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Leadtime Management in a Periodic-Review Inventory System: A State-Dependent Base-Stock Policy

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ABSTRACT

We examine a single-item, periodic-review inventory system with stochastic leadtimes, in which a replenishment order is delivered immediately or one period later, depending probabilistically on costly effort. The objective is to determine an inventory policy and an effort-choice strategy that minimize the expected total costs. Our analytical and computational analysis suggests that (i) a state-dependent base-stock policy is optimal, (ii) the optimal effort strategy is such that the marginal cost of effort is equal to the value of immediate delivery, and (iii) the cost impact of leadtime reduction can be very large. We also provide some counter-intuitive results, compared with the traditional multi-period newsvendor model.

Key Words: inventory, leadtime management, state-dependent base-stock policy

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1. INTRODUCTION

In this paper we examine a single-item, periodic-review inventory system with stochastic leadtimes, in which a replenishment order is delivered either immediately or one period later, depending probabilistically on costly effort. The effort might take the form of expediting, the use of better equipment, and so on. The other assumptions are standard: Unmet demands are backordered, ordering costs are linear, and one-period expected inventory-holding and shortage cost is a convex function of initial inventory. The objective is to determine an inventory policy and an effort-choice strategy that minimize the expected total costs over T time periods.

Characteristically, replenishment leadtimes have both a fixed and variable components. The length of the fixed component typically depends on process-design choices (e.g., the extent of automation, choice of common carrier) and environmental factors (e.g., the nature of the product, the geographic distance from the supplier). The length of the variable component also depends on choices, in particular, how each realized leadtime can be decreased or increased by choices made and actions taken during each replenishment cycle. Finally, each realized replenishment leadtime has random variation, some of which is manageable; and some variation is strictly random. In this paper we focus on the variable components: choices made and actions taken during each replenishment cycle. These choices and actions are represented implicitly by "effort", wherein better choices and actions (i.e., more effort) yield shorter leadtimes.

We analyze the single-period ($T=1$), multi-period ($T<\infty$), and infinite-horizon ($T=\infty$) problem. In the single-period problem, despite the non-unimodality of the cost function, we prove the optimality of a state-dependent base-stock policy: There exists a critical number such that it is optimal to place an order if and only if initial inventory is less than the critical number; and the optimal order-up-to level may depend on initial inventory. For the multi-period problem, we provide simple bounds for the optimal order-up-to levels. In all of the problems, we show that the optimal effort strategy is such that the marginal cost of effort is equal to the value of immediate delivery. Our computational results indicate that a state-dependent base-stock policy is also optimal for the multi-period problem. They also suggest that the cost impact of leadtime reduction can be very large. In our computational study the total costs were reduced 24.7 % on average, compared with no effort case. We also discovered that, unlike in the traditional multi-period newsvendor model, given the same initial inventory, the optimal order-up-to levels may *increase* as the end of the horizon approaches. In the infinite-horizon problem, by solving the problem using a policy-iteration procedure, we were able to compute not only the optimal policy but also the end-of-horizon inventory-valuation function that induces a stationary optimal policy in the finite-horizon problem. However, we found that, unlike in the traditional model, the valuation function is non-linear.

Most of the leadtime-management related literature has used continuous-review inventory models to examine one-time effort in leadtime reduction and its impact on inventory costs (e.g., Hariga and Ben-Daya, 1999; Ray et al., 2004; Yang et al., 2005; and Lee and Schwarz, 2006). Only a few papers have studied leadtime management in periodic-review inventory scenarios (Ouyang and Chuang, 1998, 2000; Chuang et al., 2004). In these models, however, the same level of effort is exerted for every replenishment order regardless of inventory level, and an approximate function of the total expected annual cost is minimized.

A related stream of research is on inventory models with two delivery modes (e.g., Fukuda, 1964; Chiang and Gutierrez, 1996, 1998; and Tagaras and Vlachos, 2001), in which the faster delivery mode is available at a higher cost than the slower mode. Each delivery mode has a deterministic leadtime. Our model is different from the above models in that it examines the problem of managing stochastic leadtimes in a periodic-review inventory model, where efforts for leadtime reduction are chosen dynamically for each replenishment order depending on inventory level.

