

# A Theory of Financial Innovation and Monetary Substitution with an Application to a Century of Data on the M1-income Ratio:

## Technical Notes (*Preliminary*)

Drew Saunders  
Purdue University

March, 2009

## 1 Economic Environment

### 1.1 Preferences

A representative consumer has preferences over consumption  $C_t$  and leisure  $L_t$  as described by the utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[ \ln C_t + \psi_0 \frac{L_t^{1-\psi_1}}{1-\psi_1} \right], \quad (1)$$

where  $\beta \in (0, 1)$ , and  $\psi_0, \psi_1 > 0$ . Here, the consumer is assumed to be endowed with a single unit of time which must be allocated in non-negative fractions to work ( $N_t$ ), shopping ( $S_t$ ), and innovation ( $X_t$ ), while leaving leisure non-negative:

$$L_t = 1 - N_t - S_t - X_t \geq 0; \quad N_t, S_t, X_t \geq 0$$

The nature of these activities will be described below.

## 1.2 Production Technology

Output is produced as

$$Y_t = W_t N_t, \quad (2)$$

where the productivity of labor at time  $t$ ,  $W_t$ , is an exogenous stochastic process with properties to be described. Output is used only for consumption, so that the aggregate resource constraint is

$$C_t = Y_t. \quad (3)$$

## 1.3 Markets and Government Policy

At the beginning of a period, the household has nominal resources in the form of current income, money and bonds (plus interest) held from the previous period, and a transfer from the government. These resources are used to purchase consumption, and accumulate money and nominal bonds to be held over into the following period. The representative consumer's budget constraint is

$$P_t C_t + M_t^1 + M_t^2 + B_t \leq P_t W_t N_t + M_{t-1}^1 + (1 + i_{t-1}^m) M_{t-1}^2 + (1 + i_{t-1}) B_{t-1} + T_t. \quad (4)$$

Here,  $M_t^1$  is non-interest-bearing (“narrow”) money,  $M_t^2$  is interest-bearing (“broad”) money,  $B_t$  is nominal bonds,  $T_t$  is a nominal transfer from the government,  $P_t$  is the price level,  $i_{t-1}^m$  is the interest earned by broad money held at the end of time  $t - 1$ , and  $i_{t-1}$  is the interest earned by bonds held at the end of period  $t - 1$ .

It is assumed that the government controls the nominal supply of each type of money, and that money and interest obligations are serviced from lump-sum tax revenues (negative transfers). The implied budget constraint for the government is

$$0 = i_{t-1}^m M_{t-1}^2 - (M_t^1 - M_{t-1}^1) - (M_t^2 - M_{t-1}^2) + T_t. \quad (5)$$

## 1.4 Transactions Technology

The technology for making transactions in the economy requires a combination of real monies  $M_t^i/P_t$  and shopping time satisfying

$$Z_{t-1}^{1-\nu} S_t^\nu \geq \left\{ \frac{C_t}{\left[ \left( \frac{M_t^1}{P_t} \right)^\alpha + \kappa_t \left( \frac{M_t^2}{P_t} \right)^\alpha \right]^{\frac{1}{\alpha}}} \right\}^\eta, \quad (6)$$

where  $\nu, \eta \geq 0$ ,  $\alpha \leq 1$ , and  $\kappa_t > 0$  for all  $t$ . Here  $Z_{t-1}$  is the level of transactions skill of the household at time  $t$ , and  $\kappa_t$  is an exogenous parameter governing the evolution of the level of transactions services yielded by broad money relative narrow money. Transactions skill is assumed to evolve according to

$$Z_t = (1 - \delta + \phi X_t) Z_{t-1}, \quad (7)$$

where  $\delta \in (0, 1)$  and  $\phi > 0$ . Note that the household effectively controls the growth rate of  $Z_t$  directly by its choice of the time investment  $X_t$ .

## 1.5 Equilibrium

An equilibrium is a collection of stochastic processes

$$\{C_t, N_t, S_t, X_t, M_t^1, M_t^2, B_t, Z_t, Y_t, T_t, P_t, W_t, i_t, i_t^m, \kappa_t\}_{t=0}^{\infty},$$

with  $(M_{-1}^1, M_{-1}^2, B_{-1}, Z_{-1}, i_{-1}, i_{-1}^m)$  taken as given, such that

1.  $\{C_t, N_t, S_t, X_t, M_t^1, M_t^2, B_t, Z_t\}_{t=0}^{\infty}$  maximize (1) subject to (4), (6), and (7) taking as given  $(M_{-1}^1, M_{-1}^2, B_{-1}, Z_{-1}, i_{-1}, i_{-1}^m)$  and stochastic processes  $\{T_t, P_t, W_t, i_t, i_t^m, \kappa_t\}_{t=0}^{\infty}$ .
2. The goods market clears, so that (3) holds.
3. The government's budget constraint (5) is satisfied.
4. The private bond market clears, so that  $B_{t-1} = 0$  for all  $t$ .

## 2 Characterization of Equilibrium

### 2.1 Simplification of the Household's Problem

Writing  $\tilde{M}_t^i \equiv M_t^i/P_t$ ,  $\tilde{B}_t \equiv B_t/P_t$ , and  $\tilde{T}_t \equiv T_t/P_t$  for real money, bond holdings, and transfers, respectively, the budget constraint can be written as

$$C_t + \tilde{M}_t^1 + \tilde{M}_t^2 + \tilde{B}_t \leq W_t N_t + (1 - \pi_t) \left[ \tilde{M}_{t-1}^1 + (1 + i_{t-1}^m) \tilde{M}_{t-1}^2 + (1 + i_{t-1}) \tilde{B}_t \right] + \tilde{T}_t, \quad (8)$$

where  $\pi_t \equiv 1 - P_{t-1}/P_t$  is the rate of inflation. Next, the law of motion for capital can be solved for time spent innovating:

$$X_t = \frac{1}{\phi} \left( \frac{Z_t}{Z_{t-1}} - 1 + \delta \right).$$

Now recognizing that the transactions constraint must hold with equality at any optimum, we use the binding constraint (6) to eliminate  $C_t$  from (8) and the objective function; and use the last equation to eliminate  $X_t$ , so that the problem of the household can be expressed as that of choosing  $\left\{ N_t, S_t, \tilde{M}_t^1, \tilde{M}_t^2, \tilde{B}_t, Z_t \right\}_{t=0}^{\infty}$  to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \frac{1-\nu}{\eta} \log Z_{t-1} + \frac{\nu}{\eta} \log S_t + \log \mathcal{M}_t + \frac{\psi_0}{1-\psi_1} \left[ 1 - N_t - S_t - \frac{1}{\phi} \left( \frac{Z_t}{Z_{t-1}} - 1 + \delta \right) \right]^{1-\psi_1} \right\}, \quad (9)$$

subject to

$$Z_{t-1}^{(1-\nu)/\eta} S_t^{\nu/\eta} \mathcal{M}_t + \tilde{M}_t^1 + \tilde{M}_t^2 + \tilde{B}_t \leq W_t N_t + (1 - \pi_t) \left[ \tilde{M}_{t-1}^1 + (1 + i_{t-1}^m) \tilde{M}_{t-1}^2 + (1 + i_{t-1}) \tilde{B}_t \right] + \tilde{T}_t, \quad (10)$$

where

$$\mathcal{M}_t \equiv \left[ \left( \tilde{M}_t^1 \right)^\alpha + \kappa_t \left( \tilde{M}_t^2 \right)^\alpha \right]^{\frac{1}{\alpha}}.$$

