OPTIMAL STORAGE ASSIGNMENT IN AUTOMATIC WAREHOUSING SYSTEMS*†

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In the past few years, increasing numbers of automatic warehousing systems using computer-controlled stacker cranes have been installed. Our research concerns the scientific scheduling and design of these systems. There are three elements to scheduling: the assignment of multiple items to the same pallet (Pallet Assignment); the assignment of pallet loads to storage locations (Storage Assignment); and rules for sequencing storage and retrieve requests (Interleaving). This paper deals with optimal storage assignment. Results are obtained which compare the operating performance of three storage assignment rules: random assignment, which is similar to the closest-open-location rule used by many currently operating systems; full turnover-based assignment; and class-based turnover assignment. It is shown that significant reductions in crane travel time (and distance) are obtainable from turnover-based rules. These improvements can, under certain circumstances, be directly translated into increased throughput capacity for existing systems, and may be used to alter the design (e.g., size and number of racks, speed of cranes, etc.) of proposed systems in order to achieve a more desirable system balance between throughput and storage capacity.

1. Introduction

Computer-directed warehousing systems using stacker cranes and palletized loads are revolutionizing the design and operation of large capacity, high volume storage facilities. Briefly, these systems work as follows: Incoming items are assigned to pallets, sometimes two or more items being assigned to the same pallet. The contents of the pallet are then communicated to a minicomputer, which assigns the pallet to a location in the storage racks and records its assigned location in computer storage. The pallets are stored using automatic stacker cranes.

Upon receipt of a request for an item, the computer obtains the pallet location from computer storage and directs a stacker crane to retrieve the pallet. After stock picking or removal the pallet is returned to the incoming stock location for storage, reconsolidation, or re-use.

There are many benefits to such systems: A substantial saving in labor costs, high floor-space utilization, improved material flow, improved inventory control, and a lower incidence of misplacement or theft. Maximal benefits of such a system are dependent upon the optimal design of the system (e.g., the number of stacker cranes, their horizontal and vertical speeds, the length and height of the storage racks, etc.) and the optimal scheduling of the system.

Our current research effort considers optimal scheduling of automatic warehousing systems, with eventual feedback to the optimal design of such systems.

There are three elements of scheduling:

(1) Pallet Assignment: The assignment of multiple items to the same pallet.
(2) Storage Assignment: The assignment of pallet loads to storage locations.
(3) Interleaving: Rules for sequencing storage and retrieve requests.

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Optimal scheduling involves an integrated approach to all three elements. The current paper deals only with optimal storage assignment, without regard to pallet assignment or interleaving. Subsequent work will emphasize these other elements of scheduling.

There are few research papers in the management science or operations research literature dealing with this subject. A very limited number of books, articles, and pamphlets dealing with the subject in a generally nontechnical manner are listed in the references.

Assumptions

The following assumptions are made throughout the paper:
1. Each pallet holds only one part number or item type.
2. All storage locations are the same size, as are the pallets themselves. Therefore all storage locations are candidates for storing any pallet load.
   (Assumptions 1 and 2 remove pallet assignment from our analysis.)
3. The system analyzed consists of a single crane serving a single two-sided aisle. Although actual systems usually consist of several cranes and aisles, each served by the same input/output conveyor system, the conveyor system itself is seldom a system bottleneck. Hence, a single M-crane system will be represented by M one-crane systems.
4. The system is bounded at the point where the crane and the I/O conveyor transfer pallets. We assume that incoming and outgoing pallets are transferred at the same point, to be denoted the I/O point, and that the I/O point is at one corner of the racks. These are common features of existing systems. We omit consideration of the I/O conveyor system itself and its associated control mechanism.
5. On each side of the aisle is a storage rack with R rows and C columns of locations. The crane is capable of moving both vertically and horizontally simultaneously, and its vertical and horizontal speeds are such that the time to reach the row most distant from the I/O point equals the time to reach the most distant column (i.e., the system is “square” in time). The crane travel measure will be time rather than distance.
6. Interleaving is ignored. All storage and retrieval functions are initiated with the crane at the I/O point. In order to execute a retrieve request the crane travels empty to the desired pallet location, removes the pallet, and returns to the I/O point. To execute a storage request the crane removes the pallet to be stored from the I/O point, travels to the assigned storage location, stores the pallet, and returns empty to the I/O point. Systems operating in this manner are sometimes called “single address” systems.
7. Actual time for the crane to load or unload a pallet at the I/O point or a storage location (called transfer time) is ignored, as is the time taken by the crane to travel from the conveyor to the I/O point. Although for most systems these times are small compared to total crane utilization time, it must be emphasized that the percentage reductions in travel time reported below refer only to the time the cranes are traveling in the aisles, and hence overestimate the percentage reductions of total crane utilization time achievable with a real system.
8. The turnover frequency of each item is known and constant through time. Turnover frequency is the number of times a given item requires storage and retrieval in some time period; e.g., day, month, year, etc. Alternatively, it is the reciprocal of the average length of storage time for the item. This assumption is relaxed when class-based turnover assignment is considered.
9. Short-run dynamic considerations are ignored. We are interested here only in the long-run average behavior of the system.
2. General Model Formulation

