Competitive Bundling and Counter-bundling with Generalist and Specialist Firms - Technical Appendix

Bikram Ghosh ¹       Subramanian Balachander²

June 2006

¹Ph. D. candidate, Krannert Graduate School of Management, Purdue University, West Lafayette, IN 47907. Tel: (765)494-6501. E-mail: ghoshb@mgmt.purdue.edu
²Krannert Graduate School of Management, Purdue University, West Lafayette, IN 47907. Tel: (765) 494-4466. E-mail: sbalacha@mgmt.purdue.edu
The following are the results of the pricing subgames.

**Subgame I**

**Lemma 1** When all firms sell unbundled products, and there is no cooperation between firms \( B \) and \( C \), the equilibrium in the price subgame is given as follows: For category \( H \), \( P_{Ah} = P_{Ch} = c \) with \( \Pi_{Ah} = \Pi_{Ch} = 0 \). In category \( D \),

(i) for \( r \in [c, c + t) \), the equilibrium has \( P_{*Ad} = P_{*Bd} = \frac{1}{2}(r + c) \); \( \Pi_{*Ad} = \Pi_{*Bd} = \frac{1}{4t}(r - c)^2 \).

(ii) for \( r \in [c + t, c + \frac{3}{2}t) \), the symmetric equilibrium is given by \( P_{*Ad} = P_{*Bd} = r - t/2 \); \( \Pi_{*Ad} = \Pi_{*Bd} = \frac{1}{4t}(2r - t - 2c) / 4 \).

(iii) for \( r \geq c + \frac{3}{2}t \), the equilibrium has \( P_{*Ad} = P_{*Bd} = c + t \); \( \Pi_{*Ad} = \Pi_{*Bd} = \frac{1}{2}t \).

**Proof:** Any equilibrium should involve \( P_{Ah} = P_{Ch} = c \) using the standard Bertrand argument.

As for category \( D \), the firms’ profit functions can be calculated in standard fashion (Hotelling, 1929; Economides, 1986) as follows:

\[
\Pi_{id} = \begin{cases} 
0 & \text{if } P_{id} > r \\
(r - P_{Ad})(P_{id} - c) & \text{if } 2r - P_{jd} - t \leq P_{id} \leq r \\
(P_{id} - P_{Ad} + t)P_{id} - c & \text{if } P_{id} \leq 2r - P_{jd} - t, \text{ and } P_{jd} - t \leq P_{id} \leq P_{jd} + t \\
(P_{id} - c) & \text{if } P_{id} \leq 2r - P_{jd} - t, \text{ and } P_{id} \leq P_{jd} - t 
\end{cases}
\]

There are three possibilities for the equilibrium in category \( D \). The first possibility is that the market is fully covered in equilibrium. The firms’ profit functions in this case are given by line 3 of the above profit function. Solving the first-order conditions yields \( P_{*Ad} = P_{*Bd} = c + t \). Since \( P_{*Ad} \leq 2r - P_{*Bd} - t \) for complete market coverage, the resulting condition on the parameters is \( r \geq c + (3/2)t \).

A second possibility is that the firm’s equilibrium prices are such that the market areas of the two firms just ‘touch’ each other. Thus, \( P_{*Ad} = 2r - P_{*Bd} - t \) in this equilibrium. For all prices lower (higher) than equilibrium prices the profit function is increasing (decreasing). Thus, we get conditions on the derivatives on either side of the equilibrium prices, which lead to the following set of inequalities:

\[
\begin{align*}
P_{Ad} &\geq \frac{1}{2}(P_{Bd} + t + c) \\
P_{Bd} &\geq \frac{1}{2}(P_{Ad} + t + c)
\end{align*}
\]

\[ P_{Ad} \leq \frac{1}{2}(r + c) \]
\[ P_{Bd} \leq \frac{1}{2}(r + c) \]

Substituting \( P_{Ad} + P_{Bd} = 2r - t \) in the above set of inequalities we obtain the condition, \( r \in [c + t, c + \frac{3}{2}t] \). Although a continuum of equilibrium prices are possible, we solve for symmetric equilibrium prices, which are given in the lemma.

Now let us consider the third possibility of an equilibrium in which the market is partially covered. The profit expressions for the firms in this equilibrium are given by line 2 of the profit function above. Solving the first-order necessary conditions yields \( P_{Ad}^* = P_{Bd}^* = \frac{1}{2}(r + c) \), and \( \Pi_{Ad}^* = \Pi_{Bd}^* = \frac{1}{4t}(r - c)^2 \) as the equilibrium profit. In order for the market to be not fully covered, we require \( r \leq c + t \). ■

Subgame II

**Lemma 2** When firm A sells a pure bundle of D and H, and firms B and C do not coordinate, the equilibrium in the price subgame is given as follows:

(i) for \( r \in [c, c + \frac{1}{4}t] \), the equilibrium has \( P_{A}^* = c + r \), \( P_{Bd}^* = \frac{1}{2}(r + c) \), \( P_{Ch}^* = r \); \( \Pi_{A}^* = \frac{1}{4}(r - c)^2 \), \( \Pi_{Bd}^* = (r - c)^2/4t \); \( \Pi_{Ch}^* = \frac{1}{4t}(r - c)(t - r) \). This equilibrium exists if \( t < 4c \), and if \( t \geq 4c \), it exists when \( r < c + \frac{1}{2}t - \frac{\sqrt{(c - 4t)^2}}{2} \).

(ii) for \( r \in [c + \frac{1}{2}t, c + \frac{56 - 24\sqrt{2}}{31}t] \), the equilibrium has \( P_{A}^* = \frac{1}{3}(5c + r + t) \), \( P_{Bd}^* = \frac{1}{2}(r + c) \), \( P_{Ch}^* = \frac{1}{3}(4c + 2t - r) \); \( \Pi_{A}^* = \frac{1}{9t}(r - c + t)^2 \); \( \Pi_{Bd}^* = \frac{1}{4t}(r - c)^2 \); \( \Pi_{Ch}^* = \frac{1}{4t}(r - c - 2t)^2 \). This equilibrium exists if \( t < \frac{4}{3}c \), and if \( t \geq \frac{4}{3}c \) it exists provided \( r \geq c - \frac{5}{9}t + \frac{3}{2}\sqrt{5t^2 - 4cd} \).

(iii) For all \( r \in [c + \frac{1}{4}(9 - 3\sqrt{2})t, \infty) \), the equilibrium has \( P_{A}^* = 2c + \frac{5}{3}t \), \( P_{Bd}^* = c + \frac{3}{4}t \), \( P_{Ch}^* = c + \frac{3}{4}t \); \( \Pi_{A}^* = \frac{22}{32}t \); \( \Pi_{B}^* = \Pi_{Ch}^* = \frac{9}{32}t \).