## 2.2 FOCs from the Household's Problem

The first-order conditions for the simplified problem are

$$\tilde{M}_t^1 : \frac{\left(\tilde{M}_t^1\right)^{\alpha-1}}{\mathcal{M}_t^\alpha} - \tilde{\lambda}_t Z_{t-1}^{(1-\nu)/\eta} S_t^{\nu/\eta} \mathcal{M}_t^{1-\alpha} \left(\tilde{M}_t^1\right)^{\alpha-1} \quad (11)$$

$$-\tilde{\lambda}_t + \beta E_t \tilde{\lambda}_{t+1} (1 - \pi_{t+1}) = 0$$

$$\tilde{M}_t^2 : \frac{\kappa_t \left(\tilde{M}_t^2\right)^{\alpha-1}}{\mathcal{M}_t^\alpha} - \tilde{\lambda}_t Z_{t-1}^{(1-\nu)/\eta} S_t^{\nu/\eta} \mathcal{M}_t^{1-\alpha} \kappa_t \left(\tilde{M}_t^2\right)^{\alpha-1} \quad (12)$$

$$-\tilde{\lambda}_t + (1 + i_t^m) \beta E_t \tilde{\lambda}_{t+1} (1 - \pi_{t+1}) = 0$$

$$\tilde{B}_t : -\tilde{\lambda}_t + (1 + i_t) \beta E_t \tilde{\lambda}_{t+1} (1 - \pi_{t+1}) = 0 \quad (13)$$

$$S_t : \frac{\nu}{\eta S_t} - \psi_0 L_t^{-\psi_1} - \frac{\tilde{\lambda}_t \nu}{\eta} Z_{t-1}^{(1-\nu)/\eta} S_t^{(\nu-\eta)/\eta} \mathcal{M}_t = 0 \quad (14)$$

$$Z_t : -\psi_0 L_t^{-\psi_1} \frac{1}{\phi Z_{t-1}} \quad (15)$$

$$+\beta E_t \left[ \frac{\omega}{Z_t} - \tilde{\lambda}_{t+1} \omega Z_t^{(1-\nu-\eta)/\eta} S_{t+1}^{\nu/\eta} \mathcal{M}_t + \psi_0 L_{t+1}^{-\psi_1} \frac{Z_{t+1}}{\phi (Z_t)^2} \right]$$

$$N_t : -\psi_0 L_t^{-\psi_1} + \tilde{\lambda}_t W_t = 0, \quad (16)$$

where  $\beta^t \tilde{\lambda}_t$  is a non-negative Lagrange multiplier on the budget constraint, and

$$L_t = 1 - N_t - S_t - \frac{1}{\phi} \left( \frac{Z_t}{Z_{t-1}} - 1 + \delta \right).$$

## 2.3 First Characterization of Equilibrium

Define  $\lambda_t \equiv \tilde{\lambda}_t W_t$ , and use (16) and the market clearing condition that

$$W_t = C_t/N_t = Z_{t-1}^{(1-\nu)/\eta} S_t^{\nu/\eta} \mathcal{M}_t/N_t$$

to re-write (14) as

$$S_t = \frac{\nu}{\eta} \left( \frac{1}{\lambda_t} - N_t \right). \quad (17)$$

Now (16) can be written as

$$\begin{aligned}\lambda_t &= \psi_0 L_t^{-\psi_1} \\ &= \psi_0 \left[ 1 - \left( 1 - \frac{\nu}{\eta} \right) N_t - \frac{\nu}{\eta \lambda_t} - \frac{1}{\phi} \left( \frac{Z_t}{Z_{t-1}} - 1 + \delta \right) \right]^{-\psi_1}.\end{aligned}\tag{18}$$

Next note that (15) can be written as

$$\lambda_t \frac{Z_t}{Z_{t-1}} = \beta E_t \left\{ \phi \left( \frac{1-\nu}{\eta} \right) + \lambda_{t+1} \left[ \frac{Z_{t+1}}{Z_t} - \phi \left( \frac{1-\nu}{\eta} \right) N_{t+1} \right] \right\}.\tag{19}$$

Now using (13) and (16), (11) and (12) may be written as

$$\frac{\left( \tilde{M}_t^1 \right)^{\alpha-1}}{\mathcal{M}_t^\alpha} \left( \frac{W_t}{\lambda_t} - Z_{t-1}^{(1-\nu)/\eta} S_t^{\nu/\eta} \mathcal{M}_t \right) = r_t$$

and

$$\frac{\kappa_t \left( \tilde{M}_t^2 \right)^{\alpha-1}}{\mathcal{M}_t^\alpha} \left( \frac{W_t}{\lambda_t} - Z_{t-1}^{(1-\nu)/\eta} S_t^{\nu/\eta} \mathcal{M}_t \right) = r_t^m,$$

where

$$r_t \equiv \frac{i_t}{1+i_t} \text{ and } r_t^m \equiv \frac{i_t - i_t^m}{1+i_t}.$$

It follows that

$$\frac{\tilde{M}_t^2}{\tilde{M}_t^1} = \left[ \frac{\kappa_t r_t}{r_t^m} \right]^{\frac{1}{1-\alpha}}.\tag{20}$$

Using (2) and (3), we may derive

$$\frac{\tilde{M}_t^1}{C_t} q_t^\alpha = \left( \frac{1}{\lambda_t N_t} - 1 \right) r_t^{-1}\tag{21}$$

where

$$q_t \equiv \left[ 1 + \kappa_t^{\frac{1}{1-\alpha}} \left( \frac{r_t}{r_t^m} \right)^{\frac{\alpha}{1-\alpha}} \right]^{\frac{1}{\alpha}}.\tag{22}$$

Finally, we may eliminate  $S_t$  using (6) to write (17) as

$$\left[ \left( \tilde{M}_t^1 / C_t \right) q_t \right]^{-\frac{\eta}{\nu}} Z_{t-1}^{-\frac{1-\nu}{\nu}} = \frac{\nu}{\eta} \left( \frac{1}{\lambda_t} - N_t \right). \quad (23)$$

Now it can be seen that, given initial conditions  $(\tilde{M}_{-1}^1, \tilde{B}_{-1}, Z_{-1})$  and processes  $\{r_t, q_t, W_t\}_{t=0}^{\infty}$ , equilibrium processes  $\{C_t, N_t, \tilde{M}_t^1, Z_t, \lambda_t\}_{t=0}^{\infty}$  are fully characterized by (18), (19), (21), (23), and

$$C_t = W_t N_t \quad (24)$$

(with an appropriate transversality condition). Notice also that  $\tilde{M}_t^2$  can be deduced, as well, from (20) using (22).