Let
\[ R = \text{number of rows in each rack}; \]
\[ C = \text{number of columns in each rack}; \]
\[ N = \text{number of pallet storage locations in one rack} \ (N = RC); \]
\[ y_i = \text{the ranked one-way time for crane to travel from I/O point to location} \ i, \ i = 1, \ldots, N; \ \text{ranked so that} \ y_i \leq y_{i+1}, \ \text{all} \ i < N. \]
\[ \lambda_j = \text{the turnover of the item on pallet} \ j, \ \text{henceforth called the turnover of pallet} \ j. \]

This turnover is the number of times that the item on pallet \( j \) will be stored (and subsequently retrieved) per unit time. For convenience the \( \lambda_j \) are ranked so that \( \lambda_j \geq \lambda_{j+1} \) for \( j < N \).

Random Storage Assignment

Consider a random storage assignment rule, in which any pallet is equally likely to be stored in any of the \( N \) rack locations. In our analysis this rule is used to approximate the "closest-open-location" rule often used in practice. The closest-open-location rule works as follows: Just before the store for a given pallet is initiated, a list of open rack locations is scanned for the one closest in time to the I/O point. The pallet is then stored in this location, regardless of its turnover.

Given stable capacity utilization over time (a constant number of full rack locations), the closest open location rule will store pallets in a square (in time) subset of the rack locations. If we redefine \( N \) to be the number of locations in this subset and assume that stores always alternate with retrieves, then under the closest open location rule, a given store will occur at the same location as the preceding retrieval (or a location of equivalent distance). Provided that the pallet stored in a given location is independent of the location’s previous occupant, in steady state all stores are equally likely to take place at any of the \( N \) utilized rack locations. Therefore, the system will behave in the same manner as if the random storage rule were being used. (A sufficient condition for this independence would be for the store queue never to be empty and for its queue discipline to be FIFO.)

The assumption of no interleaving implies that each storage/retrieve involving location \( i \) requires crane travel of \( 4y_i \) (travel to location \( i \) (full); travel back to the I/O point (empty); later, travel to location \( i \) for the corresponding retrieve (empty); and travel back to I/O point (full)). To avoid carrying constants we shall henceforth restrict our analysis to only the one-way travel time, \( y_i \).

Under random storage assignment the expected one-way travel time for a pallet, \( T_R \), is simply
\[
T_R = \bar{y} = \frac{1}{N} \sum_{i=1}^{N} y_i. \tag{1}
\]

Turnover-Based Assignment

It can be easily shown by a simple switching argument that expected one-way travel time is minimized if the highest-turnover pallet is assigned to the closest location (in time); i.e., pallet 1 to location 1, pallet 2 to location 2, etc. Under this scheme, the expected one-way travel time, \( T_T \), would be
\[
T_T = \frac{\sum_{i=1}^{N} y_i \lambda_i}{\sum_{i=1}^{N} \lambda_i}. \tag{2}
\]

\(^1\) This definition of \( N \) will be used for the remainder of the paper.
Using the random storage assignment rule as a benchmark, the percentage reduction in expected one-way travel time achieved with turnover-based assignment, $I_T$, would be

$$I_T = \left( \frac{(T_R - T_T)}{T_R} \right) \times 100. \quad (3)$$

Equation (3) can be evaluated for any specified $y_i$ and $\lambda_j, i, j = 1, \ldots, N$. However, before proceeding, we will introduce continuous representations of the distance and turnover functions. These representations considerably reduce the difficulty of the subsequent analysis and, as we shall see in §4, yield results which are sufficiently close to the discrete analysis to merit our attention.

3. The Continuous Representation

In the continuous representation the discrete functions $y_i$ and $\lambda_j$ are approximated by continuous functions $y(i)$ and $\lambda(j)$. Furthermore, the indices $i$ and $j$ themselves become continuous on the interval $[0, 1]$. The rack is represented as a square (see Figure 1) because the rack was assumed to be square in time. The $x_1$ and $x_2$ co-ordinates of each point in the square are the horizontal and vertical travel times, respectively, for the crane to reach the given point. Without loss of generality we will select a unit of travel time measurement to yield a rack with unit sides.

