THE JOURNAL OF FINANCE • VOL. XXXVI, NO. 3 • JUNE 1981

Valuation of GNMA Mortgage-Backed Securities

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ABSTRACT

GNMA mortgage-backed pass-through securities are supported by pools of amortizing, callable loans. Additionally, mortgagors often prepay their loans when the market interest rate is above the coupon rate of their loans. This paper develops a model for pricing GNMA securities and uses it to examine the impact of the amortization, call, and prepayment features on the prices, risks and expected returns of GNMA's. The amortization and prepayment features each have a positive effect on price, while the call feature has a negative impact. All three features reduce a GNMA security's interest rate risk and, consequently, its expected return.

Introduction

IN THIS PAPER WE present a model for the valuation of Government National Mortgage Association (GNMA) mortgage-backed pass-through securities. We then use the model to evaluate various facets of the pricing, returns, and risks of GNMA securities relative to those of other types of fixed rate securities. The paper is motivated by the considerable interest among portfolio managers, financial analysts, security dealers, and government officials in the pricing and investment performance of GNMA securities ([9], [17], [19], [20], [22], [23]).

In Section I we describe the unique characteristics of the GNMA security. In Section II we summarize and recapitulate the essential features of the generic model for pricing interest dependent securities developed by Brennan and Schwartz [2] and Cox, Ingersoll, and Ross [5]. In Section III we extend the generic bond pricing model to incorporate the unique characteristics of GNMA mortgage-backed pass-through securities. In Section IV we present numerical solutions for the prices of three types of default-free bonds: (1) nonamortizing, noncallable coupon bonds; (2) nonamortizing, callable coupon bonds; and (3) amortizing, noncallable bonds. We then compare these with solutions for GNMA mortgage-backed pass-through securities. The solutions are presented for alternative assumptions about the shape of the term structure of interest rates, the remaining terms to maturity of the securities, and the rate at which the individual mortgage loans that back the GNMA security are expected to be "prepaid." These comparisons are designed to highlight the impact of the call, amortization, and prepayment features on the pricing, returns, and risks of GNMA securities. A final section contains a conclusion.

* This paper has benefited from helpful comments by J. Ang, M. Brennan, P. Hendershott, R. Johnson, W. Kracaw, M. Long, G. Schlarbaum, C. Spratt, R. Thompson, and Eduardo Schwartz. We are especially grateful to J. Ingersoll, S. Richard, and G. Wright for many helpful discussions and suggestions.

I. GNMA Mortgage-backed Pass-through Securities

GNMA mortgage-backed pass-through securities are issued by mortgagees, generally mortgage bankers, who are approved by the Federal Housing Administration (FHA). Prior to issuing the security, a mortgage banker must generate a pool of new individual residential mortgage loans. GNMA requires that all the loans in a pool have the same coupon interest rate and original term to maturity and that each be insured by the FHA or guaranteed by the Veterans Administration (VA). Once GNMA approves the mortgage loans in the pool, the issuer can either sell GNMA securities (i.e. participations in the pool) directly to individual investors or sell the entire issue to a GNMA dealer. Subsequently, the issuer is responsible for servicing the loans in the pool. For providing this service the issuer receives a monthly administration fee of .0367 percent per month (.44 percent per year) of the remaining principal balances of the loans in the pool. For guaranteeing the pool GNMA charges a fee of .005 percent per month (.06 percent per year) of the remaining principal balances of the loans in the pool. Thus, a GNMA security is issued with an annual coupon interest rate that is .50 percent less than the contract rate on the underlying mortgage loans.

Each month the issuer of a GNMA security must "pass through" the scheduled interest and principal payments on the underlying mortgage loans to the holder of the security, whether or not the issuer has actually collected those payments from the individual mortgagors. Each month the issuer must also pass through any additional amounts which are received from the mortgagors for loan prepayments and/or from the FHA or VA for settlements on those loans in the pool which have been foreclosed. If the security issuer defaults on the monthly payments, GNMA assumes responsibility for the timely payment of principal and interest. Because GNMA monitors the performance of the security issuers and because the securities are backed by the "full faith and credit" of the U.S. Treasury, GNMA pass-through securities are generally considered to be riskless in terms of default.

The mortgage loans which back GNMA securities are fully amortizing. Each of the equal monthly payments on the loans includes interest on the outstanding principal balance and a partial repayment of principal.[1] Because the fee for servicing and guaranteeing the loans is a fixed percentage of the declining principal of the loans, the scheduled monthly payment to the holders of the security increases slightly through time, approaching the total monthly payment on the underlying loans at maturity.

All FHA and VA mortgage loans can be prepaid (i.e. called by the mortgagor) at any time without a prepayment penalty (i.e. without the payment of a call premium). Furthermore, the loans are assumable. That is, the mortgagor may transfer his obligation for the debt. Hence, with FHA and VA mortgage loans there are no contractual restrictions which limit mortgagors' call strategies. Thus, when markets are frictionless, mortgagors will exercise their call option only when

[1] Recently GNMA began guaranteeing securities backed by graduated mortgage loans. Although the pricing model derived in this paper can price securities backed by graduated payment loans and other nonstandard mortgage loans, we focus on securities backed by standard 30-year amortizing loans because they are by far the most widely issued securities to date.

they can refinance their existing loan with a similar loan that has a lower contract interest rate.