## 2.4 Characterization in Terms of Equilibrium Ratios

Let

$$q_t = \bar{\gamma}_q^t u_t,$$

where  $\bar{\gamma}_q$  is a positive constant; and define also

$$\begin{aligned} m_t &\equiv \left( \tilde{M}_t^1 / C_t \right) \bar{\gamma}_q^{\alpha t} \\ z_t &\equiv Z_{t-1} \bar{\gamma}_q^{(1-\alpha)\eta/(1-\nu)t} \\ \gamma_{z,t} &\equiv Z_t / Z_{t-1}. \end{aligned}$$

Now note that, given initial conditions  $(\tilde{M}_{-1}^1, Z_{-1}, i_{-1})$ , an equilibrium may be characterized by stochastic processes  $\{\lambda_t, N_t, m_t, \gamma_{z,t}, z_t, r_t, u_t\}_{t=0}^\infty$  satisfying

$$0 = -\lambda_t + \psi_0 \left[ 1 - \frac{\nu}{\eta \lambda_t} - \left( 1 - \frac{\nu}{\eta} \right) N_t - \frac{1}{\phi} (\gamma_{z,t} - 1 + \delta) \right]^{-\psi_1} \quad (25)$$

$$0 = \frac{\nu}{\eta} \left( \frac{1}{\lambda_t} - N_t \right) - m_t^{-\frac{\eta}{\nu}} z_t^{-\frac{1-\nu}{\nu}} u_t^{-\frac{\eta}{\nu}} \quad (26)$$

$$0 = -m_t + r_t^{-1} \left( \frac{1}{\lambda_t N_t} - 1 \right) u_t^{-\alpha} \quad (27)$$

$$0 = -\lambda_t \gamma_{z,t} + \beta E_t \left\{ \lambda_{t+1} \left[ \gamma_{z,t+1} - \phi \left( \frac{1-\nu}{\eta} \right) N_{t+1} \right] + \phi \left( \frac{1-\nu}{\eta} \right) \right\} \quad (28)$$

$$0 = -\gamma_{z,t} + \left( \frac{z_{t+1}}{z_t} \right) \bar{\gamma}_q^{-\frac{(1-\alpha)\eta}{1-\nu}}. \quad (29)$$

## 2.5 A Balanced Growth Path of Sorts

In this subsection, assume that  $r_t = \bar{r}$ , and  $u_t = \bar{u}$  for all  $t$ .

Given the assumptions described in the previous paragraph, I will characterize an equilibrium in which  $N_t$ ,  $S_t$ , and  $X_t$  are each constant over time; and let us write  $\bar{N}$ ,  $\bar{S}$ , and  $\bar{X}$  for the respective constant values. In this case, it is immediate from (7) that  $\gamma_{z,t}$  is constant along such a path, with

$$\gamma_{z,t} = \bar{\gamma}_z = (1 - \delta + \phi \bar{X}).$$

Next, (??) shows that  $\lambda_t$  must be constant, as well; and I write  $\bar{\lambda}$  for its value in the steady-state. Now (??) shows that  $m_t$  is constant, and so (??) implies that  $z_t$  is constant; and I write  $\bar{m}$  and  $\bar{z}$  (respectively) for their values.

We have thus shown that the dynamics of the transformed system are consistent with a steady-state equilibrium when the opportunity cost is held constant, and the substitution term  $q_t$  grows deterministically at a constant rate. The values of the constants  $(\bar{\lambda}, \bar{N}, \bar{m}, \bar{\gamma}_z, \bar{z})$  may now be obtained by solving the system of five equations (??)-(??) under the steady-state assumption.

From (??), the results above show that

$$\bar{\gamma}_z = \bar{\gamma}_q^{-\frac{(1-\alpha)\eta}{1-\nu}}.$$

Now using this result and (??), we can derive

$$(1 - \beta) \gamma_z = \frac{\beta \phi (1 - \nu)}{\eta} \left( \frac{1}{\bar{\lambda}} - \bar{N} \right). \quad (30)$$

Then considering additionally (??), we have that

$$\bar{\lambda} = \psi_0 \left[ 1 - \frac{\nu}{\eta \bar{\lambda}} - \left( 1 - \frac{\nu}{\eta} \right) \bar{N} - \frac{1}{\phi} (\bar{\gamma}_z - 1 + \delta) \right]^{-\psi_1}.$$

Now given values of exogenous variables  $\bar{\gamma}_q$  and  $\bar{u}$ , the last three equations may be solved for  $\bar{\lambda}$  and  $\bar{N}$ , so that output and the household's consumption may be evaluated. Moreover, from (7), (6), and (??), the steady-state values of  $X_t$  and  $S_t$  may be evaluated as

$$\bar{X} = \frac{\bar{\gamma}_z - 1 + \delta}{\phi}$$

and

$$\bar{S} = \frac{\nu}{\eta} \left( \frac{1}{\bar{\lambda}} - \bar{N} \right), \quad (31)$$

respectively; so that the household's leisure time in the steady-state, say  $\bar{L} \equiv 1 - \bar{N} - \bar{S} - \bar{X}$ , is also independent of  $\bar{r}$ ,  $\bar{m}$ , and  $\bar{z}$ .

The steady-state values of  $\bar{m}$  and  $\bar{z}$  may now be obtained by plugging into the steady-state representations of (??) and (??) the values  $\bar{\lambda}$  and  $\bar{N}$  and solving; this gives

$$\bar{m} = \left( \frac{1}{\bar{\lambda} \bar{N}} - 1 \right) \bar{r}^{-1} \bar{u}^{-\alpha}$$

and

$$\bar{z} = \left( \frac{\eta}{\nu} \right)^{\frac{\nu}{1-\nu}} \left( \frac{1}{\bar{\lambda} \bar{N}} - 1 \right)^{-\frac{\eta+\nu}{1-\nu}} \bar{N}^{-\frac{\nu}{1-\nu}} (\bar{r})^{\frac{\eta}{1-\nu}} \bar{u}^{\frac{(1-\alpha)\eta}{1-\nu}}.$$

## 2.6 Summary of Equations Characterizing the Non-stochastic Steady-state (NSSS)

I will refer to the collection of constant ratios characterizing the dynamic equilibrium discussed in the last subsection as the non-stochastic steady-state (NSSS) of the system. For convenience the equations characterizing these ratios are summarized

here:<sup>1</sup>

$$0 = -\bar{\lambda} + \psi_0 \left[ 1 - \frac{\nu}{\eta\bar{\lambda}} - \left( 1 - \frac{\nu}{\eta} \right) \bar{N} - \frac{1}{\phi} (\bar{\gamma}_z - 1 + \delta) \right]^{-\psi_1} \quad (32)$$

$$0 = \frac{\nu}{\eta} \left( \frac{1}{\bar{\lambda}} - \bar{N} \right) - \bar{m}^{-\frac{\eta}{\nu}} \bar{z}^{-\frac{1-\nu}{\nu}} \bar{u}^{-\frac{\eta}{\nu}} \quad (33)$$

$$0 = -\bar{m} + \bar{r}^{-1} \left( \frac{1}{\bar{\lambda}\bar{N}} - 1 \right) \bar{u}^{-\alpha} \quad (34)$$

$$0 = -(1 - \beta) \bar{\lambda} \bar{\gamma}_z - \beta \phi \left( \frac{1 - \nu}{\eta} \right) \bar{N} + \beta \phi \left( \frac{1 - \nu}{\eta} \right) \quad (35)$$

$$0 = -\bar{\gamma}_z + \gamma_q^{-\frac{(1-\alpha)\eta}{1-\nu}}. \quad (36)$$

As suggested by the analysis in subsection 2.5, the system admits no closed form solution in general, and the NSSS values will need to be obtained numerically given the structural parameters. The recursive algorithm developed in subsection 2.5 is an efficient way to calculate the NSSS given a set of parameters.

