Microeconomic Principles
with
Econport Experiments

Steven Gjerstad

Economics Department

Purdue University

June 12, 2006

Contents

Preface

These lecture notes examine the economic problem of how resources are allocated in a market economy. Traditionally, a course on microeconomic principles examines the competitive market model which has many buyers and sellers, the monopoly model with only a single seller, and the oligopoly model with a small number of sellers. These models are then applied to a number of problems, such as the impact of taxes on market efficiency, the responsiveness of demand to a price change, and the way that supply in a competitive market adjusts when demand increases. These notes cover all of these standard topics: the usual models and applications are emphasized. But in the standard microeconomics course, all of the analysis is based on the idea that the market price and quantity of exchange are the result of equating supply and demand. These lecture notes emphasize this equilibrium analysis, but in addition, we compare the predictions of the market models to experiments where you will act as a seller or a buyer in a market, and trade with your classmates.

In an experiment, some students act as buyers and others act as sellers. Trade is computerized, and moves quickly. Sellers make offers to sell and buyers make bids to purchase. In lectures, the results of these experiments are compared to the predictions of the models. There are several benefits to these experiments. Students enjoy them. The pace of the experiments is fast, and you can feel the pull of market forces. Experiments turn your perspective on the market models inside out. Rather than looking from the outside at a model as a graph or as demand and supply equations, in an experiment, you become one of the actors in the model. The trades in the experiment result from the interaction between you and dozens of your classmates. After the experiment, the results are compared to the predictions of one of the market models. Typically, the result of a market experiment conforms closely to the predictions of the relevant model, so the experiment should increase your confidence in the model. Experiments should help you learn in other ways. The experiments illustrate many concepts that are covered in lectures, so they should help you to recognize key concepts. Students find the experiments interesting. Finally, the experiments go one step beyond the basic market models. The models predict what the price will be in a competitive market or in a monopolized market, but they don't describe how the price comes about. In the experiment, you can observe the trading process, and watch as prices fluctuate at first and eventually settle down near the predicted price.

Experiments for this course are conducted from the Econport web site (http://www.econport.org). The market experiments that we use are run in an applications called MarketLink and Normal-Form

Game (NFG), which are two of several economics experiment software applications on Econport. Instructions for logging on to Econport experiments involve only a few simple steps, which will be projected on the overhead during the experiments. MarketLink has data review features that will allow us to compare market outcomes to the predictions of the microeconomic models.

Many microeconomic principles courses substitute graphs for models of supply and demand, and also for the models of production and utility that underlie models of supply and demand. The rationale for this seems to be that careful development of microeconomic models requires calculus. These notes strike a compromise between the casual development that has become common in microeconomic principles courses and a more formal development using calculus. Most of the key ideas in the course can be developed using college algebra. These notes assume a familiarity with math at the level of college algebra. One common problem that we encounter is solving two simultaneous equations in two unknowns, where the equations are supply and demand and the unknowns are price and quantity. The other common problem is finding the maximum or minimum of a quadratic equation using the technique of "completing the square." These techniques are fairly straightforward and are used frequently throughout the lecture notes. Both techniques are reviewed in an appendix.

Use of explicit market models helps with the comparison between experiment results and market models. Development of specific models also helps prepare students for subsequent economics courses. My view is that the benefits of a careful analysis of microeconomic models are worthwhile. The economic content of the course is substantial, while the mathematical level is relatively modest.

1 Introduction

A significant portion of a typical microeconomic principles course is devoted to descriptions of market performance. We study four key market models in this course. These models are: (1) the short-run competitive market model with many sellers who have a fixed capital investment and fixed supply; (2) the long-run competitive market model with many sellers, each with adjustable capital investment level; (3) the monopoly model of a market with a single seller; and (4) the duopoly market model with two sellers. All four of these models rely on the ideas of a demand function for buyers and either a supply function for sellers (in the two competitive market models) or a marginal cost function for sellers. Another significant portion of the course is devoted to a description of how demand can be obtained from buyers' preferences and how supply is obtained from the production capabilities of firms.

Section 2 treats the unit costs of sellers and the unit values of buyers as basic concepts. A seller's schedule of unit costs determines a relationship between quantity of sales and the costs of the units sold. Since a seller should make units available as long as the unit cost is below the price, the schedule of unit costs determines the amount the seller should supply at each price. This is the well-known supply function. Market supply is determined by summing the supplies of individual firms. A similar relationship between the unit values of a buyer and the buyer's demand function is developed in that section as well. Market demand is determined by summing up the demand function of all buyers. These two constructions create an explicit connection between the values and costs used in market experiments and the supply and demand commonly used in economic models.

The competitive market model is described using both the unit value and unit cost formulation typical of market experiments, and the supply and demand functions from the typical microeconomic principles course. Typical market models in microeconomic principles courses use linear supply and demand. This approach simplifies analysis considerably, because the intersection of supply and demand is then determined by solving two linear equations (supply and demand) in two unknowns (price and quantity). The connection between market experiments and models of linear supply and demand is drawn explicitly by showing the schedules of values (with equal step sizes) can be approximated with a linear demand function, and schedules of costs (again, with equal step sizes) can be approximated with a linear supply function. With these linear approximations, we can then determine properties of the models that bear on the experiment predictions, and the experiment results can be used to test the market models, so that the two approaches reinforce one another.

After a thorough discussion of the competitive market model in the short-run with fixed capital investments by firms, we'll examine the long-run model. The long-run equilibrium model is based on the idea that supply in a market will adjust when shifts to demand occur that lead to unusually high profits for firms. At that point, we'll want a model that provides some insight into where demand comes from, so that we'll know possible causes of the shift to market demand. We'll also want to develop a model of where market supply comes from, so that we can trace out the responses by firms to a shift in their profits.

A consumer's demand function is derived from the consumer's utility function. A utility function orders the desirability of different possible consumption alternatives that a consumer has available. A utility function represents preferences, which are a natural and intuitive notion. Preferences simply indicate that, if presented with any two consumption alternatives, a person can decide which she likes better. A utility function summarizes all of the information in preferences by assigning a number (or utility level) to each consumption alternative. The economic model of choice states that, among the affordable alternatives, the person will choose the alternative with the highest utility.

Supply is derived through a sequence of steps that begins with a production function for the firm. The production function relates quantities of inputs to quantities of output. From the prices of inputs and the production function, we determine the cost function for the firm. The supply of a firm in a competitive market is easily determined from its cost function. A competitive firm that sees the market price p will want to sell any unit that has a unit cost (or marginal cost) at or below p. So the firm's supply is determined from the condition that price is equal to marginal cost. This model of the determination of firm supply is developed in Section 5.6. The relationships between these model elements are depicted in figure 1.

The two competitive market models – short-run and long-run – are two variants of one key market structure model: perfect competition. In the other two types of market structure – monopoly and oligopoly – the firms also use their marginal cost information to determine the output level that will maximize their profit, but the conditions are slightly different. The specific decision rules for these two market structures are described in Sections 6 and 7.

These notes provide a detailed description of the logical structure of the microeconomic model of supply and demand. That's only possible with a mathematical development of the model elements. The mathematical requirements though are basic. Microeconomics texts typically either forego math-

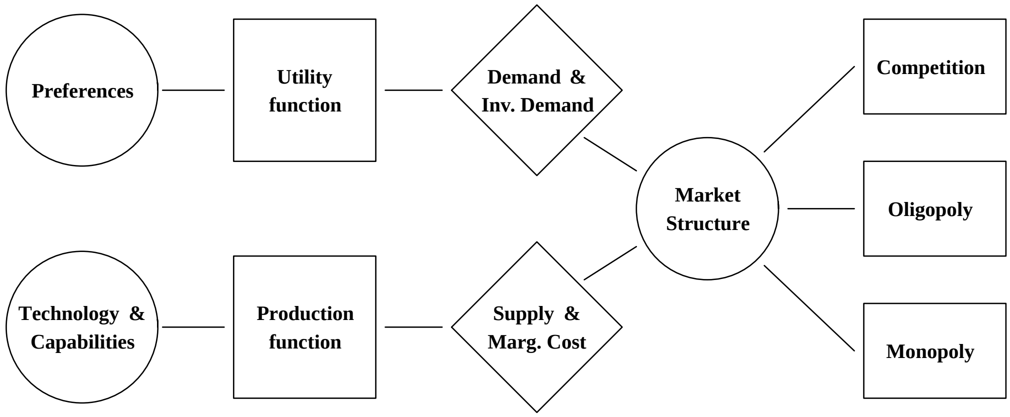


Figure 1: Schematic representation of microeconomic models of markets

ematical models and rely on a graphical descriptions of concepts, or else they use calculus to determine the utility maximization conditions for consumers and the profit maximization conditions for firms. These notes take an approach that lies between these two common approaches: key concepts are developed formally, but with only a few ideas from college algebra. This facilitates emphasis of the economic concepts unobscured by excessive reliance on mathematical techniques.

2 Values, costs, and competitive equilibrium in market experiments

One of the fundamental concepts of microeconomics is that each buyer has a marginal value (or unit value) for each unit consumed. In this section, we'll consider examples of these values and interpret them loosely as the result of a consumer's preferences. (In Section 4 we'll develop the connection between preferences and demand more systematically.) We'll then use the values to determine the demand function for an individual consumer. Market demand functions result from adding together the demand functions of all of the consumers in a market.

Marginal costs (or unit costs) are the parallel concept for sellers. In this section, we treat marginal costs as a fundamental concept, but in Section 5.3 the total cost function and marginal costs are derived from a firm's production function.

2.1 Marginal values and demand

This section begins with the idea that buyers have values for units of the commodities that they consume. Values typically decline as more of a good is consumed. The reason that values decline is intuitive: think about the most money that you would pay for the first four ounces of food that you eat in a day, the most you'd pay for the next four ounces, and so on until you reach about 24 ounces of food. The first four have great value, the next four are still very valuable, but at some point, the value will be nil.

A schedule of unit values for a consumer describes how the maximum amount that a person would pay for a unit of a commodity varies as the quantity of the commodity consumed increases. This relationship can be inverted to look at how much of the commodity the person wants at each price. That relationship is called the buyer's demand function. After the schedule of marginal values for a buyer is described, the relationship between values and demand is described.

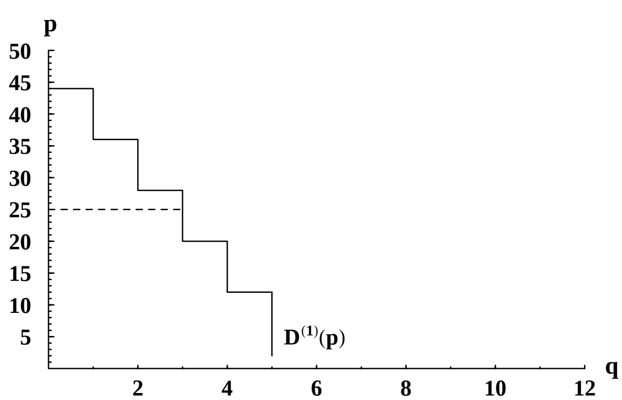
The section on marginal values and demand concludes by showing that the demand functions of buyers can be summed up to obtain a market demand function.

2.1.1 Marginal values and individual demand

The first two columns of the table in figure 2 show the values that a consumer has for each of five units of consumption of a commodity. The marginal value is the most that a buyer would be willing to pay for a unit of that commodity. A consumer who wants several units of a commodity would naturally value the first unit the most, so the list of values is decreasing.

Think of these values as something like concert tickets purchased per year. This consumer would pay at most \$44 for a ticket, and if tickets cost that much, he would buy only one. He values the second ticket less, at \$36. There are at least two good reasons that he would value the second unit less than the first. One reason is that he would buy his first ticket to see his favorite band. The second reason that unit values decline is that other activities become more appealing once you've seen a lot of concerts. Because the factors that determine values are so subjective, specific unit values can vary from person to person, depending on their preferences.

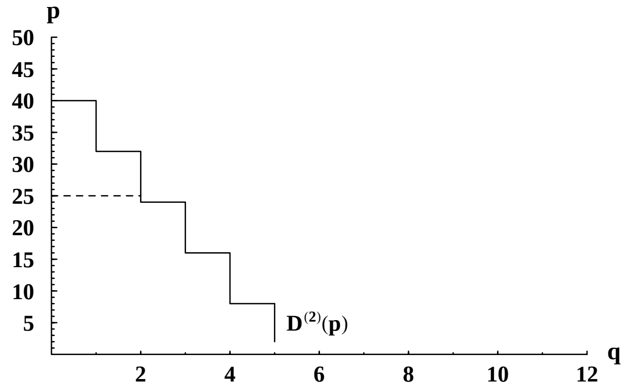
When we look at a list of unit values, we're describing the value that a consumer has for a commodity as the quantity of the commodity purchased varies. We can also describe the number of units that the consumer wants to purchase as the price varies. That's called the consumer's *demand*. The list of unit values for a buyer and the buyer's demand are really just two ways of looking at the same information. From the dashed line in the graph from figure 2 we can see that when the price is $p = \$25$, the buyer will want to purchase three units of the commodity. The demand for any other price can be determined in the same way by tracing a line to the right from the price. So we can interpret the step function in figure 2 as the consumer's demand function $D^{(1)}(p)$. (Later we will discuss market demand. The superscript "(1)" in $D^{(1)}(p)$ indicates that this is the demand function for person 1.)



Unit #	Unit Value	Price	Demand
		45 – ∞	0
1	44	37 – 44	1
2	36	29 – 36	2
3	28	21 – 28	3
4	20	13 – 20	4
5	12	0 – 12	5

Figure 2: A buyer's marginal values and demand

Figure 3 shows the unit values and the demand for another buyer. This buyer is similar to the first, but her values are all lower than the values of the first buyer. For example, when the price is $p = 25$, we saw that buyer 1 wanted to purchase three units. Buyer 2 only wants to purchase two units when the price is $p = 25$.



Unit #	Unit Value	Price	Demand
		41 – ∞	0
1	40	33 – 40	1
2	32	25 – 32	2
3	24	17 – 24	3
4	16	9 – 16	4
5	8	0 – 8	5

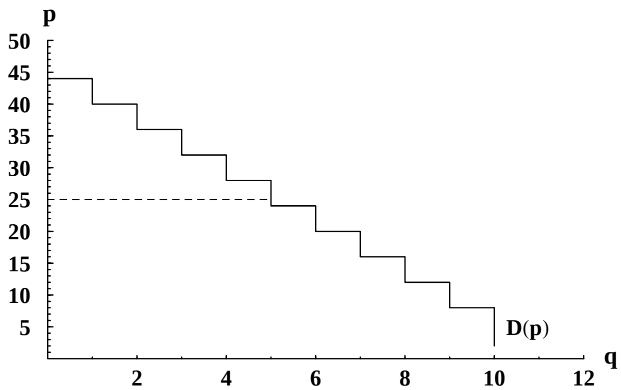
Figure 3: Marginal values and demand for another buyer

A comment on graph conventions

Perhaps the most peculiar feature of graphs in economics is that they often have the independent variable (price) on the vertical axis. In math and in sciences, the independent variable is almost always on the horizontal axis. There is a simple reason that economists sometimes have the independent variable on the vertical axis. We often move between marginal values and demand, or as we'll see in a later section, we also move between marginal costs and supply. Rather than switching the price and quantity axes back and forth, we switch the independent variable from the horizontal to the vertical axis. In this course, graphs that include price and quantity have quantity on the horizontal axis and price on the vertical axis.

2.1.2 Marginal values and market demand

Our first simple example of constructing a market demand from individual demand includes only two buyers. This example has been used because the main ideas of constructing market demand from individual demand are most easily described with only two buyers. The second column of the table in figure 4 combines all of the values of buyer 1 and the values of buyer 2 from the tables in figures 2 and 3. The values are sorted from highest to lowest. These values are also shown in the graph in figure 4. Each of the value steps from the tables in figures 2 and 3 appear in figure 4. In the graph, we can read off the market demand in the same way that we did with individual demand. The market demand is determined by adding together the demand of buyer 1 with the demand of buyer 2. For example, when the price is $p = 25$ we found that buyer 1 wants three units and buyer 2 wants two units. We can see in the graph in figure 4 that the market demand is five units when the price is $p = 25$. By doing this for a variety of prices, you can verify that the graph of demand is the same as the graph of the unit values.



Unit #	Unit Value	Price	Demand
		$45 - \infty$	0
1	44	41 – 44	1
2	40	37 – 40	2
3	36	33 – 36	3
4	32	29 – 32	4
5	28	25 – 28	5
6	24	21 – 24	6
7	20	17 – 20	7
8	16	13 – 16	8
9	12	9 – 12	9
10	8	0 – 8	10

Figure 4: Combined marginal values and demands of two buyers

2.2 Marginal cost and supply

This section begins with the idea that sellers have costs for units of the commodities that they produce. Costs of individual units will typically vary for a variety of reasons. The employees of a firm who produce its output may have some variation in their skills. If the firm's output were temporarily reduced, the firm's owner could use the most productive employee to reduce the per unit cost. At the other extreme, if the firm's output is temporarily very high, many employees, including some of the less productive ones, would have to be paid overtime. This is a very simplified story, but it illustrates the idea that factor productivity is higher at low outputs. (Later, in Section 5.3, we'll see how increasing marginal costs arise by looking at the production function and cost function of a firm.)

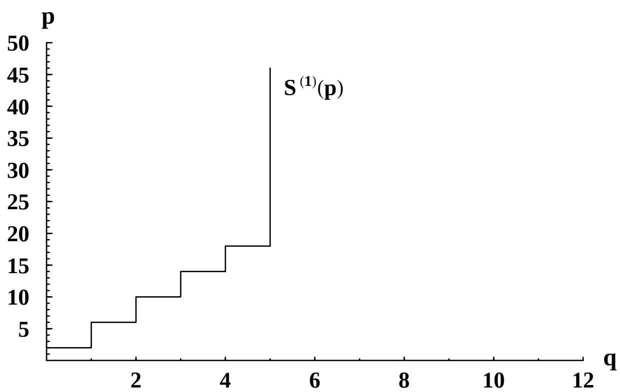
Unit costs describe how production costs vary with the quantity of the commodity produced. This relationship can be inverted to look at how much of the commodity the firm produces at each price. That relationship is called the firm's supply function. After the schedule of marginal costs for a firm is described, the relationship between marginal costs and supply is described.

As with market demand, market supply is obtained by summing up the individual supply functions of the firms.

2.2.1 Marginal costs and individual supply

A seller incurs additional cost when he produces a unit, which we call the marginal cost for the unit. We'll begin by assuming that the marginal costs for a firm are given as a list of numbers. (Later, in Section 5.3, we'll derive a cost function for a firm from its production function and determine marginal costs and the increment to the total cost as quantity increases.) The schedule of marginal costs is used to determine the supply of the firm.

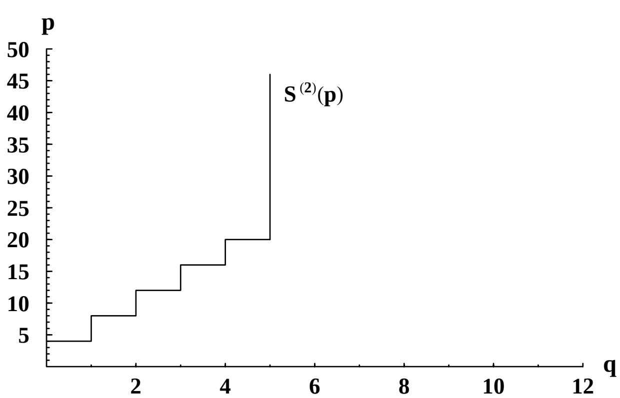
As with unit values, unit costs can be graphed, and the graph of the unit costs is the same as the graph of the seller's supply function. The graph in figure 5 shows the unit costs and the supply function for the seller.



Unit #	Unit Cost	Price	Supply
		0 – 1	0
1	2	2 – 5	1
2	6	6 – 9	2
3	10	10 – 13	3
4	14	14 – 17	4
5	18	18 – ∞	5

Figure 5: A seller's marginal costs and supply

Suppose that a second seller has the schedule of marginal costs and the supply shown in figure 6. The marginal costs for this seller are all higher that those for the first seller.

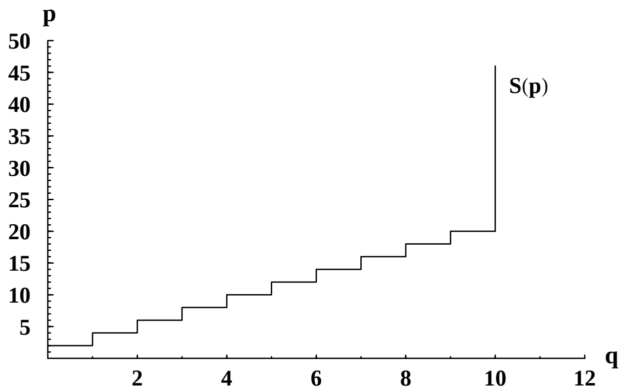


Unit #	Unit Cost	Price	Supply
		0 – 3	0
1	4	4 – 7	1
2	8	8 – 11	2
3	12	12 – 15	3
4	16	16 – 19	4
5	20	20 – ∞	5

Figure 6: Marginal costs and supply for another seller

2.2.2 Marginal costs and market supply

As with the buyers' market demand, we want to determine the market supply function from the marginal cost functions and the supply functions of individual sellers. The unit costs in the tables from figures 5 and 6 are combined in the table in figure 7 by sorting the unit costs from lowest to highest.



Unit #	Unit Cost	Price	Supply
		0 – 1	0
1	2	2 – 3	1
2	4	4 – 5	2
3	6	6 – 7	3
4	8	8 – 9	4
5	10	10 – 11	5
6	12	12 – 13	6
7	14	14 – 15	7
8	16	16 – 17	8
9	18	18 – 19	9
10	20	20 – ∞	10

Figure 7: Combined marginal costs and supplies of two sellers

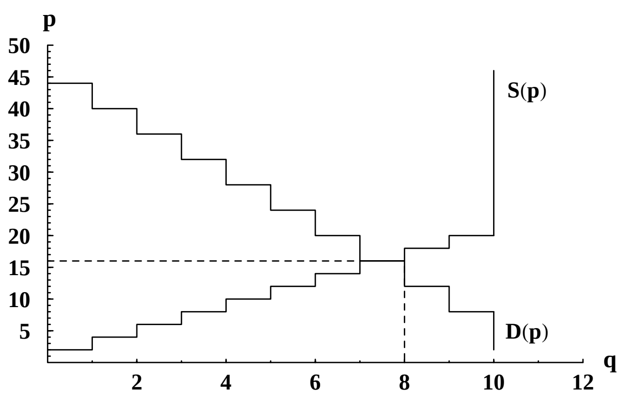
2.3 The competitive market model

In a competitive market, the predicted price and quantity of trade are determined by finding the price that equates supply and demand. At a competitive equilibrium price p^*, supply and demand are equal. For the example of market demand in figure 4 and the example of market supply in figure 7, if we combine the demand function and supply function in the same graph, it is easy to find the price that clears the market.