**Proof:** If \( P_{A}^* < P_{Ch} \), firm A sells to the whole market, including those customers who see no value in firm A’s product D. The latter type of customers will also buy product D from firm B if such a purchase yields positive surplus. However, firm C makes zero sales in this case. Thus, there is a discontinuity in firm A’s profit functions at \( P_{A}^* = P_{Ch} \). On the other hand, if \( P_{Ch} \leq P_{A}^* \), firm C cannot get all of firm A’s customers. Thus, if \( P_{Ch} \leq P_{A}^* \) in equilibrium, there can be up to three customer segments depending on their purchase. Customers in the region \([0, x_1]\) buy the bundle from firm A, customers in \([x_1, x_2]\) buy H alone from firm C, and customers in \([x_2, 1]\) buy both H from firm C and D from firm B. Since the customer at \( x_1 \) derives equal surplus from buying the bundle from firm A and buying H alone from firm C, we can obtain \( x_1 \) as follows:
\[
x_1 = \frac{r - P^b_A + P_{Ch}}{t}
\]

Similarly, we can obtain \( x_2 \) by equating the customer surplus in buying \( D \) from firm \( B \) to zero. Thus we get \( x_2 \) as follows:

\[
x_2 = 1 - \frac{r - P_{Bd}}{t}
\]

For the segment \([x_1, x_2]\) to be nonzero, we require \( x_2 > x_1 \). This condition leads to the following inequality:

\[
P^b_A > 2r - P_{Bd} + P_{Ch} - t \tag{1}
\]

Note also that \( x_1 \geq x_2 \) when \( x_2 \leq 0 \) which is true if \( P_{Bd} \leq r - t \). When \( x_1 \geq x_2 \), we will have just two segments based on customer purchase. Customers in \([0, x_3]\) buy the bundle from firm \( A \) while those in \([x_3, 1]\) buy \( H \) from firm \( C \) and \( D \) from firm \( B \). Equating the surplus in buying the bundle from firm \( A \) to the surplus from buying \( H \) from firm \( C \) and \( D \) from firm \( B \), we can obtain \( x_3 \) as follows:

\[
x_3 = \frac{P_{Bd} + P_{Ch} - P^b_A + t}{2t}
\]

We can now obtain the profit functions for the firms as follows:

\[
\Pi^b_A = 0 \text{ if } P^b_A \geq \text{Min}\{P_{Ch} + r, P_{Bd} + P_{Ch} + t\}
\]

\[
= x_1(P^b_A - 2c) \text{ if } 2r - P_{Bd} + P_{Ch} - t \leq P^b_A \leq P_{Ch} + r \text{ and}
\]

\[
= x_3(P^b_A - 2c) \text{ if } P_{Bd} + P_{Ch} - t \leq P^b_A \leq 2r - P_{Bd} + P_{Ch} - t
\]

\[
= (P^b_A - 2c) \text{ if } P^b_A \leq \text{Max}\{P_{Bd} + P_{Ch} - t, P_{Ch}\}
\]

\[
\Pi_{Bd} = 0 \text{ if } P_{Bd} \geq P^b_A - P_{Ch} + t
\]

\[
=(1 - x_2)(P_{Bd} - c) \text{ if } 2r - P^b_A + P_{Ch} - t \leq P_{Bd} \leq r
\]

\[
= (1 - x_3)(P_{Bd} - c) \text{ if } P^b_A - P_{Ch} - t \leq P_{Bd} \leq 2r - P^b_A + P_{Ch} - t
\]

\[
= (P_{Bd} - c) \text{ if } P_{Bd} \leq P^b_A - P_{Ch} - t
\]

\[
\Pi_{Ch} = 0 \text{ if } P_{Ch} \geq \text{Min}\{P^b_A - P_{Bd} + t, P^b_A\}
\]

\[
= (1 - x_3)(P_{Ch} - c) \text{ if } P^b_A + P_{Bd} - 2r + t \leq P_{Ch} \leq P^b_A - P_{Bd} + t
\]

\[
= (P_{Ch} - c) \text{ if } P^b_A + P_{Bd} - 2r + t \leq P_{Ch} \leq P^b_A - P_{Bd} - t
\]
Lemma. Since, $P_{ch}$ expressions. Solving the first order conditions we get the equilibrium prices in part (iii) of the Lemma. Since, $P_{bA}^b \leq 2r - P_{bd}^* + P_{ch}^* - t$, we have $r \geq c + \frac{9}{3\sqrt{t}}$ as a condition for this equilibrium. The second-order conditions are satisfied locally. Since the profit functions for firms $A$ and $B$ are concave, the above prices represent the global maximum in their case, except for the point of discontinuity in firm $A$’s profit function at $P_{A}^b = P_{ch}$, which needs further checking. However, it can be shown that firm $A$ does not gain by charging $P_{ch}^* - \epsilon$ and getting the whole market. On the other hand, the profit function for firm $C$ is not globally concave. Hence, we check if firm $C$ can do better by unilaterally deviating to a price below $P_{bA}^b + P_{bd} - 2r + t$. If firm $C$ deviates in this fashion, its best deviating price can be shown to be $(12c - 4r + 9t)/8$. Such a deviation by firm $C$ is feasible only if $r \leq c + \frac{5}{2}t$. However, this deviation is not profitable as long as $r \geq c + (9 - 3\sqrt{2})t/4$. Thus, this equilibrium exists for $r \geq c + (9 - 3\sqrt{2})t/4$. When $x_1 \geq x_2$, there may also be a corner equilibrium with firm $C$ pricing at $r$. However, it can be shown that such an equilibrium does not exist due to profitable deviations for firm $C$.

The second possibility for an equilibrium is one in which $x_2 \geq x_1$. In this case, there can be two types of equilibria. First, there can be a corner solution for small values of $r$ at which $P_{ch}^* = r$. Second, there can be an interior solution for which $P_{ch}^* \leq r$. Considering first the case of the corner solution, we solve the first-order conditions for firms $A$ and $B$ with $P_{ch} = r$ and get the equilibrium prices as given in part (i) of the Lemma. $P_{ch}^* = r$ requires $\partial \Pi_{ch}/\partial P_{ch} \leq 0$ at $P_{ch} = r_-$. This gives us the condition $r \leq c + \frac{1}{2}t$. We check if firm $A$ can deviate by reducing $P_{bA}^b$ below $P_{ch}^*$ to capture the whole market. We can show that such a deviation is not profitable except when $t < 4c$ together with $r < c + \frac{1}{2}t - \frac{\sqrt{c^2 - 4ct}}{2}$. We now solve for the second case where $P_{ch}^*$ is an
interior solution. Solving the first order condition we get the equilibrium prices given in part (ii) of the Lemma. Setting \( P_{Ch}^* \leq r \) yields the boundary condition \( r \geq c + \frac{t}{2} \). The condition \( x_2 \geq x_1 \) at equilibrium yields the boundary condition, \( r \leq c + \frac{4}{5}t \). Firm C can potentially deviate to a \( P_{Ch} \geq P_b^* + P_{Bd}^* - 2r + t \). We can show that this deviation is not profitable if \( r \leq c + (56 - 24\sqrt{2})t/31 \).

Likewise, firm A can deviate to a \( P_b^A \) just below \( P_{Ch}^* \) to capture the entire market. We can show that such a deviation is not profitable except when \( t > \frac{4}{5}c \) in conjunction with \( r < c - \frac{5}{2}t + \frac{5}{2}\sqrt{5t^2 - 4ct} \).

