

Economics 380

Homework #2 solutions

Friday October 12, 2007

Problem 4.1 Verify that the equilibrium price in the Levantine economy is $(p^L)^* = y$ by substituting the aggregate endowment in that region into equation (3.5), setting the aggregate demand equal to the supply of X_2 in that region, and solving for the equilibrium price.

Solution

In the Levant, the endowment is $\omega^L = (y, 1)$. The equilibrium price in the Levant is determined by setting the demand for X_2 equal to the supply of X_2 in that economy. By equation (3.5) the demand there is

$$\begin{aligned}x_2^L(p) &= (\omega_1^L + p\omega_2^L)/(2p) \\ &= (y + p \cdot 1)/(2p) \\ &= y/(2p) + 1/2.\end{aligned}$$

The supply of commodity X_2 there is $\omega_2^L = 1$, so the equilibrium equation is $y/(2p) + 1/2 = 1$. This reduces to $y/p = 1$ so the equilibrium price in the Levant, $(p^L)^*$, is $(p^L)^* = y$.

Problem 4.2 In equation (4.4) we found the quantity \bar{x}_T of trade that eliminates the profit of the merchants. Use this value \bar{x}_T and equation (4.1) for the price in the Levant to determine the price there when profit has been driven to zero.

Solution

The price of commodity X_2 in the Levant is $p(x_T) = \frac{y}{1+2x_T}$. We found that when the Italian merchants carry

$$\bar{x}_T = \frac{1}{2} \cdot \frac{yx-1}{1+y}$$

units of commodity X_2 to the Levant, their profit is zero. If we substitute this value of \bar{x}_T into the equation for the price of commodity X_2 in the Levant, we'll know the price in the Levant when the merchants' profit is zero.

The price in the Levant when profit is zero is

$$\begin{aligned}p(\bar{x}_T) &= \frac{y}{1 + 2\bar{x}_T} \\ &= \frac{y}{\frac{1+y+yx-1}{1+y}} \\ &= \frac{y}{\frac{y+yx}{1+y}} \\ &= \frac{1}{\frac{1+x}{1+y}} \\ &= \frac{1+y}{1+x}.\end{aligned}$$

Problem 4.3 Use the value \bar{x}_T and equation (4.2) for the price in Italy to determine the price there when profit has been driven to zero. How does this compare to the price in the Levant when profit is zero? What accounts for this relationship between the prices in the two locations?

Solution

The price of commodity X_2 in Italy is

$$q^*(x_T) = \frac{1 + 2x_T + yx_T}{(1 + 2x_T)(x - x_T)}.$$

We found that when the Italian merchants carry $\bar{x}_T = \frac{1}{2} \cdot (yx - 1)/(1 + y)$ units of commodity X_2 to the Levant, their profit is zero. If we substitute this value of \bar{x}_T into the equation for the price of commodity X_2 in Italy, we'll know the price in Italy when the merchants' profit is zero.

The algebra for this problem is more complicated than it was for the previous problem, so it is helpful to substitute the value of \bar{x}_T into the numerator of the equation for $q^*(x_T)$ and also into each of the two terms in the denominator of that equation before simplifying the equation. The first term in the denominator is

$$\begin{aligned} 1 + 2\bar{x}_T &= 1 + \frac{yx - 1}{1 + y} \\ &= \frac{1 + y + yx - 1}{1 + y} \\ &= \frac{y + yx}{1 + y} \\ &= \frac{y(x + 1)}{1 + y}. \end{aligned}$$

The numerator can be simplified using this expression for $1 + 2\bar{x}_T$ and also making another substitution for \bar{x}_T in the term $y\bar{x}_T$:

$$\begin{aligned} 1 + 2\bar{x}_T + y\bar{x}_T &= \frac{y(x + 1)}{1 + y} + \frac{y}{2} \cdot \frac{yx - 1}{1 + y} \\ &= \frac{y(x + 1)}{1 + y} + \frac{y^2x/2 - y/2}{1 + y} \\ &= \frac{y(x + 1)}{1 + y} + \frac{y^2x/2 - y/2}{1 + y} \\ &= y \frac{yx/2 + x + 1/2}{1 + y}. \end{aligned}$$