Figure 8 combines the buyers' unit values and the sellers' unit costs. For the first unit, the difference between the unit value and the unit cost is $\$44 - \$2 = \$42$, for the second it is $\$40 - \$4 = \$36$, and so forth. At the eighth unit the unit value equals the unit cost: both are \$16. Beyond the eighth unit, surplus is negative for each unit, so these units are not produced or traded at the competitive equilibrium. So for a price of $p^* = \$16$ buyers want to purchase eight units and producers want to sell eight units. That means that $p^* = 16$ is the equilibrium price.

Consumers' surplus (denoted CS in the table in figure 8) is the difference between the value and the price for every unit purchased. For this example, the total consumers' surplus is 112. The producers' surplus is the difference between the price and the unit cost. The total producers' surplus is 56.

The values and costs in the table in figure 8 are shown in the supply and demand graph in figure 8. The graph shows the equilibrium price as a horizontal line at $p^* = 16$ and it shows the equilibrium trade quantity as a vertical line at 8. The consumers' surplus is the area above the equilibrium price and below the market demand. The producers' surplus is the area below the equilibrium price and above the market supply.



Unit #	Unit Value	Unit Cost	CS	PS
1	44	2	28	14
2	40	4	24	12
3	36	6	20	10
4	32	8	16	8
5	28	10	12	6
6	24	12	8	4
7	20	14	4	2
8	16	16	0	0
9	12	18	0	0
10	8	20	0	0

Figure 8: Supply, demand, and competitive equilibrium with four traders

The competitive market experiment from the first day of class had many buyers and many sellers. On the second day, the experiment was similar, but the class was divided into many groups, each with four buyers and four sellers. This accomplishes two goals. Typically, prices in the large group experiment converge to the competitive equilibrium price quickly and sharply. In the smaller groups as in the large group, prices are similar to the competitive equilibrium price, but there is more variability. When a market has a lot of trade, that is often called 'liquidity.' The two experiments combine to demonstrate the important effect that liquidity has on convergence. The second goal of the small experiment sessions is that analysis of the experiment data is simpler with the small session. The first homework set involves a sequence of simple steps that lead you through some basic analysis of competitive market experiment data for one of the small group sessions.

2.4 Induced values and induced costs in market experiments

In market experiments that are conducted for research, unit values are induced by the experimenter. The amount that the experimenter pays to a buyer is proportional to the difference between the unit value and the price that the buyer pays for the unit. With that technique, a buyer should be willing to pay any amount up to the unit value (although the buyer should also be motivated to pay as low a price as possible). Inducement of costs is similar: the payment to a seller is proportional to the difference between the price paid for the unit by the buyer and the cost of the induced unit cost. The seller then makes a profit as long as he sells a unit for a price that exceeds its induced cost. The seller though should try to obtain as high a price as possible in order to increase his profit.

2.5 Values, costs, and equilibrium with many sellers and buyers

The competitive model relies on the assumption that there are many sellers and many buyers. In order to describe how the values of two buyers are combined to get a schedule of values for the market, and to describe how individual demands are combined to get the market demand, we've worked until now with two buyers. For similar reasons, our example also had two sellers. If we add two new buyers that have the same values as buyers 1 and 2 (with the values shown in figures 2 and 3) and two new sellers with the same costs as sellers 1 and 2 (with the costs shown in figures 5 and 6), then the equilibrium price will be unaffected and the quantity traded will double. The reason for this is straightforward.

We can see from figure 8 that at the market price $p = 16$ the first pair of buyers wants to purchase $q^d = 8$ units and the first pair of sellers supplies $q^s = 8$ units. So the market is in equilibrium. So the equilibrium price is $p^* = 16$ and the equilibrium quantity traded is $q^* = 8$ units. When two new buyers and two new sellers are added that are identical to the first pair or buyers and the first pair of sellers, then at $p = 16$ the four buyers will want to purchase $q^d = 16$ units and the four sellers supply $q^s = 16$ units. Once again the the market equilibrium price is $p^* = 16$ but the equilibrium quantity traded doubles to $q^* = 16$. The graph on the left side of figure 9 shows the supply, demand, and the equilibrium price and quantity for this market. The graph is similar in many respects to the graph in figure 8, but there are two unit values at 44, two at 40, and so on instead of a single value at 44, a single one at 40, and so on. The market is twice as large, but the equilibrium price is unaffected.

The consumers' surplus is also doubled when the market is replicated in this way, because the second buyer of type 1 has the same surplus as the first buyer of type 1. The same is true for the producers' surplus, for the similar reason. In summary, when the market is doubled in size by adding new buyers and sellers with characteristics that are identical to the existing buyers and sellers, the equilibrium price is unaffected, the equilibrium trade quantity doubles, and the consumers' and producers' surplus doubles.

If instead of adding one new buyer like buyer 1 and one buyer like buyer 2 we add three buyers of

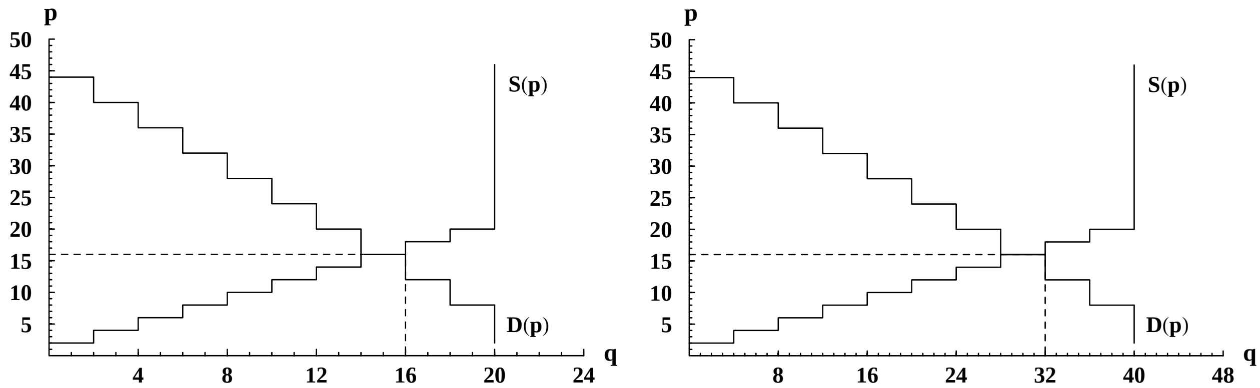


Figure 9: Market with eight traders (left) and sixteen traders (right).

each type, and if we also add three sellers like seller 1 and three sellers like seller 2, then the supply and demand graph will be the one shown on the right side of figure 9. In this case the market price is again unaffected but the quantity is scaled up by a factor of four from the original market. The consumers' surplus and producers' surplus are also quadrupled from the original surplus measures.

In general, the size of the market can be scaled up by adding new buyers and new sellers that are like those already in the market. This is valuable for the classroom demonstrations, because we have determined the equilibrium price and quantity for the market shown in figure 8 with four market participants, but the market predictions are easily determined when the number of students in the experiment is eight, twelve, sixteen, or any multiple of four.

As the number of traders gets large, the price variability tends to decrease and the market outcome looks more like the outcome predicted by the competitive model. Although we began with a market with only two buyers and two sellers, which doesn't conform to the competitive model, that was primarily done to simplify the explanation of how the competitive equilibrium price and quantity are determined, and how the consumers' and producers' surplus are calculated. But we see that we can construct a market that does conform to the assumptions of the competitive model by just replicating the values and costs of buyers in the small market.

2.6 Shifts to values and costs in market experiments

There is one other useful idea for market experiments that is a part of the preliminary background. If we add a constant k to every unit value and add the same constant k to every unit cost, then the equilibrium market price shifts up by k, but the equilibrium trade quantity and the consumers' and producers' surplus is unaffected. This is easy to verify. For example, if $k = 12$ is added to every unit value and every unit cost in the table in figure 8, then the result is the table on the right side of figure 10.

Unit #	Unit Value	Unit Cost	CS	PS
1	44	2	28	14
2	40	4	24	12
3	36	6	20	10
4	32	8	16	8
5	28	10	12	6
6	24	12	8	4
7	20	14	4	2
8	16	16	0	0
9	12	18	0	0
10	8	20	0	0

Unit #	Unit Value	Unit Cost	CS	PS
1	56	14	28	14
2	52	16	24	12
3	48	18	20	10
4	44	20	16	8
5	40	22	12	6
6	36	24	8	4
7	32	26	4	2
8	28	28	0	0
9	24	30	0	0
10	20	32	0	0

Figure 10: Values and costs before (left) and after (right) an increase of 12 to all values and costs.

From the tables, it is easy to see that each unit that was traded before the shift will also trade after the shift. The last unit that trades is still the eighth, but the price is $p^* = 28$ rather than $p^* = 16$. The equilibrium price increased by the amount of the shift. The consumers' surplus on each unit is the same, because the value and the equilibrium price both shift up by an equal amount and consumers' surplus is the difference between value and equilibrium price, summed up over each unit that trades. For a similar reason, producers' surplus also remains the same after the shift to unit costs.

There is one cautionary note that deserves mention at this point. If only the values shift and not the costs, then the equilibrium price, the equilibrium quantity, and the consumers' and producers' surplus will all change. This is also true if the sellers' costs change but the buyers' values remain fixed. We'll study examples of these later. The price remains fixed if the costs and values and shift by the same amount. From an economic point of view, that is something that would be very unlikely, but as a technique for conducting market experiments it turns out to be useful.

3 The competitive market model in the short-run

The short-run competitive market model is based on the idea that sellers have either a fixed schedule of unit costs, or a fixed supply schedule. By contrast, in the long-run competitive market model, sellers can change their capital investment, which affects their unit cost schedule or supply function. This section describes some characteristics of the short-run competitive market model. Later, in Section 5, we'll examine the long-run competitive market model.

Section 2 demonstrates that a list of unit values can be represented as a demand function, by looking at the quantity demanded at each price. This is usually written $q^d = D(p)$. Similarly, the list of sellers' costs can be represented as a supply function, written as $q^s = S(p)$.

Linear supply and demand functions are often used in economic models because they are easy to work with. These linear functions can be interpreted as approximations to the step function supply and demand described in the previous section and utilized in market experiments. The connection between the step functions and linear supply can be described explicitly by constructing a linear supply function that corresponds to the unit costs or supply in figures 7 and 8. The connection between unit values and a linear demand function is described by determining the linear demand function that corresponds to the step function in figures 4 and 8. Of course, not all step functions have equal step sizes, so cost and value schedules from an experiment aren't always approximated by linear supply and demand. Linear approximations to unit values and costs connect the experiments that you participate in to the analysis of linear models of supply and demand. The connection between the experiments and the linear models allows us to move back and forth between discussion of experiment results and model predictions confidently. The next section connects the unit values and costs from market experiments to the linear supply and demand functions that are commonly used to evaluate properties of the competitive market model.

3.1 Approximation of unit values and costs with linear demand and supply

The market demand function in figure 4 has equal step sizes from unit to unit, so we can draw a line that passes through the middle of each step. When we draw a linear approximation to the demand, and also draw a linear approximation to the supply function in figure 7, the competitive equilibrium price that we determine from the intersection of the linear supply and demand equals the competitive equilibrium price from the step functions (which we determined in Section 2.3). Other important economic measures, such as the quantity of trade and the consumers' surplus and producers' surplus, are approximately the same when we use these step functions and when we use linear approximations to the step functions.

3.1.1 Approximation of unit values with linear demand

A simple procedure determines the linear demand that approximates the unit values in figure 4. Linear demand has the form $q = D(p) = a - b\,p$. We can pick two points on the demand in figure 4 and use the points to determine the coefficients a and b for the linear demand that provides the best approximation to the step function market demand that comes from the unit values. The graph on the left side of figure 11 replicates figure 4, and also shows two points (q_0, p_0) and (q_1, p_1) on the demand function. The best approximation is obtained by taking each of these two points at the midpoint of one of the horizontal segments of the demand.

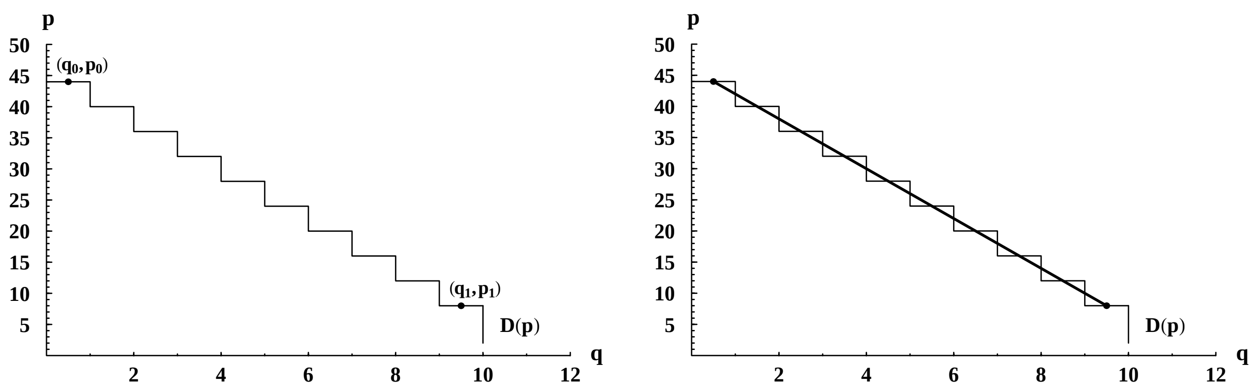


Figure 11: Points used to approximate values (left) and linear approximation (right)

The point $(q_0, p_0) = (0.5, 44)$ lies on our linear approximation $q = D(p) = a - b\,p$. With $(q_0, p_0) = (0.5, 44)$ substituted into the demand equation the result is $0.5 = a - b\,44$. The point $(q_1, p_1) = (9.5, 8)$ also lies on our linear approximation so $q_1 = D(p_1) = a - b\,p_1$, which is equivalent to $9.5 = a - b\,8$. Subtract the first equation from the second:

$$\begin{array}{rl} 9.5 = & a - \;\;8\,b \\ 0.5 = & a - 44\,b \\ \hline 9.0 = & \quad\;\; 36\,b. \end{array}$$

The coefficient b is therefore $b = 1/4$. The coefficient a can be determined from the first equation with $b = 1/4$: $0.5 = a - 44/4$ so $a = 11.5$. The demand equation is therefore $q = D(p) = 11.5 - p/4$. The graph on the right side of figure 11 shows the the portion of the linear approximation to the demand that lies between (q_0, p_0) and (q_1, p_1).

3.1.2 Approximation of unit costs with linear supply

A similar procedure determines the linear supply that approximates the unit costs in figure 7. Two points on the sellers' market supply function are $(q_0, p_0) = (0.5, 2)$ and $(q_1, p_1) = (9.5, 20)$. Using the equation $q = S(p) = c + d\,p$ with these two points results in the equations $0.5 = c + d\,2$ and $9.5 = c + d\,20$. If the first equation is subtracted from the second the result is

$$\begin{array}{rl} 9.5 = & c + 20\,d \\ 0.5 = & c + \ \ 2\,d \\ \hline 9.0 = & \qquad 18\,d \end{array}$$

so $d = 1/2$ and c can be determined by substituting d into the first equation: $0.5 = c + 2/2$. So $c = -0.5$ and the market supply function is $S(p) = -1/2 + p/2 = (p-1)/2$.

In the next section, the competitive market model is evaluated with linear supply and demand. The linear approximation to the supply and demand from the experiments is useful, because the predictions for the experiment can be compared to the predictions obtained with the linear supply and demand functions. Since the calculations are much easier with linear supply and demand and the predictions are almost identical, the exercise of carrying out the approximation connects the experiments to the simpler linear supply and demand equations that we work with frequently throughout the lecture notes.

3.2 The competitive market model with linear supply and demand

To illustrate ideas simply, linear demand and supply functions are convenient, so we'll work with demand functions with the form $D(p) = a - b\,p$ and supply functions of the form $S(p) = c + d\,p$. The left side of figure 12 shows an example of linear supply and demand schedules $S(p) = 2\,p$ and $D(p) = 48 - p$.

Competitive equilibrium example The equilibrium price p^* in a competitive market is determined by the condition that supply and demand are equal. This can be written $D(p^*) = S(p^*)$. For the example supply and demand functions in the figure, this condition is $48 - p^* = 2\,p^*$ which has the solution $p^* = 16$. The quantity traded at the competitive equilibrium price can be determined from

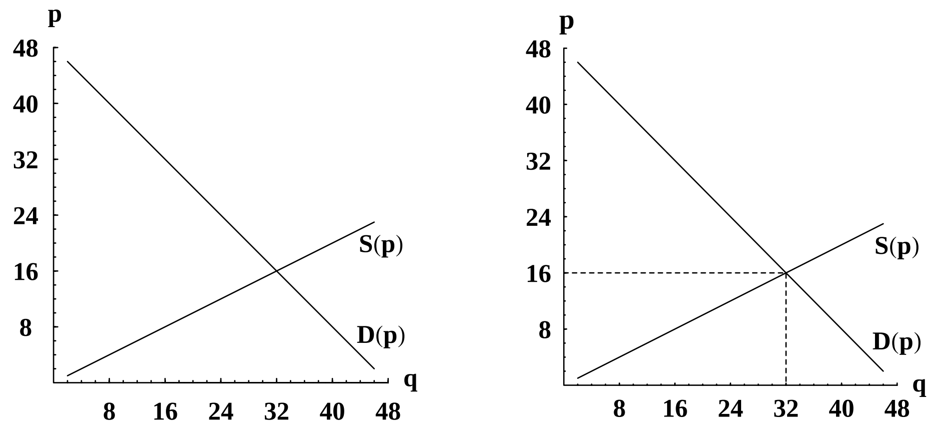


Figure 12: Supply and demand example.

either the demand or the supply since they are equal at the equilibrium price. The equilibrium quantity traded in the example is $q^* = S(p^*) = 2\,p^* = 32$. The dashed lines in the graph on the right side of figure 12 show the equilibrium price and quantity for the example.

Equilibrium price with linear supply and demand For linear demand and supply, we can solve for the equilibrium price and quantity even when a and b aren't specified in the demand equation $D(p) = a - b\,p$, and c and d aren't specified in the supply equation $S(p) = c + d\,p$. This is useful when we examine properties of competitive equilibrium. When $D(p) = a - b\,p$ and $S(p) = c + d\,p$, the equilibrium price is the solution to the equation $c + d\,p^* = a - b\,p^*$. This is equivalent to $(b+d)\,p^* = a - c$, so

$$p^* = (a - c)/(b + d). \tag{1}$$

3.3 Consumers' and producers' surplus

Consumers' surplus is a measure of the value obtained by consumers as a result of trade. In a competitive market, there is a single price for the commodity. In the graph in figure 12, the demand – which also represents consumers' values – is above the competitive equilibrium price up to the point where the supply and demand intersect. This means that those units all have a value to consumers that exceeds the price. The difference between value and price is the consumers' surplus. Graphically, the total consumers' surplus is the area below the demand and above the price. The left side of figure 13, which has the same supply and demand as the competitive market example from figure 12, shows the consumers' surplus (CS) for this example.

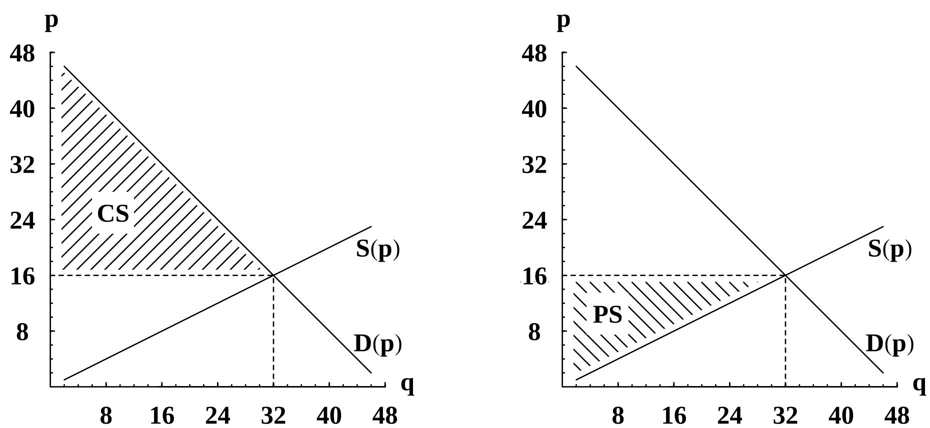


Figure 13: Consumers' surplus (left) and producers' surplus (right).

The producers' surplus is the difference between the price received and the producers' costs. The producers' surplus is shown graphically on the right side of figure 13. It is the area below the price and above the supply function.

Surplus in the competitive equilibrium example For the example shown in figure 13, the consumers' surplus is the area of the triangle labeled 'CS', which is $CS = \frac{1}{2}\,32\,(48 - 16) = 512$. The producers' surplus is the area of the triangle labeled 'PS', which is $PS = \frac{1}{2}\,32 \cdot 16 = 256$.

Calculation of consumers' surplus From the definitions of consumers' and producers' surplus and their graphical representations as triangles it is easy to write down formulas for calculating CS and PS. The consumers' surplus is

$$\begin{aligned} CS &= \frac{1}{2}\,q^*\,(d_0 - p^*) \\ &= \frac{1}{2}\,D(p^*)\,(d_0 - p^*). \end{aligned} \tag{2}$$

In this equation, p^* and q^* are the equilibrium price and quantity and d_0 is the price when demand is zero. Demand is zero when $a - b\,p = 0$, which is the same as $p = a/b$, so $d_0 = a/b$.

Calculation of producers' surplus The producers' surplus is

$$\begin{aligned} PS &= \frac{1}{2}\,q^*\,(p^* - s_0) \\ &= \frac{1}{2}\,S(p^*)\,(p^* - s_0). \end{aligned} \tag{3}$$

In this case, s_0 is the price when supply is zero. Supply is zero when $c + d\,p = 0$, which is the same as

$p = -c/d$, so $s_0 = -c/d$. We'll usually assume that $c \leq 0$ so that the price s_0 where quantity is zero will be positive. This just means that the sellers have a positive cost for their first unit of output.