The third possibility is that the equilibrium is a corner solution with \( x_2 = x_1 \) holding exactly. This requires the equilibrium prices to satisfy the following:

\[
P_A^{bs} + P_{Bd}^* - P_{Ch}^* = 2r - t
\]  

(2)

However, we can then show that

\[
\frac{\partial \Pi_{Ch}}{\partial P_{Ch}} |_{P_{Ch} \to P_{Ch}^*} - \frac{\partial \Pi_{Ch}}{\partial P_{Ch}} |_{P_{Ch} \to P_{Ch}^*} = \frac{1}{2t}(P_{Ch}^* - c) > 0
\]

This implies that the only candidate for a corner equilibrium is when

\[
P_{Ch}^* = r
\]  

(3)

Substituting (3) in (2) we get

\[
P_A^{bs} + P_{Bd}^* = 3r - t
\]  

(4)

Further, in such a corner equilibrium we have \( \frac{\partial \Pi_{Ch}}{\partial P_{Ch}} |_{P_{Ch} \to P_{Ch}^*} \geq 0 \), which leads to

\[
P_A^{bs} \geq 3r - c - t
\]  

(5)

Further, we have

\[
\frac{\partial \Pi_{Bd}}{\partial P_{Bd}} |_{P_{Bd} \to P_{Bd}^*} = \frac{1}{2}(c - 2P_{Bd} + r) \leq 0,
\]

leading to

\[
P_{Bd}^* \geq \frac{1}{2}(r + c)
\]  

(6)

From (4), (5), and (6), we get \( 3r - c - t + \frac{1}{2}(r + c) \leq 3r - t \) or \( r \leq c \), which contradicts our assumption that \( r > c \). Hence there does not exist a corner solution in equilibrium with \( x_2 = x_1 \).
Subgame III

**Lemma 3** When firm A sells a pure bundle of D and H, and firms B and C cooperate but sell unbundled products, the equilibrium in the price subgame is given as follows:

(i) for \( r \in [c, c+\frac{1}{2}t] \), the equilibrium has \( P_{A}^{b*} = c + r, P_{Bd}^{*} = \frac{1}{2}(r+c), P_{Ch}^{*} = r; \Pi_{A}^{b*} = \frac{1}{4}(r-c)^{2}, \Pi_{BC}^{*} = (r-c)^{2}/4t + \frac{1}{t}(r-c)(c+t-r) \). This equilibrium exists if \( t < 4c \), and if \( t \geq 4c \), it exists when \( r < c + \frac{1}{2}t - \frac{\sqrt{t^{2}-4ct}}{2} \).

(ii) for \( r \in [c+\frac{1}{2}t, c+\frac{4}{5}t] \), the equilibrium has \( P_{A}^{b*} = \frac{1}{3}(5c+r+t), P_{Bd}^{*} = \frac{1}{2}(r+c), P_{Ch}^{*} = \frac{1}{9}(4c+2t-r); \Pi_{A}^{*} = \frac{1}{9}(r-c+t)^{2}, \Pi_{BC}^{*} = \frac{1}{36}(r-c)^{2} + \frac{1}{9}(r-c-2t)^{2}. \) This equilibrium exists if \( t < \frac{4}{5}c \), and if \( t \geq \frac{4}{5}c \), it exists when \( r \in [c - \frac{5}{2}t + \frac{3}{2}\sqrt{5t^{2} - 4ct}, c + \frac{4}{5}t] \).

(iii) for \( r \in [c+t, \infty) \), the equilibrium has \( P_{A}^{b*} = 2c + t; P_{Bd}^{*} + P_{Ch}^{*} = 2c + t; \Pi_{A}^{*} = \frac{1}{2}t; \Pi_{BC}^{*} = \frac{1}{2}t \)

**Proof:** We can derive the profit functions for this case from those developed in the proof of Lemma 2. The only difference is that the profits of firms B and C are combined for the alliance BC. The profit functions are as follows:

\[
\Pi_{A}^{b} = 0 \text{ if } P_{A}^{b} \geq \text{Min}\{P_{Ch} + r, P_{Bd} + P_{Ch} + t\}
\]

\[
= x_{1}(P_{A}^{b} - 2c) \text{ if } 2r - P_{Bd} + P_{Ch} - t \leq P_{A}^{b} \leq P_{Ch} + r
\]

\[
= x_{3}(P_{A}^{b} - 2c) \text{ if } P_{Bd} + P_{Ch} - t \leq P_{A}^{b} \leq 2r - P_{Bd} + P_{Ch} - t
\]

\[
= (P_{A}^{b} - 2c) \text{ if } P_{Bd} + P_{Ch} \geq P_{A}^{b} - t
\]

\[
\Pi_{BC} = 0 \text{ if } P_{Bd} + P_{Ch} \geq P_{A}^{b} + t
\]

\[
= (1 - x_{2})(P_{Bd} - c) + (1 - x_{1})(P_{Ch} - c) \text{ if } 2r - P_{A}^{b} + P_{Ch} - t \leq P_{Bd} \leq r
\]

\[
\text{and } P_{A}^{b} - r \leq P_{Ch} \leq \text{Min}\{P_{A}^{b} + P_{Bd} - 2r + t, r\}
\]

\[
= (1 - x_{2})(P_{Bd} - c) + (P_{Ch} - c) \text{ if } 2r - P_{A}^{b} + P_{Ch} - t \leq P_{Bd} \leq r
\]

\[
\text{and } P_{Ch} \leq P_{A}^{b} - r
\]

\[
= (1 - x_{3})(P_{Bd} + P_{Ch} - 2c) \text{ if } P_{A}^{b} - P_{Ch} - t \leq P_{Bd} \leq 2r - P_{A}^{b} + P_{Ch} - t
\]

\[
\text{and } P_{A}^{b} + P_{Bd} - 2r + t \leq P_{Ch} \leq P_{A}^{b} - P_{Bd} + t
\]

\[
= (P_{Bd} + P_{Ch} - 2c) \text{ if } P_{Bd} + P_{Ch} \leq P_{A}^{b} - t
\]

There are three possibilities for an equilibrium. First, there may be an equilibrium in which \( x_{1} \geq x_{2} \). In this case, the profit for firm A is given by the third line of its function and that for firm BC is given by the fourth line of its function. Solving for the first order conditions, we get the equilibrium prices given by part (iii) of the Lemma. Once again, we check for profitable unilateral
deviations by firms. Firm $A$ cannot gain by deviating since the profit function for firm $A$ is concave. However, firm $BC$ can potentially deviate in such a way that the following holds after deviation

$$2r - P_{Bd}^{dev} + P_{Ch}^{dev} - t - P_A^{bs} < 0 \quad (7)$$

In any deviation, firm $BC$ cannot do better than its monopoly profit in category $D$ obtained by charging $P_{Bd}^{dev} = \frac{1}{2}(r + c)$. Thus, a possible deviation is that firm $BC$ fixes $P_{Bd}^{dev} = \frac{1}{2}(r + c)$ and maximizes profit from product $H$. Solving the first-order condition we get $P_{Ch}^{dev} = \frac{1}{2}(3c + 2t - r)$. Since (7) holds after deviation, we obtain after substitution the condition for a valid deviation as $r < c + t$. With such a deviation, the joint profit for firm $BC$ can be shown to exceed its equilibrium profit. Therefore, the condition for non-deviation is $r \geq c + t$.

