Notes on the Mussa-Rosen duopoly model

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In the most general version of the model, n consumers uniformly distributed over $[\theta_L, \theta_H]$, where $\theta_L \geq 0$. Without loss of generality, normalize $n = 1$.

Firm A produces a variety of quality χ_A, firm B produces a variety of quality χ_B; without loss of generality, let $\chi_A \leq \chi_B$.

Individual utility function:

$$u(\theta, \chi) = \theta\chi - p. \tag{1}$$

Quality-adjusted prices:

$$\theta_{\phi A} = \frac{p_A}{\chi_B} \tag{2}$$

$$\theta_{\phi B} = \frac{p_B}{\chi_B} \tag{3}$$

Consumers with $\theta_{\phi A} \leq \theta \leq \theta_H$ get positive net utility buying from A; consumers with $\theta_{\phi B} \leq \theta \leq \theta_H$ get positive net utility buying from B; in both cases, if the indicated intervals exist.

Marginal utility of quality of the consumer who receives the same utility buying from A or B:

$$\theta_{AB}\chi_A - p_A = \theta_{AB}\chi_B - p_A$$

$$\theta_{AB} = \frac{p_B - p_A}{\chi_B - \chi_A} = \frac{\theta_{\phi B}\chi_B - \theta_{\phi A}\chi_A}{\chi_B - \chi_A}. \tag{4}$$

Note that

$$\theta_{AB} = \frac{(\chi_B - \chi_B)\,\theta_{\phi B} + (\theta_{\phi B} - \theta_{\phi A})\,\chi_A}{\chi_B - \chi_A}$$

$$= \theta_{\phi B} + (\theta_{\phi B} - \theta_{\phi A})\,\frac{\chi_A}{\chi_B - \chi_A} \tag{5}$$

Hence

$$\theta_{AB} - \theta_{\phi B} = (\theta_{\phi B} - \theta_{\phi A})\,\frac{\chi_A}{\chi_B - \chi_A} \tag{6}$$

and

$$\theta_{\phi B} \geq \theta_{\phi A} \Longrightarrow \theta_{AB} \geq \theta_{\phi B} \tag{7}$$

$$\theta_{\phi A} \geq \theta_{\phi B} \Longrightarrow \theta_{\phi B} \geq \theta_{AB} \tag{8}$$

θ_{AB} is the marginal utility of quality of the consumer who gets the same net utility buying from A or B; nothing requires that utility to be positive.

1.1 Firm A

(1) When would firm A supply the entire market?

$$\theta\chi_A - p_A \geq \theta\chi_B - p_B \ \forall\ \theta \text{ and } \theta\chi_A - p_A \ \geq 0 \ \forall\ \theta$$

$$p_B - \theta(\chi_B - \chi_A) \geq p_A \ \forall\ \theta \text{ and } \theta\chi_A \geq p_A \ \forall\ \theta$$

$$p_A \leq \min\left[p_B - \theta_H(\chi_B - \chi_A), \theta_L\chi_A\right]$$

$$\theta_{\phi A} \leq \min\left[\frac{\chi_B}{\chi_A}\theta_{\phi B} - \theta_H\left(\frac{\chi_B}{\chi_A} - 1\right), \theta_L\right] \tag{9}$$

(2) When would firm A sell nothing?

$$\theta_{AB} \leq \theta_L$$

$$\frac{\chi_B\theta_{\phi B} - \chi_A\theta_{\phi A}}{\chi_B - \chi_A} \leq \theta_L$$

$$\chi_B\theta_{\phi B} - \chi_A\theta_{\phi A} \leq \theta_L\left(\chi_B - \chi_A\right)$$

$$\chi_B\theta_{\phi B} - \theta_L\left(\chi_B - \chi_A\right) \leq \chi_A\theta_{\phi A}$$

$$\frac{\chi_B}{\chi_A}\theta_{\phi B} - \theta_L\left(\frac{\chi_B}{\chi_A} - 1\right) \leq \theta_{\phi A}$$

$\theta_{\phi A} \geq \theta_{\phi B}$ or equivalently $p_A \geq \frac{\chi_A}{\chi_B}p_B$; $\theta_{AB} \leq \theta_{\phi B} \leq \theta_{\phi A} \implies q_a = 0$; $\theta_{\phi A} \geq \theta_H \implies q_a = 0$; hence

$$\theta_{\phi A} \geq \min\left(\frac{\chi_B}{\chi_A}\theta_{\phi B} - \theta_L\left(\frac{\chi_B}{\chi_A} - 1\right), \theta_{\phi B}, \theta_H\right) \implies q_a = 0. \tag{10}$$