## 2.7 Stochastic Processes

The stochastic processes  $r_t$  and  $u_t$  are assumed to be governed individually by the evolution of autoregressive processes as follows:

$$\hat{r}_t \equiv \log r_t - \log \bar{r} = \rho_1 \hat{r}_{t-1} + \rho_2 \hat{r}_{t-2} + \varepsilon_t^r \quad (37)$$

$$\hat{u}_t \equiv \log u_t - \log \bar{u} = \chi \hat{u}_t + \varepsilon_t^u, \quad (38)$$

where  $\bar{r}$  and  $\bar{u}$  are positive constants (as in previous sections), and  $\varepsilon_t \equiv (\varepsilon_t^r, \varepsilon_t^u)'$  is distributed as  $N(0, \Sigma)$ .

---

<sup>1</sup>Note that the time path of  $\tilde{M}_t^2$  may be determined from (??) and

$$\left[ 1 + \xi_t^{\frac{1}{1-\alpha}} \left( \frac{r_t^1}{r_t^2} \right)^{\frac{\alpha}{1-\alpha}} \right]^{\frac{1}{\alpha}} = \bar{\xi} \gamma_{\xi}^t.$$

## 2.8 Linearization of the System of Equilibrium Ratios around the NSSS

I will write variables with carets (“hats”) to denote log deviations from the respective element of the NSSS point; for example, I write

$$\hat{\lambda}_t \equiv \log \tilde{\lambda}_t - \log \bar{\lambda}$$

and

$$\hat{N}_t \equiv \log N_t - \log \bar{N}.$$

Now, the system of interest may be approximated by log-linearizing (25)-(29):

$$0 = - \left[ 1 + \frac{\psi_0 \psi_1 (\nu/\eta)}{\bar{L} \bar{\lambda}} \right] \hat{\lambda}_t + \frac{\psi_0 \psi_1 (1 - \nu/\eta) \bar{N}}{\bar{L}} \hat{N}_t + \frac{\psi_0 \psi_1 \bar{\gamma}_z}{\bar{L} \phi} \hat{\gamma}_{z,t} \quad (39)$$

$$0 = -\theta \hat{\lambda}_t - (\theta - 1) \hat{N}_t + (\eta/\nu) \hat{m}_t + [(1 - \nu)/\nu] \hat{z}_t + (\eta/\nu) \hat{u}_t \quad (40)$$

$$0 = -\hat{m}_t - \hat{r}_t - \theta \hat{\lambda}_t - \theta \hat{N}_t - \alpha \hat{u}_t \quad (41)$$

$$0 = -\hat{\lambda}_t - \hat{\gamma}_{z,t} + \beta E_t \left[ \left( 1 - \frac{\phi(1 - \nu) \bar{N}}{\eta \bar{\gamma}_z} \right) \hat{\lambda}_{t+1} + \hat{\gamma}_{z,t+1} - \frac{\phi(1 - \nu) \bar{N}}{\eta \bar{\gamma}_z} \hat{N}_{t+1} \right] \quad (42)$$

$$0 = -\hat{\gamma}_{z,t-1} + \hat{z}_t - \hat{z}_{t-1} \quad (43)$$

where

$$\bar{L} \equiv 1 - \bar{N} - \bar{S} - \frac{1}{\phi} (\bar{\gamma}_z - 1 + \delta).$$

and

$$\theta \equiv \frac{1}{1 - \bar{\lambda} \bar{N}}.$$

Notice that (43) is an approximation of the *lag* of equation (29).

## 3 Solving the Model Analytically

The fundamental step in “solving” the model is to characterize the evolution of a sufficient set of endogenous state variables as a function of contemporaneous values of the exogenous processes (exogenous state variables) and of lags of themselves. Having done so, the model’s remaining variables may be described as functions of this (sufficient) set of (endogenous and exogenous) state variables. In this section, I show how to characterize the laws of motion of these state variables analytically.

To proceed, I first derive a single equation characterizing the evolution of the endogenous state variable  $\hat{z}_t$  (given the exogenous processes).

Writing (39) as

$$0 = -(1 + \Psi_1) \hat{\lambda}_t + \Psi_2 \hat{N}_t + \Psi_3 \hat{\gamma}_{z,t}$$

to economize on space, we first solve (39)-(41) for  $\hat{N}_t$ ,  $\hat{\lambda}_t$ , and  $\hat{m}_t$ . This procedure gives

$$\begin{aligned} \hat{N}_t &= \frac{(1 + \Psi_1) \omega \left(\frac{1-\nu}{\nu}\right)}{1 + \Psi_1 + (1 + \omega) \Psi_2} \hat{z}_{t-1} - \frac{\Psi_3 (1 + \omega)}{1 + \Psi_1 + (1 + \omega) \Psi_2} \hat{\gamma}_{z,t} - \frac{(1 + \Psi_1) \omega \left(\frac{\eta}{\nu}\right)}{1 + \Psi_1 + (1 + \omega) \Psi_2} \hat{r}_t \\ &\quad + \frac{(1 + \Psi_1) \omega (1 - \alpha) \left(\frac{\eta}{\nu}\right)}{1 + \Psi_1 + (1 + \omega) \Psi_2} \hat{u}_t, \end{aligned}$$

$$\begin{aligned} \hat{\lambda}_t &= \frac{\Psi_2 \omega \left(\frac{1-\nu}{\nu}\right)}{1 + \Psi_1 + (1 + \omega) \Psi_2} \hat{z}_{t-1} + \frac{\Psi_3 (1 + \omega)}{1 + \Psi_1 + (1 + \omega) \Psi_2} \hat{\gamma}_{z,t} - \frac{\Psi_2 \omega \left(\frac{\eta}{\nu}\right)}{1 + \Psi_1 + (1 + \omega) \Psi_2} \hat{r}_t \\ &\quad + \frac{\Psi_2 \omega (1 - \alpha) \left(\frac{\eta}{\nu}\right)}{1 + \Psi_1 + (1 + \omega) \Psi_2} \hat{u}_t, \end{aligned}$$

and

$$\begin{aligned} \hat{m}_t &= \left[ \frac{(1 + \Psi_1 + \Psi_2) \theta \omega \frac{\eta}{\nu}}{1 + \Psi_1 + (1 + \omega) \Psi_2} - 1 \right] \hat{r}_t - \frac{(1 + \Psi_1 + \Psi_2) \theta \omega \left(\frac{1-\nu}{\nu}\right)}{1 + \Psi_1 + (1 + \omega) \Psi_2} \hat{z}_{t-1} \\ &\quad - \left[ \frac{(1 + \Psi_1 + \Psi_2) \theta \omega (1 - \alpha) \frac{\eta}{\nu}}{1 + \Psi_1 + (1 + \omega) \Psi_2} + \alpha \right] \hat{u}_t, \end{aligned} \quad (44)$$

where

$$\omega \equiv \frac{1 - \bar{\lambda} \bar{N}}{\frac{\eta}{\nu} + \bar{\lambda} \bar{N}}.$$