One of the notable characteristics of mortgagors is that, in practice, many of them call their loans even when the market interest rate is above the contract rate on their existing loans. These prepayments are generally associated with one of the following events: (1) a mortgagor changes his residence and the obligation for the existing mortgage is not assumed by the purchaser of his house; (2) the present house is refinanced so that the owner can withdraw equity; or (3) the mortgagor defaults on his loan.[2]

The fact that GNMA requires all loans in a pool to be approximately homogenous is especially convenient for our purposes. This requirement allows us to value a GNMA security as if it were a single default-free mortgage loan.[3]

II. The Generic Pricing Model

The model for valuing GNMA mortgage-backed pass-through securities is based on the generic model for pricing interest contingent securities developed in [2] and [5]. The generic model is derived from the following assumptions:

A.1: *The value of a default-free fixed interest rate security, $V(r, \tau)$, is a function only of the current value of the instantaneous risk-free rate, $r(t)$, and its term to maturity τ.*

A.2: *The interest rate for instantaneous riskless borrowing and lending follows a continuous stationary Markov process given by the stochastic differential equation*

$$dr = \mu(r)\,dt + \sigma(r)\,dz \tag{1}$$

where

$$\mu(r) \equiv k(m - r), \qquad k, m > 0,$$
$$\sigma(r) \equiv \sigma\sqrt{r},\ \sigma \text{ constant, and}$$

dz is a Wiener process with $E(dz) = 0$ and $dz^2 = dt$ with probability 1. The function $\mu(r)$ is the instantaneous drift of the process, k is the speed of adjustment parameter, m is the steady-state mean of the process, and the function $\sigma^2(r)$ is the instantaneous variance. Negative interest rates are precluded with this mean reverting interest rate process and the variance of the process increases with the interest rate.

A.3: *The risk adjustment term, $p(r)\sigma\sqrt{r}$ is proportional to the spot interest rate, i.e.*

$$p(r)\sigma\sqrt{r} = qr, \tag{2}$$

[2] Because of the mortgage insurance, default of an individual loan is equivalent to a loan prepayment from the perspective of a GNMA security holder.

[3] The mortgage loans which back a GNMA security are composed of three values—default-free financing, default insurance, and servicing. With a GNMA security, the servicing is provided by the security issuer, while the U.S. Government provides the default protection. As a consequence, the value of a GNMA security is the value of the default-free financing.

where q is the proportionality factor and $p(r)$, the price of interest rate risk, equals the equilibrium expected instantaneous return in excess of the riskless return per unit of risk for securities which satisfy A.1.

A.4: *Individuals are nonsatiated, have risk preferences consistent with* (2), *and agree on the specification of Equation* (1).

A.5: *The capital market* (*including the market for individual junior and senior mortgage loans*) *is perfect and competitive*; *trading takes place continuously.*

A.6: *The cash flows $C(\tau)$ from any security* (*including a GNMA security*) *are paid continuously.*

Assumption A.1 means that a single state variable, the current risk-free interest rate, completely summarizes all information which is relevant for the pricing of fixed-rate securities. Because changes in the value of all default-free fixed-rate securities are governed by the same random variable, the returns on all fixed-rate securities are locally perfectly correlated. Assumptions A.1 to A.5 lead to the model of the term structure of interest rates derived by Cox, Ingersoll, and Ross [5] in a general equilibrium framework for an economy with a single source of uncertainty.[4] This model of the term structure provides the foundation for the GNMA pricing model.

Assumptions A.4 and A.5 ensure that a borrower will prepay his loan according to the optimal call policy. Specifically, a borrower will never let the market value of this existing loan exceed its outstanding principal balance. If this condition were violated, the loan could be refinanced with an otherwise identical loan which has a lower effective rate of interest than the rate on the existing loan.

Although the cash payments from most fixed-rate securities occur at discrete intervals, most securities are traded with interest that accures daily. Thus, the assumption of continuous cash flows, A.6, is a convenient means of approximating the way in which fixed-rate securities (including GNMAs) are actually traded.

Given the assumptions above and the hedging arguments developed by Black and Scholes [1] and Merton [15], it follows that the value of a default-free security must satisfy the nonstochastic parabolic partial differential equation (PDE)

$$\tfrac{1}{2}\,\sigma(r)^2 V_{rr} + [\mu(r) - p(r)\sigma(r)]V_r - V_\tau - rV + C(\tau) = 0, \tag{3}$$

where subscripts on V denote partial derivatives. This equation is a special case of the fundamental valuation equation derived by Cox, Ingersoll, and Ross [5] for the value of any contingent claim and differs from the PDE derived by Brennan and Schwartz [2] for valuing several types of bonds only with respect to the functional forms of $\sigma(r)$, $\mu(r)$, and $p(r)$.

According to the generic bond pricing model, differences among interest-dependent claims are reflected in the form of their cash flows and the boundary conditions which Equation (3) must satisfy. At maturity, $\tau = 0$, the value of a

[4] Cox, Ingersoll, and Ross [5] derive a general equilibrium model of the term structure for an economy with many sources of uncertainty of which the model assumed in this paper is a special case. In a preliminary report on their joint work, Ingersoll [10] derives the model where the risk-free interest rate is the only state variable. Brennan and Schwartz [3], Dothan [6], Langetieg [13], Richard [18], and Vasicek [21] also derive continuous time models of the term structure of interest rates.

default-free bond must equal its face value or remaining principal balance $F(0)$. This provides the initial condition

$$V(r, 0) = F(0). \tag{4}$$

For a bond with continuous amortization payments, $F(0)$ is zero. For a nonamortizing bond, $F(0)$ is equal to the face value of the bond.

The value of an interest-dependent security goes to zero as the interest rate approaches infinity. This yields the boundary condition

$$\lim_{r \to \infty} V(r, \tau) = 0. \tag{5}$$

With the assumed interest rate process, $r = 0$ is a natural boundary. Setting $r = 0$ in (3) and substituting from (1) for $\sigma(0) = 0$ and $\mu(0) = km$, we obtain

$$kmV_r + C(\tau) = V_\tau \tag{6a}$$

which is the boundary condition for noncallable bonds at $r = 0$.