Our work is also related to the literature on state-dependent (i.e., modified) base-stock policies. Two of the earliest references are Karlin (1958) and Karlin and Scarf (1958). Karlin (1958) shows that if the ordering cost is convex, then the optimal base-stock level is a function of initial inventory, since the total costs, including the convex ordering costs, are reduced by smoothing order quantities between time periods. Karlin and Scarf (1958) show that if excess demand is lost and there is a time lag in delivery, then the optimal base-stock level should be modified by initial inventory, since the cost for future time periods depends jointly on the current order and initial inventory. Federgruen and Zipkin (1986a, b) prove that a modified base-stock policy is optimal for an inventory system with limited production capacity, where it is optimal to order up to a target base-stock level or as close to it as possible. Henig and Gerchak (1990) and Anupindi and Akella (1993) show the optimality of a state-dependent base-stock policy when yield is random and when orders are allocated between two suppliers with different supply uncertainties, respectively. Our model is similar to some of these models in a sense that the optimal base-stock level is modified by initial inventory. However, the reason for the modification is different: In our model, if inventory is low, then we want an order to be received sooner and hence exert a higher effort. But as the probability of short leadtime increases (through the increased effort), we would choose to order either less or more.

The rest of this paper is organized as follows: In section 2 we present the basic model. In section 3 and 4 we analyze the single-period and the multi-period problem, respectively. In section 5 we provide computational results for the multi-period problem. In section 6 the infinite-horizon problem is analyzed. Finally, in section 7 we summarize the main results.

2. THE BASIC MODEL

Consider a single-item, stochastic, periodic-review inventory system over T time periods. Demands D_t, $t = 1, 2, ..., T$, for the item in each period are independent and identically distributed. Unmet demands in every period except T are backordered; in period T they are lost. At the end of period T leftover inventory is worthless. The acquisition cost is \$$c$ per unit. There is no fixed order cost. Let $A(x)$ denote the one-period expected inventory-holding and shortage cost given initial inventory x. We assume that $A(x)$ is a convex function of x. Let s^0 be its minimizer; i.e., $A(s^0) \le A(x)$ for all x.

A replenishment order will either be delivered immediately or one period later, depending probabilistically on costly effort. The effort exerted for an order in period t determines the probability p_t of immediate delivery, and, hence, the probability ($1 - p_t$) of delivery next period. Thus, the choice of effort can be modeled as choosing p_t. Assume that we can choose any p_t, $\underline{p} \le p_t \le \overline{p}$, at cost of $W(p_t)$, where $\underline{p} \ge 0$ and $\overline{p} \le 1$ represent the smallest and the largest effort that we can choose, respectively. We assume

$$W(\underline{p}) = 0;\ W'(p) > 0;\ W'(\underline{p}) = 0;\ W''(p) > 0;\ \lim_{p \to \overline{p}} W(p) = \infty . \tag{1}$$

That is, no cost is incurred if no effort is exerted; the cost of effort increases as the effort increases; the marginal cost of effort is zero at no effort, but increases as the effort increases; and it is extremely costly to choose $p = \overline{p}$. There is no carryover of effort from period to period.

The objective is to determine an inventory policy and an effort-choice strategy that minimize the expected total costs over the T time periods. In the infinite-horizon problem, we minimize the expected long-run average total costs per period.

3. THE SINGLE-PERIOD PROBLEM

In the single-period $(T=1)$ problem, given initial inventory x, we have to choose an order-up-to level s and an effort level p to:

$$\underset{s\geq x;\underline{p}\leq p\leq\overline{p}}{\text{Minimize}} \quad c(s-x)+W(p)+pA(s)+(1-p)A(x) \tag{2}$$

Note that if we choose s and p, then with probability p the expected inventory-holding and shortage cost will be $A(s)$, and with probability $(1-p)$ it will be $A(x)$.

It is easy to show that the optimal p for any given s and x, denoted $p(s,x)$, satisfies:

$$W'(p(s,x))=\left[A(x)-A(s)\right]^{+} \tag{3}$$

where $[\omega]^{+}=\max\{\omega,0\}$. Equation (3) means that p should be chosen such that the marginal cost of the effort, p, is equal to the value of immediate delivery; i.e., the difference between the one-period inventory costs when the replenishment order is delivered immediately and one period later.

Now we have to find an optimal order-up-to level $s^{*}(x)$ to:

$$\underset{s\geq x}{\text{Minimize}} \quad y(s,x)\equiv c(s-x)+W(p(s,x))+p(s,x)A(s)+(1-p(s,x))A(x) \tag{4}$$

If $-y(s,x)$ were a unimodal function of s for any given x, then it would be easy to show that a state-dependent base-stock policy is optimal. However, $-y(s,x)$ may not be unimodal in s. See Figure 1 for an example of the shape of $y(s,x)$. Nonetheless, the following proposition states that a state-dependent base-stock policy is optimal: There exists a critical number such that it is optimal to place an order if and only if initial inventory is less than the critical number; and the optimal order-up-to level may depend on initial inventory.

Proposition 1. *Assume that demand is a continuous random variable and $A(\cdot)$ is a continuous, differentiable, and strictly convex function.*

(a) For any x and $s>x$, if $\partial y(s,x)/\partial s>0$, then $\partial y(s,\tilde{x})/\partial s>0$ for all $x<\tilde{x}<s$.