The first term in the denominator is $1 + 2\bar{x}_T$ again, so we only need to find the second term in the denominator. That is

$$\begin{aligned} x - \bar{x}_T &= x - \frac{1}{2} \frac{yx - 1}{1 + y} \\ &= \frac{x + xy}{1 + y} - \frac{1}{2} \frac{yx - 1}{1 + y} \\ &= \frac{x + xy}{1 + y} - \frac{1}{2} \frac{yx - 1}{1 + y} \\ &= \frac{x + \frac{1}{2}xy + \frac{1}{2}}{1 + y}. \end{aligned}$$

These intermediate results can be combined when we solve for $q^*(\bar{x}_T)$. The price of X_2 in Italy is

$$\begin{aligned} q^*(\bar{x}_T) &= \frac{1}{1 + 2\bar{x}_T} \cdot \frac{1 + 2\bar{x}_T + y\bar{x}_T}{x - \bar{x}_T} \\ &= \frac{1}{1 + 2\bar{x}_T} \cdot y \\ &= \frac{1 + y}{y(x + 1)} \cdot y \\ &= \frac{1 + y}{x + 1}. \end{aligned}$$

The price in Italy is the same as the price in the Levant. This makes sense, because the profit of the merchants is the product of the profit margin times the number of units traded. Since the trade volume is positive, the profit per unit must be zero. Profit per unit is the difference between the prices in the two locations.

Once the prices have been driven together in the two locations, the two economies function as though they are a single economy. In that case, we can follow the same steps that we did in problem 4.1 to find the equilibrium price for the integrated economy. The total endowment of X_1 in the two economies is $1 + y$, and the total endowment of X_2 in the two economies is $x + 1$. We can denote the endowment in the combined economy ω^C . Then $\omega^C = (1, x) + (y, 1) = (1 + y, x + 1)$. The equilibrium price in the combined economy is determined by setting the demand in the combined economy, which is

$$\begin{aligned} x_2^L(p) &= \frac{1 + y + p(x + 1)}{2p} \\ &= \frac{1 + y}{2p} + \frac{x + 1}{2}, \end{aligned}$$

equal to the total supply of X_2 in the combined economy. Since the total supply is $x + 1$, the equilibrium equation is $(1 + y)/(2p) = (x + 1)/2$. This reduces to $(1 + y)/p = x + 1$ so the equilibrium price in the combined economy is $p^* = \frac{1+y}{x+1}$. This is the same price that we found in each economy when trade had expanded to the point where the profits from trade had been driven to zero. When profits have been driven to zero, the two economies are fully integrated: they function as though they are a single economy.

Problem 4.4 Suppose that this person has risk aversion parameter $\alpha = 0.5$. What is this person's utility level for the St. Petersburg gamble?

Solution

The expected utility of the St. Petersburg gamble for a person with the CRRA expected utility function in equation (4.10) is

$$E[u(G)] = \frac{1}{2 - 2^{1-\alpha}}.$$

If $\alpha = 0.5$ then the expected utility is

$$\begin{aligned} E[u(G)] &= \frac{1}{2 - 2^{1-0.5}} \\ &= \frac{1}{2 - 2^{0.5}}. \end{aligned}$$

Problem 4.5 For the same agent with risk aversion parameter $\alpha = 0.5$, find the certainty equivalent of the the St. Petersburg gamble. The value of the St. Petersburg gamble is its certainty equivalent. What value does this person place on the St. Petersburg gamble?

