{2}\,q^*\,(MR(q^*) - MC(0))$. If you use the first of these two formulas the result is the same as the one above. If you use the second formula the result is $PS = 287$. A close look at the graph on page 63 of the new lecture notes should convince you that the first formula is correct, and the second isn't.

6. (3 point) Suppose that a competitive market has the demand function

$$\begin{aligned} q &= D(p) \\ &= 45 - p \end{aligned}$$

and the market supply $q = S(p) = p - 3$. Graph the demand and supply and compare it to the inverse demand and the marginal cost from the monopoly problem in the lecture notes. How do the surplus measures for consumers and for producers in the competitive market compare to those measures that you found in problem 5 for consumers and the producer in the monopoly market?

Solution When supply equals demand, $p - 3 = 45 - p$, so $p^* = 24$. The market quantity traded is $q^* = 21$. Consumers' surplus is

$$\begin{aligned} CS &= \frac{1}{2} q^* \left(D^{-1}(0) - p^*\right) \\ &= \frac{1}{2} 21 \,(45 - 24) \\ &= 220.5. \end{aligned}$$

Producers' surplus is

$$\begin{aligned} PS &= \frac{1}{2} q^* \left(p^* - MC(0) - p^*\right) \\ &= \frac{1}{2} 21 \,(24 - 3) \\ &= 220.5. \end{aligned}$$

In monopoly the surplus split was highly unequal. The monopolist's surplus was three times the consumers' surplus. In the competitive market with the same marginal costs and marginal values, the consumers' surplus and the producers' surplus are equal.

7. (3 point) For the monopoly market with

$$\begin{aligned} p &= D^{-1}(q) \\ &= 45 - q \end{aligned}$$

and $MC(q) = 3 + q$ we found the monopoly price and quantity in the lecture notes. Suppose instead that the market inverse demand was

$$\begin{aligned} p &= D^{-1}(q) \\ &= 61 - q \end{aligned}$$

and the market monopolist's marginal cost was $MC(q) = 19 + q$. What would the monopoly quantity and price be in this market?

Solution To find the monopoly output, we need to equate marginal revenue to marginal cost. The revenue function for the monopolist is

$$\begin{aligned} R(q) &= p\,q \\ &= D^{-1}(q)\,q \\ &= (61-q)\,q. \end{aligned}$$

Marginal revenue for the monopolist is

$$\begin{aligned} MR(q) &= \frac{1}{\delta}\,(R(q)-R(q-\delta)) \\ &= \frac{1}{\delta}\,((61-q)\,q-(61-q+\delta)\,(q-\delta)) \\ &= \frac{1}{\delta}\,((61-q)\,q-(61-q+\delta)\,q+(61-q+\delta)\,\delta)) \\ &= \frac{1}{\delta}\,((61-q)\,q-(61-q)\,q-\delta\,q+(61-q+\delta)\,\delta)) \\ &= \frac{1}{\delta}\,(-\delta\,q+(61-q+\delta)\,\delta)) \\ &= -q+61-q+\delta \\ &\doteq 61-2\,q. \end{aligned}$$

The equilibrium output for the monopolist is determined from the condition that $MC(q) = MR(q)$. Marginal cost for the monopolist is $MC(q) = 19 + q$ so the equilibrium condition is $61 - 2\,q = 19 + q$. This can be written as $42 = 3\,q$ so $q^* = 14$. The equilibrium price can be determined from the inverse demand evaluated at q^*: $D^{-1}(q^*) = D^{-1}(14) = 61 - 14 = 47$.

The inverse demand in this market is shifted up by 16 units on the price axis in this example compared to the example in the notes. In the notes the inverse demand was $p = D^{-1}(q) = 45 - q$ so that when the quantity is $q = 0$, the price is $p = 45$. So $p = 45$ is approximately equal to the maximum of the buyers' values. In this problem the inverse demand is $p = D^{-1}(q) = 61 - q$ so that when the quantity is $q = 0$, the price is $p = 61$. So $p = 61$ is approximately equal to the maximum of the buyers' values. All values have been shifted up by 16.

The marginal cost in this market is also shifted up by 16 units on the price axis in this example compared to the example in the notes. In the notes the marginal cost was $p = MC(q) = 3 + q$ so that when the quantity is $q = 0$, the price is $p = 3$. So $p = 3$ is approximately equal to the minimum of the sellers' costs. In this problem the marginal cost is $p = MC(q) = 19 + q$ so that when the quantity is $q = 0$, the price is $p = 19$. So $p = 19$ is approximately equal to the minimum of the sellers' costs. All costs have been shifted up by 16.

The equilibrium quantity doesn't change, but the equilibrium price shifts up by an amount equal to the shift to the values and costs.

8. (3 point) For the duopoly market with

$$\begin{aligned} p &= D^{-1}(q) \\ &= 45 - q \end{aligned}$$

and $MC_i(q_i) = 3 + 2\,q_i$ we found the monopoly price and quantity in the lecture notes. Suppose instead that the market inverse demand was

$$\begin{aligned} p &= D^{-1}(q) \\ &= 61 - q \end{aligned}$$

and the marginal cost of each firm is $MC_i(q_i) = 19 + 2 * q_i$. What would the duopoly quantity and price be in this market?

Solution If you repeat all of the calculations in Section 7.1 of the new notes with $D^{-1}(q) = 61 - q$, you'll find that firm 1 revenue is

$$\begin{aligned} R_1(q_1, q_2) &= p\,q_1 \\ &= D^{-1}(Q)\,q_1 \\ &= (61 - (q_1 + q_2))\,q_1 \end{aligned}$$

and firm marginal revenue is For Firm 1 the marginal revenue is

$$\begin{aligned} MR_1(q_1) &= \frac{1}{\delta}(R(q_1, q_2) - R(q_1 - \delta, q_2)) \\ &= \frac{1}{\delta}(61\,q_1 - q_1^2 - q_1\,q_2 - (61\,(q_1 - \delta) - (q_1 - \delta)^2 - (q_1 - \delta)\,q_2)) \\ &= \frac{1}{\delta}(61\,\delta - q_1^2 - q_1\,q_2 + (q_1 - \delta)^2 + (q_1 - \delta)\,q_2) \\ &= \frac{1}{\delta}(61\,\delta - q_1^2 + q_1^2 - 2\,q_1\,\delta + \delta^2 - \delta\,q_2) \\ &= \frac{1}{\delta}(61\,\delta - 2\,q_1\,\delta - \delta\,q_2 + \delta^2) \\ &= 61 - 2\,q_1 - q_2 + \delta \\ &\doteq 61 - 2\,q_1 - q_2. \end{aligned}$$

The profit maximization condition for firm 1 is that its marginal revenue equals its marginal cost, so or

$$61 - 2\,q_1 - q_2 = 19 + 2\,q_1.$$

This equation can be simplified to get the condition $4\,q_1 = 42 - q_2$ or

$$q_1 = BR_1(q_2) = 10.5 - q_2/4. \tag{1}$$

This is exactly the same response function that the firm had before the shift to the inverse demand and the shift to the marginal cost. Firm 2 also has the same response function. When they are solved simultaneously, the result is still $q_1^* = 8.4$ and $q_2^* = 8.4$. There is no change to the firm outputs or to the market output. Market output remains $Q^* = q_1^* + q_2^* = 16.8$. The market price is $p = 61 - (q_1^* + q_2^*) = 44.2$. The market price increases by an amount equal to the shifts to the inverse demand and to the marginal costs. (The same result held in problem 7 for monopoly.)