(b) For any x and $s>x$, if $\partial y(s,x)/\partial s\leq 0$, then $\partial\Upsilon(s,x)/\partial x<0$, where

$$\Upsilon(s,x) \equiv y(x,x) - y(s,x)$$

(c) *If* $\bar{x}$ *is the value such that* $s^*(\bar{x}) = \bar{x}$ *and* $s^*(x) > x$ *for all* $x < \bar{x}$*, then* $s^*(x) = x$ *for all* $x \geq \bar{x}$.

Proof. (a) It is straightforward to show that that for any x and $s > x$,

$$\partial y(s,x)/\partial s = c + p(s,x)A'(s).$$

Consider any $\tilde{x}$, $x < \tilde{x} < s$. If $A'(s) \geq 0$, then clearly $\partial y(s,\tilde{x})/\partial s = c + p(s,\tilde{x})A'(s) > 0$. Suppose $A'(s) < 0$. Then, since $A(x) \geq A(\tilde{x}) \geq A(s)$ from the (strict) convexity of $A(\cdot)$, this implies that $p(s,\tilde{x}) \leq p(s,x)$ and thus

$$\partial y(s,\tilde{x})/\partial s = c + p(s,\tilde{x})A'(s) \geq c + p(s,x)A'(s) > 0.$$

(b) It is straightforward to show that for any x and $s > x$,

$$\partial \Upsilon(s,x)/\partial x = c + p(s,x)A'(x).$$

Since $A'(x) < A'(s)$ by the strict convexity of $A(\cdot)$,

$$\partial \Upsilon(s,x)/\partial x = c + p(s,x)A'(x) < c + p(s,x)A'(s) = \partial y(s,x)/\partial s \leq 0.$$

(c) Suppose that $s^*(\tilde{x}) > \tilde{x}$ for some $\tilde{x} > \bar{x}$. If we let $\tilde{s} \equiv s^*(\tilde{x})$, then it must be $\partial y(\tilde{s},\tilde{x})/\partial s = 0$. This implies that $\partial y(\tilde{s},x)/\partial s \leq 0$ for all $x \leq \tilde{x}$ from (a). This, in turn, implies that $\partial \Upsilon(\tilde{s},x)/\partial x < 0$ for all $x \leq \tilde{x}$ from (b). But, note that $\Upsilon(\tilde{s},\bar{x}) = y(\bar{x},\bar{x}) - y(\tilde{s},\bar{x}) \leq 0$ since $s^*(\bar{x}) = \bar{x}$; and that $\Upsilon(\tilde{s},\tilde{x}) = y(\tilde{x},\tilde{x}) - y(\tilde{s},\tilde{x}) \geq 0$ since $\tilde{s} = s^*(\tilde{x})$. This gives a contradiction. □

Note that Proposition 1(a) and 1(b) show that stationary points, e.g., $P_{x,\hat{s}} \equiv (x, y(\hat{s},x))$ in Figure 1, move upward relative to $P_x \equiv (x, y(x,x))$ as x increases, which means that once a stationary point becomes higher than P_x, then it always stays above P_x as x increases. Proposition 1(c) states that if we define $\bar{x}$ as the smallest value of initial inventory such that $s^*(\bar{x}) = \bar{x}$, then $s^*(x) > x$ *if and only if* $x < \bar{x}$.

4. THE MULTI-PERIOD PROBLEM

In the multi-period $(T < \infty)$ problem, if we define $TC_t(x)$ as the minimum expected total costs from period t to T given initial inventory x, then we have to solve:

$$TC_t(x) = \min_{s\geq x, \underline{p}\leq p\leq \overline{p}} \left\{ c(s-x) + W(p) + pA(s) + (1-p)A(x) + E\left[TC_{t+1}(s-D)\right]\right\}, \quad t = 1,2,...,T \quad (5)$$

where $TC_{T+1}(x) \equiv 0$. As in the single-period ($T=1$) problem, the optimal p in any period t for any given s and x, denoted $p_t(s,x)$, satisfies equation (3); i.e., $W'(p_t(s,x)) = \left[A(x) - A(s)\right]^+$. Consequently, problem (5) becomes

$$TC_t(x) = \min_{s\geq x} \left\{ y(s,x) + E\left[TC_{t+1}(s-D)\right]\right\} \quad (6)$$

where $y(s,x)$ is the single-period expected total costs, given by (4). Define

$$J_t(s,x) \equiv y(s,x) + E\left[TC_{t+1}(s-D)\right] \quad (7)$$

Note that, although $-y(s,x)$ is not unimodal, we were able to prove the optimality of a state dependent base-stock policy in the single-period problem. However, for the multi-period problem, it is difficult to characterize its optimal policy, since the addition of the minimum expected future costs (i.e., the second term of $J_t(s,x)$) makes the problem much harder. Instead, we shall provide simple bounds for the optimal order-up-to levels, which can be used for numerical analysis. In this section and afterwards we assume that demand and inventory can take only on integer values