The Distance Distribution

We derive the continuous distance function $y(i)$ as follows. Consider a location in the "i-th" fractile or percentile of the distance distribution. By definition $i\%$ of the locations are closer to the I/O point than the location under consideration. These $i\%$ locations must be arranged in a square, since the time taken by the crane to move from the I/O point to any point $(x_1, x_2)$ is $\text{Max}[x_1, x_2]$. The dimensions of this square must therefore be $i^{1/2}$ by $i^{1/2}$ (so that the total area is $i$). Thus the distance to the location under consideration must be $i^{1/2}$. In general, then, the travel time to the location in the $i$th percentile is

$$y(i) = i^{1/2}, \quad 0 < i \leq 1. \quad (4)$$

2 A proof of convergence from the discrete to the continuous case is given in [1], available from the authors. This reference also contains details of other derivations omitted from the current paper.
The Turnover Distribution

For a given application one would directly estimate the distribution of pallet turnover. In order to present our results we will derive a general distribution for turnover using the well-known “ABC” phenomenon for inventories and the basic EOQ model.

ABC analysis ranks all items in an inventory by their contribution to total demand, with “A” items representing the high-volume items, “B” the medium-volume items, and “C” the low-volume items. It is typical to find that a small percentage of the items represents a large percentage of the total demand. The “ABC” curve is a plot of ranked cumulative % demand versus % of inventoried items.

We will represent the “ABC” curve by the function

$$G(i) = i^s \quad \text{for} \quad 0 < s < 1,$$

where demand is measured in full pallet loads. Examples of these curves are shown in Figure 2.

Now let

$D(i) =$ demand rate (pallets per unit time) of item $i$,

$Q(i) =$ the economic order quantity of item $i$.

By definition,

$$G(i) = i^s = \int_0^i D(j) dj / \int_0^1 D(j) dj. \quad (6)$$

Letting $\int_0^1 D(j) dj = 1$ for convenience, we have

$$i^s = \int_0^1 D(j) dj \quad (7)$$

which has the solution

$$D(i) = si^{s-1}, \quad 0 < i < 1. \quad (8)$$

If all items are ordered using the standard EOQ model, then $Q(i) = (2KD(i))^{1/2}$, where $K$ is the ratio of order cost to holding cost, which, for simplicity, is assumed constant for all items. Given $Q(i)$, the average inventory in pallet loads of item $i$ is

![Figure 2. ABC Curves.](image-url)
$Q(i)/2 = (2 KD(i))^{1/2}/2$, each with average turnover $2D(i)/Q(i) = (2D(i)/K)^{1/2}$. The total number of rack locations required for the average inventory of all items, $L$, is, using (8),

$$L = \int_{i=0}^{1} \left( (2KD(i))^{1/2}/2 \right) di = (2Ks)^{1/2}/(s + 1). \quad (9)$$

In order to determine the time index, $i$, of the $j$th pallet, $i(j)$, we must solve

$$j = \int_{0}^{i(j)} \left( (2KD(k))^{1/2}/2 \right) dk \quad \text{or} \quad i(j) = \left( (s + 1)^{2}j^{2}/2Ks \right)^{1/(s+1)} \quad (10)$$

for $j \in (0, L)$. If we redefine $j$ as $j = j/L$ in order to rescale the index to the interval $[0, 1]$, we obtain

$$i(j) = \left[ \frac{(s + 1)^{2}[(2Ks)^{1/2}/(s + 1)]^{2}}{2Ks} \right]^{1/(s+1)} \quad (12)$$

for $j \in [0, 1]$. Substituting (12) into (8) yields the demand rate for the item on pallet $j$, denoted by $D'_j$:

$$D'_j = D_{i(j)} = sj^{2(s-1)/(s+1)}. \quad (13)$$

Finally, the turnover of the $j$th pallet, $\lambda(j)$, is

$$\lambda(j) = (2D'_j/K)^{1/2} = (2s/K)^{1/2}(j^{(s-1)/(s+1)}). \quad (14)$$

Results for Random Storage Assignment

Given (4), the expected one-way travel time under random storage assignment is

$$T'_R = E[y(i)] = \int_{0}^{1} y(i) di = 2/3 \quad (15)$$

corresponding to the discrete equation (1).