A second possibility of an equilibrium is one in which $x_1 \leq x_2$. As in the case of Lemma 2, there are two cases: $P_{Ch}^* = r$ and $P_{Ch}^* < r$. In both these cases, the profit to firm $BC$ from products $H$ and $D$ are independent of the price it charges for the other product. Thus, the equilibrium prices presented in Lemma 2 are also the equilibrium prices in these cases, since $P_{Ch}^*$ and $P_{Bd}^*$ from that Lemma are still the best responses for firm $BC$ to $P_A^{bs}$. As in the Proof of Lemma 2, we obtain the condition $r \in [c, c + \frac{t}{2}]$ when $P_{Ch}^* = r$, and the condition $r \in [c + \frac{t}{2}, c + \frac{t}{6}]$ when $P_{Ch}^* < r$. In checking if firm $A$ can deviate by reducing $P_A^b$ below $P_{Ch}^*$ to capture the whole market, we obtain similar conditions for existence of equilibrium as in Lemma 2. However, in checking potential deviations by firm $BC$ on the price of either product, we need to consider the impact on the total profit from $H$ and $D$ to firm $BC$. A deviation by firm $BC$ to a $P_{Bd}^{dev}$ and $P_{Ch}^{dev}$ such that $2r - P_{Bd}^{dev} + P_{Ch}^{dev} - t - P_A^{bs} > 0$ is not profitable when $r$ is in the lowest interval (part (i) of the Lemma) with $P_{Ch}^* = r$, as firm $BC$ is charging the maximum price for $H$ and is making the maximum possible profit from $D$. However, when $r$ is in the intermediate range with $P_{Ch}^* < r$ (part (ii) of the Lemma), firm $BC$ may consider a similar deviation. However, it can be shown that such a deviation is not profitable.

Lastly we consider the case of a corner solution where $x_2 = x_1$ holds exactly. Once again, this requires the equilibrium prices to satisfy (2) exactly. The following are necessary conditions for this equilibrium:

$$\left. \frac{\partial \Pi_{BC}}{\partial (P_{Bd})} \right|_{P_{Ch} = P_{Ch}^*} = \left. \frac{P_{Bd}^{bs} - P_{Bd}^{bs}}{2c + P_A^{bs} - 2P_{Bd}^{bs} - 2P_{Ch}^{bs} + t} \right|_{P_{Ch}^*} > 0 \quad (8)$$
\[ \frac{\partial \Pi_{BC}}{\partial (P_{Ch})} \bigg|_{P_{Ch} = P_{Ch}^*} = \frac{2c + P_{A}^{bs} - 2P_{Bd}^* - 2P_{Ch}^* + t}{2t} < 0 \quad (9) \]

As can be seen above, inequality (8) contradicts (9). Thus, the only feasible point at which such an equilibrium may exist is when \( P_{Ch}^* = r \). However, we can use the same steps as we used in the proof of Lemma 2, beginning with equation (3), to show that such an equilibrium cannot exist.

**Subgame IV**

**Lemma 4** In the pricing subgame when all firms sell pure bundles in equilibrium,\(^2\)

(i) for \( r \in \left[ \text{Max}\{c, \frac{1}{2}(c + t)\}, c + \frac{1}{2}t \right] \), the equilibrium has \( P_{A}^{bs} = P_{BC}^{bs} = (r + c) \); \( \Pi_{A}^{bs} = \Pi_{BC}^{bs} = \frac{1}{t}(r - c)^2 \)

(ii) for \( r \in [c + \frac{1}{2}t, c + \frac{3}{4}t] \), the symmetric equilibrium has \( P_{A}^{bs} = P_{BC}^{bs} = 2r - t/2 \); \( \Pi_{A}^{bs} = \Pi_{BC}^{bs} = r - c - t/4 \).

(iii) for \( r \in [c + \frac{3}{4}t, \infty) \), the equilibrium has \( P_{A}^{bs} = P_{BC}^{bs} = (2c + t) \); \( \Pi_{A}^{bs} = \Pi_{BC}^{bs} = \frac{1}{2}t \)

**Proof:** If prices of the bundle are low enough that the market areas of the firms overlap, we obtain the location \( x_3 \) of the customer indifferent between the two bundles by equating the surplus from the competing bundles.

\[ x_3 = \frac{P_{BC}^b - P_{A}^b + t}{2t} \]

Customers in \([0, x_3]\) buy the bundle from firm \( A \) while those in \([x_3, 1]\) buy the bundle from firm \( BC \). If the prices of the competing bundles are high enough, the market areas do not overlap. Then there exists a region \([0, x_1]\) in which customers buy from firm \( A \), a region \([x_1, x_2]\) in which customers do not buy either bundle, and a region \([x_2, 1]\) in which customers buy the bundle from firm \( BC \). \( x_1 \) and \( x_2 \) are given by the following equations:

\[ x_1 = \frac{2r - P_{A}^b}{t} \]

---

\(^2\)When \( r \in \left[ \frac{1}{2}(c + t), c + \frac{1}{2}t \right] \), a pure strategy equilibrium may not exist as a firm may deviate and sell its bundle under \( r \) so as to sell to the whole market. In this case, customers who are far away from the firm on the Hotelling line buy the bundle only to consume the homogeneous product.
The market areas do not overlap if \( x_1 + (1 - x_2) \leq 1 \), which yields the condition:

\[
P_A^b + P_{BC}^b \leq 4r - t
\]

There is a third possibility when either or both of the competing bundles is offered at a price below \( r \). In this case, the market areas overlap but customers in the middle buy the cheaper of the two competing bundles but use only the homogeneous product. This is because the disutility from the differentiated product in the bundle exceeds the customer’s reservation price. Moreover, if both bundles are priced below \( r \), the cheaper bundle gets those customers in the middle who consume only the homogeneous product. Thus, if \( P_A^b \leq r \), and \( P_A^b < P_{BC}^b \), there are three possible regions. Customers in \([0,x_4]\) buy the bundle from firm A and consume both products, and those in \([x_5,1]\) do so with the bundle from firm BC. Customers in \([x_4,x_5]\) buy the bundle from firm A and consume only the homogeneous product. Equating the surplus from both products in firm A’s bundle to that from consuming its homogeneous product alone, we get \( x_4 = r/t \). Since \( x_4 < 1 \) for a middle region of customers consuming only the homogeneous product in the bundle to exist, we get \( r < t \). The value of \( x_5 \) is obtained by equating the surplus from consuming the homogeneous product in firm A’s bundle and the surplus from both products in firm BC’s bundle as follows:

\[
x_5 = \frac{P_{BC}^b - P_A^b - r + t}{t}
\]

The bundle from firm BC has positive sales provided \( x_5 < 1 \), which leads to the condition, \( P_{BC}^b < P_A^b + r \). The middle region exists if \( x_5 > x_4 \), which leads to the following condition:

\[
P_{BC}^b > 2r + P_A^b - t
\]