3.4 Taxes in the competitive model

There are two simple taxes that we can consider to determine the effects of taxes in competitive markets. These are a sales tax paid by the buyer, and an excise tax paid by the seller. Typically, a sales tax that is a percentage of the purchase price. The analysis of a sales tax is simpler though if we consider a per unit tax rather than a tax that is a percentage of the purchase price. The basic results are similar for a percentage tax and a per unit tax. After evaluation of a sales tax on consumers, we'll consider an excise tax on sellers. This will also be treated as a per unit tax that the seller pays on each unit produced. The goal of the section is to compare the amount of the tax that is paid by the buyers and the amount of the tax that is paid by sellers with these two forms of taxation.

3.4.1 Sales taxes

If a tax is added on to each unit that the consumers purchase, the demand function is shifted. If the amount of the tax on each unit purchased is denoted by t, and if the price paid to the seller per unit is p, then the total price paid by the buyer is $p + t$ after the tax is added. Suppose that the demand is $D(p) = a - b\,p$ before the tax. Then the demand after the tax is

$$\begin{aligned} D_t(p) &= D(p+t) \\ &= a - b\,(p+t). \end{aligned} \tag{4}$$

Example Suppose that the demand is the same as in the previous examples, $D(p) = 48 - p$, and that the tax per unit is $t = 12$. Then from equation (4) the new demand is $D_t(p) = 48 - (p + 12)$, where the subscript t on $D_t(p)$ indicates that this is the demand after the tax is added. The new demand has the same slope as the original demand, but the maximum price that anyone will pay is 36 instead of 48, so the demand is shifted down by the amount of the tax. If the price paid to the seller is 36 and the tax is 12, then the total price that the consumers pay is 48, which was the maximum value for any buyer before the sales tax was imposed. The buyer should only care about the total price, so the maximum that a buyer will pay to the seller is 36 when the buyer also has to pay 12 per unit to the government for the sales tax. Before the tax, the equilibrium price was $p^* = 16$ (as shown in the graph on the left side figure 14). In the next subsection, we'll compare the prices received by sellers when there is a sales tax and when there is an excise tax. At that point it'll be convenient to have notation for the equilibrium price with a per unit sales tax that is distinct from the notation for a per unit excise tax, so let p_s^* be the new equilibrium price after the sales tax is imposed.

After the tax is imposed, supply equals demand when

$$D_t(p_s^*) = S(p_s^*). \tag{5}$$

If we put the new demand equation $D_t(p_s^*)$ and the supply equation $S(p_s^*)$ into equation (5), the result is $36 - p_s^* = 2\,p_s^*$. So the equilibrium price shifts down to $p_s^* = 12$. The share of the tax burden that falls to the producers is easy to determine. Before the tax was imposed on the consumers, the producers received the price $p^* = 16$. After the tax of $t = 12$ per unit was added, the producers received only $p_s^* = 12$. The price obtained by the sellers drops by \$4 per unit, and the tax is \$12 per unit, so the sellers pay $4/12 = 1/3$ of the tax. The buyers pay the remaining $2/3$ of the tax.

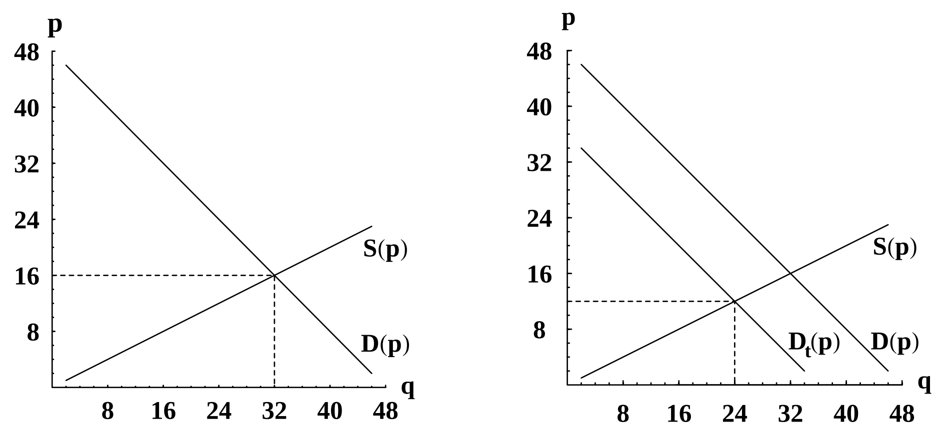


Figure 14: Demand, supply, and equilibrium before and after a sales tax is added.

Surplus in the sales tax example

In the example with the sales tax, the producers' surplus is

$$\begin{aligned} PS &= \tfrac{1}{2}\, S(p_s^*)\,(p_s^* - s_0) \\ &= \tfrac{1}{2}\, 24\,(12 - 0) \\ &= 144. \end{aligned}$$

The consumers' surplus is

$$\begin{aligned} CS &= \tfrac{1}{2} D_t(p_s^*)\,(d_0 - p_s^*) \\ &= \tfrac{1}{2}\, 24\,(36 - 12) \\ &= 288. \end{aligned}$$

We've already seen that in the example, buyers pay two-thirds of the tax. In Section 3.3 we found that consumers' surplus before the tax was $CS = 512$ and producers' surplus was $PS = 256$. Consumers' surplus declines from 512 to 288, a decrease of 224. Producers' surplus falls from 256 to 144, which is a decrease of 112. So consumers' also incur two-thirds of the surplus reduction.

Equilibrium price with a sales tax

In Section 3 on the competitive model, we found the competitive equilibrium price p^* when there is a linear demand $D(p) = a - b\,p$ and a linear supply $S(p) = c + d\,p$. The equilibrium price for this model, given in equation (1), is $p^* = (a - c)/(b + d)$. We can do the same thing when the demand is shifted by the tax to $D_t(p) = a - b\,(p + t)$.

When the sales tax is in effect, the supply and demand are equal when $c + d\,p_s^* = a - b\,(p_t^* + t)$. This is equivalent to $(b + d)\,p_s^* = a - c - b\,t$, so

$$p_s^* = \frac{a - c - b\,t}{b + d}. \tag{6}$$

The difference between the price before the tax and after the tax is the amount of the tax that is paid by the sellers. This difference is

$$\begin{aligned} p^* - p_s^* &= \frac{a - c}{b + d} - \frac{a - c - b\,t}{b + d} \\ &= \frac{b}{b + d}\,t. \end{aligned} \tag{7}$$

This formula shows that the fraction of the tax paid by the sellers depends on the slope b of the demand function and the slope d of the supply function.

3.4.2 Excise taxes

In the analysis of sales taxes, we found the new equilibrium price, quantity of exchange, and split of the tax burden between sellers and buyers when demand shifts as the result of a tax. In this section, we determine the effect of an excise tax, which is paid by the sellers rather than the buyers.

When sellers pay a tax, it is the supply that shifts rather than the demand. If sellers receive the price p from the buyers, but an amount t is paid as a tax, then the effective price received by the sellers is $p - t$. So the amount supplied at price $p - t$ after the tax is the same as the amount supplied at price p before the tax. This can be written as $S_t(p) = S(p - t)$. In the case of the linear supply function that we've been using, supply after the tax is

$$S_t(p) = S(p - t) \tag{8}$$

$$= c + d\,(p - t). \tag{9}$$

Example Suppose that the supply is the same as in the previous examples, $S(p) = 2\,p$, and that the tax per unit is $t = 12$. Then from equation (8) the new supply is $S_t(p) = 2\,(p - 12)$, where the subscript t on $S_t(p)$ indicates that this is the supply after the tax is added. The new supply has the same slope as the original supply, but the minimum price that any seller will accept is 12 instead of 0, so the intercept of the supply with the price axis is shifted up by the amount of the tax. Before the tax, the equilibrium price was $p^* = 16$ (as shown in the graph on the left side of figure 15). In the previous subsection, we denoted the equilibrium price with the sales tax by p_s^*. In this subsection, we'll denote the equilibrium price with the excise tax by p_e^*.

After the excise tax is imposed, supply equals demand when

$$D(p_e^*) = S_t(p_e^*). \tag{10}$$

If we put the new supply equation $S_t(p_e^*)$ and the demand equation $D(p_e^*)$ into equation (10), the result is $48 - p_e^* = 2\,(p_e^* - 12)$. So the equilibrium price shifts up to $p_e^* = 24$. The share of the tax burden that falls to the producers is easy to determine. Before the tax was imposed on the producers, they received the price $p^* = 16$. After the tax of $t = 12$ per unit was added, the price the producers received increased to $p_e^* = 24$, but they paid 12 in tax, so they have net receipts of $p_e^* - t = 24 - 12 = 12$ per unit. The price obtained by the sellers increases by \$8 per unit, and the tax is \$12 per unit, so the sellers pay \$4 per unit which is 1/3 of the tax.

Surplus in the excise tax example

In the example with the excise tax, the producers' surplus is

$$PS = \tfrac{1}{2}\,S(p_e^*)\,(p_e^* - s_0)$$

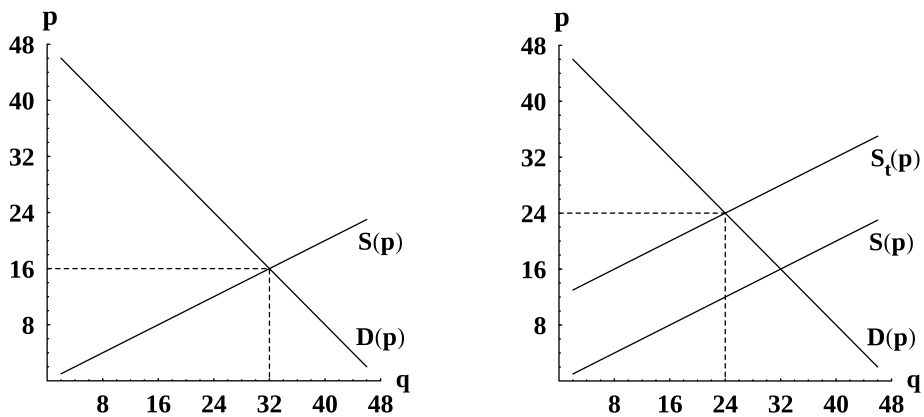


Figure 15: Demand, supply, and equilibrium before and after an excise tax is added.

$$= \tfrac{1}{2}\, 24\,(24 - 12)$$
$$= 144.$$

The consumers' surplus is

$$\begin{aligned} CS &= \tfrac{1}{2}\, D_t(p_e^*)\,(d_0 - p_e^*) \\ &= \tfrac{1}{2}\, 24\,(48 - 24) \\ &= 288. \end{aligned}$$

The changes to the consumers' surplus and to the producers' surplus with the excise tax are identical to the changes that we found with the sales tax. With both the sales tax and the excise tax, the consumers' surplus fell from 512 to 288, and the producers' surplus fell from 256 to 144. With either tax, the consumers' paid two-thirds of the tax, and also incurred two-thirds of the surplus loss.

Equilibrium price with an excise tax

Suppose, as in the previous section, that demand is a linear function $D(p) = a - b\,p$. Let p_e^* be the new equilibrium price after the excise tax is imposed. In this case, the supply and demand are equal when $c + d\,(p_e^* - t) = a - b\,p_e^*$. This is equivalent to $(b + d)\,p_e^* = a - c + d\,t$, so

$$p_e^* = \frac{a - c + d\,t}{b + d}.$$

The amount that the seller keeps is the new equilibrium price p_e^* minus the amount of the tax. This amount is

$$p_e^* - t \;=\; \frac{a - c + d\,t}{b + d} - t$$

$$
\begin{aligned}
&= \frac{a - c + d\,t}{b + d} - \frac{b + d}{b + d}\,t \\
&= \frac{a - c - b\,t}{b + d}. \qquad (11)
\end{aligned}
$$

3.4.3 Comparison of sales and excise taxes

With the sales tax, the amount received by the seller is the new equilibrium price p_s^*, since the tax is added on to the price paid by the buyer. The equilibrium price received by the seller with the sales tax is given in equation (6).

The amount received by the seller after the excise tax is imposed is the new equilibrium price minus the per unit tax, which is $p_e^* - t$. We found this difference in equation (11). When we compare the two equations, we see that the seller earns exactly the same amount in either case.

We can make a similar comparison between the price paid by the buyers in the two cases. In the case of the sales tax, the buyers pay the equilibrium price p_s^* plus the per unit tax t for each unit. With the excise tax the buyers pay the equilibrium price p_e^* because the tax is paid by the seller.

3.5 Elasticity

The elasticity of demand is a measure of how responsive demand is to a change in the price of the commodity. The definition of elasticity is "the percentage change in the quantity demanded divided by the percentage change in the price." If the original price is p_0 then the original quantity demanded is $q_0^d = D(p_0)$. When the price changes to p_1 the new quantity demanded is $q_1^d = D(p_1)$. The change in demand that results from this price change is $q_1^d - q_0^d$. The percentage change in demand is the change in demand divided by the original amount of the demand. So the percentage change in demand can be written $(q_1^d - q_0^d)/q_0^d$. Similarly, the percentage change in price is $(p_1 - p_0)/p_0$. This is enough information to write out the formula for the elasticity of demand.

Elasticity of demand The elasticity of demand is

$$
\begin{aligned}
\epsilon_d &= \left| \frac{(q_1^d - q_0^d)/q_0^d}{(p_1 - p_0)/p_0} \right| \\
&= \left| \frac{q_1^d - q_0^d}{p_1 - p_0} \cdot \frac{p_0}{q_0^d} \right|. \qquad (12)
\end{aligned}
$$

Elastic demand Demand is called elastic if $\epsilon_d > 1$. When $\epsilon_d > 1$, the percentage change in quantity demanded exceeds the percentage change in price, so the consumers are very responsive to price changes.

Include examples.

Unit elastic demand Demand is called unit elastic if $\epsilon_d = 1$. When $\epsilon_d = 1$, the percentage change in demand is equal to the percentage change in price. We'll examine an important implication of this case below when we consider revenue maximization for a monopolist or for a government that attempts to maximize tax revenue.

Inelastic demand Demand is called inelastic if $\epsilon_d < 1$. When $\epsilon_d < 1$, the percentage change in demand is less than the percentage change in price, so the consumers are not so responsive to price changes.

Include examples.

Example We can return to the example that we used in Section 3 on the competitive market model. In that example, the demand was $q^d = D(p) = 48 - p$. If the original price is $p_0 = 16$ and the price increases to $p_1 = 17$, then the quantity will change from $q_0^d = 32$ to $q_1^d = 31$. From equation (12), the elasticity of demand is

$$\epsilon_d = \left| \frac{31 - 32}{17 - 16} \frac{16}{32} \right| = \frac{1}{2}.$$

The demand at this point is inelastic.

Elasticity of supply

The elasticity of supply determines how responsive supply is to changes in the price. The definition is analogous to the definition of elasticity of demand, and the formula is the same, except that quantities supplied are substituted for quantities demanded. For the original price p_0, the original quantity supplied is $q_0^s = S(p_0)$. When the price changes to p_1 the new quantity supplied is $q_1^s = S(p_1)$. The elasticity of supply is

$$\begin{aligned} \epsilon_s &= \frac{(q_1^s - q_0^s)/q_0^s}{(p_1 - p_0)/p_0} \\ &= \frac{q_1^s - q_0^s}{p_1 - p_0} \cdot \frac{p_0}{q_0^s}. \end{aligned} \tag{13}$$

Supply is also called elastic if $\epsilon_s > 1$ and it is called inelastic if $\epsilon_s < 1$.

Example For the example from Section 3, the supply was $q^s = S(p) = 2p$. For the original price

$p_0 = 16$, the supply is $q_0^s = 2 \cdot 16 = 32$. When the price changes to $p_1 = 17$, the quantity supplied changes to $q_1^s = 34$. The elasticity at the original price is

$$\epsilon_s = \frac{34 - 32}{17 - 16} \cdot \frac{16}{32} = 1.$$

Elasticity and the split of the tax burden With linear supply and demand schedules, the split of the tax burden between sellers and buyers can be expressed in terms of elasticities. The fraction of the tax T_s paid by the sellers is $T_s = \epsilon_d/(\epsilon_s + \epsilon_d)$. This is not too difficult to show, but you can skip to the example below if you aren't curious, or if you believe me.

As usual, the demand is $q^d = D(p) = a - b\,p$ and the supply is $q^s = S(p) = c + d\,p$. From equation (1) the equilibrium price is $p^* = (a - c)/(b + d)$ and we use this as the original price p_0. Suppose that price changes to $p^* + \delta$ where δ is some small number. Then the quantities demanded are $q_1^d = D(p^* + \delta) = a - b\,(p^* + \delta)$ and $q_0^d = D(p^*) = a - b\,p^*$. So the difference in the quantities demanded is

$$\begin{aligned} q_1^d - q_0^d &= a - b\,(p^* + \delta) - (a - b\,p^*) \\ &= -b\,\delta. \end{aligned}$$

The difference in the prices is $p_1 - p_0 = p^* + \delta - p^* = \delta$. The elasticity of demand is therefore

$$\begin{aligned} \epsilon_d &= \left| \frac{-b\,\delta}{\delta} \cdot \frac{p_0}{q_0^d} \right| \\ &= b\,p_0/q_0^d. \end{aligned}$$

The quantity supplied at $p_0 = p^*$ is $q_0^s = S(p^*) = c + d\,p^*$ and the quantity supplied at $p_1 = p^* + \delta$ is $q_1^s = S(p^* + \delta) = c + d\,(p^* + \delta)$. The difference in the quantities supplied is $q_1^s - q_0^s = d\,\delta$, so the elasticity of supply is

$$\begin{aligned} \epsilon_s &= \frac{d\,\delta}{\delta} \cdot \frac{p_0}{q_0^s} \\ &= d\,p_0/q_0^s. \end{aligned}$$

Since we are calculating the elasticities at the equilibrium, $q_0^s = q_0^d$ so we can write $\epsilon_s = d\,p_0/q_0^d$.

Finally, calculate the ratio

$$\begin{aligned} \frac{\epsilon_d}{\epsilon_d + \epsilon_s} &= \frac{b\,p_0/q_0^d}{b\,p_0/q_0^d + d\,p_0/q_0^d} \\ &= b/(b + d) \end{aligned}$$

where the last equality holds because p_0/q_0^d can be factored out of the numerator and the denominator. This fraction, $\frac{\epsilon_d}{\epsilon_d+\epsilon_s} = \frac{b}{b+d}$, is the same fraction of the tax that we found is paid by the seller in equation (7).

Example In the first example in this section, we found that for the demand $q_d = D(p) = 48 - p$, the elasticity of demand at $p_0 = 16$ was $\epsilon_d = 1/2$. In the second example we found that the elasticity of supply for $q^s = S(p) = 2p$ at $p_0 = 16$ was $\epsilon_s = 1$. Then $\frac{\epsilon_d}{\epsilon_d+\epsilon_s} = \frac{1/2}{1+1/2} = 1/3$. In the example in Section 3.4.1, we found that the tax share paid by the sellers in this example was 1/3, which agrees with the split that we found using elasticities.

Elasticity and revenue maximization When revenue is maximized, then a small change in price has no effect (or only a small effect) on total revenue. When there is a single price, revenue is the price per unit times the number of units that buyers purchase. This can be written as $R(p) = p\,D(p)$. Suppose that at p_0 revenue $R(p) = p\,D(p)$ is at its maximum. If p_1 is close to p_0, then revenue is maximized if $p_0\,D(p_0) \doteq p_1\,D(p_1)$. If we write $q_0 = D(p_0)$ and $q_1 = D(p_1)$, then the revenue is approximately equal at p_0 and at p_1 if $p_1\,q_1 \doteq p_0\,q_0$. Rewrite this as

$$\begin{aligned}
(p_0 + p_1 - p_0)\,q_1 &= p_0\,q_0 \\
p_0\,q_1 + (p_1 - p_0)\,q_1 - p_0\,q_0 &= 0 \\
p_0\,(q_1 - q_0) &= -(p_1 - p_0)\,q_1 \\
\frac{p_0(q_1 - q_0)}{q_1(p_1 - p_0)} &= -1 \\
\frac{(q_1 - q_0)/q_1}{(p_1 - p_0)/p_0} &= -1.
\end{aligned}$$

The left hand side of the last equation differs from the definition of the demand elasticity only because there is a q_1 in the denominator instead of q_0. If p_1 is close to p_0 though, then q_1 and q_0 are also close, so the left hand side is approximately equal to the demand elasticity. This argument demonstrates that for linear demand revenue is maximized when the elasticity of demand is equal to 1.

4 Utility and demand

Demand is a useful notion. It connects price changes to changes in quantities desired by consumers. One of the central ideas of demand functions – that the quantity demanded of a good decreases when the price of the good increases – seems natural. Other properties of demand functions can be determined though if we start with the preferences of a consumer and determine what combination of commodities is preferred by the consumer over all other available combinations of commodities. One important property of demand functions is how the demand for a good changes when the price of a related commodity changes. For example, we may want to know how much demand for movie theater tickets decreases if the price of high definition televisions decreases, or how much the demand for DVDs increases if the price of HDTVs decreases, or how much demand for HDTVs increases as income increases. The first of these examples is for two commodities – movie tickets and home entertainment – that are substitutes. The second example – digital movie recordings and home entertainment systems – are products that are complements.

Up to this point in the lecture notes, the demand for a product has been treated independently of the demand for other products. The examples above connect demand for different products, and also connect demand to income. Models of demand that include these connections across markets are useful because we can see how changes in a market are propagated across related markets.

The economic model of individual demand arises from the interaction of several natural and intuitive elements. Faced with a number of consumption alternatives, we assume that a consumer can choose her favorite alternative. As a convenient terminology, we call a specific combination of commodities, like three DVDs and two movie tickets, a *consumption bundle*. The available consumption bundles are determined by the income of the individual and the prices of the commodities. Prices and income together determine a budget set. The choice problem for a consumer is to select the most desirable consumption bundle from among those in the budget set. The solution to the choice problem determines – for a fixed level of income and fixed prices of the alternative commodities – the amount of each commodity that a consumer will select. This is very close to what we've previously called a demand function. A demand function relates price to the quantity consumed. We've already described how the quantity consumed is determined as the solution to the choice problem. If price is systematically varied then the solution to the choice problem – that is, the amount of the commodity consumed – will change also. So that traces out a relationship between the price of a good and the amount of the good consumed, that is, a demand curve.