Results for Turnover-Based Assignment

For full turnover-based assignment the corresponding expected one-way travel time is

$$T'_T = \frac{\int_{j=0}^{1} \lambda(j)y(j) dj}{\int_{j=0}^{1} \lambda(j) dj} \quad (16)$$

which, when evaluated using (4) and (14), becomes

$$T'_T = 4s/(5s + 1). \quad (17)$$

The percentage improvement over random storage assignment is

$$I'_T = (100) \frac{(2/3) - (4s/(5s + 1))}{(2/3)} = (100)(1 - s)/(5s + 1). \quad (18)$$

Table 1 tabulates values of $I'_T$ for the four different $ABC$ curves and their corresponding "s" values.
TABLE 1
Percentage Improvement of Turnover-Based Assignment over Random Assignment.

<table>
<thead>
<tr>
<th>ABC Curve*</th>
<th>Corresponding &quot;s&quot; Parameter</th>
<th>I_I (%) Improvement</th>
</tr>
</thead>
<tbody>
<tr>
<td>20%/60%</td>
<td>0.318</td>
<td>26.3%</td>
</tr>
<tr>
<td>20%/70%</td>
<td>0.222</td>
<td>36.9%</td>
</tr>
<tr>
<td>20%/80%</td>
<td>0.139</td>
<td>50.8%</td>
</tr>
<tr>
<td>20%/90%</td>
<td>0.065</td>
<td>70.6%</td>
</tr>
</tbody>
</table>

* 20%/60% implies that 20% of the items in inventory represent 60% of the total demand; et cetera. The corresponding "s" value may be found by solving $G(i = 0.2) = 0.6 = 0.2^s$ for $s$.

As Table 1 indicates, for typical inventory distributions, the percentage reduction in crane travel times ranges from 26% to 71%. The improvement $I_I$ increases considerably as the skewness of the inventory distribution increases.

Class-Based Turnover Assignment

It is unrealistic to assume that the turnover of every pallet to be stored in the system will be known and/or constant over time. Hence, it is desirable to develop a storage assignment rule which permits weaker assumptions concerning pallet turnover. One such rule is the class-based turnover assignment rule, under which the racks and pallets are partitioned into $K$ classes based on one-way travel times, $y_p$, and turnover, respectively. Pallets are then assigned to a class of storage according to their class of turnover (e.g., highest-turnover class to closest-location class, etc.). Within any given class, pallets are assigned to locations randomly.

Consider a two-class system, as in Figure 3. The symbol $R$ represents the partitioning of the unit square into two classes, with the Class I region to be used for the higher-turnover pallets, and the Class II region for the lower-turnover pallets.

The expected one-way travel time under this two-class system as a function of the partitioning value $R$, $T_2(R)$, is

$$T_2(R) = \int_0^{R^2} \lambda(j) \overline{y}_1 \, dj + \int_{R^2}^1 \lambda(j) \overline{y}_II \, dj \int_0^1 \lambda(j) \, dj

$$

where $\overline{y}_K$ = average travel time to region $K$.

It can be shown (see [1]) that

$$\overline{y}_1 = \frac{2}{3} R,$$

and

$$\overline{y}_II = \frac{2}{3} (1 - R^2)/(1 - R^2).$$

By substituting (14), (20) and (21) into (19) one can derive

$$T_2(R) = \frac{2}{3} \left[ R^{(s+1)/(s+1)} + (1 - R^3)(1 - R^{4s/(s+1)})/(1 - R^2) \right].$$

The derivative of (22) with respect to $R$ leads to an equation in $R$ which has been solved by numerical means for the $s$ values of interest. Table 2 contains, for various inventory distributions, the optimal value $R^*$ for partitioning into two classes, and the percentage improvement over random storage assignment. It also indicates for the
two-class system the fraction of $I_T$ (improvement from full turnover-based assignment) which is obtained with the two-class system.

From Table 2 it is apparent that, over a wide range of inventory distributions, the two-class system yields about 70% of the potential gain of a fully turnover-based system. The improvement obtained, like the full turnover-based improvement, depends upon the inventory distribution, with the most highly skewed distribution yielding the largest improvement.

Table 3 presents similar results for a three-class turnover system, where now $R_1$ and $R_2$ indicate the partitioning points between classes (see Figure 4).