Likewise, there may exist a middle region of customers who buy the bundle from firm BC just to consume the homogeneous product if \( P_{BC}^b \leq r \), and \( P_{BC}^b < P_A^b \). In this case, customers up to \( x_6 \) buy and consume the bundle from firm A, where \( x_6 \) is given by the following:

\[
x_6 = \frac{r - P_A^b + P_{BC}^b}{t}
\]
Thus, the profit function for firm $A$ is given by the following:

$$\Pi_A^b = \begin{cases} 0 & \text{if } P_A^b > P_{BC}^b + t \\
= x_3(P_A^b - 2c) & \text{if } P_{BC}^b - t \leq P_A^b \leq P_{BC}^b + t \text{ and } P_A^b \leq 4r - P_{BC}^b - t \\
= (P_A^b - 2c) & \text{if } P_A^b \leq P_{BC}^b - t \\
= x_1(P_A^b - 2c) & \text{if } 4r - P_{BC}^b - t \leq P_A^b \leq 2r \\
= x_5(P_A^b - 2c) & \text{if } P_A^b \leq r, P_A^b < P_{BC}^b \text{ and } P_{BC}^b - r < P_A^b < P_{BC}^b - 2r + t \\
= (P_A^b - 2c) & \text{if } P_A^b \leq r, P_A^b < P_{BC}^b \text{ and } P_A^b \leq P_{BC}^b - r \\
= x_6(P_A^b - 2c) & \text{if } P_{BC}^b \leq r, \text{ and } P_{BC}^b < P_A^b < P_{BC}^b + r \\
= 0 & \text{if } P_{BC}^b \leq r, \text{ and } P_A^b \geq P_{BC}^b + r \\
\end{cases}$$

The profit function for firm $BC$ is similar. There are four possibilities for the equilibrium. The first possibility is that the market is fully covered in equilibrium. The firms’ profit function in this case is given by line 2 of the above profit function. Solving the first-order conditions yields the equilibrium prices in part (iii) of the Lemma. Since $P_{A}^{bs} \leq 4r - P_{BC}^{bs} - t$ for complete market coverage for both segments, the resulting condition on the parameters is $r \geq c + (3/4)t$. We assume that this condition holds and check for potential deviations by firms. Firm $A$ may deviate to a price $P_A^{bdev}$ below $r$ such that $x_5 > x_4$ after deviation. For this deviation, $P_A^{bdev}$ should satisfy

$$P_A^{bdev} < P_{BC}^{bs}; P_{BC}^{bs} \geq 2r + P_A^{bdev} - t \text{ and } P_A^{bdev} \leq r \quad (10)$$

If $r$ is sufficiently low, firm $A$ could consider $P_A^{bdev} = r$ so that (10) is satisfied. However (10) with $P_A^{bdev} = r$ can be shown to require $r \leq \frac{2}{3}(c+t)$. This contradicts our assumption that $r \geq c + (3/4)t$. Therefore, firm $A$ needs to deviate to a price below $r$ in order to satisfy (10). Firm $A$ will deviate if will be profitable to price below $P_{BC}^{bs} + 2r - t$, and this can be shown to be possible only if $r \leq \frac{2}{3}t$. However, this requirement contradicts our assumption that $r \geq c + (3/4)t$.

A second possibility is that the firms’ equilibrium prices are such that the market areas of the two firms just ‘touch’ each other. Thus, the candidate equilibrium prices, $P_A^{b0}$ and $P_{BC}^{b0}$, are such that $P_A^{b0} = 4r - P_{BC}^{b0} - t$. This requires that for all prices lower (higher) than equilibrium prices the profit function is increasing (decreasing) in price for either firm. Thus, we get conditions on the derivatives on either side of the equilibrium prices, which lead to the following set of inequalities:

$$P_A^{b0} \leq \frac{1}{2}(P_{BC}^{b0} + t + 2c)$$
$$P_{BC}^{b0} \leq \frac{1}{2}(P_A^{b0} + t + 2c)$$
\[ P_{A}^{b0} \geq (r + c) \]
\[ P_{BC}^{b0} \geq (r + c) \]

Substituting \( P_{A}^{b0} + P_{BC}^{b0} = 4r - t \) in the above set of inequalities we get \( r \in [c + \frac{1}{2}t, c + \frac{3}{4}t] \).

Although, a continuum of equilibrium prices are possible, we choose the symmetric equilibrium prices given by part (ii) of the Lemma. Again we check for potential deviations by firms as we did above and show that firms do not have any incentive to deviate.

Now let us consider the third possibility of an equilibrium in which the market is partially covered. The profit expressions for the firms in this equilibrium are given by line 2 of the profit function above. Solving the first-order conditions, we obtain the equilibrium prices in part (i) of the Lemma. Since \( P_{A}^{br} \geq 4r - P_{BC}^{br} - t \) for incomplete market coverage, we obtain \( r \leq c + t/2 \) as the condition for this equilibrium. We assume that this condition holds. We check if firm A can deviate to a price equal to \( r \) such that (10) is satisfied. At \( P_{A}^{br} = r \), satisfying (10) requires \( r \leq \frac{1}{2}(c + t) \). Since, \( r \in [c, c + \frac{1}{2}t] \) by assumption, the above deviation is feasible if \( t > c \) Thus, a sufficient condition for existence of the equilibrium is \( r \geq \frac{1}{2}(c + t) \), and we assume that this condition holds.

It is possible for firm A to deviate strictly below \( r \) such that (10) is satisfied. However, it can be shown that such a deviation is not profitable. Due to symmetry of the equilibrium, deviation by firm BC can be addressed in a similar fashion.

The fourth possibility of an equilibrium is one in which \( x_5 > x_4 \) in equilibrium. However, it can be shown that such an equilibrium cannot exist if \( r \geq \frac{1}{2}(c + t) \) as we have assumed.

Proof of Propositions 1, 2 and 3

Propositions 1, 2 and 3 give the equilibrium product lines chosen by the firms. These are established by a straightforward comparison of profit. The lower limit for \( r \) in Proposition 2 and the upper limit for \( r \) in Proposition 1 need some explanation. Since firm A makes more profit if bundling unilaterally, unbundled product lines cannot be an equilibrium. We need to see if firm BC will bundle, given firm A bundles. Assuming that firm A bundles, and comparing firm BC’s bundling profit with its unbundling profit for intermediate \( r \), we can show that the bundling profit of firm BC exceeds its unbundling profit if and only if \( r \geq c + (2 - 3\sqrt{39}/13)t \).

Proof of Proposition 4

We first exclude mixed bundling strategies that offer product D separately. Table A.1 expands Table 2 of the paper to include mixed bundling strategies that involve selling H separately. Of the new subgames that involve mixed bundling, subgame XII is similar to subgame VIII, while
subgame XI is similar to subgame IX. The following lemma shows that in subgames VII, VIII, and X, firms’ profits do not exceed the profits in subgame I.

**Lemma 5**

(i) In subgame VII, firms’ profits are equal to those in subgame I.

(ii) In subgame VIII, firms’ profits are equal to those in subgame I.

(iii) In subgame X, either competitor’s profits are equal to those in subgame I.