Define $\Delta J_t(s,x)$ as the marginal change of the expected total costs as the order-up-to level is increased from s to $s+1$, i.e.,

$$\Delta J_t(s,x) \equiv J_t(s+1,x) - J_t(s,x) = \Delta y(s,x) + E\left[TC_{t+1}(s+1-D) - TC_{t+1}(s-D)\right] \quad (8)$$

where

$$\Delta y(s,x) \equiv y(s+1,x) - y(s,x) \quad (9)$$

Lemma 1. *For any x and $s \geq x$,*

$$c + p(s+1,x)\left[A(s+1) - A(s)\right] \leq \Delta y(s,x) \leq c + p(s,x)\left[A(s+1) - A(s)\right] \quad (10)$$

Proof. For any x and $s \geq x$,

$$\begin{aligned}\Delta y(s,x) &= y(s+1,x) - y(s,x) \\ &= c + W\left(p(s+1,x)\right) - W\left(p(s,x)\right) \\ &\quad + p(s+1,x)A(s+1) - p(s,x)A(s) - \left[p(s+1,x) - p(s,x)\right]A(x)\end{aligned}$$

Since $W\left(p(s+1,x)\right)-W\left(p(s,x)\right)\geq W'\left(p(s,x)\right)\left[p(s+1,x)-p(s,x)\right]$,

$$\begin{aligned}\Delta y(s,x)\geq\ & c+p(s+1,x)\left[A(s+1)-A(s)\right]\\ & +\left[p(s+1,x)-p(s,x)\right]\left\{W'\left(p(s,x)\right)-\left[A(x)-A(s)\right]\right\}\end{aligned}$$

If $A(x)\geq A(s)$, then $W'\left(p(s,x)\right)=A(x)-A(s)$. If $A(x)<A(s)$, this implies that $s^0<s$. Since $A(x)<A(s)\leq A(s+1)$, we have $p(s,x)=p(s+1,x)=\underline{p}$. Hence, in any case,

$$\left[p(s+1,x)-p(s,x)\right]\left\{W'\left(p(s,x)\right)-\left[A(x)-A(s)\right]\right\}=0.$$

Therefore,

$$\Delta y(s,x)\geq c+p(s+1,x)\left[A(s+1)-A(s)\right].$$

Similarly, it can be shown that $\Delta y(s,x)\leq c+p(s,x)\left[A(s+1)-A(s)\right]$. □

The bounds of $\Delta y(s,x)$ provided by Lemma 1 depend on the value of initial inventory, x. To obtain *x-free* lower and upper bounds of $\Delta y(s,x)$, we define $\Delta\underline{y}(s)$ and $\Delta\overline{y}(s)$ as follows:

$$\Delta\overline{y}(s)\equiv\sup_{x\leq s}\left\{c+p(s,x)\left[A(s+1)-A(s)\right]\right\}=\begin{cases}c+\underline{p}\left[A(s+1)-A(s)\right], & \text{if } s<s^0\\ c+\overline{p}\left[A(s+1)-A(s)\right], & \text{if } s\geq s^0\end{cases}\tag{11}$$

$$\Delta\underline{y}(s)\equiv\inf_{x\leq s}\left\{c+p(s+1,x)\left[A(s+1)-A(s)\right]\right\}=\begin{cases}c+\overline{p}\left[A(s+1)-A(s)\right], & \text{if } s<s^0\\ c+\underline{p}\left[A(s+1)-A(s)\right], & \text{if } s\geq s^0\end{cases}\tag{12}$$

where s^0 is the minimizer of $A(\cdot)$.

If we define

$$\Delta\overline{J}_t(s)\equiv\Delta\overline{y}(s)+E\left[TC_{t+1}(s+1-D)-TC_{t+1}(s-D)\right]\tag{13}$$

$$\Delta\underline{J}_t(s)\equiv\Delta\underline{y}(s)+E\left[TC_{t+1}(s+1-D)-TC_{t+1}(s-D)\right]\tag{14}$$

then it is easy to see that

$$\Delta\underline{J}_t(s)\leq\Delta J_t(s,x)\leq\Delta\overline{J}_t(s).\tag{15}$$

Define l_t as the largest integer value of s such that $\Delta\overline{J}_t(\tilde{s}-1)<0$ for all $\tilde{s}\leq s$; and u_t as the smallest integer value of s such that $\Delta\underline{J}_t(\tilde{s})\geq 0$ for all $\tilde{s}\geq s$. Then, the following proposition says that the optimal order-up-to level, $s^*(x)$, in period t is bounded by l_t and u_t.