For callable bonds, the region of the interest rate is limited by the optimal call policy. Optimal calls are driven by the stochastic process governing the risk-free interest rate. For each τ there is some level of the risk-free interest rate, say $r_c(\tau)$, for which $V[r_c(\tau), \tau] = F(\tau)$ and the call option will be exercised. Risk-free interest rates below $r_c(\tau)$ are not relevant for pricing callable bonds. The effect of the optimal call policy is to preclude the market value of a bond from exceeding its remaining principal balance; therefore, the boundary condition for a callable bond is

$$V(r, \tau) \leq F(\tau). \tag{6b}$$

Given the boundary conditions above and the relevant functional form of the cash flows, Equation (3) can be solved for the value of any default-free interest-dependent security for which Assumptions A.1 through A.6 are appropriate.

III. The GNMA Pricing Model

As we discussed above, one of the notable characteristics of mortgagors (or at least those whose loans are pooled to support GNMA securities) is that they often call their loans at times other than those that would be dictated by the optimal call policy. We differentiate between the two types of prepayments by referring to those which occur when r is above r_c as "suboptimal" prepayments.[5] In an efficient market, the price of a GNMA security will reflect the possible occurrence of suboptimal prepayments and the generic pricing model must be modified to incorporate them. To do so, we add the following two assumptions:

[5] We use the term "suboptimal" in a casual sense. The prepayments are suboptimal only in the sense that the amount of the prepayment (i.e. the outstanding balance of the loan) exceeds the market value of the debt. Mortgagors cannot repurchase the debt at its market value and a perfect market for the "capital gain" (i.e. the face value less the market value) does not exist; therefore, the "suboptimal" prepayments are constrained maximum. Hence, the prepayment decisions of mortgagors are not suboptimal, but the prepayments are a suboptimal relative to those which would be observed if mortgagors had direct access to the capital market or if there were a perfect market for the capital gain on mortgage loans.

A.7: *Prepayments which occur when the value of a GNMA security is less than its remaining principal balance follow a Poisson-driven process. The Poisson random variable, y, is equal to zero until the loan is called suboptimally. If y jumps to one, there is a suboptimal prepayment and the security ceases to exist. The Poisson process, dy, is given by*

$$dy = \begin{cases} 0 \text{ if a suboptimal prepayment does not occur} \\ 1 \text{ if a suboptimal prepayment occurs} \end{cases}$$

where

$$E(dy) = \lambda(r, \tau)\, dt \tag{7}$$

and $\lambda(r, \tau)$ is the probability per unit of time of a suboptimal prepayment at a time to maturity τ and interest rate r.

A.8: *Prepayments which occur when the value of a GNMA security is less than its remaining principal balance are uncorrelated with all relevant market factors and are, therefore, purely nonsystematic.*

With the addition of Assumption A.7, the value of a GNMA security $V(r, \tau, y)$, is a function of two state variables, r and y, and is governed by the mixed process

$$dV = [a(r, \tau)V - C(\tau) - \lambda(r, \tau)(F(\tau) - V)]\, dt + s(r, \tau)Vdz + [F(\tau) - V]\, dy. \tag{8}$$

In (8), $a(r, \tau)$ is the total instantaneous expected rate of return on the security and $s(r, \tau)$ is the instantaneous standard deviation of the return, conditional on the Poisson event not occurring. From Ito's lemma and an analogous lemma for Poisson processes (Merton [14]), we obtain

$$a(r, \tau) = [\tfrac{1}{2}\sigma(r)^2 V_{rr} + \mu(r)V_r - V_\tau + C(\tau) + \lambda(r, \tau)(F(\tau) - V)]/V$$

and

$$s(r, \tau) = \sigma(r)V_r/V. \tag{9}$$

A portfolio containing a GNMA security and any other interest-dependent security can be constructed so that the uncertainty due to unexpected changes in the interest rate is completely eliminated. Let $b(r, \tau)$ denote the instantaneous expected rate of return and $g(r, \tau)$ denote the standard deviation of the return on the other security. The interest rate risk can be eliminated by investing the proportion $g/(g - s)$ in the GNMA security and by investing the proportion $-s/(g - s)$ in the other security. The rate of return on this portfolio is

$$\frac{dP}{P} = \left(\frac{g}{g - s}\right)\left[[a - \lambda\left(\frac{F - V}{V}\right) - \frac{s}{g}b]\, dt + \left(\frac{F - V}{V}\right) dy\right]. \tag{10}$$

Most of the time the realized return on this portfolio will equal the coefficient of dt in (10), but, when there is a suboptimal prepayment, there will be an unexpected return equal to the proportion of the portfolio invested in the GNMA security times $(F - V)/V$.

Because of the importance of Assumption A.8 to our model, some additional discussion is appropriate. From A.7 the prepayment probabilities depend only on the time to maturity and the interest rate at that time. By introducing the dynamics for other market factors, the prepayment probabilities could be made to depend on additional state variables. Assumption A.8 means that given the state of the economy at the beginning of any time interval, the Poisson process is uncorrelated with changes in the state variables during that time interval. Therefore, prepayments are unique to each security and the uncertainty due to the suboptimal prepayments can be costlessly diversified away. As a consequence, there is not a risk premium associated with the suboptimal prepayments and the expected return on the portfolio must be the riskless rate of return, r.[6] Setting the expected value of dP/P equal to $r\,dt$ and rearranging, we obtain

$$\frac{a-r}{s}=\frac{b-r}{g}\equiv p(r). \tag{11}$$

Thus, if the risk associated with suboptimal prepayments is diversifiable, a GNMA security must be priced so that its equilibrium expected excess return per unit of risk equals the price of interest rate risk, $p(r)$, for interest-dependent securities.