**Proof.**

(i) In any equilibrium, we cannot have $P_{Ch} \geq P_{Ah} > c$, as firm C could lower its price to $P_{Ah} - \varepsilon$, for $\varepsilon > 0$ and arbitrarily small to make a positive profit. Similarly, we cannot have an equilibrium with $P_{Ah} \geq P_{Ch} > c$, as firm A could lower its price to $P_{Ch} - \varepsilon$, for $\varepsilon > 0$ and arbitrarily small to increase its profits from $H$ with arbitrarily small effect on its profits from $D$. If $P_{Ah} > P_{Ch} = c$ in equilibrium, firm C can increase $P_{Ch}$ to increase profit. That leaves us with three remaining possibilities. The first is an equilibrium with $P^*_{Ah} = P^*_{Ch} = c$. In this case firm C cannot deviate and increase its profit. In such an equilibrium, the customer’s reservation price for firm A’s bundle $DH$ is $r + c$, while firm A’s cost of its bundle is $2c$. We can then derive similar to subgame I the equilibrium profits for the firms from market $D$ using the modified reservation price of $r + c$, and the modified marginal cost of $2c$ for firm A. We can show that the equilibrium profits in this case coincide with those in subgame I. Thus, firm A’s profits equal its profits when using the product line $\{D, H\}$. Moreover, this is a valid equilibrium as firm A cannot deviate from $P^*_{Ah}$ and increase profit. (If it charges $P_{Ah} < c$, the reservation price of $DH$ will become $r + P_{Ah} < r + c$ leading to a loss in market $H$ and lower profit in $D$ for firm A). A second possibility is an equilibrium in which $P_{Ch} > P_{Ah} = c$. In this case, firm A can make a positive profit in $H$ by raising $P_{Ah}$ and in doing so can increase its profit in $D$ as well, since it increases the reservation price for the bundle from $r + c$ to $r + P_{Ah}$. A similar argument rules out a third possibility of an equilibrium with $P_{Ah} < P_{Ch} = c$.

(ii) As in (i) above, we can rule out an equilibrium with $P_{Ch} \geq P_{Ah} > c$ or $P_{Ah} \geq P_{Ch} > c$. Now consider an equilibrium with $P^*_{Ah} = P^*_{Ch} = c$. Using arguments similar to those used in (i) above, we can show that this is an equilibrium. The equilibrium prices and profits in market $D$ in this case will be similar to those given in (i) above using the modified reservation price of $r + c$, and the modified marginal cost of $2c$ for firm A. Thus, firms’ profits are equal to those in subgame I. A second possibility is an equilibrium in which $P_{Ch} > P_{Ah} = c$. In this case, firm A can make a positive profit in $H$ by raising $P_{Ah}$ and in doing so can increase its profit in $D$ as well, since it
increases the reservation price for the bundle from \( r + c \) to \( r + P_{Ah} \). A similar argument rules out a third possibility of an equilibrium with \( P_{Ah} > P_{Ch} = c \). Finally, we can similarly rule out an equilibrium with \( P_{Ah} < P_{Ch} = c \) or with \( P_{Ch} < P_{Ah} = c \).

(iii) As in (i) above, we can rule out an equilibrium with \( P_{BCh} \geq P_{Ah} > c \) or \( P_{Ah} \geq P_{BCh} > c \). Now consider an equilibrium with \( P_{Ah}^* = P_{BCh}^* = c \). Using arguments similar to those used in (i) above, we can show that this is an equilibrium. The equilibrium prices and profits in market for the bundles will be similar to those in subgame I with the exception of a modified reservation price of \( r + c \), and a modified marginal cost of \( 2c \) for both firms. It can be seen from these results that both firms’ profits are equal to their profits in subgame I. A second possibility is an equilibrium in which \( P_{BCh} > P_{Ah} = c \). In this case, firm A can make a positive profit in \( H \) by raising \( P_{Ah} \) and in doing so can increase its profit in the bundle market as well, since it increases the reservation price for the bundle from \( r + c \) to \( r + P_{Ah} \). A similar argument rules out a third possibility of an equilibrium with \( P_{Ah} > P_{BCh} = c \). Finally, we can similarly rule out an equilibrium with \( P_{Ah} < P_{BCh} = c \) or with \( P_{BCh} < P_{Ah} = c \). ■

The following lemma deals with the one remaining subgame, XI, that involves mixed bundling with \( H \) being sold separately.

**Lemma 6** In subgame XI,

(i) for \( r \in [\text{Max}\{c, \frac{1}{3}(c+2t)\}, c+\frac{1}{2}t] \), the equilibrium has \( P_{Ah}^* = c+r \), \( P_{BCh}^* = \frac{1}{2}(r+c) \), \( P_{Ch}^* = r \);

\[ \Pi_{Ah}^* = \frac{1}{4}(r-c)^2, \quad \Pi_{BCh}^* = (r-c)^2 / 4t + \frac{1}{4}(r-c)(c+t-r). \]

(ii) for \( r \in [c+\frac{1}{2}t, c+\frac{1}{2}t] \), the equilibrium has \( P_{Ah}^* = \frac{1}{3}(5c+r+t) \), \( P_{BCh}^* = \frac{1}{2}(r+c) \), \( P_{Ch}^* = \frac{1}{3}(4c+2t-r) \);

\[ \Pi_{Ah}^* = \frac{1}{9}(r-c+t)^2, \quad \Pi_{BCh}^* = \frac{1}{9t}(r-c)^2 + \frac{1}{9t}(r-c-2t)^2. \]

(iii) for \( r \in [c+t, \infty) \), the equilibrium has \( P_{Ah}^* = 2c + t \), \( P_{BCh}^* = 2c + t \), \( P_{Ch}^* = r \);

\[ \Pi_{Ah}^* = \frac{1}{2}t; \quad \Pi_{BCh}^* = \frac{1}{2}t. \]

**Proof.** We will let \( P_{Ch} \) denote the price of the product \( H \) separately offered by firm \( BC \). We can define \( x_1, x_2 \) as in case of subgame IV such that customers in the region \([0, x_1] \) buy the bundle from firm \( A \), customers in \([x_1, x_2] \) buy \( H \) alone from firm \( C \), and customers in \([x_2, 1] \) buy bundle from firm \( B \). Since the customer at \( x_1 \) derives equal surplus from buying the bundle from firm \( A \) and buying \( H \) alone from we can obtain \( x_1 \) as follows:

\[ x_1 = \frac{r - P_{Ah}^* + P_{Ch}}{t} \]
Similarly, we can obtain \( x_2 \) by equating the customer surplus in buying bundle from \( BC \) and \( H \) from firm \( C \). Thus we get \( x_2 \) as follows:

\[
x_2 = 1 - \frac{r - P_{BC}^b + P_{Ch}}{t}
\]

For the segment \([x_1, x_2]\) to be nonzero, we require \( x_2 > x_1 \). This condition leads to the following inequality:

\[
P_A^b > 2r - P_{BC}^b + 2P_{Ch} - t
\]

Note also that \( x_1 \geq x_2 \) when \( x_2 \leq 0 \) which is true if \( P_{Bd} \leq P_{Ch} + r - t \). When \( x_1 \geq x_2 \), we will have just two segments based on customer purchase. Customers in \([0, x_3]\) buy the bundle from firm \( A \) while those in \([x_3, 1]\) buy the bundle from firm \( BC \). Equating the surplus in buying the bundle from firm \( A \) to the surplus from buying \( H \) from firm \( C \) and \( D \) from firm \( B \), we can obtain \( x_3 \) as follows:

\[
x_3 = \frac{P_{BC}^b - P_A^b + t}{2t}
\]