Normally we would automatically limit attention to $0 \leq \theta_{\phi A} \leq \theta_H$, so we would write

$$\theta_{\phi A} \geq \min\left(\frac{\chi_B}{\chi_A}\theta_{\phi B} - \theta_L\left(\frac{\chi_B}{\chi_A} - 1\right), \theta_{\phi B},\right) \implies q_a = 0 \tag{11}$$

(3) now consider $\theta_{\phi A} \leq \theta_{\phi B}$
$\theta_{\phi A} \leq \theta_{\phi B} \Longrightarrow \theta_{\phi A} \leq \theta_{\phi B} \leq \theta_{AB}$
$\theta_L \leq \theta_{\phi A}$ and $\theta_{\phi A} \leq \theta_{\phi B}$ or equivalently $\theta_L \chi_A \leq p_A$ and $p_A \leq \frac{\chi_A}{\chi_B} p_B$;
$\theta_{\phi A} \leq \theta_{\phi B} \Longrightarrow \theta_L \leq \theta_{\phi A} \leq \theta_{\phi B} \leq \theta_{AB}$,

$$q_A = \frac{N}{\theta_H - \theta_L} \left[\min(\theta_{AB}, \theta_H) - \theta_{\phi A}\right]. \tag{12}$$

Together, the conditions are $\theta_L \leq \theta_{\phi A} \leq \theta_{\phi B}$ or $\theta_L \chi_A \leq p_A \leq \frac{\chi_A}{\chi_B} p_B$
(3a) When is $\theta_{AB} \leq \theta_H$?

$$\frac{p_B - p_A}{\chi_B - \chi_A} \leq \theta_H$$

$$p_B - p_A \leq \theta_H(\chi_B - \chi_A)$$

$$p_B - \theta_H(\chi_B - \chi_A) \leq p_A$$

Since we already require $\theta_L \chi_A \leq p_A \leq \frac{\chi_A}{\chi_B} p_B$, the overall condition for $\theta_{AB} \leq \theta_H$ and

$$q_A = \frac{N}{\theta_H - \theta_L} \left(\theta_{AB} - \theta_{\phi A}\right)$$

is

$$\max\left[\theta_L \chi_A, p_B - \theta_H(\chi_B - \chi_A)\right] \leq p_A \leq \frac{\chi_A}{\chi_B} p_B \tag{13}$$

Expressed in terms of quality-adjusted prices

$$\max\left[\theta_L, \frac{\chi_B}{\chi_A}\left(\frac{p_B}{\chi_B}\right) - \theta_H\left(\frac{\chi_B}{\chi_A} - 1\right)\right] \leq \frac{p_A}{\chi_A} \leq \frac{p_B}{\chi_B}$$

$$\max\left[\theta_L, \frac{\chi_B}{\chi_A}\theta_{\phi B} - \theta_H\left(\frac{\chi_B}{\chi_A} - 1\right)\right] \leq \theta_{\phi A} \leq \theta_{\phi B}. \tag{14}$$

(3b) $p_A \leq p_B - \theta_H(\chi_B - \chi_A) \Longrightarrow \theta_H \leq \theta_{AB}$
Since we already require $\theta_L \chi_A \leq p_A \leq \frac{\chi_A}{\chi_B} p_B$, the overall condition is

$$\theta_L \chi_A \leq p_A \leq \min\left[\frac{\chi_A}{\chi_B} p_B, p_B - \theta_H(\chi_B - \chi_A)\right] \tag{15}$$

or, expressed in terms of quality-adjusted prices,

$$\theta_L \leq \theta_{\phi A} \leq \min\left[\theta_{\phi B}, \frac{\chi_B}{\chi_A}\theta_{\phi B} - \theta_H\left(\frac{\chi_B}{\chi_A} - 1\right)\right], \tag{16}$$

and for $p_A/\theta_{\phi A}$ in this range,

$$q_A = \frac{\theta_H - \theta_{\phi A}}{\theta_H - \theta_L}.. \tag{17}$$

When is

$$\frac{\chi_B}{\chi_A}\theta_{\phi B} - \theta_H\left(\frac{\chi_B}{\chi_A} - 1\right) \leq \theta_{\phi B}$$

$$\left(\frac{\chi_B}{\chi_A} - 1\right)\theta_{\phi B} \leq \theta_H\left(\frac{\chi_B}{\chi_A} - 1\right)$$

$$\theta_{\phi B} \leq \theta_H$$

so condition reduces to

$$\theta_L \leq \theta_{\phi A} \leq \frac{\chi_B}{\chi_A}\theta_{\phi B} - \theta_H\left(\frac{\chi_B}{\chi_A} - 1\right). \tag{18}$$