Now substituting these into (42) and simplifying, we have

$$\begin{aligned} &\Psi_2 \omega \left(\frac{1-\nu}{\nu}\right) \hat{z}_{t-1} - \Psi_2 \omega \frac{\eta}{\nu} \hat{r}_t + \Psi_2 \omega (1 - \alpha) \frac{\eta}{\nu} \hat{u}_t + [1 + \Psi_1 + (\Psi_2 + \Psi_3) (1 + \omega)] \hat{\gamma}_{z,t} \\ &= \beta E_t \left\{ \left[ \Psi_2 - \frac{\phi (1 - \nu) \bar{N}}{\eta \bar{\gamma}_z} (1 + \Psi_1 + \Psi_2) \right] \omega \left[ \left(\frac{1-\nu}{\nu}\right) \hat{z}_t - \frac{\eta}{\nu} \hat{r}_{t+1} + (1 - \alpha) \frac{\eta}{\nu} \hat{u}_{t+1} \right] \right. \\ &\quad \left. + [1 + \Psi_1 + (\Psi_2 + \Psi_3) (1 + \omega)] \hat{\gamma}_{z,t+1} \right\}. \end{aligned}$$

Next substitute for  $\hat{\gamma}_{z,t}$  and  $\hat{\gamma}_{z,t+1}$  using (43) to derive

$$\begin{aligned}
& \left[ 1 + \Psi_1 + (\Psi_2 + \Psi_3)(1 + \omega) - \Psi_2\omega \left( \frac{1 - \nu}{\nu} \right) \right] \hat{z}_{t-1} \\
& - \left\{ (1 + \beta) \left[ 1 + \Psi_1 + (\Psi_2 + \Psi_3)(1 + \omega) - \Psi_2\omega \left( \frac{1 - \nu}{\nu} \right) \right] \right. \\
& \quad \left. + \beta(1 + \Psi_1 + \Psi_2) \frac{\phi(1 - \nu)\bar{N}}{\eta\bar{\gamma}_z} \right\} \hat{z}_t \\
& + \beta [1 + \Psi_1 + (\Psi_2 + \Psi_3)(1 + \omega)] E_t \hat{z}_{t+1} \\
& + \Psi_2\omega \frac{\eta}{\nu} \hat{r}_t - \Psi_2\omega(1 - \alpha) \frac{\eta}{\nu} \hat{u}_t \\
& + \beta \left[ (1 + \Psi_1 + \Psi_2) \frac{\phi(1 - \nu)\bar{N}}{\eta\bar{\gamma}_z} - \Psi_2 \right] \omega \left[ \frac{\eta}{\nu} E_t \hat{r}_{t+1} - (1 - \alpha) \frac{\eta}{\nu} E_t \hat{u}_{t+1} \right].
\end{aligned}$$

Dividing through by the coefficient on  $\hat{z}_{t-1}$ , say

$$A_0 \equiv 1 + \Psi_1 + (\Psi_2 + \Psi_3)(1 + \omega) - \Psi_2\omega \left( \frac{1 - \nu}{\nu} \right),$$

write this as

$$\begin{aligned}
0 = & \hat{z}_{t-1} - \left( 1 + \beta + \beta \frac{A_1}{A_0} \right) \hat{z}_t + \beta \frac{A_2}{A_0} E_t \hat{z}_{t+1} + \frac{\Psi_2\omega}{A_0} \left( \frac{\eta}{\nu} \right) \hat{r}_t - \frac{\Psi_2\omega}{A_0} (1 - \alpha) \left( \frac{\eta}{\nu} \right) \hat{u}_t \\
& + \frac{\beta(A_1 - \Psi_2)\omega}{A_0} \left[ \frac{\eta}{\nu} E_t \hat{r}_{t+1} - (1 - \alpha) \frac{\eta}{\nu} E_t \hat{u}_{t+1} \right]
\end{aligned}$$

where

$$\begin{aligned}
A_1 & \equiv (1 + \Psi_1 + \Psi_2) \frac{\phi(1 - \nu)\bar{N}}{\eta\bar{\gamma}_z} \\
A_2 & \equiv 1 + \Psi_1 + (\Psi_2 + \Psi_3)(1 + \omega).
\end{aligned}$$

Now following the approach of Sargent (1987), the lag operator polynomial on  $z_{t-1}$  can be factored to write the difference equation as

$$\beta \frac{A_2}{A_0} (L^{-1} - \xi_1) (L^{-1} - \xi_2) E_t z_{t-1} = E_t Q_{t+1},$$

where

$$Q_{t+1} \equiv \frac{\Psi_2 \omega \eta}{A_0 \nu} \hat{r}_t - \frac{\Psi_2 \omega (1 - \alpha) \eta}{A_0 \nu} \hat{u}_t + \frac{\beta (A_1 - \Psi_2) \omega \eta}{A_0 \nu} \hat{r}_{t+1} - \frac{\beta (A_1 - \Psi_2) \omega (1 - \alpha) \eta}{A_0 \nu} \hat{u}_{t+1},$$

and  $\xi_1$  and  $\xi_2$  are the distinct solutions to

$$\begin{aligned} \xi &= \frac{1}{2} \left( \frac{1 + \beta + \beta A_1 / A_0}{\beta A_2 / A_0} \right) \\ &\pm \sqrt{\frac{1}{4} \left( \frac{1 + \beta + \beta A_1 / A_0}{\beta A_2 / A_0} \right)^2 - \frac{1}{\beta A_2 / A_0}}. \end{aligned} \quad (45)$$

Note that, equating powers in the lag operator, we have

$$\beta \frac{A_2}{A_0} \xi_1 \xi_2 = 1 \text{ and } \beta \frac{A_2}{A_0} (\xi_1 + \xi_2) = 1 + \beta + \beta \frac{A_1}{A_0}, \quad (46)$$

so that

$$\xi_2 = \frac{1}{\xi_1 \beta A_2 / A_0} \quad (47)$$

and each root satisfies

$$\xi_k \beta A_2 / A_0 + \frac{1}{\xi_k} = 1 + \beta + \beta \frac{A_1}{A_0}. \quad (48)$$

Thus, for saddle-path stability and uniqueness of the equilibrium path, it is necessary and sufficient that (w.l.o.g.)

$$0 < \xi_1 < 1 < \xi_2. \quad (49)$$

Figure 1 plots the expression on the LHS of this equation for the case that  $\beta A_2 / A_0 < 1$ . In this case, it is clear that (49) will hold if and only if

$$1 + \beta + \beta \frac{A_1}{A_0} > 1 + \beta A_2 / A_0; \quad (50)$$

otherwise it will hold that  $1 \leq \xi_1 \leq \xi_2$ . Figure 2 plots the expression for the case that  $\beta A_2 / A_0 > 1$ . Once again, we see that (49) will hold if and only if (50) does; otherwise it will be the case that  $\xi_1 \leq \xi_2 \leq 1$ . (This analysis can be compared to that on pages 201-202 of Sargent (1987). )

Although it seems difficult to interpret (50) generally, it can be seen to hold for  $\psi_0$  close enough to zero.