There is a third possibility when either or both of the competing bundles is offered at a price below \( P_{Ch} \leq r \). In this case, the market areas overlap but customers in the middle buy the cheaper of the two competing bundles but use only the homogeneous product. This is because the disutility from the differentiated product in the bundle exceeds the customer’s utility from it. Moreover, if both bundles are priced below \( P_{Ch} \), the cheaper bundle gets those customers in the middle who consume only the homogeneous product. Thus, if \( P_A^b < P_{Ch} \), and \( P_A^b < P_{BC}^b \), there are three possible regions. Customers in \([0, x_4]\) buy the bundle from firm \( A \) and consume both products, and those in \([x_5, 1]\) do so with the bundle from firm \( BC \). Customers in \([x_4, x_5]\) buy the bundle from firm \( A \) and consume only the homogeneous product. Equating the surplus from both products in firm \( A \)’s bundle to that from consuming its homogeneous product alone, we get \( x_4 = r/t \). Since \( x_4 < 1 \) for a middle region of customers consuming only the homogeneous product in the bundle to exist, we get \( r < t \). The value of \( x_5 \) is obtained by equating the surplus from consuming the homogeneous product in firm \( A \)’s bundle and the surplus from both products in firm \( BC \)’s bundle as follows:
\[ x_5 = \frac{P_{BC}^b - P_A^b - r + t}{t} \]

The bundle from firm BC has positive sales provided \( x_5 < 1 \), which leads to the condition, \( P_{BC}^b < P_A^b + r \). The middle region exists if \( x_5 > x_4 \), which leads to the following condition:

\[ P_{BC}^b > 2r + P_A^b - t \] (11)

Likewise, there may exist a middle region of customers who buy the bundle from firm BC just to consume the homogeneous product if \( P_{BC}^b < P_{Ch}^b \), and \( P_{BC}^b < P_A^b \). In this case, customers up to \( x_6 \) buy and consume the bundle from firm A, where \( x_6 \) is given by the following:

\[ x_6 = \frac{r - P_A^b + P_{BC}^b}{t} \]

We can now obtain the profit functions for firm A and firm BC as follows:

\[ \Pi_A^b = 0 \text{ if } P_A^b \geq \text{Min}\{P_{Ch} + r, P_{BC}^b + t\} \]

\[ = x_1(P_A^b - 2c) \text{ if } 2r - P_{BC}^b + 2P_{Ch} - t \leq P_A^b \leq P_{Ch} + r \text{ and} \]

\[ = x_3(P_A^b - 2c) \text{ if } P_{BC}^b - t \leq P_A^b \leq 2r - P_{BC}^b + 2P_{Ch} - t \]

\[ = x_5(P_A^b - 2c) \text{ if } P_A^b < P_{Ch}, P_A^b < P_{BC}^b - r \leq P_A^b \leq P_{BC}^b < 2r + t \]

\[ = (P_A^b - 2c) \text{ if } P_A^b < P_{Ch}, P_A^b < P_{BC}^b \text{ and } P_A^b \leq P_{BC}^b - r \]

\[ = x_6(P_A^b - 2c) \text{ if } P_{BC}^b < P_{Ch}, \text{ and } P_{BC}^b \leq P_A^b \leq P_{BC}^b + r \]

\[ = 0 \text{ if } P_{BC}^b < r, \text{ and } P_A^b \geq P_{BC}^b + r \]

\[ \Pi_{BC} = 0 \text{ if } P_{BC}^b \geq P_A^b + t \text{ and } P_{Ch} \geq P_A^b + t - r \]

\[ = (1 - x_2)(P_{BC}^b - 2c) + (x_2 - x_1)(P_{Ch} - c) \text{ if } 2r - P_A^b + 2P_{Ch} - t \leq P_{BC}^b \leq 2r \]

and \( \text{Max}\{P_A^b - r, P_{BC}^b - r\} \leq P_{Ch} \leq r \)

\[ = (1 - x_3)(P_{BC}^b - 2c) \text{ if } P_A^b - t \leq P_{BC}^b \leq 2r - P_A^b + 2P_{Ch} - t \]

\[ = (P_{BC}^b - 2c) \text{ if } P_{BC}^b \leq P_A^b - t \text{ and } P_{Ch} \geq P_{BC}^b + t - r \]

\[ = (1 - x_2)(P_{BC}^b - 2c) + (x_2)(P_{Ch} - c) \text{ if } P_{BC}^b \leq P_A^b - t \text{ and } P_{Ch} \leq P_{BC}^b + t - r \]

\[ = (1 - x_6)(P_{BC}^b - 2c) \text{ if } P_{BC}^b < P_{Ch}, P_{BC}^b < P_A^b \text{ and } P_A^b - r \leq P_{BC}^b \leq P_A^b - 2r + t \]

\[ = (P_{BC}^b - 2c) \text{ if } P_{BC}^b < P_{Ch}, P_{BC}^b < P_A^b \text{ and } P_{BC}^b \leq P_A^b - r \]

\[ = (1 - x_3)(P_{BC}^b - 2c) \text{ if } P_A^b < P_{Ch}, \text{ and } P_A^b \leq P_{BC}^b \leq P_A^b + r \]

\[ = 0 \text{ if } P_A^b < r, \text{ and } P_{BC}^b \geq P_A^b + r \]
As before, we can argue that there can be no equilibrium with firm A making zero sales, and firm B making zero sales for its bundle. Then there are four possibilities for equilibrium. First, there may be an equilibrium in which \( x_1 > x_2 \). Thus, in this case, firm BC makes zero sales of its separate product H. We therefore set \( P^*_C = r \). The profits for firm A and firm BC in this case are given by the third line of their respective profit functions. Solving the first order conditions we get \( p^{bs}_A = p^{bs}_BC = 2c + t \); \( \Pi^{bs}_A = \Pi^{bs}_BC = \frac{1}{4}t \). Since \( 4r - p^{bs}_A - p^{bs}_BC - t \geq 0 \) for this equilibrium, we have the condition \( r \geq c + \frac{3}{4}t \) for this equilibrium. We now check for potential deviations by firms. Firm A cannot gain by deviating as its profit function is concave. However, firm BC can potentially reduce \( P^*_C \) and deviate in a way that the following holds after deviation.

\[
2r - P^{dev}_BC + 2P^{dev}_C - t - P^{bs}_A \leq 0
\]  

(12)

We can show that such a deviation by firm BC is not possible as long as \( r \geq c + t \). Thus, the above equilibrium exists for \( r \in [c + t, \infty) \).

A second possible equilibrium is when \( x_1 = x_2 \) exactly so that the market areas for the two bundles just touch. Thus firm BC makes zero sales of its separate product H in this equilibrium as well. The results for this case are identical to those in part (ii) of subgame IV with \( P^*_C = r \) and assuming symmetric prices for the competitors’ bundles. However, firm BC may potentially deviate by lowering \( P^*_C \) below \( r \). It can be shown that such a deviation is always profitable so that such a touching equilibrium cannot exist.