Solution

Denote the certain “gamble” that pays \$c with probability 1 by C . Then

$$\begin{aligned} E[u(C)] &= 1 \cdot u(c) \\ &= 1 \cdot c^{1-\alpha} \\ &= 1 \cdot c^{1-0.5} \\ &= 1 \cdot c^{0.5}. \end{aligned}$$

If the person values the certain amount \$c just as much as she values the St. Petersburg gamble, then her utility for each is the same. Therefore $E[u(C)] = E[u(G)]$. We’ve already found the value of $E[u(G)]$ in problem 4.4. Substituting the values of $E[u(C)]$ and $E[u(G)]$ that we’ve already calculated into the equation $E[u(C)] = E[u(G)]$ we get

$$c^{0.5} = \frac{1}{2 - 2^{0.5}},$$

which can be squared to get

$$\begin{aligned} c &= \left(\frac{1}{2 - 2^{0.5}} \right)^2 \\ &\doteq 2.914. \end{aligned}$$

Problem 4.6 How does the value of the gamble for this person compare to the value for the person with CRRA utility with $\alpha = 0.5$?

Solution

The argument on page 51 shows that the person with logarithmic utility has expected utility $E[u(G)] = \ln 2$ for the St. Petersburg gamble. We can follow the same general approach that we used in the previous problem to find the certainty equivalent for this person. Denote the certain “gamble” that pays \$c with probability 1 by C . Then

$$\begin{aligned} E[u(C)] &= 1 \cdot u(c) \\ &= 1 \cdot \ln c \\ &= \ln c. \end{aligned}$$

If the person values the certain amount \$c just as much as he values the St. Petersburg gamble, then his utility for each is the same. Therefore $E[u(C)] = E[u(G)]$. In the lecture notes we found that this person has expected utility $E[u(G)] = \ln 2$ for the St. Petersburg gamble. Substituting the values of $E[u(C)]$ and $E[u(G)]$ that we’ve calculated into the equation $E[u(C)] = E[u(G)]$ we get

$$\ln c = \ln 2.$$

This equation only holds if $c = 2$, so the person has the same utility for \$2 received with certainty as he has for the St. Petersburg gamble. We can interpret that fact by saying that the person values the St. Petersburg gamble at \$2.

Problem 4.7 Show that the demand for loans in equation (4.15) by this risk averse merchant is lower than the demand by a merchant with the credit constraint in equation (4.7). Assume that everything else in the decision problems faced by the two merchants is identical, so they have the same wealth w , and face the same venture V .

Solution

The demand for loans by the merchant with logarithmic expected utility is

$$x_T = \frac{(E[R] - r) w}{(r - R^L)(R^H - r)}.$$

(There was a typographical error in the lecture notes. As I mentioned in class on Oct. 4 and on Oct. 11, the factor w was omitted in equation (4.15).) The demand for loans by a merchant who only restricts his borrowing to the extent imposed by the credit constraint is

$$x_T = \frac{w}{r - R^L}.$$

We want to compare these two loan amounts. Assume that

$$\frac{w}{r - R^L} > \frac{(E[R] - r) w}{(r - R^L)(R^H - r)}.$$

We've already argued that the expressions $r - R^L$, $R^H - r$, w , and $E[R] - r$ are all positive, so we don't need to change the sign of the inequality as we proceed through the algebra steps in this problem. Both sides of the inequality can be multiplied by $\frac{r - R^L}{w}$ to get

$$1 > \frac{E[R] - r}{R^H - r}.$$

Both sides of this inequality can be multiplied by $R^H - r$ to get

$$R^H - r > E[R] - r$$

and r can be added to both sides to get

$$R^H > E[R].$$

We assumed in the statement of the problem that the two possible rates of return R^H and R^L are equally likely, so

$$E[R] = \frac{1}{2} (R^H + R^L).$$

Therefore the inequality $R^H > E[R]$ can be written as

$$R^H > \frac{1}{2} (R^H + R^L).$$

Subtract $\frac{1}{2} R^H$ from both sides of the inequality to get

$$\frac{1}{2} R^H > \frac{1}{2} R^L.$$

Finally, multiply both sides of this inequality by 2 to get $R^H > R^L$. Since we had two possible rates of return on the investment, R^H and R^L , and we assumed that $R^H > R^L$, our assumption that

$$\frac{w}{r - R^L} > \frac{(E[R] - r) w}{(r - R^L)(R^H - r)}$$

is correct.