Proposition 4. *For any period* $t = 1, 2, ..., T$ *and any given value of* x, $l_t \le s_t^*(x) \le u_t$ *if* $x < u_t$, *and* $s_t^*(x) = x$ *if* $x \ge u_t$.

Proof. Note that $\Delta J_t(s,x) \le \Delta \bar{J}_t(s) < 0$ for all $s < l_t$ and $0 \le \Delta \underline{J}_t(s) \le \Delta J_t(s,x)$ for all $s \ge u_t$. Hence, $l_t \le s_t^*(x) \le u_t$ for $x < u_t$ and $s_t^*(x) = x$ for $x \ge u_t$. □

Now, to provide a bound for the bounds $\{(l_t, u_t)\}$, we use the following lemma:

Lemma 2. $TC_t(x+1) - TC_t(x) \ge M(x)$ *for any* x *and* t, *where*

$$M(x) \equiv \begin{cases} -c + (1-\underline{p})\left[A(x+1) - A(x)\right], & \text{if } x < s^0 \\ -c + (1-\bar{p})\left[A(x+1) - A(x)\right], & \text{if } x \ge s^0 \end{cases} \tag{16}$$

Proof. For any x and t, since $J_t(s_t^*(x), x) \le J_t(s_t^*(x+1), x)$

$$\begin{aligned} TC_t(x+1) - TC_t(x) &= J_t(s_t^*(x+1), x+1) - J_t(s_t^*(x), x) \\ &\ge J_t(s_t^*(x+1), x+1) - J_t(s_t^*(x+1), x) \\ &= -c + (1 - p_t^*(x+1))\left[A(x+1) - A(x)\right] \\ &\ge M(x). \quad \square \end{aligned}$$

Lemma 2 means that the marginal value of a unit of initial inventory in any period is at most the negative of $M(x)$ given by (16). This result is very intuitive because the marginal value of a unit of initial inventory should be its acquisition cost plus its contribution to the expected inventory-holding and shortage cost multiplied by the probability of an order being delayed.

If we define $\bar{s}$ as the smallest integer value of s such that

$$c + \underline{p}\left[A(s+1) - A(s)\right] + E\left[M(s-D)\right] \ge 0 \tag{17}$$

and $\underline{s}$ as the largest integer value of s such that $\Delta \bar{y}(s-1) < 0$, then the following proposition provides a lower and upper bound for the bounds $\{(l_t, u_t)\}$.

Proposition 5. *For any period* $t = 1, 2, ..., T$, $\underline{s} \le l_t \le u_t \le \bar{s}$.

Proof. Note that $u_t \le \bar{s}$ if and only if $\Delta \underline{J}_t(s) \ge 0$ for all $s \ge \bar{s}$. But, for any $s \ge \bar{s}$,

$$\begin{aligned}\Delta \underline{J}_t(s) &= c + \underline{p}\left[A(s+1) - A(s)\right] + E\left[TC_{t+1}(s+1-D) - TC_{t+1}(s-D)\right] \\ &\geq c + \underline{p}\left[A(s+1) - A(s)\right] + E\left[M(s-D)\right] \geq 0\end{aligned}$$

Hence, $u_t \leq \bar{s}$ for all t. To prove $\underline{s} \leq l_t$ for all t, note that $\underline{s} \leq l_t$ if and only if $\Delta \bar{J}_t(s) < 0$ for all $s < \underline{s}$. We shall use an induction to prove $\Delta \bar{J}_t(s) < 0$ for all $s < \underline{s}$. For period T, note that $\underline{s} = l_T \leq s^0$. For any $x < \underline{s}$, since $s_T^*(x) \geq \underline{s} \geq x+1$,

$$\begin{aligned}TC_T(x+1) - TC_T(x) &= J_T\left(s_T^*(x+1), x+1\right) - J_T\left(s_T^*(x), x\right) \\ &\leq J_T\left(s_T^*(x), x+1\right) - J_T\left(s_T^*(x), x\right) \\ &= -c + \left(1 - p_T^*(x)\right)\left[A(x+1) - A(x)\right] \\ &\leq -c\end{aligned}$$

Now, suppose that for period $t+1$, (i) $\underline{s} \leq l_{t+1}$ and (ii) $TC_{t+1}(x+1) - TC_{t+1}(x) \leq -c$ for all $x < \underline{s}$. For any $s < \underline{s}$,

$$\begin{aligned}\Delta \bar{J}_t(s) &= c + \underline{p}\left[A(s+1) - A(s)\right] + E\left[TC_{t+1}(s+1-D) - TC_{t+1}(s-D)\right] \\ &\leq \underline{p}\left[A(s+1) - A(s)\right] < 0\end{aligned}$$

Hence, $\underline{s} \leq l_t$. It can be also shown that $TC_t(x+1) - TC_t(x) \leq -c$ for all $x < \underline{s}$. This completes the proof. □

5. COMPUTATIONAL STUDY

We conducted a computational study for the multi-period problem in order to (i) compare the total costs under leadtime management with no effort case to assess the cost impact of leadtime reduction and (ii) examine the optimality of a state-dependent base-stock policy in the multi-period problem.