The partial differential equation for the value of a GNMA security is obtained by substituting from (9) for $a(r, \tau)$ and $s(r, \tau)$ in (11). Making these substitutions and rearranging yields

$$\tfrac{1}{2}\sigma(r)^2 V_{rr} + [\mu(r) - p(r)\sigma(r)]V_r - V_\tau - rV + C(\tau) + \lambda(r, \tau)[F(\tau) - V] = o.^7 \tag{12}$$

Comparing (12) with (3) shows that (12) contains the additional term $\lambda(r, \tau)[F(\tau) - V(r, \tau, y)]$. This additional term is the expected value of a suboptimal prepayment when the remaining time to maturity is τ and the riskless interest rate is r. If the Poisson event occurs, investors will receive $F(\tau)$. At that point, the market value of the security will "jump" by the amount $F(\tau) - V(r, \tau, y)$. Hence, $\lambda(r, \tau)[F(\tau) - V]$ is an additional component of the expected change in the value of the GNMA security. Like (11), (12) requires that the expected risk-adjusted return on a GNMA security be equal to the instantaneous risk-free return.

Substituting for $\mu(r)$ and $\sigma(r)$ from (1) and for $p(r)$ from (2), we obtain

$$\tfrac{1}{2}\sigma^2 r V_{rr} + [km - (k+q)r]V_r - V_\tau - rV + C(\tau) + \lambda(r, \tau)[F(\tau) - V] = 0. \tag{13}$$

With the initial condition, (4), the boundary conditions, (5) and (6b), (13) can be solved for the value of a GNMA mortgage-backed pass-through security.

IV. Comparison of GNMA Mortgage-backed Securities with other types of Fixed-rate Bonds

A. Preliminaries

The mean of the Poisson process driving suboptimal prepayments is equal to zero for all securities except a GNMA security with suboptimal prepayments. Further,

[6] Ingersoll [11] and Merton [16] have used this approach previously to deal with similar problems.

[7] This is similar to equation (7.15) in Brennan and Schwartz [4].

with $\lambda = 0$, (13) coincides with (3). Thus, by setting $\lambda = 0$ in (13) and changing either the boundary conditions and/or the functional form of the future cash flows, (13) can be solved for the prices of each of the other fixed-rate securities of concern. We use an implicit finite difference method, as described by Brennan and Schwartz [2], to solve (13) with the boundary conditions (4) through (6a) or (6b) for the price of: (1) a nonamortizing, noncallable coupon bond; (2) a nonamortizing callable coupon bond; (3) an amortizing, noncallable bond; (4) a GNMA security when the optimal call policy is followed; and (5) a GNMA security with suboptimal prepayments. Comparison of the solutions for the various types of securities illustrates the effects of the amortization feature, the call option, and the suboptimal prepayments on the value, risk, and expected return of a GNMA security.

For the models presented in this paper, the value of every interest-dependent security is a function of the risk-adjustment parameter, q, and the parameters k, m, and σ^2 of the stochastic process which governs the instantaneous interest rate. From Cox, Ingersoll, and Ross [5] we know that the price of every interest-dependent security and, therefore, the price of a GNMA security decreases with increases in the instantaneous interest rate, r, the long run mean of the current interest rate, m, and the risk premium (which is the product of the risk-adjustment parameter, q, and the interest rate elasticity of the security's price). Further, the price of noncallable security increases with increases in the variance of the current interest rate, σ^2. Because of the call option, however, an increase in σ^2 can either increase or decrease the value of a callable security such as a GNMA. When the term structure is falling (rising), prices increase (decrease) as the speed of the adjustment parameter, k, increases. For the numerical solutions presented, we assume $k = .8$, $m = .056$, $\sigma^2 = .008$, and $q = .247$.[8] The value of q is calculated by assuming that the long run interest rate, $R(\infty)$ is .08 per year.[9] When $k = .8$ the current interest rate is expected to revert halfway back to m in 10.4 months.

In the numerical illustrations we assume that the mean of the Poisson process driving the suboptimal prepayments, $\lambda(r, \tau)$, is a function only of the remaining term to maturity of the loans supporting to the GNMA security. The $\lambda(\tau)$'s are estimated from the historical FHA actuarial data in [17]. With those data it is not possible to estimate the expected prepayment rates as a function of both r and τ.

Tables I and II contain selected numerical solutions for the four types of interest-dependent securities described above. To facilitate comparisons among the securities, the prices shown are stated per $100 of remaining principal balance.

[8] These parameters are similar to those estimated by Ingersoll [12].

[9] The absence of arbitrage requires that the expected excess return per unit of risk, $p(r)$, be the same for all interest-dependent securities. Therefore, the risk adjustment term, $p(r)\sigma\sqrt{r} \equiv qr$, is not a function of maturity and one maturity is as good as another for the purpose of estimating q. Cox, Ingersoll, and Ross [5] show that the yield-to-maturity on a discount bond, $R(r, \tau)$, approaches a limiting value which is independent of the current interest rate as the time to maturity goes to infinity. This limiting yield is $R(\infty) = 2km/(g + k + q)$ where $g = \sqrt{(k + q)^2 + 2\sigma^2}$. Solving for q we obtain

$$q = k\left(\frac{m}{R(\infty)} - 1\right) - \frac{\sigma^2 R(\infty)}{2km}.$$

Each of the securities is assumed to have an original term to maturity of 30 years and a coupon interest rate of 8 percent per year. The probabilities of a suboptimal prepayment are stated relative to the historical FHA experience. For example, 100 percent FHA experience indicates that the $\lambda(\tau)$'s equal the historical FHA prepayment rates, while 200 percent FHA experience means that they are twice the FHA rate.

In Table I the current instantaneous interest rate, which determines the entire term structure, is varied from zero to 20 percent per year. When the current interest rate, r, is below the long run interest rate of 8 percent per year, the term structure is ascending. The term structure is humped when r is between $R(\infty)$ and $km/(k+q)$ and falling when r is above $km/(k+q)$. This table indicates the impact of the amortization feature, the call option, and the suboptimal prepayments on the price of a GNMA security at different levels of the current interest rate when the remaining term to maturity of each security is 30 years.