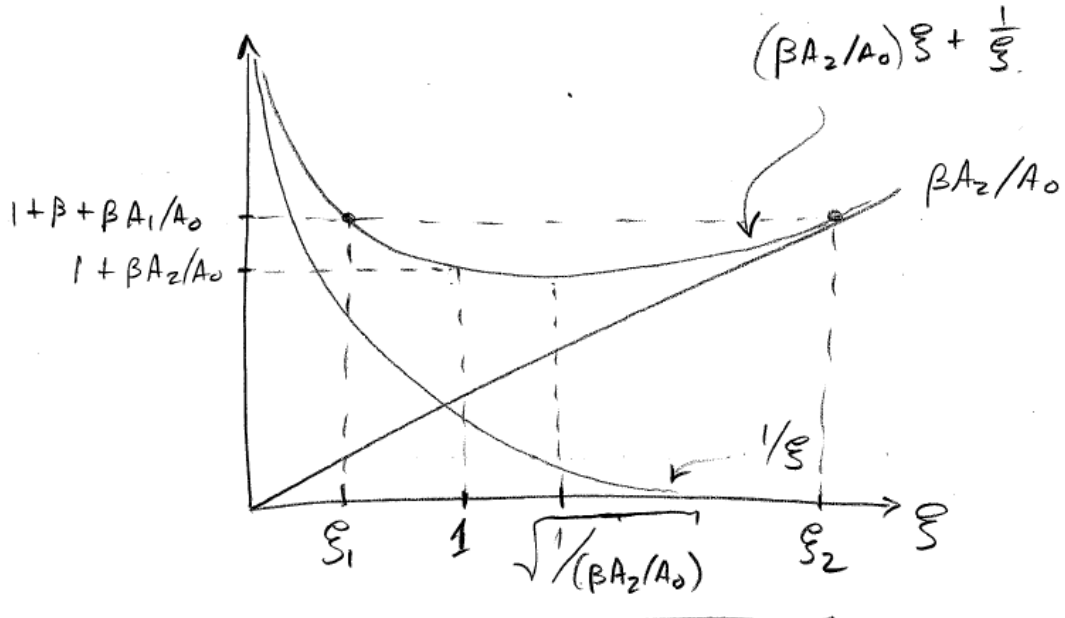


Figure 1:  $\beta A_2/A_0 < 1$  case.

Solving the “unstable root” forward, we have<sup>2</sup>

$$\begin{aligned} (L^{-1} - \xi) \hat{z}_{t-1} &= \frac{1}{\beta A_2/A_0} \left( L^{-1} - \frac{1}{\beta \xi} \right)^{-1} E_t Q_{t+1} \\ &= \frac{\beta \xi}{\beta A_2/A_0} (1 - \beta \xi L^{-1})^{-1} E_t Q_{t+1} \end{aligned}$$

<sup>2</sup>Hereafter, I will write  $\xi$  for the “stable root”  $\xi_1$  where there can be no confusion.

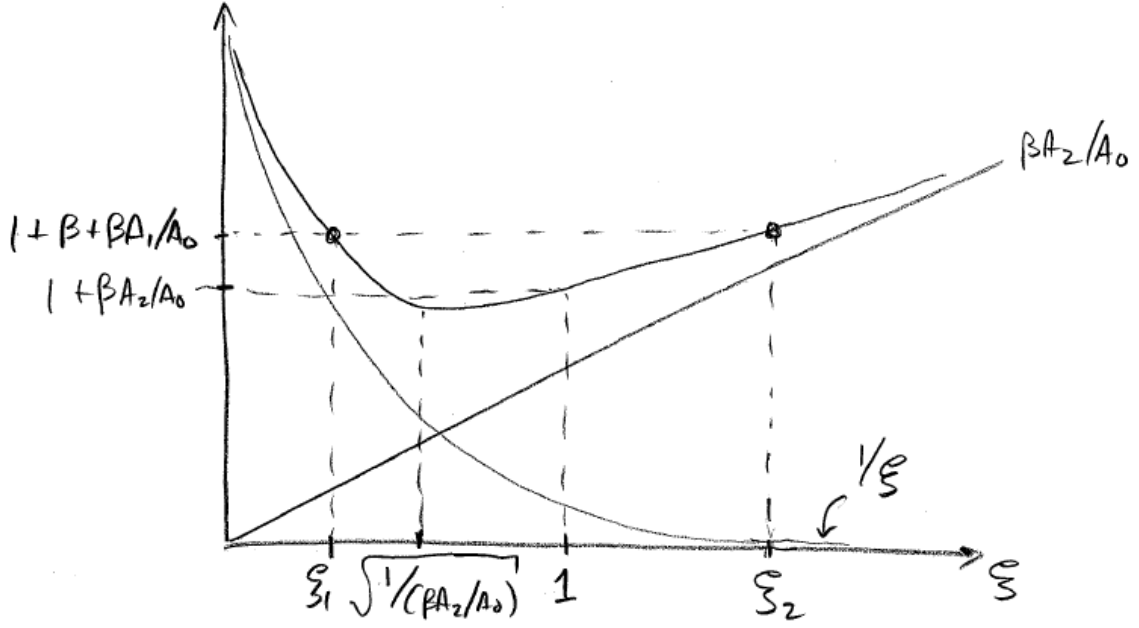


Figure 2:  $\beta A_2/A_0 > 1$  case.