### Table 2

<table>
<thead>
<tr>
<th>$ABC$ Curve (see Figure 2)</th>
<th>$R^*$ (optimal partitioning into 2 classes)</th>
<th>$I$ (% improvement over random storage assignment)</th>
<th>Fraction of Maximum possible improvement</th>
</tr>
</thead>
<tbody>
<tr>
<td>20%/60%</td>
<td>0.408</td>
<td>18.1%</td>
<td>68.6%</td>
</tr>
<tr>
<td>20%/70%</td>
<td>0.355</td>
<td>25.5%</td>
<td>69.1%</td>
</tr>
<tr>
<td>20%/80%</td>
<td>0.285</td>
<td>35.9%</td>
<td>70.6%</td>
</tr>
<tr>
<td>20%/90%</td>
<td>0.181</td>
<td>52.9%</td>
<td>75.1%</td>
</tr>
</tbody>
</table>

### Table 3

<table>
<thead>
<tr>
<th>$ABC$ Curve (see Figure 2)</th>
<th>$R^*_1$</th>
<th>$R^*_2$</th>
<th>$I$ (% improvement over random storage assignment)</th>
<th>Fraction of maximum possible improvement</th>
</tr>
</thead>
<tbody>
<tr>
<td>20%/60%</td>
<td>0.10</td>
<td>0.575</td>
<td>22.4%</td>
<td>84.7%</td>
</tr>
<tr>
<td>20%/70%</td>
<td>0.10</td>
<td>0.525</td>
<td>31.4%</td>
<td>85.0%</td>
</tr>
<tr>
<td>20%/80%</td>
<td>0.075</td>
<td>0.45</td>
<td>43.7%</td>
<td>86.0%</td>
</tr>
<tr>
<td>20%/90%</td>
<td>0.05</td>
<td>0.40</td>
<td>62.5%</td>
<td>88.7%</td>
</tr>
</tbody>
</table>

† Note: $R^*_1$, $R^*_2$ determined numerically by a grid search method.
From Table 3 one can observe that the three-class turnover system produces approximately 85% of the potential gain of the full turnover-based system. For the 20%/80% ABC curve, the gain obtained is approximately 44% over random storage assignment.

4. Testing the Continuous Approximation

Since actual warehousing systems deal with discrete pallets and discrete storage locations, it is important to determine the relationship of the continuous models analyzed to the corresponding discrete models as initially formulated. For this

<table>
<thead>
<tr>
<th>Storage Assignment System</th>
<th>Continuous Model:</th>
<th>Discrete Model:</th>
</tr>
</thead>
<tbody>
<tr>
<td>CASE I: 20%/60%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Random</td>
<td>0.66667</td>
<td>0.665</td>
</tr>
<tr>
<td>Full Turnover</td>
<td>0.491</td>
<td>0.523</td>
</tr>
<tr>
<td>Two-Class</td>
<td>0.546</td>
<td>0.565</td>
</tr>
<tr>
<td>Three-Class</td>
<td>0.518</td>
<td>0.558</td>
</tr>
</tbody>
</table>

| CASE II: 20%/70%           |                   |                 |
| Random                     | 0.66667           | 0.665           |
| Full Turnover              | 0.421             | 0.481           |
| Two-Class                  | 0.497             | 0.534           |
| Three-Class                | 0.457             | 0.519           |

| CASE III: 20%/80%          |                   |                 |
| Random                     | 0.66667           | 0.665           |
| Full Turnover              | 0.328             | 0.436           |
| Two-Class                  | 0.427             | 0.509           |
| Three-Class                | 0.375             | 0.481           |

| CASE IV: 20%/90%           |                   |                 |
| Random                     | 0.66667           | 0.665           |
| Full Turnover              | 0.196             | 0.389           |
| Two-Class                  | 0.314             | 0.503           |
| Three-Class                | 0.250             | 0.439           |

TABLE 4
Discrete Expected Travel Times vs. Continuous Times.
purpose a computer program was written to compute directly the travel times from
the I/O point to each possible location and average them, weighted by appropriate
turnover characteristics as determined by the various storage assignment policies
under consideration.

Table 4 contains selected exact results for two sizes \((R, C)\) of racks. The time
function from the I/O point to location \((i, j)\) is defined as

\[
y = \max \left\{ \frac{2i - 1}{2R}, \frac{2j - 1}{2C} \right\} \quad \text{for } i = 1, \ldots, R; j = 1, \ldots, C.
\]

Table 4 results indicate that as the measure of skewness of the inventory distribu-
tion increases, the error involved in the continuous approximation increases signif-
ically; the true potential improvement is less than that indicated by the continuous
approximation. The reason that the approximation becomes worse for more-skewed
inventory distributions relates to the increasing importance given to the first few
pallets (the high runners) and their "discreteness" relative to the continuous
approximation. Note, however, that although the continuous approximation is rela-
tively poor in some cases, the actual improvement from class-based storage assign-
ment remains quite substantial in those cases.

5. Conclusions

Significant potential reductions in crane travel times in automatic warehousing
systems seem to be possible, based on class-based turnover assignment policies rather
than closest-open-location (essentially random) policies. However, the current results
must be viewed as preliminary and explorative, because the interrelationship between
storage assignment and interleaving (the topic of our future research) has not been
addressed here.

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