Column 1 of Table I gives the level of the current interest rate. For each level of the current interest rate, Column 2 shows the corresponding yield-to-maturity on a pure discount bond with a 30-year term to maturity. Together, these two columns provide an impression of the term structure of interest rates, given the assumed market parameters. Column 3 gives the values of the nonamortizing, noncallable bond. Column 4 shows the prices of the nonamortizing, callable bond. Column 5 presents the prices of the amortizing, noncallable bond. Columns 6, 7, and 8 contain the prices of GNMA securities under the optimal call policy and when the prepayment rates are 100 and 200 percent of the FHA experience, respectively.

B. The Shape of the Term Structure

B. 1. The Call Option

Table I shows that the noncallable bonds are more valuable than the otherwise identical callable ones. The price of each bond declines as the current interest rate is increased from zero to 20 percent. However, the magnitude of the decrease in value is greater for the noncallable than for the callable securities. Unlike a noncallable bond, the value of a callable bond cannot exceed its call price, here \$100. With 30 years to maturity, the level of the current interest rate at which the 8 percent nonamortizing callable bond (Column 4) will be called, r_c, is between 4 and 5 percent. For each of the GNMA securities (Columns 6, 7, and 8), r_c, is between 5 and 6 percent. When the current interest rate is below r_c, a callable security will have been called at its call price of \$100.

At every level of the instantaneous interest rate the value of the call option can be computed by subtracting the value of a callable security from the value of an otherwise identical noncallable one. At "high" levels of the current interest rate, the call option has a smaller impact on the total value of the security than when the interest rate is low. This is because there is a smaller probability that the option will eventually be exercised optimally when the current interest rate is high. For example, when the current interest rate is zero the difference in the values of otherwise identical callable and noncallable bonds (i.e. Column 3 less

Table I

Prices of Various Fixed-rate Securities as a Function of the Current Interest Rate when the Coupon Rate of the Bonds is 8 percent and the Term to Maturity is 30 years

(1) Percentage Current Interest Rate	(2) Percentage 30 Year Interest Rate	(3) Nonamortizing Noncallable Bond	(4) Nonamortizing Callable Bond	(5) Amortizing Noncallable Bond	(6) GNMA Security with the Optimal Call Policy	(7) GNMA Security with Suboptimal Prepayments at 100% of FHA Experience	(8) GNMA Security with Suboptimal Prepayments at 200% of FHA Experience
0.0000	7.5268	113.29805	100.00000	113.11888	100.00000	100.00000	100.00000
1.0000	7.5863	111.52800	100.00000	111.37348	100.00000	100.00000	100.00000
2.0000	7.6459	109.78614	100.00000	109.65568	100.00000	100.00000	100.00000
3.0000	7.7054	108.07378	100.00000	107.96676	100.00000	100.00000	100.00000
4.0000	7.7649	106.38961	100.00000	106.30542	100.00000	100.00000	100.00000
5.0000	7.8244	104.73284	99.98574	104.67093	100.00000	100.00000	100.00000
6.0000	7.8840	103.10400	99.27023	103.06378	99.37159	99.62100	99.75924
7.0000	7.9435	101.50229	98.13678	101.48319	98.29127	98.69237	98.92891
8.0000	8.0030	99.92686	96.85764	99.92836	97.05001	97.55658	97.86710
9.0000	8.0625	98.37738	95.51956	98.39893	95.74209	96.33082	96.70242
10.0000	8.1221	96.85347	94.15900	96.89456	94.40717	95.06459	95.48951
11.0000	8.1816	95.35476	92.79359	95.41488	93.06444	93.78187	94.25497
12.0000	8.2411	93.88086	91.43269	93.95950	91.72411	92.49552	93.01308
13.0000	8.3006	92.43135	90.08169	92.52803	90.39207	91.21296	91.77215
14.0000	8.3602	91.00581	88.74380	91.12006	89.07187	89.93871	90.53723
15.0000	8.4197	89.60381	87.42101	89.73514	87.76572	88.67562	89.31155
16.0000	8.4792	88.22487	86.11456	88.37284	86.47499	87.42552	88.09717
17.0000	8.5387	86.86852	84.82519	87.03268	85.20054	86.18959	86.89548
19.0000	8.5982	85.53428	83.55332	85.71419	83.94289	84.96857	85.70734
20.0000	8.6578	84.22222	82.29961	84.41745	82.70275	83.76333	84.53372
	8.7173	82.93240	81.06464	83.14252	81.48074	82.57460	83.37541
Selected Interested Rate Elasticities of the Bond Prices							
6.0000	7.8840	−0.09398	−0.05968	−0.09277	−0.05557	−0.04376	−0.03650
8.0000	8.0030	−0.12508	−0.10862	−0.12346	−0.10565	−0.09755	−0.09180
12.0000	8.2411	−0.18683	−0.17803	−0.18434	−0.17488	−0.16673	−0.16026

NOTE—The prices equal to $100.00 in Columns 4, 6, 7, and 8 indicate that the securities have been called optimally at their call prices of $100.00.

Column 4 and Column 5 less Column 6) is about $13.00. When the current interest rate is 20 percent, the difference in values is about $2.00.

B. 2. The Amortization Feature

The impact of the amortization feature on value can be seen by comparing the nonamortizing, noncallable bond (Column 3) with the amortizing, noncallable bond (Column 5) and by comparing the nonamortizing, callable bond (Column 4) with the GNMA security under the optimal call policy (Column 6). With the assumed market parameters, the amortization feature has a relatively small impact on the values of the securities when their remaining terms to maturity are 30 years.[10] The differences between the prices in Columns 2 and 4 and between those in Columns 3 and 5 range in absolute value from about $.01 to about $.42.