or

$$\begin{aligned}
\hat{z}_t &= \xi \hat{z}_{t-1} + \frac{\beta \xi}{\beta A_2/A_0} \left\{ \frac{\Psi_2 \omega \eta}{A_0 \nu} \hat{r}_t - \frac{\Psi_2 \omega (1 - \alpha) \eta}{A_0 \nu} \hat{u}_t \right\} \\
&+ \frac{\beta \xi}{\beta A_2/A_0} \sum_{j=1}^{\infty} (\beta \xi)^j E_t \left\{ \frac{\Psi_2 \omega \eta}{A_0 \nu} \hat{r}_{t+j} - \frac{\Psi_2 \omega (1 - \alpha) \eta}{A_0 \nu} \hat{u}_{t+j} \right\} \\
&+ \frac{1}{\beta A_2/A_0} \sum_{j=1}^{\infty} (\beta \xi)^j E_t \left\{ \frac{\beta (A_1 - \Psi_2) \omega \eta}{A_0 \nu} \hat{r}_{t+j} - \frac{\beta (A_1 - \Psi_2) \omega \eta (1 - \alpha)}{A_0 \nu} \hat{u}_{t+j} \right\} \\
&= \xi z_{t-1} + \frac{\xi}{A_2} \left\{ \frac{\Psi_2 \omega \eta}{\nu} \hat{r}_t - \frac{\Psi_2 \omega (1 - \alpha) \eta}{\nu} \hat{u}_t \right\} \\
&+ \frac{1}{A_2} \sum_{j=1}^{\infty} (\beta \xi)^j E_t \left\{ \left[ \frac{(A_1 - \Psi_2) \omega \eta}{\nu} + \frac{\xi_1 \Psi_2 \omega \eta}{\nu} \right] \hat{r}_{t+j} \right. \\
&\quad \left. - \left[ \frac{(A_1 - \Psi_2) \omega \eta (1 - \alpha)}{\nu} + \frac{\xi_1 \Psi_2 \omega (1 - \alpha) \eta}{\nu} \right] \hat{u}_{t+j} \right\}.
\end{aligned}$$

To evaluate these expectations, we may apply formula (91) on page 304 of Sargent (1987). Letting  $\rho(L)\hat{r}_t = (1 - \rho_1 L - \rho_2 L^2)\hat{r}_t$ , we have that

$$E_t \sum_{j=0}^{\infty} (\beta\xi)^j \hat{r}_{t+j} = g(L)\hat{r}_t$$

where

$$g_0 = \rho(\beta\xi) = \frac{1}{1 - \rho_1\beta\xi - \rho_2(\beta\xi)^2}$$

$$g_1 = \beta\xi g_0 \rho_2 = \frac{\beta\xi\rho_2}{1 - \rho_1\beta\xi - \rho_2(\beta\xi)^2};$$

thus

$$E_t \sum_{j=0}^{\infty} (\beta\xi)^j \hat{r}_{t+j} = \frac{1}{1 - \rho_1\beta\xi - \rho_2(\beta\xi)^2} \hat{r}_t + \frac{\beta\xi\rho_2}{1 - \rho_1\beta\xi - \rho_2(\beta\xi)^2} \hat{r}_{t-1}$$

Then

$$E_t \sum_{j=1}^{\infty} (\beta\xi)^j \hat{r}_{t+j} = -\hat{r}_t + E_t \sum_{j=0}^{\infty} (\beta\xi)^j \hat{r}_{t+j}$$

$$= \frac{\rho_1\beta\xi + \rho_2(\beta\xi)^2}{1 - \rho_1\beta\xi - \rho_2(\beta\xi)^2} \hat{r}_t + \frac{\beta\xi\rho_2}{1 - \rho_1\beta\xi - \rho_2(\beta\xi)^2} \hat{r}_{t-1}.$$

Similarly, it is easily seen that

$$E_t \sum_{j=1}^{\infty} (\beta\xi)^j \hat{u}_{t+j} = \frac{\chi\beta\xi}{1 - \chi\beta\xi} \hat{u}_t.$$

It follows that

$$\begin{aligned}
\hat{z}_t &= \xi z_{t-1} + \frac{\xi}{A_2} \left\{ \frac{\Psi_2 \omega \eta}{\nu} \hat{r}_t - \frac{\Psi_2 \omega (1 - \alpha) \eta}{\nu} \hat{u}_t \right\} \\
&+ \frac{1}{A_2} \left[ \frac{(A_1 - \Psi_2) \omega \eta}{\nu} + \frac{\xi_1 \Psi_2 \omega \eta}{\nu} \right] \left[ \frac{\rho_1 \beta \xi_1 + \rho_2 (\beta \xi)^2}{1 - \rho_1 \beta \xi - \rho_2 (\beta \xi)^2} \hat{r}_t + \frac{\beta \xi \rho_2}{1 - \rho_1 \beta \xi - \rho_2 (\beta \xi)^2} \hat{r}_{t-1} \right] \\
&- \frac{1}{A_2} \left[ \frac{(A_1 - \Psi_2) \omega \eta (1 - \alpha)}{\nu} + \frac{\xi_1 \Psi_2 \omega (1 - \alpha) \eta}{\nu} \right] \frac{\chi \beta \xi}{1 - \chi \beta \xi} \hat{u}_t \\
&= \xi z_{t-1} + \left\{ \frac{\xi \Psi_2 \omega \eta}{A_2 \nu} + \frac{1}{A_2} \left[ \frac{(A_1 - \Psi_2) \omega \eta}{\nu} + \frac{\xi_1 \Psi_2 \omega \eta}{\nu} \right] \left[ \frac{\rho_1 \beta \xi_1 + \rho_2 (\beta \xi)^2}{1 - \rho_1 \beta \xi - \rho_2 (\beta \xi)^2} \right] \right\} \hat{r}_t \\
&- \left\{ \frac{\xi \Psi_2 \omega (1 - \alpha) \eta}{A_2 \nu} - \frac{1}{A_2} \left[ \frac{(A_1 - \Psi_2) \omega \eta (1 - \alpha)}{\nu} + \frac{\xi_1 \Psi_2 \omega (1 - \alpha) \eta}{\nu} \right] \frac{\chi \beta \xi}{1 - \chi \beta \xi} \right\} \hat{u}_t.
\end{aligned}$$

This solution for the endogenous state variable can be plugged into (44) to derive an expression for  $\hat{m}_t$  as a function of exogenous variables. To understand the nature of the implied reduced form econometric “money demand” equation, one then needs to obtain the projection of  $\hat{u}_t$  upon observables, current and past  $\hat{m}_t$  and  $\hat{r}_t$ . Such an analysis is carried out for the case that  $\psi_0 = 0$ , and where  $\hat{r}_t$  and  $\hat{u}_t$  are serially and contemporaneously uncorrelated. Apparently, the reduced form will have the same general configuration in the general case, except that the reduced form coefficients will be functions of an expanded constellation of parameters. Of note, however, is the fact that the vector consisting of the log-money income ratio and the log opportunity cost follow a VARMAX(3,1) representation, where the exogenous variables (the “X”) are a constant and trend.