4.1 Indifference curves

When we analyze a utility function and derive a demand function from utility, or when we want to consider the shape of a utility function, one of the most common and useful representations of utility is an *indifference curve*. An indifference curve is the set of all combinations of inputs that lead to the same level of utility. The concept is the same as a contour line on a topographical map. A contour line on a topographic map runs through places that all have the same elevation. An indifference curve runs through combinations of two commodities that all lead to the same level of satisfaction or utility. Suppose that we have a utility function over two commodities X and Y. We can take a simple example where the consumer's satisfaction is ranked by simply multiplying together the amount x of commodity X consumed with the amount y of commodity Y consumed. Then $u(x, y) = x\,y$. Combinations of x and y that lead to the same value for $u(x, y)$ lie on a single indifference curve. An example illustrates this idea. Many combinations (x, y) all lead to a utility level of $u(x, y) = 225$. The graph on the left side of figure 16 shows five combinations of x and y that have utility level $u(x, y) = x\,y = 225$. The combinations shown are $(x, y) = (5, 45)$, $(x, y) = (9, 25)$, $(x, y) = (15, 15)$, $(x, y) = (25, 9)$, and $(x, y) = (45, 5)$. It's easy to verify that for each of these five points $u(x, y) = x\,y = 225$.

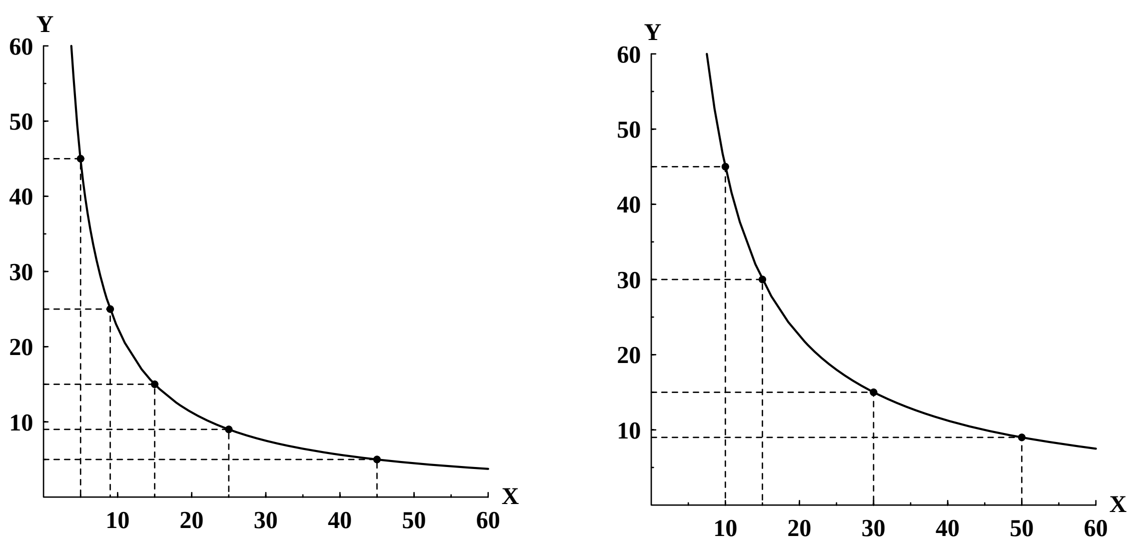


Figure 16: Two indifference curves for the example utility function $u(x, y) = x\,y$.

The indifference curve shown in the graph on the left side of figure 16 shows all the combinations of

X and Y that lead to the same level of utility: the consumer likes each of these combinations equally well. All of the combinations of commodities that lie above the indifference curve are preferred to those on the indifference curve. The combinations of commodities on and above the indifference curve are preferred to combinations of commodities that are below the indifference curve.

The graph on the right of the figure shows some combinations of x and y that lead to a higher level of satisfaction for this consumer. In the graph on the right, each of the points on the indifference curve has the utility level $u(x, y) = x\,y = 450$. This graph also shows some combinations of x and y that lie on the indifference curve. These points are $(x, y) = (10, 45)$, $(x, y) = (15, 30)$, $(x, y) = (30, 15)$, and $(x, y) = (45, 10)$. It's easy to verify that for each of these points $u(x, y) = x\,y = 450$.

Although a specific utility number is assigned to each combination of x and y, these numbers don't represent a degree of satisfaction. The utility level for the indifference curve on the left side of figure 16 is 225, which is half of the utility level for combinations of x and y on the indifference curve shown on the right side of the figure. But that doesn't mean that the consumer is only half as happy with the combinations that lead to 225 as with those that lead to 450. The utility numbers only order the curves. When we derive a demand function, it isn't the utility numbers that matter. The consumer chooses the available alternative that has the highest utility. So it is only the ordering of the indifference curves that matters, not the utility level itself.

4.2 The marginal rate of substitution

The slope of an indifference curve at a point is called the *marginal rate of substitution.* The marginal rate of substitution represents the tradeoff between X and Y that keeps a consumer equally satisfied. The graph on the left side of figure 17 shows two different points on an indifference curve for a consumer with the utility function $u(x, y) = x\,y$. The utility at $(x, y) = (6, 20)$ and the utility at $(x, y) = (5, 24)$ both lie on the indifference curve $u(x, y) = x\,y = 120$. Moving from $(x, y) = (5, 24)$ to $(x, y) = (6, 20)$ the consumer maintains the same utility level, so the consumer substitutes four units of Y to get one additional unit of X. This number, $-\Delta Y/\Delta X$, is called the marginal rate of substitution at $(x, y) = (5, 24)$. Since $\Delta Y = -4$ and $\Delta X = 1$, the marginal rate of substitution is $-\Delta Y/\Delta X = 4$ at $(x, y) = (5, 24)$.

The graph on the right side of figure 17 shows two more points on the same indifference, but for a larger value of X. The utility at $(x, y) = (15, 8)$ and the utility at $(x, y) = (30, 4)$ also lie on the

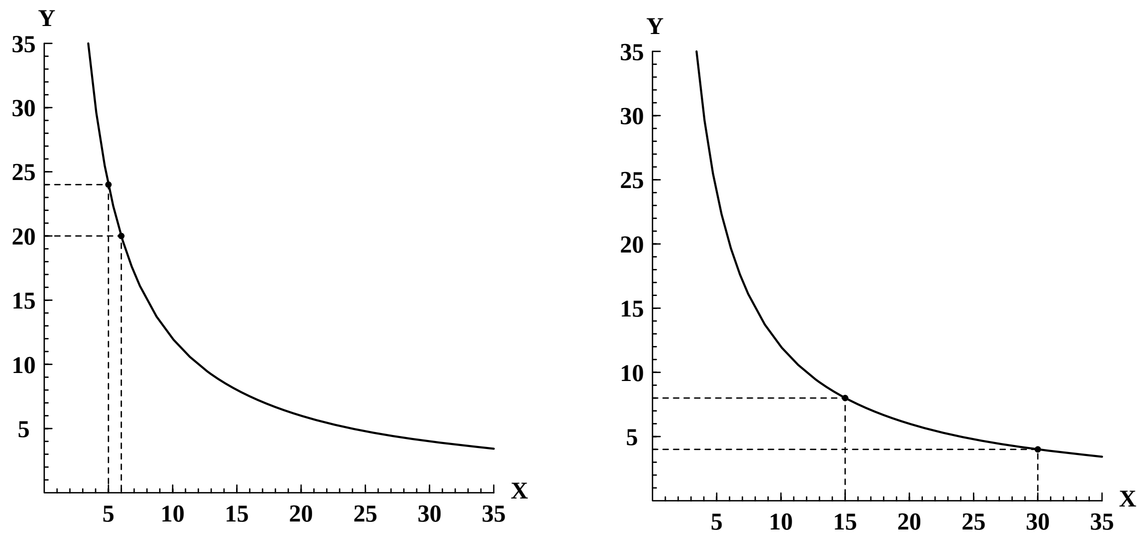


Figure 17: The marginal rate of substitution diminishes along an indifference curve.

indifference curve $u(x, y) = x\,y = 120$. Moving from $(x, y) = (15, 8)$ to $(x, y) = (30, 4)$ the consumer maintains the same utility level, so the consumer substitutes four units of Y to get fifteen additional units of X. The marginal rate of substitution at $(x, y) = (15, 8)$ is $-\Delta Y/\Delta X = 4/15$ (since $\Delta Y = -4$ and $\Delta X = 15$). This is an example of a property of preferences which is considered fundamental in the economic models of choice. As the amount of commodity X increases, the marginal rate of substitution between Y and X decreases. This is usually called the *diminishing marginal rate of substitution.*

Diminishing marginal rate of substitution means that as you get more of a commodity (for example, more of X), the amount of another commodity that you'll be willing to give up to get an additional unit of X decreases. In the graph on the left hand side of figure 17, the consumer was willing to give up four units of Y to get an additional unit of X. In the graph on the right hand side, when the consumer already has more X, he is only willing to give up about 4/15 of a unit of Y to get an additional unit of X. This property of preferences implies the convex shape of indifference curves, and also implies that demand will respond continuously to price changes.

4.3 The budget set

A consumer's budget determines all of her possible consumption levels for each commodity. All of the possible consumption levels form the *budget set.* The budget set is easily determined from prices and income. An example illustrates the idea. Suppose that a consumer has income $M = 120$ and the prices of X and Y are $p_x = 2$ and $P_y = 4$. From this we can determine the budget set. Two steps help to graph the budget set. If the consumer spends all of her income on X, then the total expenditure on X is $p_x\, x$. The total income available is $M = 120$. If expenditure on X equals the income then $p_x\, x = M$. In our example we know that income is $M = 120$ and we know that the price of X is $p_x = 2$, so we can determine the maximum amount of X that the consumer can purchase if she spends her entire income on X. That amount is $x = M/p_x = 120/2 = 60$. If instead she spends all of her income on Y, then $p_y\, y = M$. So the most Y she can purchase is $y = M/p_y = 120/4 = 30$. So the points $(x, y) = (60, 0)$ and $(x, y) = (0, 30)$ are two points in the budget set. Typically, a consumer spends money on both commodities. The expenditure on X is $p_x\, x$ and the expenditure on Y is $p_y\, y$. Total expenditure is $p_x\, x + p_y\, y$. If the budget is all spent[1] then the budget equation is

$$p_x\, x + p_y\, y = M. \tag{14}$$

This is the equation of a line. We've already found two points on this line. Her consumption level is $(x, y) = (60, 0)$ when she spends all her income on X and it is $(x, y) = (0, 30)$ when she spends all of her income on Y. The graph on the left side of figure 18 shows these two points and the budget line between them for our example.

The graph on the right side of figure 18 shows the budget line from the example on the left side of the figure, and it also shows a second budget line. The lower budget line corresponds to a higher price of X, $p_x = 4$. Since the price of X has increased to $p_x = 4$, the total amount of X that the consumer can purchase if she spends her entire income on X has decreased. If all her income is used to purchase X, then $x = M/p_x = 120/4 = 30$. Budget lines for other prices of X can all be drawn in the same way.

4.4 The consumer's choice problem

The consumer's choice problem is to select, from among those that are available (or in the budget set), the combination of commodities that maximizes utility. This choice problem can be viewed

[1] When we derive demand functions we'll assume that the entire budget is used. There are two steps in the argument that all of the budget is used. First, we are working with a simple model of consumption that doesn't include savings behavior and future consumption, so income only has value for present consumption. Second, we consider utility functions that increase in both commodities, so that if part of the budget isn't used, then utility could be increased by spending the remaining part of the budget on one or both commodities.

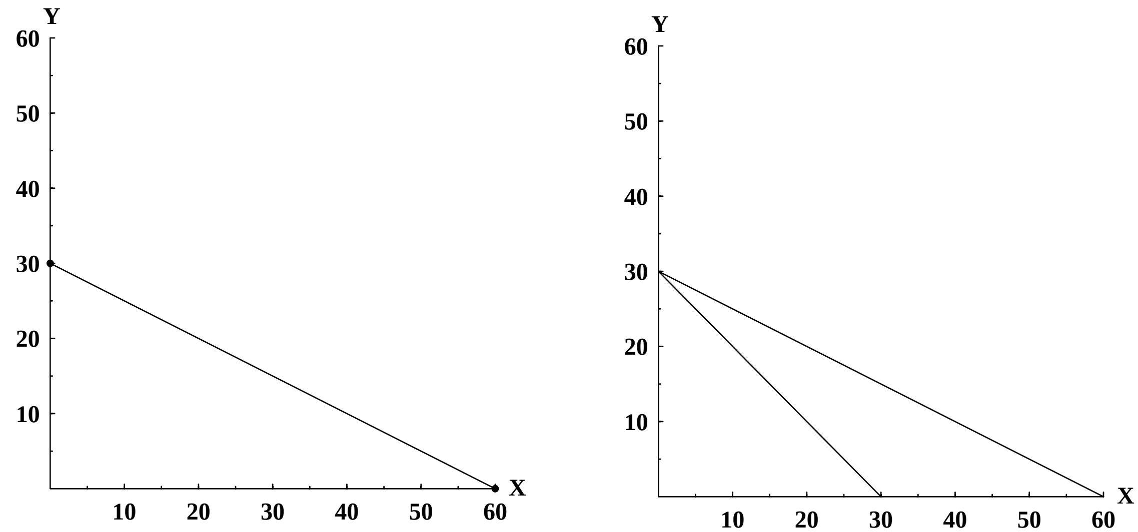


Figure 18: Budget line with $p_x = 2$ (left); budget lines for $p_x = 2$ and $p_x = 4$ (right)

graphically by combining the budget set, which represents the available alternatives, with indifference curves, which provide information about preferences.

The graph on the left side of figure 19 shows the two indifference curves $u(x, y) = x\,y = 225$ and $u(x, y) = x\,y = 450$ from the example in Section 4.1. The graph also shows the budget line when $M = 120$, $p_x = 2$, and $p_y = 4$. Any point on the budget line is available to the consumer, including the points labeled A and B. Point A lies on a lower indifference curve than point B though, so B is preferred to A. Any point above the indifference curve through point B is preferred to point B, but every point that is above the indifference curve through B is also above the budget line, so no point that is preferred to B is available to this consumer. The chosen combination of X and Y lies on the budget line and also lies on the highest indifference curve that touches any point on the budget line. For the budget line shown, this occurs at point B.

We can also look at the consumption bundle chosen for other budget lines. For example, the graph on the right side of figure 19 includes a second budget line that corresponds to the same values of income and p_y, but the higher price $p_x = 4$ for X. In this case, the highest indifference curve that touches the budget line is the one that passes through point C. So point C is the chosen point for this budget.

The two choice points B and C contain the key idea of a demand function. The price change from $p_x = 2$ to $p_x = 4$ led to a change in the choice point from B to C. When we determine the amount of

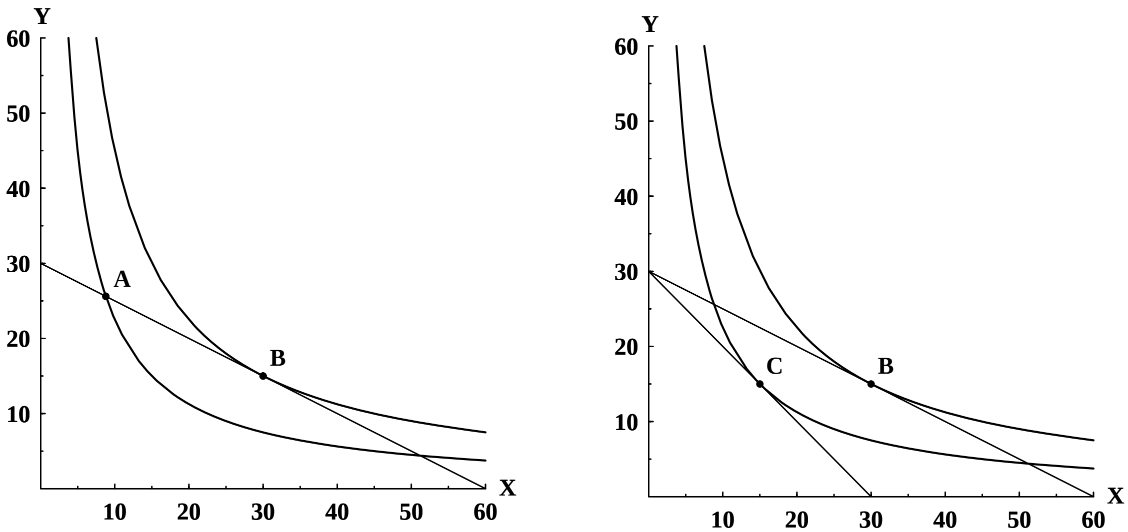


Figure 19: Combined budget lines and indifference curves

X consumed at point B and graph that against $p_x = 2$ and determine the amount of X consumed at point C and graph that against $p_x = 4$, that will be the start of a demand function. The next section develops this idea in greater detail.

4.5 Demand functions

Demand functions relate the price of a commodity to the amount of the commodity that a consumer chooses to purchase. In this section, demand functions are derived for two different utility functions. The first derivations of demand are graphical; the second derivations are analytical. The section concludes with discussion of three other demand functions and discussion of some general properties of demand functions.

Graphical derivation of a demand curve from a utility function, income, and prices proceeds as described in the "choice problem" section. When income and prices are specified, that determines a budget line. The choice point is the one that lies on the highest indifference curve that has a point in common with the budget line. The demand function for a commodity specifies how the quantity of the commodity desired by the consumer varies with the price of the good. The way to determine this graphically is to fix income and the price of the other good (commodity Y in our examples) and then consider the choice problem for several different prices of X. As p_x varies, the optimal choice point will vary, and the result then determines a relationship between the price of X and the quantity of X

desired by the consumer.

In each of the four examples in this section, we'll fix the income of the consumer at $M = 120$ and the price of Y at $p_y = 4$. Each example determines the demand for a different utility function. Four prices of X, $p_x = 2, 4, 6$, and 8, are considered in each example, in order to facilitate comparison of properties of the four different demand functions.

4.5.1 Perfect complements

Perfect complements are an extreme case in which a consumer always wants two different commodities in the same proportion, regardless of the relative price of the two commodities. The utility function that represents these preferences is $u(x, y) = \min\{x, y\}$. Of course, if two goods really led to a utility function like this for many consumers, then the commodities would just be sold together as a single good, like left and right shoes. Many goods are approximately like this though, and the perfect complements case is simple to analyze, so it is a useful example.

Graphical demand derivation for perfect complements In the graph on the left side of figure 20, the income is $M = 120$ and the price of Y is $p_y = 4$ for all four of the budget lines shown. To find the intercept of the budget line, we can consider what would happen if the consumer were to spend her entire income on commodity Y. (In most cases the consumer won't actually spend all of her income on one commodity or the other. We consider this case in order to determine where the budget line intersects the vertical axis.) Expenditure on Y is $p_y\, y$ and if the entire budget is spent on Y, then $p_y\, y = M$. In the example $4\, y = 120$ so $y = 30$. Each of the four budget lines intersects the vertical axis at $y = 30$. We can find where the budget line intersects the horizontal axis by considering the amount of X that the consumer would be able to purchase if she were to spend her entire income on X. (Once again, we consider this not because that is what the consumer typically does, but because that will help us graph the budget line easily.)

In our example, when the price of X is $p_x = 2$, if the consumer were to spend her entire budget on X then $p_x\, x = M$ so $2\, y = 60$. The X intercept is therefore $x = 60$ when $p_x = 2$. The choice point for this budget line is shown in the graph on the left side of figure 20 as point **A**. At point **A**, the consumer wants to purchase $x = 20$ units of X (and $y = 20$ units of Y). Since she wants $x = 20$ units of X when $p_x = 2$, point **A** in the consumption space graph on the left (with budgets and indifference curves) corresponds to point **A** in the demand graph on the right. Point **A** in the right graph shows that the consumer wants to purchase $x = 20$ units when $p_x = 2$.

When the price of X increases to $p_x = 4$, then the X intercept of the budget line is again determined from the equation $p_x\, x = M$ but with $p_x = 4$ this time. So $4\,x = 120$ so the X-intercept is at $x = 30$. The choice point for the budget line with X-intercept at $x = 30$ and Y-intercept at $y = 30$ is shown in the consumption space graph as **B**. The amount of X desired at this price is $x = 15$. Consumption $x = 15$ at price $p_x = 4$ is shown as point **B** in the demand graph on the right side of figure 20.

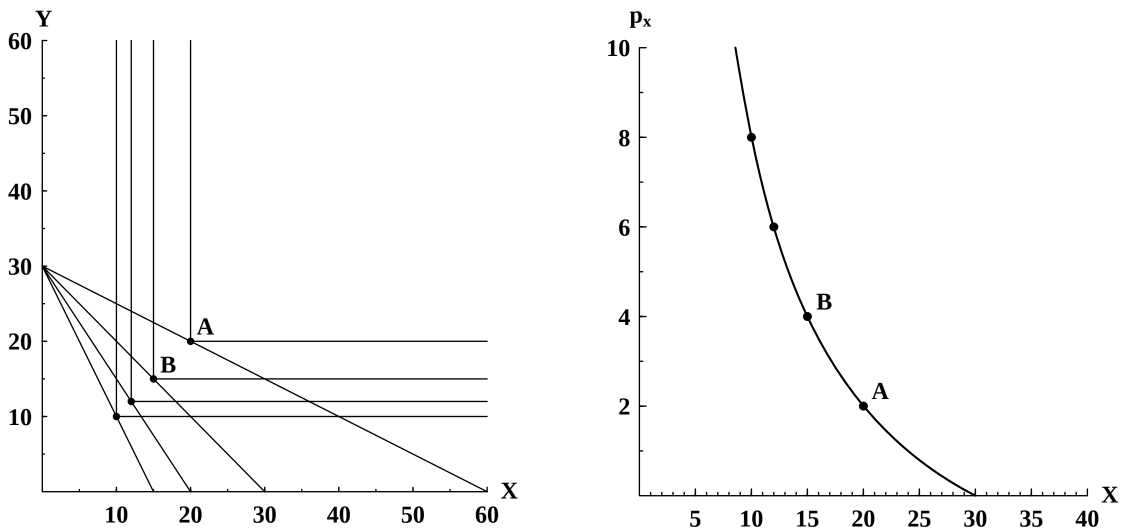


Figure 20: “Consumption space” graph (left) and demand graph(right) for perfect complements

The two unlabeled choice points in the consumption space graph correspond to $p_x = 6$ and $p_x = 8$. The amounts of X desired at these two prices are shown as the two unlabeled points in the demand graph.

Graphical analysis of demand is useful. It provides an intuitive description of the relationship between budgets, indifference curves, choice points, and demand. Frequently though it is useful to derive the formula for a demand function. The steps required to do this are fairly simple, and the perfect complements example has one of the simplest demand derivations.