Note, however, (by comparing Columns 3 and 5) that for high levels of the current interest rate an amortizing, noncallable bond is more valuable than a nonamortizing, noncallable one, but the difference in value declines as the current interest rate declines so that the value of the nonamortizing bond eventually exceeds the value of the amortizing one. This phenomenon occurs because the level cash flows from the amortizing bond are always greater than those from the nonamortizing one until maturity when the total principal of the nonamortizing bond is repaid. When the current interest rate is high, relative to the contract rate on the securities, the final payment on the nonamortizing bond is severely discounted so that the amortizing bond is more valuable than the nonamortizing one.

The valuation relationship is reversed when the current interest rate passes below the long-term interest rate of 8 percent (which is the coupon rate of the bonds). In other words, the amortizing, noncallable bond is more (less) valuable than the nonamortizing, noncallable one when the discount rates given by the term structure are above (below) the coupon rate on the securities. However, for equal absolute differences between the current interest rate and 8 percent, the absolute value of the differences in the prices of the two bonds are, in general, smaller when r is above 8 percent than when it is below 8 percent. For example, the absolute value of the difference in the prices is .08419 when r is 4 percent and .07864 when r is 12 percent. This is because when $k = .8$, the current interest rate is expected to revert rapidly to its steady-state mean of 5.6 percent. Thus, the term structure has a "natural" tendency to be ascending and below the 8 percent coupon rate of these securities. Hence, there is a "natural" tendency for a nonamortizing, noncallable bond to be more valuable than an amortizing, noncallable one.

We should note, however, that there is an interactive effect between the amortization feature and the call option. The nonamortizing, noncallable security is more valuable than the amortizing, noncallable one when both of them are selling at a premium to their face values. However, when they are both selling at a discount, the amortizing security is more valuable. Because the call option

[10] The difference in the prices of an amortizing bond and a nonamortizing bond would be larger if we had assumed a lower value for the speed of adjustment parameter k or if we also allowed for uncertainty in the long run interest rate (e.g. see [3], [5] and [18]).

prevents a callable security from selling at a premium, an amortizing, callable security is more valuable than an otherwise identical nonamortizing, callable security. Comparing Columns 6 and 4 shows that the GNMA security with the optimal call policy is more valuable than the nonamortizing callable bond for all relevant levels of the current interest rate. Further, comparing the difference between Columns 3 and 4 with the difference between Columns 5 and 6 shows that a call option on a nonamortizing security is more valuable than a call option on an amortizing security.

A comparison of Columns 6 and 3 shows that the GNMA security with the optimal call policy is less valuable than the nonamortizing, noncallable bond for all levels of the current interest rate. As discussed above, most of the difference in value is due to the callability feature and very little is due to the amortization feature.

B. 3. Suboptimal Prepayments

The last three columns of Table I show that the effect of suboptimal prepayments is to increase the value of a GNMA security and that the effect is greater the higher the current interest rate. This occurs because the increase in an investor's wealth due to a suboptimal prepayment is greater the larger the discount of the security's price from face value. The increase in value due to suboptimal prepayments also increases with increases in the expected rate of suboptimal prepayments.[11]

For example, as the current interest rate rises from 5 percent to 20 percent, the difference between the value of the GNMA security with the optimal call policy and the one with an expected prepayment rate that is 100 percent of the FHA experience (i.e. Column 6 vs. Column 7) increases from zero to slightly over $1.00. When the expected prepayment rate is 200 percent of the FHA experience, the additional value due to suboptimal prepayments (i.e. Column 6 vs. Column 8) increases from zero to almost $2.00 as the interest rate rises from 5 percent to 20 percent.

C. Risk and Return

The information contained in Table I can also be used to examine the effect of the call option, the amortization feature, and the suboptimal prepayments on the risk and instantaneous expected return of the GNMA security. Let $a(r, \tau)$ denote the expected return of the securities. From Equations (2), (9), and (11), $a(r, \tau) = r + q[rV_r/V]$. Thus, the expected return equals the current risk-free rate plus a risk premium proportional to the interest rate elasticity of a security's price. Because the interest rate elasticity of each bond and the risk-adjustment parameter, q, are both negative, the expected return increases with increases in the absolute value of a security's interest rate elasticity. By using a centered finite difference approximation V_r, the interest rate elasticity of the price of each

[11] If the prepayment probabilities were assumed to decrease with increases in the interest rate, the increase in value due to suboptimal prepayments would be reduced somewhat. This is because there would be an interactive effect between the prepayment probabilities and the security's discount from face value as the current interest rate increased.

security, can be computed at any level of the current interest rate. The interest rate elasticity of each security is given at the bottom of Table I for current interest rates of 6 percent, 8 percent, and 12 percent.[12]

C. 1. The Amortization Feature

The impact of the amortization feature on the risk and expected returns of amortizing bonds relative to otherwise identical nonamortizing ones can be seen by comparing the elasticities in Columns 5 and 6 with those in Columns 3 and 4, respectively. These comparisons show that the prices of amortizing securities are slightly less sensitive to interest rate fluctuations than their nonamortizing counterparts, but the impact of the amortization feature on expected return is small (for the assumed parameters of the interest rate process and when the term to maturity of the securities is 30 years).

When the current interest rate is 12 percent, the absolute value of the interest rate elasticity of the GNMA security with the optimal call policy is .00315 less than that of the nonamortizing, callable bond. This means that the expected return on the GNMA security is 8 basis points per year (.00315 x .247) less than the expected return on the nonamortizing, callable bond.