Third, there may be an equilibrium with \( x_1 \leq x_2 \). Note that in this case, profits to both firms are given by the second line of their profit functions. As we saw in case of subgame III, there are two types of equilibria - corner and interior. In the case of the corner solution, setting \( P^*_C = r \), and solving for the first order conditions, we get \( p^{bs}_A = (r + c); \quad p^{bs}_BC = \frac{1}{2}(3r + c) \) and \( \Pi^{bs}_A = \frac{1}{t}(r - c)^2; \quad \Pi^{bs}_BC = \frac{1}{4t}(r - c)(3c + 4t - 3r) \). As in the proof of subgame III, we can obtain the initial boundary condition for this equilibrium as \( r \in [c, c + \frac{t}{2}] \). We assume that this condition holds. However, firm A could potentially deviate below \( P^*_C \) to sell to customers who may buy the bundle from firm A only to consume product H. In order to do so, firm A’s deviating price, \( p^{dev}_A \), should satisfy (11). There are two possibilities for \( p^{dev}_A \) that could satisfy (11): (1) \( p^{dev}_A = r \) and (2) \( p^{dev}_A < r \). If \( p^{dev}_A = r \), satisfying (11) requires \( r < \frac{1}{3}(c + 2t) \). Since \( r \in [c, c + \frac{1}{2}t] \) by assumption, the above deviation is feasible if \( t > c \). Thus, a sufficient condition for existence of the equilibrium
is \( r \geq \frac{1}{3}(c + 2t) \), and we assume that this condition holds as well. It is also possible for firm A to deviate strictly below \( r \) such that (11) is satisfied. However, it can be shown that such a deviation is not profitable.

The second case of an equilibrium with \( x_1 \leq x_2 \) is with \( P^*_{Ch} < r \) representing an interior solution. Solving the first-order conditions, we get the equilibrium prices and profits as given in part (ii) of the Lemma. As in the proof of subgame III, we obtain the boundary condition \( r \in [c + \frac{t}{2}, c + \frac{4t}{5}] \) for this equilibrium. We assume that this condition holds. Firm A can potentially deviate below \( P^*_{Ch} \) to sell to customers who may buy the bundle from firm A only to consume product \( H \). In order to do so, firm A’s deviating price, \( P^{bdev}_A \), should satisfy (11). There are two possibilities for \( P^{bdev}_A \) that could satisfy (11): \( P^{bdev}_A = P^*_{Ch} \) and (2) \( P^{bdev}_A < P^*_{Ch} \). However, it can be shown that no such deviation is profitable.

The implication of Lemma 6 is that firms’ profits in subgame XI are equal to those in subgame III except in some cases when \( \frac{1}{3}(c + 2t) > c \), the equilibrium may not exist in subgame XI. The combined import of Lemmas 5 and 6 is that any unilateral deviation by a firm to the unshaded region in Table A.1 from a stage 1 equilibrium in the shaded region is not profitable. Thus the equilibrium identified in Propositions 1, 2 and 3 continues to be an equilibrium when allowing for mixed bundling strategies that involve selling \( H \) separately without selling \( D \) separately. Next we consider product line strategies that involve selling \( D \) separately with or without \( H \) being offered separately. The following lemma consider mixed bundling strategies with \( D \) being sold separately.

**Lemma 7** The equilibrium payoffs in the pricing subgame to firm A or the alliance of firms B and C from the mixed bundling strategies \{\( D H, H, D \)\} or \{\( D H, D \)\} do not exceed the firm’s payoffs when it unilaterally deviates to the product lines \{\( D H, H \)\}, \{\( D H, D \)\}, \{\( D, H \)\}, or \{\( D \)\}.

**Proof.** We assert that in the equilibrium of the price subgame, a firm or alliance, \( i \), using a mixed bundle that involves selling product \( D \) separately cannot have positive sales of both the bundle and the separate product. Suppose not. Then there exists some customer located at say, \( x_1 \), who prefers the bundle to buying the product \( D \) separately. In comparing these two options, the customer will also consider the surplus from buying product \( H \) when buying product \( D \) separately, provided product \( H \) is offered separately by one or both firms. Let \( P^*_h \) be the lowest price of product \( H \) if offered. If product \( H \) is not offered separately by either firm, it is equivalent to product \( H \) being available at a price of \( r \) for calculating customer surplus. Then, for the customer at \( x_1 \), we should
have the following inequality.

\[ 2r - P_i^b - tx_1 > 2r - P_{id} - \text{Min}(P_{h}, r) - tx_1 \]  \hspace{1cm} (13)

Similarly, given our assumption of positive sales of the separate product \( D \) from firm \( i \), there exists a customer \( x_2 \), who prefers buying the product \( D \) from firm \( i \) separately to the bundle. Then we should have the following condition:

\[ 2r - P_{id} - \text{Min}(P_{h}, r) - tx_2 > 2r - P_i^b - tx_2 \]  \hspace{1cm} (14)

However (13) implies \( 2r - P_i^b - tx_2 > 2r - P_{id} - \text{Min}(P_{h}, r) - tx_2 \), which contradicts (14) thereby establishing the above assertion. This assertion leads to the conclusion that firm \( i \)'s equilibrium payoff in the game (call it \( G \)) where it uses a mixed bundling strategy with product \( D \) sold separately does not exceed its payoffs in the game (call it \( G' \)) where it sells all the same products as in \( G \) without the bundle or in one (call it \( G'' \)) in which it sells the same products as in \( G \) except for the separate product \( D \). To prove this, assume the contrary that firm \( i \)'s payoff in \( G \) strictly exceeds its payoffs in \( G' \) and \( G'' \). Then, note that the firms' payoffs in \( G \) at equilibrium depends only on firm \( i \)'s price of the bundle or product \( D \), whichever has positive sales, due to (13) and (14). In addition, the firms' payoffs depend on the prices of all other products of all the firms in the equilibrium in \( G \). Thus, firm \( i \)'s equilibrium price for the bundle or product \( D \), whichever has positive sales, and the equilibrium prices of all other products of all the firms in this equilibrium will also constitute an equilibrium in either \( G' \) or \( G'' \) as the case may be. This contradicts the assumption that firm \( i \)'s payoff in \( G \) exceeds its payoffs in \( G' \) and \( G'' \). This contradiction establishes the Lemma.

Lemma 7 implies that a firm in an equilibrium given by Propositions 1, 2, and 3 will not unilaterally deviate to mixed bundling strategies with \( D \) sold separately. Thus, the subgame perfect equilibrium identified in Propositions 1, 2, and 3 is also an equilibrium when mixed bundling strategies are considered, establishing part (a) of the proposition.\(^3\) Lemmas 5, 6 and 7 also show that in any equilibrium involving mixed bundling, the firm using the mixed bundling strategy will

\(^3\)We also note that in the game in which a firm offers \( D \) separately in addition to the bundle \( DH \), an equilibrium in pure strategies will exist whenever such an equilibrium exists in a game where the firm offers only \( DH \). This is because the equilibrium prices in the latter game are also an equilibrium in the former game since both products \( D \) and \( DH \) from a firm cannot simultaneously have positive sales as argued in the proof of Lemma 7.
not make a higher profit than in one in which mixed bundling is not allowed. This establishes part (b) of Proposition 4. ¶
Table A.1. Possible Pricing Subgames

<table>
<thead>
<tr>
<th>Firm A’s Product Line</th>
<th>( \phi_A = {D} ), ( \phi_C = {H} )</th>
<th>{D,H}</th>
<th>{DH}</th>
<th>{DH,H}</th>
</tr>
</thead>
<tbody>
<tr>
<td>{D,H}</td>
<td>I</td>
<td>V</td>
<td>VI</td>
<td>XII</td>
</tr>
<tr>
<td>{DH}</td>
<td>II</td>
<td>III</td>
<td>IV</td>
<td>XI</td>
</tr>
<tr>
<td>{DH,H}</td>
<td>VII</td>
<td>VIII</td>
<td>IX</td>
<td>X</td>
</tr>
</tbody>
</table>