Demand derivation for perfect complements The demand derivation for perfect complements starts from the observation that when the utility function is $u(x, y) = \min\{x, y\}$, then at every possible choice point $x = y$. If this weren't so then there are two other possibilities: either $x > y$ or $x < y$. If $x > y$ and the prices of both X and Y are positive, then the amount of X purchased in excess of the amount of Y adds to expense but adds nothing to utility. If $x < y$ a similar argument applies to the amount of Y purchased in excess of the amount of X purchased. So at the optimal point (or choice point), $x = y$.

The next two steps in the derivation of the demand function for perfect complements are very typical of the derivation of demand from utility functions. The budget equation is $p_x\, x + p_y\, y = M$. Since $x = y$, we can eliminate y from the budget equation with the substitution $y = x$, so $p_x\, x + p_y\, x = M$. Now the only step left is to solve for x. When x is factored from the left hand side of this equation then $(p_x + p_y)\, x = M$. Finally, divide through by $p_x + p_y$ on both sides of this equation to get

$$x(p_x, p_y) = \frac{M}{p_x + p_y}. \tag{15}$$

In this example, the demand for Y is the same as the demand for X, because we found that at our choice point, $y = x$. So

$$y(p_x, p_y) = \frac{M}{p_x + p_y}. \tag{16}$$

It is easy to verify that the demand equation (15) and (16) are consistent with the four points in the graph on the left side of figure 20. For example, the budget line through point **A** came from $M = 120$, $p_x = 2$, and $p_y = 4$. In that case, $x(p_x, p_y) = 120/6 = 20$ and $y(p_x, p_y) = 20$ also. Those are the levels of X and Y purchased at choice point **A**. Consistency of the other three choice points with the demand equations are verified in the same way.

4.5.2 Cobb-Douglas utility

The utility function $u(x, y) = x\, y$ is a special case of a class of utility functions that is called “Cobb-Douglas” utility. A Cobb-Douglas utility function has one property that is often useful in economic modeling. We’ll see when we derive demand that the percentage of income spent on each commodity is independent of prices with this utility function. There are important examples where this is approximately what is found in the data. For example, it is commonly recommended that people spend 25 to 30% of their disposable (after tax) income on housing, and this is approximately what has been found empirically.

Graphical demand derivation for Cobb-Douglas utility Indifference curves in figure 19 are from a Cobb-Douglas utility function. In Section 4.4 we’ve already considered the choice problem for Cobb-Douglas utility with with two different prices of X ($p_x = 2$ and $p_x = 4$) for income $M = 120$ and $p_y = 4$. The graph on the left side of figure 21 redraws the graph from the right side of figure 19 but with two additional budget lines and choice points that correspond to prices $p_x = 6$ and $p_x = 8$. (The first two points are also relabeled.)

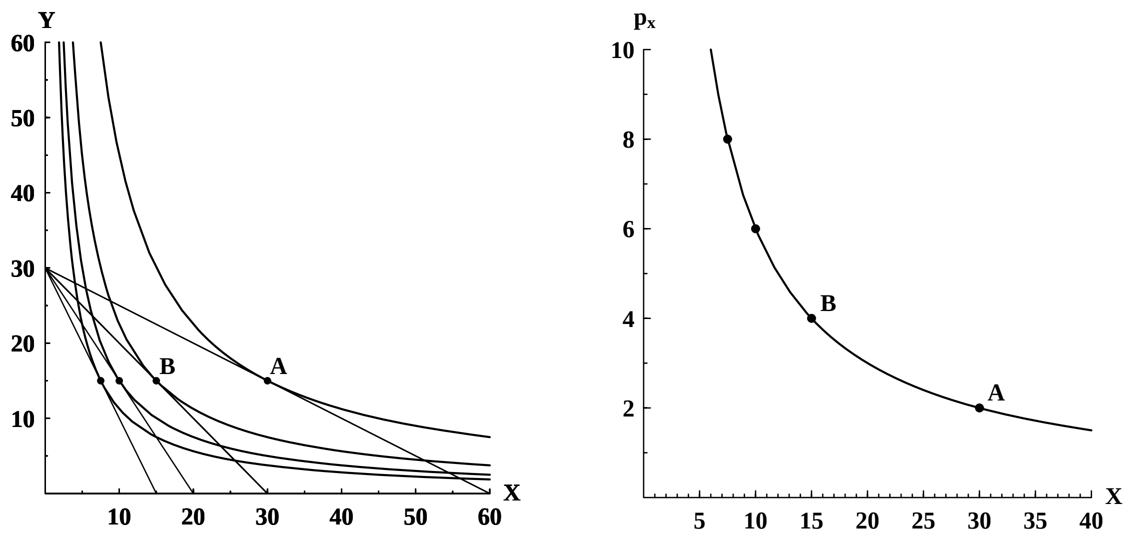


Figure 21: "Consumption space" graph (left) and demand graph(right) for Cobb-Douglas utility

Construction of the budget lines and the correspondence between the choice points in the consumption space graph on the left side of the figure and the demand graph on the right side of the figure is similar to the perfect complements example. At point **A** in the consumption space graph the price of X is $p_x = 2$ and the amount of X purchased at the choice point **A** is $x = 30$. This appears in the demand graph as the point labeled **A** at $p_x = 2$ and $x = 30$. The relationships between the other three choice points and the other three points on the demand graph are determined in the same way.

Demand derivation for Cobb-Douglas utility As with the perfect complements example, the demand function for the Cobb-Douglas example can be derived. The technique is different though from the technique used to solve for demand in the perfect complements case, and the technique used for this example is unusual because it uses only elementary techniques from college algebra.

The first step in the derivation is to eliminate y from the utility function. The budget equation is $p_x\, x + p_y\, y = M$, which can be solved for y to get

$$y = \frac{M}{p_y} - \frac{p_x}{p_y}\, x. \tag{17}$$

When this expression for y is substituted into the utility function the result is an expression for utility as a function only of x, which we can write as

$$\begin{aligned}\tilde{u}(x) &= x\left(\frac{M}{p_y} - \frac{p_x}{p_y}\,x\right) \\ &= -\frac{p_x}{p_y}\,x^2 + \frac{M}{p_y}\,x.\end{aligned}$$

From the formula for "completing the square" (which is provided in Appendix A), utility is maximized where

$$\begin{aligned}x &= -\frac{M/p_y}{2\,(-p_x/p_y)} \\ &= \frac{M}{2\,p_x}.\end{aligned}$$

So the demand function for x is

$$x(p_x) = \frac{M}{2\,p_x}. \tag{18}$$

We can use equation (17) once again to get the demand for y easily:

$$\begin{aligned}y &= \frac{M}{p_y} - \frac{p_x}{p_y}\,x \\ &= \frac{M}{p_y} - \frac{p_x}{p_y}\,\frac{M}{2\,p_x} \\ &= \frac{M}{p_y} - \frac{M}{2\,p_y} \\ &= \frac{M}{2\,p_y}.\end{aligned}$$

So the demand function for Y is

$$y(p_y) = \frac{M}{2\,p_y}. \tag{19}$$

The amounts of X and Y demanded at each of our four price pairs can be compared to the four choice points in the consumption space graph on the left side of figure 21, as we did in the perfect complements example. For our example with $M = 120$ and $p_y = 4$, when the price of X is $p_x = 2$ then $x(2) = 120/4 = 30$ and $y(4) = 120/8 = 15$. These demands for X and Y are the same as the choice point **A** in the graph on the left side of figure 21. The other three points can be compared in the same way by calculating the demand for X and the demand for Y at the prices that corresponds to each of the other three budget lines.

4.5.3 Perfect substitutes

Perfect substitutes, like perfect complements, are an unusual case economically, but they also illustrate the importance of the role of the diminishing marginal rate of substitution. With perfect substitutes, either commodity provides the same level of satisfaction to the consumer, so the utility function that represents these preferences has a particularly simple form: $u(x, y) = x + y$. Section 4.2 provides a brief description of the marginal rate of substitution. The example in that section shows that as the consumer obtains more of commodity X, the amount of Y that he would be willing to give up to acquire another unit of X decreases. Perfect substitutes are unusual because they don't have this property: the marginal rate of substitution is the same at every point along an indifference curve. A consumer with preferences represented by the utility function $u(x, y) = x + y$ is always willing to give up one unit of Y to acquire one unit of X. These preferences lead to an unusual demand function. When the price of X is lower than the price of Y, the consumer spends his entire income on X; when the price of X is above the price of Y, the consumer spends his entire income on Y. These properties of the demand function are shown graphically and then algebraically.

Graphical demand derivation for perfect substitutes With perfect substitutes, indifference curves are straight lines. The consumption space graph on the left side of figure 22 shows the indifference curves $u(x, y) = 30$ and $u(x, y) = 60$ as dashed lines. (For this utility function, the indifference curves look the same as the budget lines, so the indifference curves are dashed lines to distinguish them from the budget lines.) When the price of X is $p_x = 2$ and the price of Y is $p_y = 4$, the budget line is the one between $(x, y) = (0, 30)$ and $(x, y) = (60, 0)$. The consumer gets utility $u(x, y) = 60$ if he spends his entire income on commodity X and consumes at the point $(x, y) = (60, 0)$. When the price of X is higher than the price of Y, then the consumer will purchase only Y, and end up at point **C** on the consumption space graph. For example, if the price of X is $p_x = 6$, then the budget line passes through $(x, y) = (0, 30)$ and $(x, y) = (20, 0)$. If any money is spent on X, then the indifference curve reached will lie below the indifference curve $u(x, y) = 30$. So the consumer purchases only Y when $p_x = 6$. This will be true for every price of X above $p_y = 4$, so the demand for X is zero when $p_x > p_y = 4$. The last case to consider is $p_x = p_y = 4$. In this case, both the budget line and the indifference curve are the line segment from $(x, y) = (0, 30)$ to $(x, y) = (30, 0)$. Every point on the budget line lies on the same indifference curve, so it doesn't matter which point is chosen. The consumption space graph (on the left) shows two points $\mathbf{B}_1$ and $\mathbf{B}_2$ that both lie on the same indifference curve and also both lie on the budget line. Since either one could be chosen, both are on the demand curve (on the right) for $p_x = 4$.

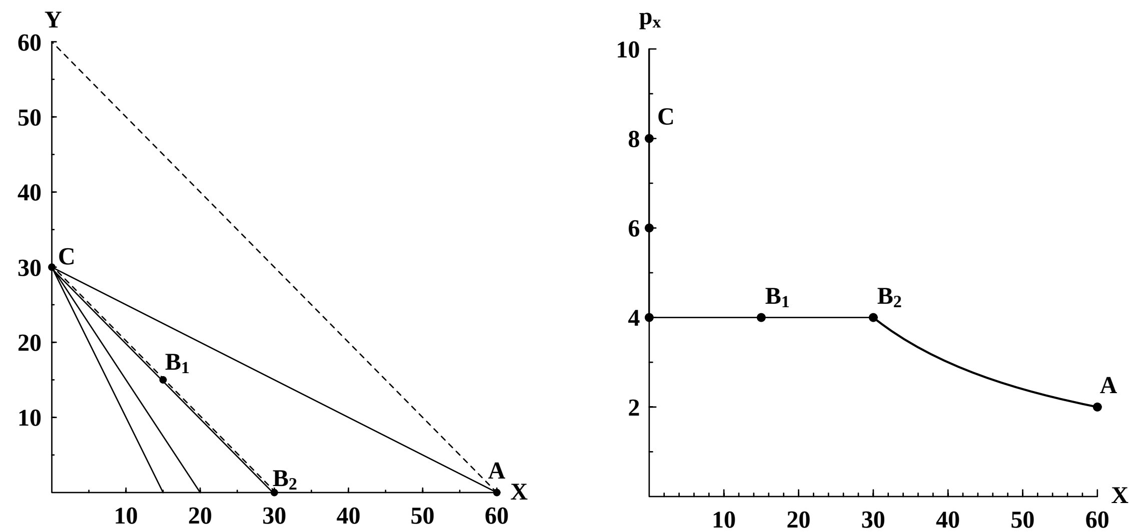


Figure 22: "Consumption space" graph (left) and demand graph(right) for perfect substitutes

Demand derivation for perfect substitutes To get the demand function for X for perfect substitutes, note that demand is zero when $p_x > p_y$. When $p_x = p_y$, the consumer is indifferent between consumption of X and consumption of Y, so the consumer will spend any amount on X between zero and his entire budget M. When he spends his entire budget on X, then $p_x x = M$ so $0 \leq x \leq M/p_x$. Finally, when $p_x < p_y$, he'll spend his entire income on X, so $x = M/p_x$. This can be summarized as

$$x(p_x, p_y) = \begin{cases} 0, & p_x > p_y; \\ (0, M/p_x), & p_x = p_y; \\ M/p_x, & p_x < p_y. \end{cases} \tag{20}$$

The demand shown in figure 22 is consistent with the demand function in equation (20) if the price of Y is $p_y = 4$ and income is $M = 120$. The graph and the demand equation both demonstrate the peculiar "all or nothing" behavior that results from a utility function that does not exhibit decreasing marginal rate of substitution.

4.5.4 Quadratic utility

Quadratic utility generates a linear demand function (for one of the two commodities). Linear demand functions are used often in economic models, so this derivation is relevant. Once we've looked at the optimal choice for several prices though, we'll see that although quadratic utility has this simple linear relationship between the price of a good and the amount of the good consumed, the quantity of commodity Y consumed is very irregular. So linear demand comes from a utility function with some peculiar characteristics.

Graphical demand derivation with quadratic utility The graphical derivation of demand with quadratic utility proceeds in the same way as with the previous utility functions. The "consumption space" graph on the left side of figure 23 shows four indifference curves for the utility function

$$u(x, y) = -(45 - x)^2 + 40y.$$

The budget lines that are tangent to these indifference curves correspond to the same income and prices as the previous examples.

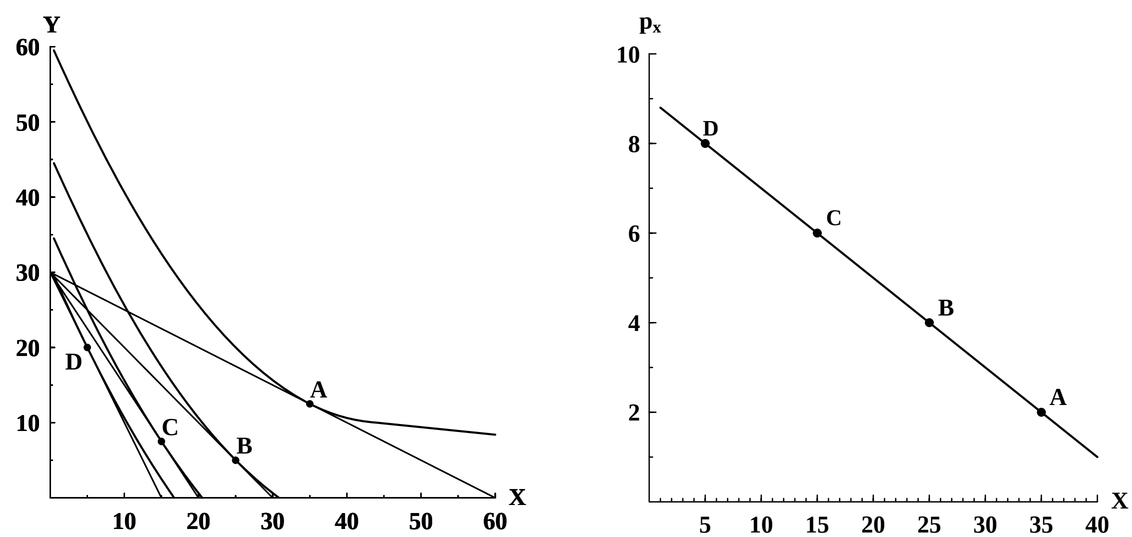


Figure 23: "Consumption space" graph (left) and demand graph(right) for quadratic utility

Demand derivation with quadratic utility A quadratic utility function has the form

$$u(x, y) = -(a - x)^2 + 2\,b\,y.$$

The budget equation is $p_x\,x + p_y\,y = M$, which can be solved for y to get $y = \frac{M}{p_y} - \frac{p_x}{p_y}\,x$. When this expression for y is substituted into the utility function the result is an expression for the utility as a function only of x, which we can write as

$$\begin{aligned}\tilde{u}(x) &= -x^2 + 2\,a\,x - a^2 + 2\,b\left(\frac{M}{p_y} - \frac{p_x}{p_y}\,x\right) \\ &= -x^2 + 2\left(a - b\,\frac{p_x}{p_y}\right)x + 2\,b\,\frac{M}{p_y} - a^2.\end{aligned}$$

From the formula for "completing the square," utility is maximized where

$$\begin{aligned}x &= -\frac{2\left(a - b\,\frac{p_x}{p_y}\right)}{2\,(-1)} \\ &= a - b\,\frac{p_x}{p_y}.\end{aligned}$$

If $p_y = 1$, then we get the usual expression for a linear demand function: $x = a - b\,p_x$. (If $p_y \neq 1$, we can still get our usual expression for a linear demand by setting $\tilde{b} = b/p_y$.) We usually write the demand for x as $x(p_x)$ or $D_x(p_x)$ so in this case

$$D_x(p_x) = a - b\,p_x.$$

Although the demand for X is linear with this utility function, the demand for Y is unusual. It can be obtained from the budget equation, with $x(p_x) = a - b\,p_x$ substituted in for x:

$$\begin{aligned}p_x\,x + p_y\,y &= M \\ p_x\left(a - b\,\frac{p_x}{p_y}\right) + p_y\,y &= M \\ p_y\,y &= M - p_x\left(a - b\,\frac{p_x}{p_y}\right) \\ y(p_y) &= \frac{1}{p_y}\left(M - a\,p_x + b\,\frac{p_x^2}{p_y}\right).\end{aligned}$$

5 Competitive equilibrium in the long-run

In previous lectures we've seen examples of shifts in the market demand. This lecture addresses the important question of how firms respond to a shift to the demand and the resulting shift in the market price. In the lecture a single example is developed and carried through, starting from an initial shift in the demand and then examining the response of a typical firm. In order to see how the firm responds, we'll consider the costs of the firm (from which we find the firm's supply function) and the profits of the firm. If profits in an industry exceed the normal rate of return on capital investment, new firms will enter the industry or existing firms will expand their capacity; if profits fall below the normal rate of return, firms will exit the industry or existing firms will reduce their investment in capital and allow their capacity to decline.

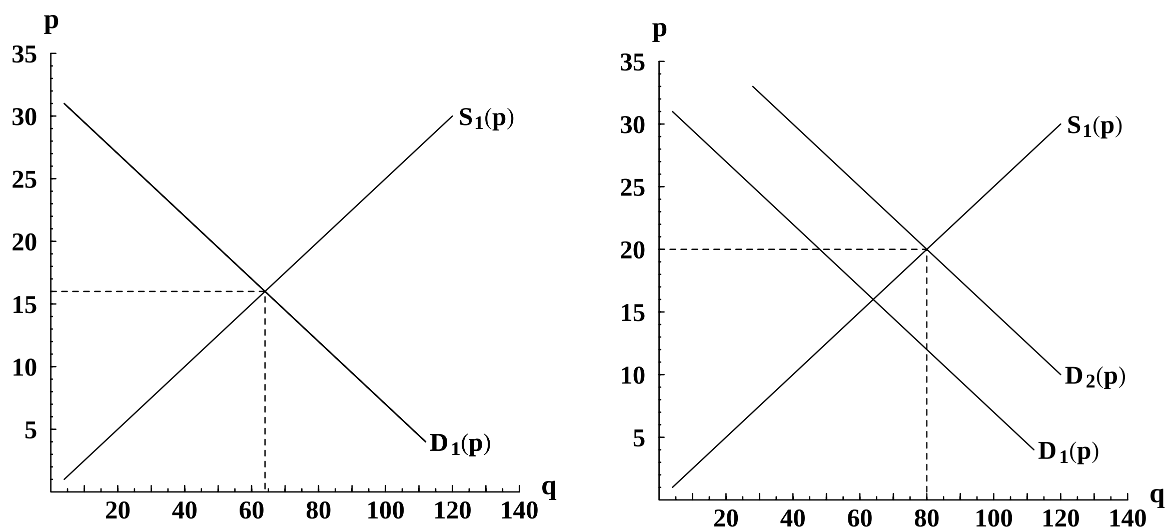


Figure 24: Original market conditions (left) and conditions after demand increase (right).

5.1 Shifts to demand and short-run price adjustment

Suppose that the demand in a market is given by the equation $D_1(p) = 128 - 4p$ and the supply is given by the equation $S_1(p) = 4p$. The equilibrium price for this market is $p_1^* = 16$, and $q_1^* = 64$ is the equilibrium quantity traded.[2] The left side of figure 24 shows supply and demand conditions from this market: the equilibrium price and quantity are indicated by the dashed lines. Suppose that

[2] The subscript on the price indicates that the price is the original one in the market. Later we'll have the demand shift and the price will change to p_2^*. The equilibrium quantities are denoted in the same way.

some event produces a demand increase to $D_2(p) = 160 - 4p$. The cause of this could be any of a number of factors. A tax on consumption of the good may have been reduced; there may be a change in consumers' preferences that increases demand; there may have been an increase in the number of consumers in the market; or the wealth of consumers may have increased. The right side of figure 24 shows the market demand both before and after the increase to the demand, along with the supply. After the increase in demand, the new equilibrium price is $p_2^* = 20$ and the equilibrium quantity traded increases to $q_2^* = 80$.

5.2 Firm supply and market supply

The market supply is comprised of the supply of individual firms. Suppose that in this market there are 16 firms, and the supply function is $S^i(p) = p/4$ for each firm.[3] The market supply at the original equilibrium price p_1^* is determined by adding up the supply of the individual firms. At this price, each firm will supply $S^i(16) = 16/4$ units, and the total quantity traded is $q_1^* = 64$ units. The market supply is therefore the sum of the supplies of each firm in the market. This is generally true, so we get market supply $S(p) = S^1(p) + S^2(p) + \cdots + S^{16}(p) = 4p$.

In order to understand how firms respond to the price change, we need to examine where the supply functions of the individual firms come from and what firm profit is at each price. In the experiments that you participated in, if you were a seller you probably noticed that in later periods, when the market price was clearly established, you continued to sell units as long as the price you received was greater than or equal to the cost of the unit. The cost of a unit is often called its *marginal cost*, so that means that you supplied units until your marginal cost was equal to the price. We will show how to calculate marginal costs, and then take the marginal cost function as the firm supply.