C. 2 The Call Option

Comparison of Column 5 with Column 6 shows that the effect of the call option is to reduce the risk and expected return of the GNMA security. This phenomenon occurs because the price of a callable security equals the price of an otherwise identical noncallable security less the value of the call option. The values of the noncallable security and the call option both decrease with increases in the current interest rate; therefore, the price of a callable security is less sensitive to changes in the current interest rate than the price of an otherwise identical noncallable one. This effect is smaller for higher levels of the current interest rate because the call feature has less effect on the value of the callable security at higher levels of the current interest rate. Again, this is because the bond is less likely to be called when the current interest rate is high.

The difference between the elasticities in Columns 5 and 6 imply that the expected return on the GNMA security with the optimal call policy is 23 basis points lower than the expected return on the noncallable, amortizing bond when the current interest rate is 12 percent and the difference is 92 basis points when the current interest rate is 6 percent.

C. 3. Suboptimal Prepayments

An increase in the expected rate of suboptimal prepayments decreases the interest rate elasticity and, therefore, the interest rate risk and expected return of the GNMA security. This phenomenon occurs because the risk associated with the suboptimal prepayments is unsystematic and, therefore, unrewarded by the

[12] For a given change in the current interest rate, the change in the yield of pure discount bonds with longer terms to maturity is larger the smaller the speed of adjustment parameter, k. Therefore, the absolute values of the interest rate elasticities increase with decreases in k.

capital market. Further, the suboptimal prepayments reduce the relevant risk of the security, i.e., $s(r, \tau)$ in Equation (9), because they reduce the sensitivity of the security's price to changes in the interest rate. This can be seen by comparing the elasticities in Columns 6, 7, and 8. When the current interest rate is 12 percent (6 percent), the expected return on the GNMA security with suboptimal prepayments at 200 percent of FHA experience is 36 (47) basis points less than the expected return on the GNMA security with the optimal call policy.[13]

D. Term to Maturity

Table II presents the solutions for the four types of securities when the term to maturity is varied from zero to 30 years and when the current interest rate is 12 percent. Column 1 of the table gives the remaining term to maturity of each security. Column 2 shows the yield-to-maturity of a pure discount bond whose term to maturity is the same as that shown in Column 1. Thus, Column 2 gives the term structure of interest rates resulting from the assumed market parameters when the current interest rate is 12 percent. Columns 3 through 8 correspond to the Columns in Table I and, for each term to maturity, Column 9 gives the mean of the Poisson process generating prepayments at 100% of FHA experience.

Table II shows that when the term structure is descending and everywhere above 8 percent, the prices of the noncallable bonds (Columns 3 and 5) and the callable bonds with the optimal call policy (Columns 4 and 6) decline and eventually approach an asymptote as the remaining term to maturity increases. However, this behavior is sensitive to the combination of the coupon interest rate and the parameters of the interest rate process considered.

We do not report the results here, but other numerical solutions show that the prices of noncallable securities which have coupon rates that are above the long run interest, but below the current interest rate, first decline and then increase with increases in the remaining terms to maturity of the bonds. This occurs because the current interest rate is expected to decrease far enough and fast enough so that a noncallable security will eventually sell at a premium. However, because the call feature precludes callable bonds from selling at a premium, the prices of nonamortizing, callable bonds and GNMA securities with the optimal call policy decline and approach an asymptote as the term to maturity is lengthened.

Examination of Columns 7 and 8 shows that the value of the GNMA security with suboptimal prepayments does not approach an asymptote as the term to maturity is lengthened to 30 years. Instead, the prices decrease rapidly as the remaining term to maturity is increased from 25 to 30 years. This phenomenon occurs because the value of a GNMA security depends on the expected rate of future prepayments and, as shown in Column 9, the empirically estimated prepayment probabilities are low in the first two years of the security's life and then increase dramatically in Years 3 and 4.

[13] Evidence on the historical rate of return experience of GNMA securities is available in Dunn and McConnell [7] and in Waldman and Baum [23].

Table II

Prices of Various Fixed-rate Securities as a Function of Term to Maturity when the Coupon Rate of the Bonds is 8 Percent and the Current Interest Rate is 12 Percent per Year