## 4 Numerical Solution and Estimation

### 4.1 Uhlig Formulation

With reference to the formulation in Uhlig (1999, subsection 6.2), I let

$$\xi_t \equiv (\hat{z}_{t-1}, \hat{\gamma}_{z,t})',$$

$$\psi_t \equiv (\hat{N}_t, \hat{m}_t, \hat{\lambda}_t)';$$

and

$$\zeta_t \equiv (\hat{r}_t, \hat{r}_{t-1}, \hat{u}_t)'.$$

The system of equations of interest are written as

$$\begin{aligned}
0 &= \mathcal{A}\xi_t + \mathcal{B}\xi_{t-1} + \mathcal{C}\psi_t + \mathcal{D}\zeta_t \\
0 &= E_t [\mathcal{F}\xi_{t+1} + \mathcal{G}\xi_t + \mathcal{H}\xi_{t-1} + \mathcal{J}\psi_{t+1} + \mathcal{K}\psi_t + \mathcal{L}\zeta_{t+1} + \mathcal{M}\zeta_t] \\
\zeta_{t+1} &= \mathcal{N}\zeta_t + w_{t+1}.
\end{aligned}$$

Here, the first matrix equation is constituted of (39)-(41) and (43); the second equation is a representation of (42); and the third equation implements the law of motion of the exogenous variables in (37) and (38). More precisely, letting  $\Psi \equiv \psi_0\psi_1/\bar{L}$ , the matrices  $\mathcal{A}$ - $\mathcal{N}$  are defined as follows.

$$\mathcal{A} = \begin{pmatrix} 0 & \Psi\bar{\gamma}_z/\phi \\ (1-\nu)/\nu & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -1 & -1 \end{pmatrix},$$

$$\mathcal{C} = \begin{pmatrix} \Psi(1-\nu/\eta)\bar{N} & 0 & -[1 + \Psi\nu/(\eta\bar{\lambda})] \\ -(\theta-1) & \eta/\nu & -\theta \\ -\theta & -1 & -\theta \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathcal{D} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \eta/\nu \\ -1 & 0 & -\alpha \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathcal{F} = (0 \ \beta), \quad \mathcal{G} = (0 \ -1), \quad \mathcal{H} = (0 \ 0),$$

$$\mathcal{J} = ( -\beta(1-\nu)\bar{N}\phi/(\eta\bar{\gamma}_z) \ 0 \ \beta[1 - (1-\nu)\bar{N}\phi/(\eta\bar{\gamma}_z)] ), \quad \mathcal{K} = (0 \ 0 \ -1),$$

$$\mathcal{L} = (0 \ 0 \ 0), \quad \mathcal{M} = (0 \ 0 \ 0),$$

$$\mathcal{N} = \begin{pmatrix} \rho_1 & \rho_2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \chi \end{pmatrix}.$$

The  $3 \times 1$  serially uncorrelated innovations vector is

$$w_{t+1} \sim N(0, \Sigma).$$

A “solution” to the linearized system is a time invariant decision rule character-

ized by matrices  $(\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{S})$  such that the recursion

$$\begin{aligned}\xi_t &= \mathcal{P}\xi_{t-1} + \mathcal{Q}\zeta_t \\ \psi_t &= \mathcal{R}\xi_{t-1} + \mathcal{S}\zeta_t\end{aligned}$$

satisfies (39)-(43), where initial conditions  $\xi_{-1}$  are taken as given. Uhlig shows (Theorem 2) how to solve this system numerically; and he shows that the system is stable if and only if all of the eigenvalues of the matrix  $\mathcal{P}$  are strictly smaller than unity in modulus.

## 4.2 Data

I use data on the logarithm of the narrow money-income ratio  $\tilde{m}_t$ , and the logarithm of the opportunity cost  $\tilde{r}_t \equiv \log [i_t / (1 + i_t)]$ , where  $i_t$  is the commercial paper rate. The data is described more fully in the text of the paper.

## 4.3 State-Space Formulation

The state-space formulation with observables vector

$$\psi_t^+ \equiv (\tilde{m}_t, \tilde{r}_t)'$$

and state vector

$$\xi_t^+ \equiv (\xi_t, \zeta_{t+1})' = (\hat{z}_{t-1}, \hat{\gamma}_{z,t}, \hat{r}_{t+1}, \hat{r}_t, \hat{u}_{t+1})'$$

has measurement equation<sup>3</sup>

$$\begin{aligned}\psi_t^+ &= \left( \begin{array}{cc|ccc} \tilde{\mathcal{R}} & & \tilde{\mathcal{S}} & & \\ \hline 0 & 0 & 1 & 0 & 0 \end{array} \right) \xi_{t-1}^+ + \mathcal{G}g_t \\ &\equiv \tilde{\mathcal{Z}}\xi_{t-1}^+ + \mathcal{G}g_t,\end{aligned}$$

where  $\tilde{\mathcal{R}}$  and  $\tilde{\mathcal{S}}$  are the second rows of the matrices  $\mathcal{R}$  and  $\mathcal{S}$  defined previously;  $g_t = (1, t)$ ;

$$\mathcal{G} = \begin{pmatrix} \mu_m & \tilde{\gamma}_q \\ \mu_r & 0 \end{pmatrix};$$

---

<sup>3</sup>The timing and the notation of the state-space form used here is meant to be conformable with that in chapter 3 of Durbin and Koopman (2001).

and the state-transition equation is

$$\begin{aligned}\xi_t^+ &= \begin{pmatrix} \mathcal{P} & \mathcal{Q} \\ 0 & \mathcal{N} \end{pmatrix} \xi_{t-1}^+ + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} w_t, \\ &\equiv \mathcal{T} \xi_{t-1}^+ + \mathcal{H} w_t\end{aligned}$$

where

$$w_t \equiv \begin{pmatrix} \varepsilon_{t+1}^r & \varepsilon_{t+1}^u \end{pmatrix}' \sim N(0, \Sigma).$$

Here,  $\mu_m$ ,  $\mu_r$ , and  $\tilde{\gamma}_q$  are parameters that implement detrending of the data.<sup>4</sup>

#### 4.4 Initialization of the Kalman Filtering Algorithm

The unconditional distribution of  $\xi_t^+$  under the model is normal with mean (5-vector-) zero and variance  $V_0$  satisfying<sup>5</sup>

$$V_0 = \mathcal{T} V_0' \mathcal{T} + \mathcal{H} \Sigma \mathcal{H}'$$

or

$$\text{vec} V_0 = [I_{25} - (\mathcal{T} \otimes \mathcal{T})]^{-1} \text{vec}(\mathcal{H} \Sigma \mathcal{H}').$$

#### 4.5 Implementation

Now given a vector of the model's parameters satisfying the appropriate restrictions (including  $\mu_m$ ,  $\tilde{\gamma}_q$ , and  $\mu_r$ ), we solve for the equilibrium policy rules  $(\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{S})$ . Together with the matrices  $\mathcal{N}$  and  $\Sigma$ , these objects completely define the state-space form for the given parameterization. Next, the Kalman filter implied by the state-space formulation is applied to the data  $(\tilde{m}_t, \tilde{r}_t)$  described above to factorize and

---

<sup>4</sup>With respect to the parameters of the model,

$$\hat{m}_t = \tilde{m}_t - \mu_m - t\tilde{\gamma}_q$$

and

$$\hat{r}_t = \tilde{r}_t - \mu_r,$$

so that  $\tilde{\gamma}_q \equiv \alpha \log \bar{\gamma}_q$ ,  $\mu_m$  is a normalizing constant, and  $\mu_r \equiv \log \bar{r}$ .

<sup>5</sup>See Hamilton (1994), chapter 13 pps. 278-9.

evaluate the likelihood of the parameterized model. The estimation algorithm involves a search over parameter vectors for the one that maximizes this likelihood. (Some restrictions on identification of these parameters are discussed in the paper.)