5.3 The firm's production function and cost function

In order to find a firm's cost function, its marginal cost function, and the firm's profits, we need to examine a firm's production capabilities. The production capabilities of a firm are represented by the firm's production function. The production function is used to determine the cost function of a firm in a way that is somewhat similar to the way that a demand function is determined from a utility function. In this course we'll work with production functions that only have two inputs: capital (K)

[3] As a convention, superscripts on the supply indicate the supply of a particular firm.

and labor (L). The cost of capital is its rental rate r; the cost of labor is the wage rate w. So in general we can write an equation for the firm's expenditure on capital and labor in terms of the amount K of capital utilized and its rental rate r, and the amount of labor employed and its wage rate w. The *expenditure equation* is the first key step in the construction of a cost function. It is

$$C = r\,K + w\,L. \tag{21}$$

In the short-run, we assume that the firm can't adjust its level of capital, and we denote its fixed level of capital as $\bar{K}$. Once the fixed level of capital is specified, we derive an equation for the firm's short-run cost in terms of only the wage rate w, the rental rate r, and the amount of output q that the firm produces. When the level of capital is fixed at $\bar{K}$, then the production function is $q = f(\bar{K}, L)$. We always assume that production increases in each of the two inputs K and L. This implies that we can always solve the short-run production function equation $q = f(\bar{K}, L)$ for L. We'll write $L(q)$ to indicate the amount of labor needed to produce q units. This function will depend on the particular production function that we use, but that will be clear from the context for each example that we consider.

Example 1 In our first example the production function is $q = f(K, L) = K^{1/2}\,L^{1/2}$. (This is called a *Cobb-Douglas production function.*) In the short-run, with a fixed capital stock $\bar{K}$, the production function is $q = f(\bar{K}, L) = \bar{K}^{1/2}\,L^{1/2}$.

[Include graph of production function with constrained capital. Use $\bar{K} = 16$ and production levels 2, 4, 6, 8, 10, and 12. The associated levels of labor required are 1, 4, 9, 16, 25, and 36.]

First solve this equation for L by squaring both sides of the equation and then dividing by $\bar{K}$ to get $L(q) = q^2/\bar{K}$. Then substitute this into the expenditure equation (21):

$$\begin{aligned} C(q) &= r\,\bar{K} + w\,L(q) \\ &= r\,\bar{K} + w\,q^2/\bar{K}. \end{aligned}$$

This is the firm's short-run cost function

$$C(q) = r\,\bar{K} + w\,q^2/\bar{K}. \tag{22}$$

For our example, we just need to assume specific values for the rental rate r, the wage rate w, and the fixed level of capital $\bar{K}$. These are $r = 2$, $w = 32$, and $\bar{K} = 16$. Then the short-run cost function for an individual firm is

$$\begin{aligned} C(q) &= r\,\bar{K} + w\,q^2/\bar{K} \\ &= 2\cdot 16 + 32\,q^2/16 \\ &= 32 + 2\,q^2. \end{aligned} \tag{23}$$

5.4 The firm's profit function and output choice

Economic analysis frequently treats a firm as a unified entity that maximizes its profit. Competitive market models typically assume that the output decisions of an individual firm has a negligible influence on the market price. When the market price p is independent of the firm's output decision, its revenue is $R(q) = p\,q$. The profit function for the firm is

$$\begin{aligned} \pi(q) &= R(q) - C(q) \\ &= p\,q - C(q). \end{aligned}$$

In our example we determined the cost function for a firm from its production function and the prices of its inputs. If we substitute the cost function from equation (23) into the profit function the result is

$$\begin{aligned} \pi(q) &= p\,q - C(q) \\ &= p\,q - (32 + 2\,q^2) \\ &= -2\,q^2 + p\,q - 32. \end{aligned}$$

When equation (A.2) from the appendix for "completing the square" is applied to this quadratic equation, the result is

$$\pi(q) = -2\,\left(q - p/4\right)^2 + p^2/8 - 32.$$

This function has its maximum at $q = p/4$. Also, when $q = p/4$ the value of the profit is $\pi(p/4) = p^2/8 - 32$. From this information it is easy to graph the profit function for the firm. Figure 25 shows graphs of the profit function for two different values of the price. The graph on the left shows the profit function when $p = 16$; the graph on the left shows the profit function when $p = 20$. In both cases, profit is maximized when output is $p/4$.

Since the firm maximizes its profit at $q = p/4$, that is the supply function for the firm. We'll use the notation $q_i = S^i(p)$ to denote the supply function of the firm. In a competitive market, there are many firms. The superscript i denotes that we've determine the supply function for firm i.

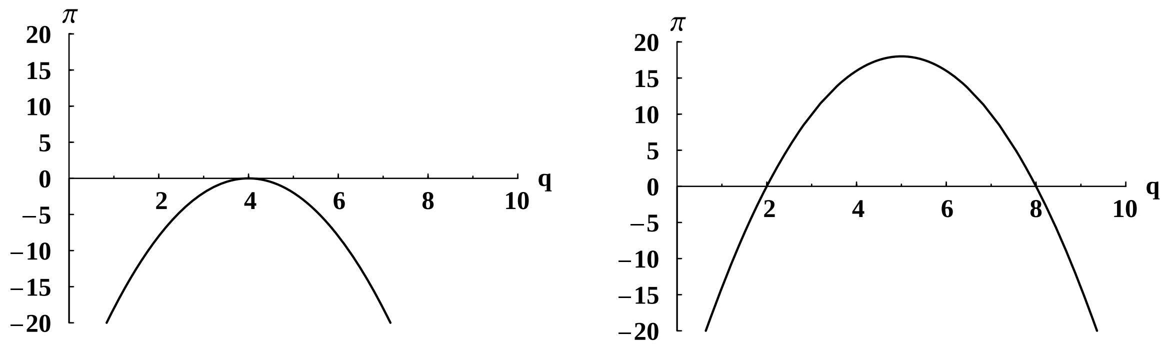


Figure 25: Profit function for $p = 16$ (left) and for $p = 20$ (right).

In this subsection, we've looked directly at the profit function in order to determine how much the firm produces. Since the output of the firm varies as the price of its product changes, we're able to determine the supply of the firm by solving for the profit maximizing output. There is another common approach to finding the profit maximizing output for a competitive firm. The next two sections describe this approach.

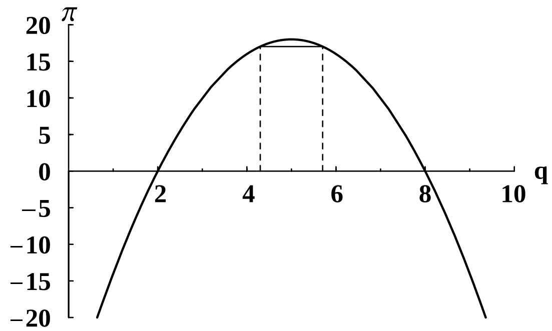
5.5 Marginal profit, marginal revenue, and marginal cost

When the idea of unit values was introduced, the terms "unit value" and "marginal value" were used interchangeably. Similarly, the terms "unit cost" and "marginal cost" were treated as synonyms. This section develops the rationale for using these terms interchangeably, and also shows how marginal values and marginal costs are connected to the profit maximization problem.

The two graphs in figure 25 show a firm's profit function at two different output prices. At either of these two prices, the profit function is flat when it reaches its maximum. The firm's profit function is typically written $\pi(q)$. When output changes from an amount q to $q + \delta$, then the change in profit is from $\pi(q)$ to $\pi(q + \delta)$. The change in profit when output goes from q to $q + \delta$ can be called the marginal profit, and it can be denoted as $M\pi(q)$.[4]

When the change in profit is zero, then the profit function is approximately at its maximum.

[4] Marginal revenue and marginal cost are terms used in almost every microeconomics text, but marginal profit isn't a common term in microeconomics. Marginal revenue and marginal cost are useful though precisely because of their connection to marginal profit, so it seems helpful to describe marginal profit first and then work with marginal revenue and marginal cost.



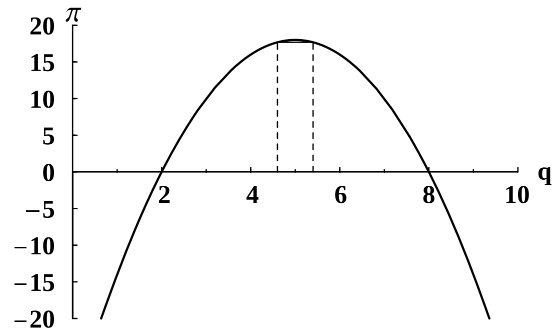


Figure 26: Marginal profit function for $p = 20$

5.6 Marginal cost and average cost

There are two key functions that we use in order to understand the behavior of firms in a competitive market. The first of these is the marginal cost function, which is used to determine the amount that a firm will supply to the market at each price. The second function is the average cost, which is used to determine whether the firm earns a profit that is greater than, less than, or equal to the market return on investments.

5.6.1 Marginal cost and firm supply

The marginal cost of the firm is the addition to cost as the firm goes from producing $q - 1$ units to producing q units. This difference is the marginal cost equation:

$$MC_D(q) \equiv C(q) - C(q-1). \tag{24}$$

The marginal cost when output is q is the increase to total cost when production increases from $q - 1$ to q units.

Equation (24) determines the marginal cost when output changes by one unit. Often it is useful to consider the marginal cost when output changes by only a small amount, so we will define the marginal cost as the change in cost when output increases from $q - \delta$ to q units. This is natural if the output is something continuous, like megawatt-hours of electricity produced by a utility. It's also natural if the quantity q in our model is something like the number gallons of milk, bicycles, automobiles, or cell phones, with all of these quantities measured in the number of thousands of units produced.

The change in cost when output increases from $q - \delta$ to q units is $\Delta C(q) = C(q) - C(q - \delta)$. The change in cost per unit of additional output is

$$MC_C(q) \equiv \frac{1}{\delta}\,(C(q) - C(q-\delta)). \tag{25}$$

When we work with models, such as a linear supply function, we'll always use the second form for the marginal cost. This form is indicated as $MC_C(q)$ with the subscript C because it is the continuous version. The first form is indicated as $MC_D(q)$ with the subscript D because it is the discrete version. We'll use the discrete form only when we discuss experiments, where the unit costs of a seller are the marginal costs of production.

When $MC(q)$ is less than the market price, the firm's profit increases if the firm produces an additional unit of output, because the increment to cost for unit q is less than the price received in the market. In order to illustrate how the marginal cost is calculated and how it is used to determine the firm's supply function, example 1 with the Cobb-Douglas production function is continued in example 2.

Example 2 For the Cobb-Douglas production function in example 1, we found the cost function $C(q) = 32 + 2\,q^2$. The marginal cost for this example is

$$\begin{aligned}
MC(q) &= \frac{1}{\delta}\,(C(q) - C(q-\delta)) \\
&= \frac{1}{\delta}\,((32 + 2\,q^2) - (32 + 2\,(q-\delta)^2)) \\
&= \frac{1}{\delta}\,(2\,q^2 - 2\,(q-\delta)^2) \\
&= \frac{2}{\delta}\,(q^2 - (q^2 - 2\,\delta\,q + \delta^2) \\
&= \frac{2}{\delta}\,(2\,\delta\,q - \delta^2) \\
&= 4\,q - 2\,\delta.
\end{aligned}$$

If δ is very small, marginal cost $MC(q)$ is equal to $4\,q$:

$$MC(q) = 4\,q. \tag{26}$$

As noted above (in the last paragraph of Section 5.2), a firm will supply units so long as the cost of the unit is below the market price. *Marginal cost* is a typical microeconomic term for what we called unit costs in the experiments, so the supply of the firm is determined by the condition that the firm selects its quantity to supply so that on the last unit, the marginal cost is equal to the price. So we write $p = MC(q)$ and solve this equation for q to find the output that maximizes the firm's profit. In the example, $p = 4\,q$. If we solve this for q we get $q = p/4$ and if we denote the supply of firm i by $q^i = S^i(p)$ then the supply function for the firm is $S^i(p) = p/4$.

5.6.2 Average cost and economic profit

Since the firm's short-run cost function in equation (22) includes the rental rate for capital, the average cost also includes the capital cost. Therefore the firm makes an economic profit if the price exceeds the short-run average cost, and the firm incurs an economic loss if the price is below the short-run average cost. The firm's *average cost* is its total cost $C(q)$ divided by its output q. The average cost function is

$$AC(q) \equiv C(q)/q. \tag{27}$$

The economic profit of a firm is the difference between its revenue and its total cost. In a competitive market, since we assume that price is independent of the output of any particular firm – that is, a firm in a competitive industry cannot affect the market price – the firm's revenue is simply the market price p times the quantity that the firm produces: $R(q) = p\,q$. The short-run cost function $C(q)$ is derived from the production function, capital level, and input prices. We can use these revenue and cost functions to determine the profit of a firm in a competitive market. This profit function is

$$\pi(q) = p\,q - C(q). \tag{28}$$

A minor rearrangement of the profit equation demonstrates that the sign of the economic profit is determined by a comparison of price and average cost. In order to obtain this representation of the profit function, solve the equation $AC(q) = C(q)/q$ for $C(q)$ to get $C(q) = q\,AC(q)$ and substitute $C(q) = q\,AC(q)$ into equation (28):

$$\begin{aligned} \pi(q) &= p\,q - C(q) \\ &= p\,q - q\,AC(q) \\ &= q\,(p - AC(q)). \end{aligned} \tag{29}$$

From this equation, it's apparent that the firm earns an economic profit if its average cost is less than the market price.

Example 3 Calculation of the average cost is illustrated by returning to the Cobb-Douglas production function from examples 1 and 2. For the example, the average cost is

$$\begin{aligned} AC(q) &= C(q)/q \\ &= (32 + 2\,q^2)/q \\ &= 32/q + 2\,q. \end{aligned} \tag{30}$$

The firm's output at the equilibrium price $p_2^* = 20$ is $q^{i\,*} = 5$, so its average cost is $AC(5) = 32/5 + 2 \cdot 5 = 16.4$. The firm receives the price $p_2^* = 20$ on each unit, so its economic profit – calculated using equation (29) – is $\pi(5) = 5\,(20 - 16.4) = 18$. This is positive, so each firm in the industry earns a return that exceeds the market rate of return. As a result of these economic profits, firms either expand their production or new firms enter the market.

In order to understand how large these economic profits are, we can compare the accounting profit of the firm to its capital costs. The accounting profit of the firm should at least equal its capital cost, because otherwise the firm would be better off to sell its capital equipment to pay off the loans that it uses to finance its equipment.

5.6.3 Accounting profit and economic profit

In order to get a reference point for the size of the accounting profit of this firm, we can calculate the accounting profit that the firm had before the shift to demand. Accounting profit is the difference between the revenue of a firm and its marginal costs. This is the same as the producer's surplus, except that it is measured for the firms individually. Before the demand increase, the equilibrium price was $p_1^* = 16$. From the firm supply function $S^i(p) = p/4$ we know that the firm produced $q^{i\,*} = S^i(16) = 4$ units before the demand increase. Its price was $p_1^* = 16$, so the accounting profit in the original equilibrium was $PS = \frac{1}{2}\,q^{i\,*}\,(p_1^* - s_0) = \frac{1}{2} \cdot 4 \cdot 16 = 32$. This is exactly equal to the capital cost, which is the rental rate of capital $r = 2$ times the amount of capital utilized $\bar{K} = 16$.

The accounting profit in the competitive equilibrium after the increase in demand is determined in the same way, but with the new equilibrium price $p_2^* = 20$ and the new firm output $q^{i\,*} = S^i(20) = 5$. From the graph on the left side of figure 27 we can see that the accounting profit or producer's surplus in this case is the area below the equilibrium price $p_2^* = 20$ and above the marginal cost. This area is $PS = \frac{1}{2} \cdot 5 \cdot 20 = 50$. This accounting profit is large relative to the accounting profit in the original equilibrium. The firm is earning a rate of return on its investment that is more than 50% above the market return.

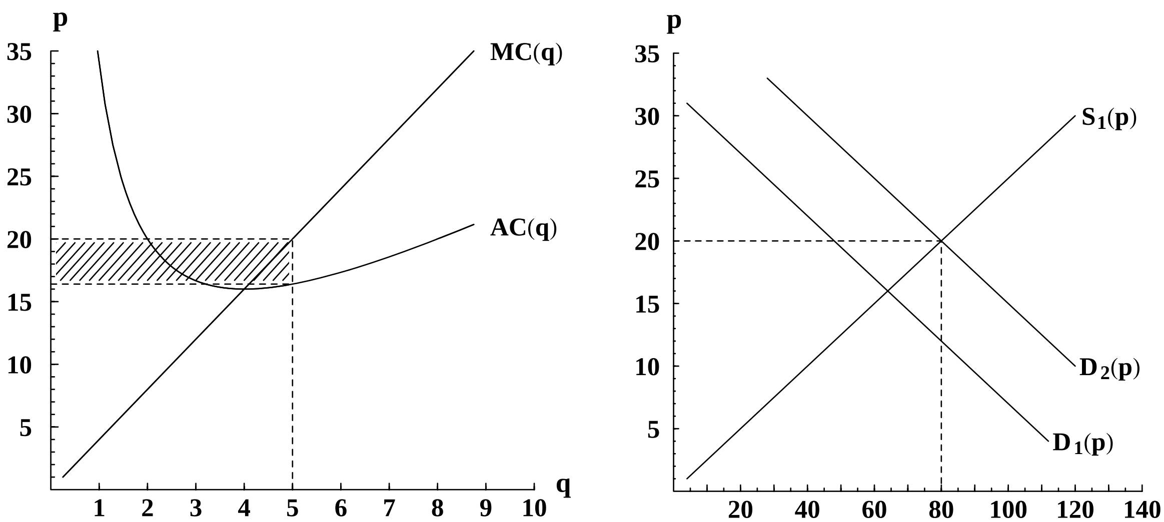


Figure 27: Firm conditions (left) and market conditions (right).

The two graphs in figure 27 illustrate most of the key ideas from Section 5.6. The graph on the left shows the marginal cost and average cost for the typical firm in the market. The marginal cost function shown is the one that we found in example 2; the average cost is from example 3. For any market price p, the firm will supply the quantity at which its marginal cost is equal to the market price. In the example in the figure, the price is $p_2^* = 20$. This is a short-run equilibrium price, because the supply and demand are equal at that price (as the graph on the right side of the figure shows). At that price, each firm produces $q^{i*} = 5$ units, because when production is 5 units, the marginal cost is equal to the market price. When production is 5 units, the average cost of production – from equation (30) and its application in example 3 – is $AC(5) \doteq 16.4$. The gap between the market price and the average cost of the firm represents the economic profit that the firm earns on each unit of production. The shaded area in the figure shows the economic profit; in example 3 we found that the economic profit is 18. Finally, the accounting profit is the profit on each unit, summed over all of the units that the firm sells. Graphically, that is the area below the price and above the marginal cost, which is 50. Accounting profit is the sum of the capital cost ($r\bar{K} = 2 \cdot 16 = 32$) and the economic profit (18).

5.7 Market adjustment

The discussion at the end of Section 5.6.3 describes the conditions of a firm in a market with economic profits, and figure 27 depicts these conditions. Since each existing firm earns a large profit relative to the cost of its capital investment, we would expect to see new firms enter this industry, or to see existing firms expand their capacity. In practice, industry expansion typically involves a combination of these two responses. In order to keep the model fairly simple, we'll only consider the two possibilities separately. Analysis of firm entry is fairly simple, so that case is evaluated first.

5.7.1 Firm entry and long-run equilibrium

Since there are large profits for the firms relative to the cost of its investment in capital, we would expect to see new firms enter this industry. Suppose that there are eight new firms that enter the industry, each identical to the sixteen that existed originally. There will then be twenty-four identical firms each with the supply function $S^i(p) = p/4$ so the market supply will be $S_3(p) = 24\,(p/4) = 6\,p$. The demand curve is $D_2(p) = 160 - 4\,p$. After the new firms enter in response to the increase in demand the equilibrium price will be $p_3^* = 16$ (because $p_3^* = 16$ is the price where the supply $6\,p$ equals the demand $160 - 4\,p$). We noted above that when the price is 16, firms earn an accounting profit equal to their capital cost. It is also easy to see that the economic profit is zero: the firm output is $S^i(16) = 4$ and the average cost at output 4 is $AC(4) = 32/4 + 2 \cdot 4 = 16$, which is equal to the price. The firms just cover their costs, including their capital costs. Therefore the firms earn zero economic profit. This means that once the price has settled back to the point where there is no more economic profit, there is no longer an incentive for new firms to enter the industry, or for existing firms to expand. That state is called *long-run equilibrium.*

5.7.2 Firm expansion and long-run equilibrium

Capacity expansion may also occur when existing firms increase their level of capital and thereby increase their supply at each price. We can evaluate firm expansion in long-run equilibrium with the Cobb-Douglas production function that we've used previously in examples 1 - 3. The example begins with analysis in the continuous case, and then the discrete case is evaluated.

The economic profits of the firm in examples 1 through 4 would naturally lead the firm to consider an expansion of its capacity. In our model this is achieved by an increase to the capital stock of the firm. The analysis proceeds in a fairly simple way. In our analysis of the short-run behavior of the firm, we started with the short-run cost function in equation (22) from Section 5.3. The capital stock of each firm was fixed at $\bar{K} = 16$ in the short-run, and this short-run cost function determines their marginal cost and average cost functions. We can repeat this analysis with an increase in the capital stock from $\bar{K} = 16$ to $\bar{K} = 24$.

Exercise 1 Substitute the input prices and new capital stock into the short-run cost equation. Show that the new short-run cost equation is $C(q) = 48 + 4/3\, q^2$.

Exercise 2 Use the cost equation from exercise 1 to determine the marginal cost function. The calculation of marginal cost is similar to the calculation in example 2. Show that the marginal cost for a firm after its expansion is $MC(q) = \frac{8}{3}\, q$. Use this to find the supply function for an individual firm and show that the market supply function is $S(p) = 6\, p$.

Exercise 3 Find the equilibrium price in the market. What is the economic profit of a firm at the new equilibrium price?