(1) Years to Maturity	(2) Percentage Yield-to-Maturity of a Discount Bond	(3) Nonamortizing Noncallable Bond	(4) Nonamortizing Callable Bond	(5) Amortizing Noncallable Bond	(6) GNMA Security with the Optimal Call Policy	(7) GNMA Security with Suboptimal Prepayments at 100% of FHA Experience	(8) GNMA Security with Suboptimal Prepayments at 200% of FHA Experience	(9) Annual Prepayment Probabilities at 100% of FHA Experience
0	12.0000	100.00000	100.00000	100.00000	100.00000	100.00000	100.00000	.00000
1	11.0864	97.12317	97.08360	98.27117	98.25939	98.43507	98.57899	.32444
2	10.4334	95.60206	95.37638	97.19669	97.12534	97.50224	97.79363	.24495
3	9.9622	94.79564	94.28749	96.44427	96.26721	96.74149	97.10418	.15885
4	9.6170	94.36739	93.55308	95.90570	95.59501	96.20273	96.66045	.15037
5	9.3597	94.13973	93.03641	95.51170	95.05344	95.79050	96.33231	.14224
6	9.1644	94.01863	92.66152	95.21722	94.60796	95.45815	96.07090	.13459
7	9.0133	93.95418	92.38434	94.99257	94.23701	95.18049	95.85009	.12733
8	8.8942	93.91988	92.17448	94.81783	93.92274	94.94301	95.65635	.12049
9	8.7987	93.90162	92.01275	94.67944	93.65248	94.73589	95.48189	.11399
10	8.7208	93.89190	91.88702	94.56804	93.41841	94.55234	95.32201	.10785
11	8.6562	93.88673	91.78890	94.47703	93.21484	94.38657	95.17380	.10208
12	8.6021	93.88398	91.71218	94.40169	93.03722	94.21396	95.00179	.09174
13	8.5560	93.88251	91.65214	94.33860	92.88167	94.04752	94.83072	.08358
14	8.5165	93.88173	91.60513	94.28521	92.74407	93.89050	94.66681	.07665
15	8.4821	93.88131	91.56832	94.23962	92.62090	93.74542	94.51423	.07099
16	8.4520	93.88109	91.53949	94.20040	92.51009	93.61551	94.37781	.06687
17	8.4255	93.88098	91.51692	94.16640	92.41020	93.50229	94.26041	.06408
18	8.4018	93.88092	91.49923	94.13676	92.32010	93.40574	94.16281	.06235
19	8.3807	93.88089	91.48537	94.11079	92.23880	93.32517	94.08464	.06145
20	8.3617	93.88087	91.47437	94.08791	92.16540	93.25891	94.02380	.06112
21	8.3444	93.88086	91.46553	94.06768	92.09910	93.20172	93.97206	.06053
22	8.3288	93.88086	91.45835	94.04972	92.03918	93.16005	93.94139	.06159
23	8.3145	93.88086	91.45250	94.03373	91.98499	93.12710	93.92032	.06221
24	8.3014	93.88086	91.44772	94.01945	91.93594	93.09666	93.89900	.06202
25	8.2893	93.88086	91.44382	94.00665	91.89151	93.06893	93.87892	.06189
26	8.2782	93.88086	91.44062	93.99516	91.85125	93.02219	93.82418	.05774
27	8.2679	93.88086	91.43801	93.98483	91.81473	92.95124	93.72754	.05128
28	8.2583	93.88086	91.43587	93.97552	91.78159	92.86122	93.59875	.04431
29	8.2494	93.88086	91.43412	93.96711	91.75148	92.72596	93.39147	.03172
30	8.2411	93.88086	91.43269	93.95950	91.72411	92.49552	93.01308	.00840

D. 1. The Amortization Feature

When the term structure is downward sloping, the values of the amortizing securities increase relative to the values of nonamortizing ones as the remaining term to maturity becomes shorter. This result occurs because the final balloon payment on a nonamortizing security is discounted at higher interest rates as the remaining term to maturity becomes shorter and the current interest rate is held constant at 12 percent. At each term to maturity, the value of the GNMA security with the optimal call policy is greater than the value of the nonamortizing, callable bond. With 30 years to maturity, the difference in values (Column 6 less Column 4) is only $.29. This difference increases to $2.05 when the term to maturity is four years and then declines to $1.18 when the term to maturity is one year. The difference between the values of the amortizing, noncallable bond (Column 5) and the nonamortizing, noncallable bond (Column 3) increases from $.08 to $1.64 as the term to maturity declines from 30 years to 3 years. This difference then declines to $1.15 when the term to maturity is one year.

D. 2. The Call Option

The effect of changes in the term to maturity on the value of the call option can be seen by comparing the noncallable bonds with their callable counterparts (i.e., Column 3 less Column 4 and Column 5 less Column 6). These comparisons show that the value of the call option declines as the term to maturity becomes shorter and that a call option on a nonamortizing security is more valuable than a call option on an amortizing one. The latter effect is due to the fact that the call option prevents the security from selling at a premium. As discussed above, the call option has a larger impact on the value of a GNMA security than the amortization feature when the remaining term to maturity is long. However, the amortizing feature has a larger impact on price than the call option when the remaining term to maturity is short. In this case the crossover occurs when the term to maturity becomes less than eight years.

D. 3. Suboptimal Prepayments

As the term to maturity is varied from 0 to 30 years, the effect of the suboptimal prepayments on the value of the GNMA security can be seen by comparing Columns 6, 7, and 8. In general, the effect of the suboptimal prepayments is positive and larger the longer the term to maturity. However, because this effect depends on both the pattern of the prepayment probabilities and the extent to which the security is selling at a discount, the effect increases rapidly as the term to maturity is increased from zero to five years and then decreases as the remaining term to maturity is lengthened from 25 to 30 years.

As the remaining term to maturity decreases, the impact of suboptimal prepayments eventually becomes greater than the impact of optimal prepayments so that the GNMA security becomes more valuable than the amortizing, noncallable bond. For example, with prepayments at 100 percent of the historical FHA experience, the GNMA security is more valuable than the amortizing, noncallable bond when the remaining term to maturity is less than 10 years. This effect probably is somewhat overstated, however, because in practice we would expect the prepayment probabilities to decrease with increases in the risk-free interest rate.[14]

V. Conclusion

In this paper we develop a model for the pricing of GNMA mortgage-backed pass-through securities. The model is based on the general model for the pricing of interest-contingent claims developed by Brennan and Schwartz [2] and Cox, Ingersoll, and Ross [5]. A GNMA security is backed by homogenous, fully-amortizing, callable mortgage loans. Additionally, mortgagors often prepay their loans even when the market value of the loan is less than the call price. We model each of the characteristics of the GNMA security and use a numerical solution technique to analyze the impact of each feature on the price, risk, and expected return of the security.

In general, the amortization and prepayment features increase the price of a GNMA security and the callability feature decreases it. In terms of the absolute magnitude, the callability feature has a greater impact on the value of the security than either of the other two features when the remaining term to maturity is long. However, the amortization feature has the largest impact on value when the term to maturity is short. The effect of all three features is to reduce the interest rate risk and, consequently, the expected return of a GNMA security relative to other securities which do not have these features.

The analysis was undertaken with the hope that it would answer questions raised by portfolio managers, financial analysts, security dealers, and government officials about the pricing and investment performance of GNMA securities. A further pressing need is an empirical study to determine if the prices generated by the model are consistent with observed market prices. If the answer is affirmative, then the model presented here should be useful to all active participants in the GNMA market.

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