In our computational study we tested problems with $T = 4$ periods and a truncated Poisson demand distribution, i.e.,

$$f(d) \equiv \Pr(D = d) = \frac{e^{-\lambda}\lambda^d}{d!\Theta}, \ d = 0, 1, \ldots, \bar{D}, \text{ where } \Theta = \sum_{d=0}^{\bar{D}} \frac{e^{-\lambda}\lambda^d}{d!}.$$

We assumed a linear inventory-holding cost h per unit per month and a linear shortage cost π per unit per month; i.e.,

$$A(x) = hE\left[(x-D)^+\right] + \pi E\left[(D-x)^+\right]$$

The value of h was fixed at \$1 per unit per month, and the value of π was set equal to \$99 per unit per month in the base case, with alternative values of 66, 82, 110, 142, and 199.

The cost-of-effort function was:

$$W(p) = \kappa\left(p - \underline{p}\right)\left(\frac{1}{\overline{p} - p} - \frac{1}{\overline{p} - \underline{p}}\right)$$

Note that the above cost function satisfies all the assumptions in (1). That is, it is an increasing, convex function of p with $W\left(\underline{p}\right) = 0$, $W'\left(\underline{p}\right) = 0$, and $\lim_{p \to \overline{p}} W(p) = \infty$. Its chosen parameter values were $\kappa = 10$ and $\underline{p} = 0.5$ in the base case and, alternatively, $\kappa = 20,\ 30,\ 50,\ 70,\ 100$ and $\underline{p} = 0.2,\ 0.3,\ 0.4,\ 0.6,\ 0.7$. The value of $\overline{p}$ was set to 1. The functions and parameters used are summarized in Tables 1 and 2.

In order to assess the cost impact of leadtime reduction, we computed the minimum expected total costs under leadtime management and under no effort, and measured the percentage cost reduction. We define:

$$IMPACT\% = \frac{TC(NE) - TC^*}{TC(NE)} \times 100$$

where TC^* and $TC(NE)$ are the expected total costs over $T = 4$ periods with zero initial inventory under optimal leadtime management and under no effort, respectively. Using the functions and parameters described above, we observed that the cost impact of leadtime reduction can be very large. IMPACT% averaged 24.7 % (range: 16.1-39.5 %). See Table 2.

Although we were not able to prove that a state-dependent base-stock policy is optimal in the multi-period model, in our computational study the optimal policy observed was always such a policy. That is, there existed a set of critical numbers, $\{\overline{x}_t\}$, such that it is optimal to place an order in period t (i.e., $s_t^*(x_t) > x_t$) if and only if initial inventory of the period is less than the critical number (i.e., $x_t < \overline{x}_t$). Further, the optimal order-up-to level depended on initial inventory. See Figure 2 for the shape of the optimal order-up-to level as a function of initial inventory.

In particular, for any given parameterization, we observed that the optimal order-up-to level is monotonic with respect to initial inventory, approaching s^0 from above or below, as initial inventory decreases. We interpret this as follows: As initial inventory decreases, we exert higher levels of effort, thereby increasing the probability of immediate delivery. As the probability of immediate delivery increases, the optimal order-up-to level becomes closer to s^0, which is the inventory level that minimizes the expected inventory-holding and shortage cost for the current period.

We also examined the time pattern of the optimal order-up-to levels across time periods. It is well known that in the traditional multi-period newsvendor model, the optimal order-up-to level is non-increasing as the end of horizon approaches because the marginal value of leftovers is non-increasing. However, in our model, this was not always the case. The following is a counter example that we found:

Non-Monotonicity Example:

Demand distribution: $\Pr\{D=0\}=\Pr\{D=50\}=0.01$, $\Pr\{D=25\}=0.98$

Inventory-holding and backorder penalty costs: $A(x)=0.2(x-25)^2$

Other parameter values: $c=4.5, \underline{p}=0.5, k=1, T=4$

The reason for this non-monotonicity is that the level of effort in a period affects the marginal value of initial inventory in that period. In particular, the value of a given level of initial inventory decreases as the effort increases (since the probability of immediate delivery increases). Hence, depending on the parameters of the business scenario, in our model, the optimal order-up-to level can be observed to increase as the end of the horizon approaches.

6. THE INFINITE-HORIZON PROBLEM

The infinite-horizon ($T=\infty$) problem is the same as the multi-period ($T<\infty$) problem, except that we minimize the expected long-run average total costs per period over an infinite horizon. The purpose of examining this problem is to (i) compute and observe the optimal policy and (ii) examine under what end-of-horizon condition the non-stationary optimal policy becomes stationary in the finite-horizon problem.