Although there is currently no long-run equilibrium experiment, we can consider how one would work, and what decisions a seller would need to make. Selection of a capital level is an important element of a seller's decision problem. The choice of a capital level affects the firm's cost function, and through the cost function it affects the marginal cost function. The cost function, with the factor prices fixed at $r = 2$ and $w = 32$, is

$$C(q) = 2\,\bar{K} + 32\, q^2/\bar{K}. \tag{31}$$

One main difference between the model and an experiment is that the seller's marginal cost function is constructed from equation (24) rather than equation (25). For the cost function in equation (31), the discrete version of the marginal cost is

$$\begin{aligned} MC(q) &= C(q) - C(q-1) \\ &= 2\,\bar{K} + 32\, q^2/\bar{K} - (2\,\bar{K} + 32\,(q-1)^2/\bar{K}) \\ &= 32\, q^2/\bar{K} - 32\,(q-1)^2/\bar{K} \\ &= \frac{32}{\bar{K}}\,(q^2 - (q^2 - 2\,q + 1)) \\ &= \frac{64}{\bar{K}}\, q. \end{aligned} \tag{32}$$

In the experiments, a seller has a fixed schedule of marginal costs. We can think of that schedule as the one that comes from a fixed level of capital investment. In a long-run equilibrium experiment, prior to the start of a trading period, a seller would choose a level of capital investment. The level of capital investment then determines the marginal cost schedule. Equation (32) defines the relationship between the capital level and the marginal cost function for the Cobb-Douglas example. Marginal costs in the equation decrease as the amount of capital increases. Table 1 shows the marginal cost schedules for several different capital levels.

	Unit Number							
$\bar{K}$	1	2	3	4	5	6	7	8
16	2	6	10	14	18	22	26	30
17	2	6	9	13	17	21	24	28
18	2	5	9	12	16	20	23	27
19	2	5	8	12	15	19	22	25
20	2	5	8	11	14	18	21	24

Table 1: Marginal cost schedules for several capital levels in the Cobb-Douglas example.

For $\bar{K} = 16$, we know that at the long-run equilibrium price $p^* = 16$ the accounting profit just covers the capital cost. (This argument is detailed in the last paragraph of Section 5.6.3.) The accounting profit is the difference between price and unit cost for each unit sold. When the capital stock is $\bar{K} = 16$, the first four units can be sold profitably at the price $p^* = 16$. The profits on these units are 14, 10, 6 and 2. These profits sum to 32, which is the cost of $\bar{K} = 16$ units of capital at the rental rate $r = 2$.

At a long-run equilibrium price, the economic profits of firms are zero, so the average cost of each firm is equal to the market price. From equation (31), the average cost function for a firm with capital stock $\bar{K}$ is

$$\begin{aligned} AC(q) &= \frac{2\,\bar{K}}{q} + \frac{32\,q}{\bar{K}} \\ &= \frac{32}{q\,\bar{K}} \left(q^2 + \frac{\bar{K}^2}{16} \right) \\ &= \frac{32}{q\,\bar{K}} \left(q^2 - \frac{2\,\bar{K}}{4}\,q + \frac{2\,\bar{K}}{4}\,q + \frac{\bar{K}^2}{16} \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{32}{q\,\bar{K}}\left(q^2 - \frac{2\,\bar{K}}{4}\,q + \frac{\bar{K}^2}{16}\right) + \frac{32}{q\,\bar{K}}\,\frac{2\,\bar{K}}{4}\,q \\
&= \frac{32}{q\,\bar{K}}\left(q^2 - \frac{2\,\bar{K}}{4} + \frac{\bar{K}^2}{16}\right) + 16 \\
&= \frac{32}{q\,\bar{K}}\left(q - \frac{\bar{K}}{4}\right)^2 + 16.
\end{aligned}$$

The average cost equation takes on its minimum at $q = \bar{K}/4$ and $AC(\bar{K}/4) = 16$ at its minimum. This minimum value is independent of the level of capital, so the long-run average cost is constant for the Cobb-Douglas production function. For our immediate problem, this has one important consequence. At every level of capital, the economic profit should be zero when the price is $p = 16$, because the price is equal to the minimum of average cost. We've already checked this for $\bar{K} = 16$.

Exercise 4 From the marginal costs in table 1, calculate the economic profit when $\bar{K} = 17$ and the market price is equal to the minimum of average cost: $p = AC(\bar{K}/4) = 16$. Calculate the economic profit for $\bar{K} = 18, 19$, and 20.

Exercise 5 From the marginal costs in table 1, calculate the economic profit when $\bar{K} = 16$ if the market price increases to $p = 20$. Calculate the economic profit for $\bar{K} = 18$ and $\bar{K} = 20$ also.

Consider the problem faced by a firm if the demand shift from figure 24 occurs. Before any adjustment to capital stocks occur, we expect that the price will adjust to the new short-run equilibrium price $p_2^* = 20$. If each firm expects this price to persist for a long time, then they would continue to make an economic profit indefinitely, and the size of the economic profit would be proportional to their capacity. After some experience with market adjustment though, they will probably come to expect expansion by other firms and a price decrease back to the minimum of average cost.

Capital markets have an important role in the long-run equilibrium model. Firms could of course accumulate their economic profits when the price increases above average cost, but that approach would increase the time required for adjustment back to the long-run equilibrium. Firms also can turn to capital markets to finance their expansion. The long-run equilibrium model suggests that even if capacity expansion is rapid, and the price quickly returns to the long-run equilibrium price (where price is equal to the minimum of average cost and economic profits are zero), the accounting profits in the new long-run equilibrium will meet the costs of borrowing for capacity expansion.

How far and how fast the price falls depends on the rate of capacity expansion. It is possible that the price will adjust only slowly in response to positive economic profits. It's also possible that the market will adjust almost instantly to the new long-run equilibrium. It's also possible that there will be overexpansion, that the price will fall below the long-run equilibrium price, and that there will be a subsequent series of capacity reductions or firm closures. The first of these three cases – slow expansion – seems to be the current situation in the energy industry. There has been a rapid expansion of demand due largely to the increase in demand from China and India, but the expansion is slow due to the natural resource constraints. The last of the three cases – over-expansion followed by rapid contraction – occurred in the technology sectors, with the expansion phase in the later half of the 1990's and the contraction beginning in 2000.

5.8 Long-run equilibrium with taxes

In Section 3.4 on taxes in the competitive model, we found that a sales tax decreases market demand and therefore leads to a drop in the market price. Buyers pay the tax but recover a portion of what they pay because the equilibrium price declines. The decrease in the equilibrium price though has important implications for firms. If the market is in long-run equilibrium before the tax is imposed, then the price decline disrupts the long-run equilibrium. Firms that were earning zero economic profit (so that their accounting profit just met their variable costs and their capital costs) begin to experience negative economic profit. These negative economic profits lead firms to reduce their investment in order to reduce the scale of their losses. When many of the firms all reduce their investment the market supply decreases and the price rises. Eventually, the price rises back to the point where firms will again make zero economic profit. At that point the market will have returned to a new long-run equilibrium. The key difference between the short-run equilibrium model with a sales tax and the long-run model is that in the long-run model, firms adjust to the price decrease that the tax causes. This section examines the implications of the responses by firms in the context of an example with a Cobb-Douglas production function and a linear demand function.

5.8.1 The firm's production function and short-run cost function

Suppose that originally the demand in a market is $D_1(p) = 300 - 5p$ and the supply in the market is $S_1(p) = 5p$. The supply and demand are shown on the left side of figure 28. Then a sales tax of \$6 per unit is placed on the buyers. Demand of the buyers then shifts to

$$D_2(p) \quad = \quad 300 - 5(p + 6)$$

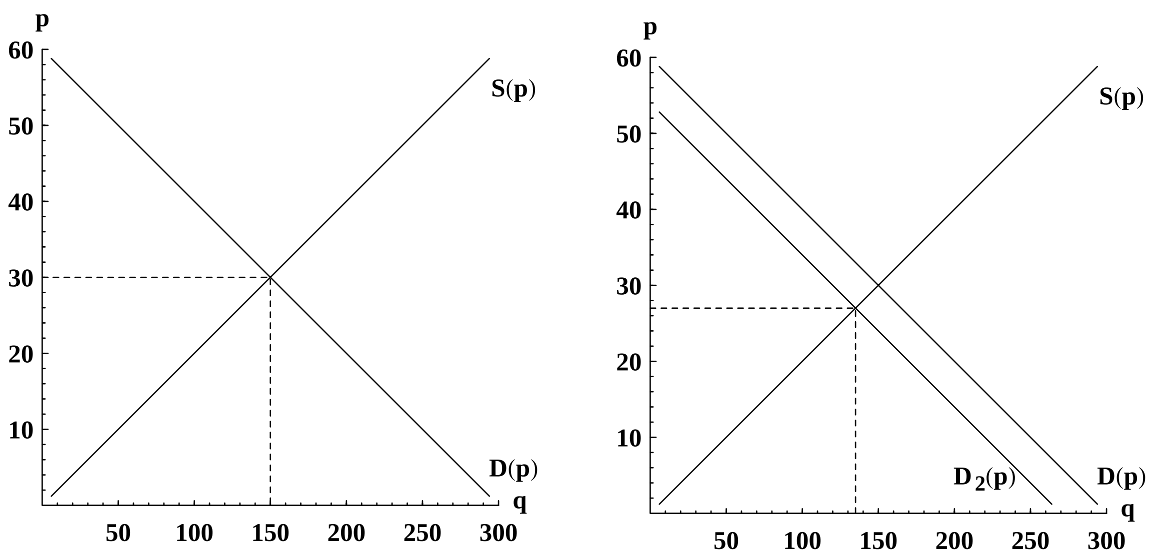


Figure 28: Market supply and demand before (left) and after (right) a sales tax is imposed.

$$= \quad 270 - 5\,p$$

as shown on the right side of figure 28.

As with our previous examples, the market supply comes from the marginal cost functions of the firms in the market. We'll define the number of firms in the market, their production functions, factor prices, and levels of capital investment for the firms that are consistent with the market supply shown in figure 28.

Assume that there are ten firms each with the production function $q = f(K, L) = \bar{K}^{1/2}\, L^{1/2}$. Assume that the factor prices are $r = 9$ and $w = 25$ and each firm originally has capital level $\bar{K} = 25$. With this production function, we've seen that the cost function for each firm is

$$\begin{aligned} C(q) &= r\,\bar{K} + w\,L(q) \\ &= r\,\bar{K} + w\,q^2/\bar{K}. \end{aligned} \tag{33}$$

With the factor prices $r = 9$ and $w = 25$ and capital level $\bar{K} = 25$, the cost function is

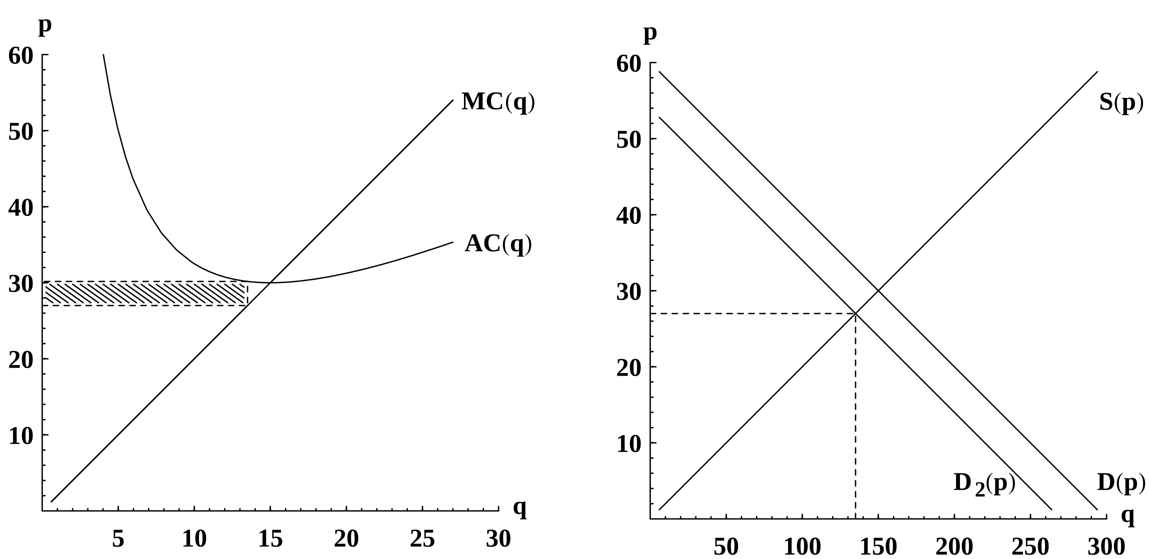
$$\begin{aligned} C(q) &= 9 \cdot 25 + 25\,q^2/25 \\ &= 225 + q^2. \end{aligned} \tag{34}$$

5.8.2 The firm's marginal cost and average cost functions

The marginal cost function for each firm is

$$\begin{aligned} MC(q) &= \frac{1}{\delta}\left(C(q) - C(q-\delta)\right) \\ &= \frac{1}{\delta}\left(225 + q^2 - (225 + (q-\delta)^2)\right) \\ &= \frac{1}{\delta}\left(q^2 - (q-\delta)^2\right) \\ &= \frac{1}{\delta}\left(q^2 - (q^2 - 2\,\delta\,q + \delta^2)\right) \\ &= \frac{1}{\delta}\left(2\,\delta\,q - \delta^2\right) \\ &= 2\,q - \delta \\ &\doteq 2\,q. \end{aligned} \tag{35}$$

The supply function for a firm in this market is determined by solving the profit maximization condition $p = MC(q)$ for $q = S^i(p)$. In this case the profit maximization condition is $p = 2\,q$ so $q = S^i(p) = p/2$. The average cost function for the firm, using the definition of the average cost $AC(q) = C(q)/q$ and the cost function from equation (34), is $AC(q) = 225/q + q$. The graph on the right side of the figure 29 shows the marginal cost and average cost for a typical firm in this market.



(a) Market conditions (b) Firm conditions

Figure 29: Firm conditions (left) and market conditions (right) after the tax is imposed.

At the original equilibrium price $p_1^* = 30$ each firm produces $q_1^* = S^i(p_1^*) = S(30) = 15$ units.

The average cost at output $q_1^* = 15$ is $AC(q_1^*) = AC(15) = 225/15 + 15 = 30$. Since price minus average cost is zero, the economic profit is zero: the original configuration of supply and demand was a long-run equilibrium.

After the tax is imposed, the equilibrium price declines. The market supply is $S(p) = 5\,p$ and market demand fell to $D_2(p) = 270 - 5\,p$ so $p_2^* = 27$. Each firm therefore supplies $q_2^* = S^i(p_2^*) = S(27) = 27/2$ units. The average cost of a firm at this level of output is

$$\begin{aligned} AC(q_2^*) &= AC(27/2) \\ &= \frac{225}{27/2} + 27/2 \\ &= \frac{450}{27} + 27/2 \\ &= \frac{50}{3} + 27/2 \\ &= 181/6. \end{aligned}$$

The economic profit for each firm in the new equilibrium is

$$\begin{aligned} \pi(q) &= (p_2^* - AC(q_2^*))\,q_2^* \\ &= \left(27 - \frac{181}{6}\right)\frac{27}{2} \\ &= \left(\frac{162}{6} - \frac{181}{6}\right)\frac{27}{2} \\ &= -\frac{19}{6}\,\frac{27}{2} \\ &= -\frac{19}{2}\,\frac{9}{2} \\ &= -42.75. \end{aligned}$$

In the original equilibrium, prior to the shift to demand, the market price was $p_1^* = 30$. Output for each firm was $q_1^* = 15$. Since $S^i(p) = p/2$, the supply function intersects the price axis at $p = 0$. So the area of the producer's surplus triangle is $PS^i = \frac{1}{2}\cdot 15\cdot 30 = 225$. The capital cost for the firm is $r\,\bar{K} = 9\cdot 25 = 225$, which is equal to the accounting profit. The firm meets its capital cost exactly, and has economic profit zero, because economic profit is accounting profit minus capital cost.

After the demand decrease due to the imposition of the sales tax, the equilibrium price declines to $p_2^* = 27$. Output for each firm declines to $q_2^* = 27/2$. Since $S^i(p) = p/2$, the supply function again intersects the price axis at $p = 0$. So the area of the producer's surplus triangle is $PS^i = \frac{1}{2}\,\frac{27}{2}\,27 = 182.25$. The capital cost is still $r\,\bar{K} = 9\cdot 25 = 225$. Accounting profit minus capital cost is $182.25 - 225 = -42.75$, which we found is the economic profit after the shift to demand. Since the market price fell below the average cost of production for the firms, they begin to experience an economic loss. Economic profit is accounting profit minus capital cost, which in this case is negative.

5.8.3 Economic losses and firm contraction

In response to the economic losses that result from the decrease to demand, firms reduce their capital investment. The loss of a firm is the loss per unit (which equals the difference between the market price and the firm's average cost) times the number of units sold. When there were positive economic profits we found that expansion led to an increase in the firm's supply and an increase in economic profit. When there are economic losses, a reduction in capital reduces the firm's supply and thereby reduces economic losses.

Suppose that each firm reduces its capital stock to $\bar{K} = 20$. For the production function and factor prices in this model, we found the cost function in equation (33). When the capital stock is $\bar{K} = 20$ the cost function for the firm is

$$\begin{aligned} C(q) &= r\,\bar{K} + w\,q^2/\bar{K} \\ &= 9 \cdot 20 + 25\,q^2/20 \\ &= 180 + 1.25\,q^2. \end{aligned} \tag{36}$$

A calculation similar to the calculation in equation (35) shows that, after the reduction to the capital level, the marginal cost function for the firm is

$$MC(q) = 2.5\,q.$$

From the profit maximization condition $p = MC(q)$ for a firm in a competitive market, the firm chooses q so that $p = 2.5q$ or $q = 0.4\,p$. This is the supply function for the firm: $S^1(p) = 0.4\,p$. There are ten firms in the market, each with an identical supply function, so the market supply after all firms reduce their capital level is $S_3(p) = 10 \cdot 0.4\,p = 4\,p$.

The contraction leads to a decrease in the market price. The demand is $D_2(p) = 270 - 5\,p$ after the tax is imposed. Once firms have all reduced their capital stock to $\bar{K} = 20$, the new market supply is $S_3(p) = 4\,p$. The new equilibrium price is the solution to $D_2(p) = S_3(p)$, which in this case is the equation $270 - 5\,p = 4\,p$. So $p_3^* = 30$.

In order to determine whether this price is a new long-run equilibrium price, we need to determine whether firms still experience an economic loss at this price. The economic profit of a firm can be written as

$$\pi_i(q) = [p - AC(q)]\, q,$$

so we need to determine what the output is for a typical firm, and what its average cost is at that output. The output at $p_3^* = 30$ is determined from the supply function for the firm, which is now $S_3(p) = 4\,p$. So $q_i^* = 0.4 \cdot 30 = 12$. We found the cost function for the firm is equation (36). Its average cost function is therefore

$$AC(q) = 180/q + 1.25q.$$

With $q = 12$, the average cost is $AC(12) = 30$. Once again, the firm's average cost is equal to the market price, so the economic profit of the firm is zero. Since the firms now meet both their variable cost and their capital cost, they have no further incentive to reduce their capital investment (nor any incentive to increase it): the market has returned to a new long-run equilibrium.

The accounting profit for the typical firm in this new equilibrium is determined from the usual formula:

$$\begin{aligned} PS &= \frac{1}{2}\, q_i^* \,(p^* - s_0) \\ &= \frac{1}{2} \cdot 12 \cdot 30 \\ &= 180. \end{aligned}$$

As always in this model, the economic profit (zero) is equal to the accounting profit (180) minus the capital cost ($r\,\bar{K} = 9 \cdot 20$).

6 Monopoly

A monopolist is the only seller of a commodity or service in its market. Competitive markets have many sellers, so a monopoly market is the opposite extreme from a competitive market. Nevertheless, in both the competitive model and the monopoly model, there is no strategic behavior required from the sellers. In a competitive market we found that each firm follows a simple decision rule to determine its output. In the competitive model, each seller simply chooses an output level so that for the market price that it observes, marginal cost equals the price. In this section, we'll find a simple decision rule for a monopolistic firm. In the monopoly model we'll find that a monopolistic firm maximizes its profit when marginal revenue equals marginal cost.

This section describes how the equilibrium output and price are determined for a monopolist. The equilibrium output and price are then used to determine consumers' surplus, monopoly profits, and the surplus forgone due to the monopolist's restriction of its output.

As with the competitive model in both the short-run and the long-run, we'll work with a linear demand function and with a linear marginal cost function, in order to simplify the analysis. The demand function has the form

$$q = D(p) = a - b\,p \tag{37}$$

and marginal cost function has the form $MC(q) = c + d\,q$. In any type of market (competitive, monopolistic, or something in between) the profit for a firm is always $Profit = Revenue - Cost$, and when a firm sells its output at a single price, $Revenue = Price \times Quantity$. For the monopolist, the key step in determining its profit function is to notice that there is an inverse relationship between the quantity that the monopolist sells and the price that it receives: as the quantity sold increases the price declines. We can solve equation (37) for p and then have the price in terms of the quantity. The result is

$$p = D^{-1}(q) = (a - q)/b,$$

which is called the *inverse demand*. The graph of the inverse demand function is the same as the graph of the demand function, since both describe the same relationship between price and quantity demanded. The only difference is that the demand function gives the quantity demanded as a function of the price, whereas the inverse demand gives the price as a function of the quantity demanded. An example of the graph of an inverse demand function is shown on the left side of figure 30.

6.1 The monopolist's revenue

Revenue, which is price times quantity, can now be written in terms of the monopolist's output q by substituting the inverse demand for the price:

$$\begin{aligned} R(q) &= p\,q \\ &= (a-q)/b\,q. \\ &= \left(a\,q - q^2\right)/b\,. \end{aligned} \tag{38}$$

6.2 The monopolist's production function and cost function

6.3 The monopolist's profit function

6.4 Marginal revenue, marginal cost, and profit maximization

The profit for a firm that produces q units can be written $\pi(q) = R(q) - C(q)$, where $C(q)$, as usual, is its cost of producing q units. The change in profit for a firm when it increases its output from $q-\delta$ to q is then $M\pi(q) = \frac{1}{\delta}\left(\pi(q) - \pi(q-\delta)\right)$. In terms of revenue and cost, the change in profit is

$$\begin{aligned} M\pi(q) &= \frac{1}{\delta}\left(R(q) - C(q) - (R(q-\delta) - C(q-\delta)\right) \\ &= \frac{1}{\delta}\left(R(q) - R(q-\delta) - (C(q) - C(q-\delta)\right) \\ &= MR(q) - MC(q). \end{aligned}$$

In order to maximize profit, the monopolist should continue to increase its output so long as the marginal revenue for the added unit of output is greater than marginal cost of the unit, because in that case, the increment to profit is positive. So we can determine the profit maximizing output once we have found the marginal revenue function.

6.5 The monopolist's marginal revenue

The marginal revenue is determined by a simple calculation based on the revenue function in equation (38). The marginal revenue is

$$MR(q) = \frac{1}{\delta}\left(\frac{1}{b}\left(a\,q - q^2\right) - \frac{1}{b}\left(a\,(q-\delta) - (q-\delta)^2\right)\right)$$

$$
\begin{aligned}
&= \frac{1}{\delta}\frac{1}{b}\left(a\,q - q^2 - a\,(q-\delta) + (q-\delta)^2\right) \\
&= \frac{1}{\delta}\frac{1}{b}\left(-q^2 + a\,\delta + (q-\delta)^2\right) \\
&= \frac{1}{b}\,(a - 2\,q + \delta).
\end{aligned}
$$

Example Suppose that the market inverse demand is the linear approximation to the market demand from the monopoly experiment: $p = D^{-1}(q) = 45 - q$. The marginal cost schedule is also a linear approximation from the monopoly experiment: $MC(q) = 3 + q$. These are both shown in the graph of the left side of figure 30. The graph on the right of the same figure also shows the marginal revenue function, which is $MR(q) = 46 - 2\,q$ for this example.

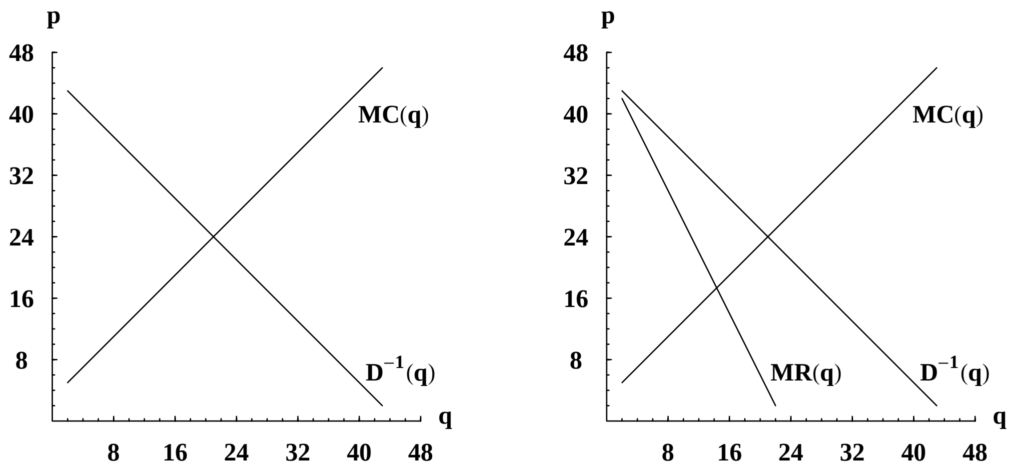


Figure 30: Monopolist's inverse demand, marginal cost, and marginal revenue

6.6 Monopoly output and price

The monopolist will want to continue to produce additional output so long as $MR(q) \geq MC(q)$. With the marginal revenue function $MR(q) = 46 - 2\,q$ and the marginal cost function $MC(q) = 3 + q$, the marginal revenue at $q^* = 14$ is $MR(14) = 18$ and the marginal cost at $q^* = 14$ is $MC(14) = 17$. For $q = 15$ marginal revenue is less than marginal cost, so $q^* = 14$ is the output level that maximizes the monopolist's profit.

6.7 Monopoly price

The monopoly price is determined by finding the highest price that the monopolist can charge and still sell the monopoly quantity of output. For the example with $p = D^{-1}(q) = 45 - q$ and $MC(q) = 3 + q$,

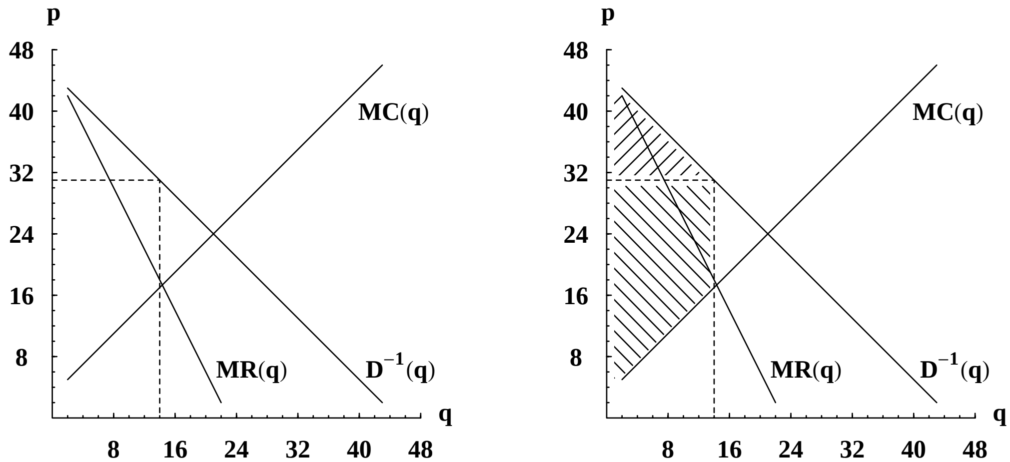


Figure 31: Monopoly price and quantity (left) and consumers' and producer's surplus (right)

the monopoly price is $p^* = D^{-1}(14) = 31$. The dashed lines in the graph on the left side of figure 31 show the monopolist's profit maximizing output and price for the example.

6.8 Surplus and efficiency

Surplus and efficiency for monopoly are defined just as they are for competitive markets. The consumers' surplus is the difference between the buyer's value for a unit and the market price, summed over all units purchased. Graphically this is the area below the demand (or inverse demand) and above the market price. The producer's surplus (which is also the monopolist's accounting profit) is the difference between the price received for the unit and the marginal cost of the unit, summed over all units sold. Graphically this is a trapezoid below the price, above the marginal cost, and to the left of the monopoly output level. In order to calculate the producer's surplus, split the trapezoid into two parts. If the equilibrium quantity is q^*, then the marginal revenue at q^* is $MR(q^*)$. The area below the monopoly price p^* and above $MR(q^*)$ is a rectangle with area $(p^* - MR(q^*))\,q^*$. The other portion of the trapezoid below $MR(q^*)$ and above $MC(q)$ is a triangle with area $\frac{1}{2}\,(MR(q^*) - MC(0))\,q^*$. The producer's surplus is the sum of the area of the rectangle and the area of the triangle.

Both of these measures are shown in the graph on the right side of figure 31.

6.9 Deadweight loss

The sum of consumers' and producer's surplus in monopoly differs from the sum of these two measures in the competitive model. The difference is the triangular area to the right of the monopoly output, below the demand, and above the marginal cost. In the example this area is $\frac{1}{2}(31-17)(21-14) = 49$. The surplus at the efficient outcome is $\frac{1}{2}(45-3)\,21 = 441$, so the deadweight loss is 1/9 of the total possible surplus.

7 Duopoly

The analysis of intermediate cases between competition and monopoly involve a feature that is absent from both monopoly and competition. When there are several sellers, but not so many that the market is competitive, each seller must take account of the effect that their output decisions will have on other sellers, and also must respond to or anticipate the output decisions made by the other sellers. The simplest case of the this type is duopoly, which has two sellers and many buyers, so this section examines a duopoly market.

In order to compare the outcome in duopoly with the outcome in competition, the same market demand function $q = D(p) = 45 - p$ is used in the example considered in this section. (The market inverse demand function also remains the same, and it is $p = D^{-1}(q) = 45 - q$.) The aggregated marginal cost schedule is also kept the same. In order that the aggregated marginal cost schedule is the same, each firm has the marginal cost function $MC_i(q_i) = 3 + 2\,q_i$, where q_i is the output of firm i. To clarify the analysis, let Q be the market output, where $Q = q_1 + q_2$ is the sum of the outputs of the two firms. From the inverse demand function we know that the market price is

$$\begin{aligned} p &= D^{-1}(Q) \\ &= 45 - Q \\ &= 45 - (q_1 + q_2). \end{aligned} \tag{39}$$

7.1 Firm 1 revenue and marginal revenue

The revenue function for Firm 1 is the market price times its own output. This can be worked out in terms of the outputs of both firms by substituting equation (39) for the price in the revenue equation. The revenue for firm 1 is then a function of the outputs of the two firms:

$$\begin{aligned} R_1(q_1, q_2) &= p\,q_1 \\ &= D^{-1}(Q)\,q_1 \\ &= (45 - (q_1 + q_2))\,q_1. \end{aligned}$$

As in the case of monopoly, marginal revenue is the change in revenue as output increases from $q - 1$ to q units. The only difference is that the marginal revenue then depends not only on its own output q_1 but also on the output q_2 of Firm 2. For Firm 1 the marginal revenue is

$$\begin{aligned}
MR_1(q_1) &= \frac{1}{\delta}\left(R(q_1, q_2) - R(q_1 - \delta, q_2)\right) \\
&= \frac{1}{\delta}\left(45\, q_1 - q_1^2 - q_1\, q_2 - (45\,(q_1 - \delta) - (q_1 - \delta)^2 - (q_1 - \delta)\, q_2)\right) \\
&= \frac{1}{\delta}\left(45\,\delta - q_1^2 - q_1\, q_2 + (q_1 - \delta)^2 + (q_1 - \delta)\, q_2\right) \\
&= \frac{1}{\delta}\left(45\,\delta - q_1^2 + q_1^2 - 2\, q_1\, \delta + \delta^2 - \delta\, q_2\right) \\
&= \frac{1}{\delta}\left(45\,\delta - 2\, q_1\, \delta - \delta\, q_2 + \delta^2\right) \\
&= 45 - 2\, q_1 - q_2 + \delta \\
&\doteq 45 - 2\, q_1 - q_2.
\end{aligned}$$

7.2 Firm 1 profit maximization and best-response function

Firm 1 will continue to produce units so long as its marginal revenue exceeds its marginal cost. So we know that the optimal quantity of output is the quantity where $MR(q_1) = MC(q_1)$ or

$$45 - 2\, q_1 - q_2 = 3 + 2\, q_1.$$

This equation can be simplified to get the condition $4\, q_1 = 42 - q_2$ or

$$q_1 = BR_1(q_2) = 10.5 - q_2/4. \tag{40}$$

Since this equation indicates how much Firm 1 should produce for each possible level of output q_2 by Firm 2, it is usually called the *best-response function*.

7.3 Firm 2 best-response function

Since Firm 2 has the same cost function as Firm 1, and both face the same market demand, the best-response function for Firm 2 will look just like the best-response function for Firm 1, except that Firm 2 responds to the output decision of Firm 1. So

$$q_2 = BR_2(q_1) = 10.5 - q_1/4. \tag{41}$$

7.4 Equilibrium price and quantity

The equilibrium outputs q_1^* and q_2^* are obtained by solving the two best-response equations simultaneously. One simple way to solve the two equations simultaneously is to substitute equation (41), which expresses the value of q_2 in terms of q_1, into equation (40). At that point, the only variable in the remaining equation is q_1, so we'll have the solution for q_1. When we make this substitution,

$$\begin{aligned} q_1 &= 10.5 - \frac{1}{4} q_2 \\ &= 10.5 - \frac{1}{4} \left(10.5 - \frac{1}{4} q_1\right) \\ &= 10.5\,(1 - \frac{1}{4}) + \frac{1}{16} q_1 . \end{aligned}$$

Subtract $\frac{1}{16} q_1$ from both sides of this equation and multiply both sides of the result by $\frac{16}{15}$ to get $q_1 = \frac{16}{15} \frac{3}{4} \frac{21}{2}$. After multiplication, we get the equilibrium level of output for firm 1:

$$q_1^* = 8.4.$$

The equilibrium level for firm 2, q_2^*, can be determined easily by substituting $q_1^* = 8.4$ into the best-response function for firm 2 in equation (41):

$$q_2^* = 10.5 - 8.4/4 = 8.4.$$

Total output is $Q^* = q_1^* + q_2^* = 16.8$. The inverse demand function relates the total output to the price, so the equilibrium market price is

$$p^* = D^{-1}(Q^*) = 45 - Q^* = 28.2.$$

7.5 Equilibrium surplus

The consumers' surplus and producers' surplus are defined in the same way for duopoly as it is in the other two market structure models. Consumers' surplus is the difference between the buyer's value and the market price for each unit purchased. In the duopoly market, the market price is $p^* = 28.2$ and the market quantity is $Q^* = 16.8$. The consumers' surplus is a triangle with base Q^* and height $d_0 - p^*$, where d_0 is the price when demand is zero. Since $p = D^{-1}(q) = 45 - q$, when $q = 0$ the price is $d_0 = 45$. So the consumers' surplus is

$$\begin{aligned} CS &= \frac{1}{2}\, 16.8\,(45 - 28.2) \\ &= \frac{1}{2}\, 16.8^2 \\ &= 141.12. \end{aligned}$$

The producers' surplus can be calculated for each firm separately from the marginal cost function of the seller and from the market price and quantity, as in the monopoly model. The output of each firm is $q^* = 8.4$. The marginal cost function for each firm is $MC_i(q_i) = 3 + 2\,q_i$ so marginal cost is $MC_i(q^*) = 3 + 2\,q^* = 3 + 2 \cdot 8.4 = 19.8$ at its equilibrium output level $q^* = 8.4$. The surplus of each of two producers can be broken into two parts. There is a rectangle with base q^* and height $p^* - MC(q^*)$ and there is a triangle with base q^* and height $MC(q^*) - MC(0)$. The producer's surplus is therefore

$$\begin{aligned} PS_i &= 8.4\,(28.2 - 19.8) + \frac{1}{2}\,8.4\,(19.8 - 3) \\ &= 8.4^2 + \frac{1}{2}\,8.4 \cdot 16.8 \\ &= 2 \cdot 8.4^2 \\ &= 141.12. \end{aligned}$$

There are two firms so the total producers' surplus is $PS = 282.24$.

7.6 Best-response dynamics

We can see how the outputs of the two firms change over time by considering effects of simple decision rules. One very simple rule that each firm can follow is to choose its output to maximize profit, assuming that the other firm will just do what it did in the previous period. This is called the *best-response dynamic.* In order to follow through the outputs from period to period, it's convenient to denote the output of Firm 1 in period t as q_1^t; the output of Firm 2 in period t is denoted q_2^t. We could trace through the responses with fractional output levels, but as in the normal-form game experiment that you participated in, we can also consider the game with only a few integer outputs possible. The idea of the best-response dynamic is conveyed just as effectively with integer quantity decisions, the best-response dynamic conforms more closely to the experiment set-up in this case, and the calculations are much easier with integer quantity decisions. For these three reasons, the example that follows uses integer quantity decisions for both firms. (In the calculations of the best-response in the integer quantity example, the results are rounded off below to the nearest integer.)

Suppose that in the first period the outputs are $q_1^1 = 3$ and $q_2^1 = 7$. Then we can use the decision rule specified in the best-response dynamic and the best-response function from equation (40) to determine the output of Firm 1 in period 2. Firm 1 chooses its output in period 2 to maximize its profit, under the assumption that Firm 2 will choose the output that it chose in the first period. So

$$\begin{aligned} q_1^2 &= 10.5 - q_2^1/4 \\ &= 10.5 - 7/4 \\ &\doteq 9. \end{aligned}$$

Similarly, Firm 2 chooses its output to maximize its profit, under the assumption that Firm 1 will choose the same output that it chose in period 1, so

$$\begin{aligned} q_2^2 &= 10.5 - q_1^1/4 \\ &= 10.5 - 3/4 \\ &\doteq 10. \end{aligned}$$

If we repeat this process for period 3, then we find that the output of Firm 1 in period 3 will be

$$\begin{aligned} q_1^3 &= 10.5 - q_2^2/4 \\ &= 10.5 - 10/4 \\ &= 8. \end{aligned}$$

Firm 2 chooses $q_2^3 = 10.5 - 9/4$ so its integer output level is $q_2^3 = 8$.

It is easy to verify that the output level of each firm will be $q_i^t = 8$ for every period t beyond $t = 3$, for both firms ($i = 1$ and $i = 2$).

7.7 Nash equilibrium

When both firms choose a strategy that is a best-response to the strategy of the other firm, that is called a *Nash equilibrium*. This is the case when each firm chooses the output level $q_i = 8$ in period t. If Firm 1 chooses $q_1 = 8$ then Firm 2 wants to choose

$$\begin{aligned} q_2 &= 10.5 - q_1/4 \\ &= 10.5 - 8/4 \\ &= 8.4. \end{aligned}$$

With the integer output restriction, this means that Firm 2 will choose $q_2 = 8$ in response. A similar argument shows that Firm 1 will choose $q_1 = 8$ in response to the output $q_2 = 8$.

So we have shown that the simple best-response dynamic reaches the Nash equilibrium very quickly in this game.

Appendices

A Connecting to Econport experiments

There are only a few simple steps needed to join an Econport experiment session. The instructions below will work for any Econport experiment session. There are only two pieces of information that your instructor needs to provide to you in order to join. These are the “Access Code,” which is used in step 5, and the session name, used in step 6.

1. Log on to a computer.
2. Open Internet Explorer, Netscape or another browser.
3. Type the URL www.econport.org into the address bar on the browser.
4. Click on the button labeled ”Join an Experiment.”
5. In the box labeled “* Access Code” type in the access code that your instructor provides and then click on “Access Experiment.”
6. Click on “Join Experiment” to the right of the experiment session name that your instructor has provided.
7. A security dialog box appears. Click the button labeled “Yes.”
8. A Login dialog box appears. Enter your initials and click ’connect’. Remember your initials just as you entered them (i.e., with or without middle initial, capitals or lower case). If you get disconnected you can rejoin the experiment if you reenter your initials.
9. Follow the instructions on the screen.

B Mathematical concepts

B.1 Solution of two equations in two unknowns

Two linear equations with two unknowns can be solved easily, and the solution to such a system of equations is often important even for basic problems in microeconomics. For example, equations for supply and demand are really a system of two equations in two unknowns. The demand equation is often written as $q = D(p) = A - B\,p$ and the supply is often written as $q = S(p) = C + D\,p$. The solution consists of values for the two unknowns (p and q) that simultaneously solve the two equations (demand and supply). With these two equations, the technique for solving for p and q is particularly simple. Since demand ($A - B\,p$) and supply ($C + D\,p$) are both equal to q, they can be set equal to each other: $A - B\,p = C + D\,p$. Now this equation can be solved for p in a few steps. First, add $B\,p$ to both sides of the equation to get $A = C + D\,p + B\,p$. Next, subtract C from both sides of this equation: $A - C = D\,p + B\,p$. When p is factored out from both terms on the right side of this equation, the result is $A - C = (D + B)\,p$. Finally, the value of p that solves the two equations, which we can call p^*, is $p^* = (A - C)/(B + D)$. The solution for q in our system of equations can be obtained by substituting p^* into either the demand equation to get

$$\begin{aligned}
q^* &= D(p^*) \\
&= A - B\,p^* \\
&= A - B\,\frac{A - C}{B + D} \\
&= \frac{A\,B + A\,D}{B + D} - \frac{A\,B - A\,C}{B + D} \\
&= A\,\frac{C + D}{B + D}.
\end{aligned}$$

Some problems with solving two linear equations in two unknowns aren't quite as transparent as the one above, but all of these problems can be transformed so that they are equivalent to the one above. If the equations are

$$a_1\,p + b_1\,q = c_1$$

and

$$a_2\,p + b_2\,q = c_2$$

then the equations can be written instead as

$$p + (b_1/a_1)\,q = c_1/a_1$$

and

$$p + (b_2/a_2)\, q = c_2/a_2.$$

Then the last two equations can be written as

$$p = c_1/a_1 - (b_1/a_1)\, q$$

$$p = c_2/a_2 - (b_2/a_2)\, q.$$

Now the equations are in the form of our original equations, so the solution can be determined as in the first example.

In some examples, such as the solution for a Cournot-Nash equilibrium in the duopoly market, the variables differ from those in the example above, but the ideas are the same. There are many techniques for solving two equations in two unknowns. The one described above is conceptually simple, but the calculations are somewhat involved. Other techniques using linear algebra are conceptually more involved, but then the calculations are simpler. The notes above are included as a review for students who aren't familiar with these other techniques.

B.2 Completing the square

It is easy to find the minimum or maximum of a quadratic equation by completing the square. A quadratic equation has the form

$$f(x) = A\,x^2 + B\,x + C. \tag{A.1}$$

If $A > 0$, then $f(x)$ is a parabola that opens upward, so it has a minimum value; if $A < 0$, then $f(x)$ is a parabola that opens downward, so it has a maximum value.

Some minor rearrangement of equation (A.1) produces an expression that can be used to find the value of x that minimizes or maximizes $f(x)$ and also the value of $f(x)$ at the minimum or maximum. The rearrangements are carried out in the next equation:

$$\begin{aligned} f(x) &= A\,x^2 + B\,x + C \\ &= A\left(x^2 + 2\,\frac{B}{2\,A}\,x\right) + C \\ &= A\left(x^2 + 2\,\frac{B}{2\,A}\,x + \frac{B^2}{4\,A^2}\right) + C - \frac{B^2}{4\,A} \\ &= A\left(x + \frac{B}{2\,A}\right)^2 + C - \frac{B^2}{4\,A}. \end{aligned} \tag{A.2}$$

The result of equation (A.2) is useful in two ways. The term

$$\left(x + \tfrac{B}{2\,A}\right)^2$$

in the last line of the equation is always either positive or zero. When A is a negative number, that means that the first term in the equation,

$$A\,\left(x + \tfrac{B}{2\,A}\right)^2,$$

is always either zero or negative. In that case, the largest value of the function $f(x)$ occurs where the squared term is zero. So the maximum of the function occurs where

$$x = -\frac{B}{2\,A}. \tag{A.3}$$

When $x = \frac{B}{2\,A}$ the first term (A times the squared term) is zero, so the value of $f(x)$ at the maximum is

$$f\left(-\frac{B}{2\,A}\right) = C - \frac{B^2}{4\,A}. \tag{A.4}$$

We'll apply this formula often. The formula is used to maximize utility for both the Cobb-Douglas utility function and the quadratic utility function. It is used to maximize profit for firms in a competitive market, for a monopolist, and for a duopolist. It is also used to find the minimum of average cost in the long-run equilibrium model.

C Marginal profit, marginal revenue, and marginal cost

These notes frequently use arguments about the effect of small changes in output on a firm's profit, on its revenue, or on its cost. These arguments are helpful. When a firm's profit is increasing as output increases, a small change to output leads to an increase in profit; when its profit is decreasing, a small change to output leads to a decrease in profit. When its profit is at its maximum, profit neither increases nor decreases as output changes: marginal profit is zero at that point.

The two other argument with regard to marginal quantities

Index