As expected, it was observed that a state-dependent base-stock policy is optimal for the infinite-horizon problem. For the second purpose of examination, note that in the traditional finite-horizon newsvendor model with fixed leadtime, the optimal policy becomes stationary if a particular end-of-horizon inventory-valuation function is added to the model. This inventory-valuation function is linear, and equal to $-c \cdot x$, where c is the acquisition cost per unit and x is the inventory at the end of the horizon. By solving the infinite-horizon problem of our model, we were able to compute the corresponding end-of-horizon inventory-valuation function, denoted $v(x)$, that induces a stationary optimal policy in the finite-horizon problem. However, as might be expected, the end-of-horizon inventory-valuation function, $v(x)$, is not linear. See Figure 3. Intuitively, this is because the marginal value of a unit of initial inventory in the infinite-horizon problem is determined by its acquisition cost plus its marginal contribution to the one-period expected inventory-holding and shortage cost multiplied by the probability of an order being delayed.

In what follows we briefly explain how to solve the infinite-horizon problem by using a policy-iteration procedure. Consider the following optimality equation:

$$g + v(x) = \min_{s \geq x} \left\{ y(s,x) + E\left[v(s-D) \right] \right\}, \quad x \in \Lambda \tag{18}$$

where g represents the long-run average total costs per period and Λ is the space of inventory levels in steady state. $v(x)$ represents the value of initial inventory x. If there exists a solution for $v(\cdot)$ and g in the optimality equation, (18), then the value of s that minimizes the right-hand side of (18), denoted $s_\infty^*(x)$, is the optimal inventory policy. The optimal effort strategy $p_\infty^*(x)$ is obtained from the equation (3). Further, if there exists a solution for the optimality equation, (18), then a stationary policy is optimal (Ross ,1970).

In our computational study, as in the previous section, we assumed that demands in each period can take on a finite number of values, i.e., $D = 0,1,2...,\bar{D}$. Then, the optimal order-up-to level $s_\infty^*(x)$ must be at most $2\bar{D}$, since the maximum demand during the current and the next period is $2\bar{D}$. Also, note that $s_\infty^*(x) \geq 0$, if we assume $\pi \underline{p} > c$ (i.e., if the expected shortage cost that can be avoided by a unit is greater than its acquisition cost). Hence, we only have to search $\max\{x,0\} \leq s \leq 2\bar{D}$ for $s_\infty^*(x)$, $x \leq 2\bar{D}$; and, the space of inventory levels in steady state is $\Lambda = \{-\bar{D},...,2\bar{D}\}$. Given this, the optimality equation, (18), can be solved in a finite number of iterations by a typical policy-iteration procedure (Howard, 1960).

7. CONCLUSION

We have examined the problem of managing stochastic replenishment leadtimes in a simple, periodic-review inventory system, where a replenishment order can be delivered immediately or one period later, depending probabilistically on costly effort. We have examined the single-period ($T = 1$), multi-period ($T < \infty$), and infinite-horizon ($T = \infty$) problem. We have provided several interesting results. In particular, our analytical and computational analysis suggests that (i) a state-dependent base-stock policy is optimal and (ii) the cost impact of leadtime reduction can be very large.

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Table 1. Functions and Parameters

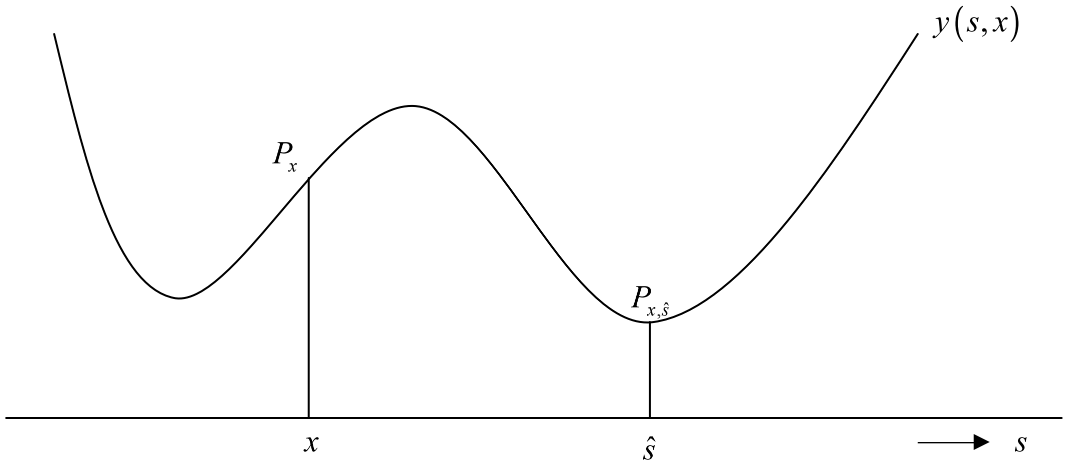
Functions	Parameters	Values
Inventory costs and demand	h	1
	π	66, 82, 99*, 110, 142, 199
	c	30
	λ	50
	$\bar{D}$	100
$W(p)=k(p-\underline{p})\left[\frac{1}{\bar{p}-p}-\frac{1}{\bar{p}-\underline{p}}\right]$	k	10*, 20, 30, 50, 70, 100
	$\underline{p}$	0.2, 0.3, 0.4, 0.5*, 0.6, 0.7
	$\bar{p}$	1

- The values with * represent the base case.

Table 2. Numerical Results on IMPACT%

		TC(NE)	TC*	IMPACT%
BASE CASE		8952.42	6761.58	24.5
π	66	8038.24	6627.02	17.6
	82	8488.11	6699.06	21.1
	110	9248.25	6796.79	26.5
	142	10096.1	6884.01	31.8
	199	11579.7	7004.83	39.5
$\underline{p}$	0.2	10400.2	6835.00	34.3
	0.3	9919.59	6813.25	31.3
	0.4	9437.72	6789.10	28.1
	0.6	8463.93	6729.72	20.5
	0.7	7970.94	6691.42	16.1
κ	20	8952.42	6882.59	23.1
	30	8952.42	6969.11	22.2
	50	8952.42	7097.00	20.7
	70	8952.42	7193.37	19.6
	100	8952.42	7306.50	18.4
			Average	24.7

Figure 1. Non-Unimodality of the Single-Period Expected Total Cost, $y(s,x)$



Example: $f(d)=\dfrac{\Gamma(6)}{100\Gamma(3)\Gamma(3)}\left(\dfrac{d}{100}\right)^2\left(1-\dfrac{d}{100}\right)^2,\quad d=0,1,\ldots,100;$

$$A(x)=E\left[(x-D)^+\right]+99E\left[(D-x)^+\right];\ W(p)=10(p-0.5)\left(\frac{1}{1-p}-2\right)$$

Figure 2. Shape of the optimal order-up-to level as a function of initial inventory

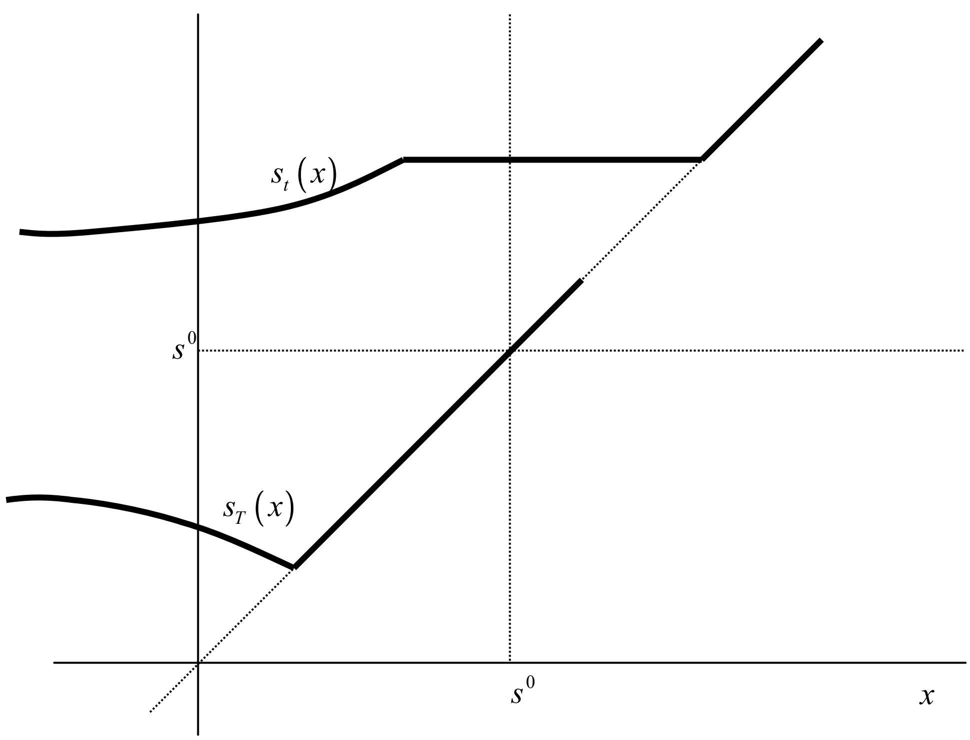


Figure 3.   End-of-horizon Inventory-valuation Function, $v(x)$