(4) Now consider $\theta_{\phi A} \leq \theta_{\phi B}, \theta_{\phi A} \leq \theta_L, \theta_{\phi A} \geq \max\left[0, \frac{\chi_B}{\chi_A}\theta_{\phi B} - \theta_H\left(\frac{\chi_B}{\chi_A} - 1\right)\right]$

$\theta_{\phi A} \leq \theta_{\phi B} \implies \theta_{\phi A} \leq \theta_{\phi B} \leq \theta_{AB}$ (as before); A sells from θ_L to $\min(\theta_{AB}, \theta_H)$,

$$q_A = \frac{1}{\theta_H - \theta_L}\left[\min(\theta_{AB}, \theta_H) - \theta_L\right]. \tag{19}$$

We know that $\theta_{AB} \leq \theta_H$ for

$$p_B - \theta_H(\chi_B - \chi_A) \leq p_A$$

or, in terms of quality-adjusted prices,

$$\frac{\chi_B}{\chi_A}\theta_{\phi B} - \theta_H\left(\frac{\chi_B}{\chi_A} - 1\right) \leq \theta_{\phi A};$$

but the latter is satisfied by assumption in the region under consideration; hence

$$q_A = \frac{\theta_{AB} - \theta_L}{\theta_H - \theta_L}. \tag{20}$$

Summary:

$$q_A = \begin{cases} 1 & 0 \leq \theta_{\phi A} \leq \theta_L, \theta_{\phi A} \leq \theta_{\phi B} \\ \frac{\theta_H - \theta_{\phi A}}{\theta_H - \theta_L} & \frac{\chi_A}{\chi_B}\theta_{\phi A} + \theta_H \left(1 - \frac{\chi_A}{\chi_B}\right) \leq \theta_{\phi B} \leq \theta_H, \theta_{\phi A} \geq \theta_L \\ \frac{\theta_{AB} - \theta_{\phi A}}{\theta_H - \theta_L} & \theta_{\phi A} \leq \theta_{\phi B} \leq \frac{\chi_A}{\chi_B}\theta_{\phi A} + \theta_H \left(1 - \frac{\chi_A}{\chi_B}\right), \theta_{\phi A} \geq \theta_L \\ 0 & \theta_{\phi A} \geq \theta_{\phi B} \end{cases} \tag{21}$$

1.2 Firm B

When would firm B supply the entire market?

$$\theta\chi_B - p_B \geq \theta\chi_A - p_A \; \forall \; \theta \text{ and } \theta\chi_B - p_B \geq 0 \; \forall \; \theta$$

or

$$\theta \left(\chi_B - \chi_A\right) + p_A \geq p_B \; \forall \; \theta \text{ and } \theta\chi_B \geq p_B \; \forall \; \theta$$

$$\theta_L \left(\chi_B - \chi_A\right) + p_A \geq p_B \text{ and } \theta_L\chi_B \geq p_B$$

$$\theta_L \left(1 - \frac{\chi_A}{\chi_B}\right) + \frac{\chi_A}{\chi_B}\frac{p_A}{\chi_A} \geq \frac{p_B}{\chi_B} \text{ and } \theta_L \geq \frac{p_B}{\chi_B}$$

$$\theta_L \left(1 - \frac{\chi_A}{\chi_B}\right) + \frac{\chi_A}{\chi_B}\theta_{\phi A} \geq \theta_{\phi B} \text{ and } \theta_L \geq \theta_{\phi B}$$

The condition for firm B to supply the entire market is

$$\theta_{\phi B} \leq \min \left[\theta_L, \theta_L \left(1 - \frac{\chi_A}{\chi_B}\right) + \frac{\chi_A}{\chi_B}\theta_{\phi A}\right] \tag{22}$$

And if $q_B = 1$ in this region, $q_A = 0$.
Note: another way to interpret the condition

$$\theta_{\phi B} \leq \theta_L \left(1 - \frac{\chi_A}{\chi_B}\right) + \frac{\chi_A}{\chi_B}\theta_{\phi A}$$

is as being derived from

$$\theta_{AB} \leq \theta_L$$

$$\frac{p_B - p_A}{\chi_B - \chi_A} \leq \theta_L$$

$$p_B \leq p_A + \theta_L(\chi_B - \chi_A)$$

Consider the line

$$\theta_{\phi B} = \theta_L\left(1 - \frac{\chi_A}{\chi_B}\right) + \frac{\chi_A}{\chi_B}\theta_{\phi A}$$

If $\theta_{\phi A} = 0$, $\theta_{\phi B} = \theta_L\left(1 - \frac{\chi_A}{\chi_B}\right)$. If $\theta_{\phi A} = \theta_L$, $\theta_{\phi B} = \theta_L$.

Hence the line $\theta_{\phi B} = \theta_L\left(1 - \frac{\chi_A}{\chi_B}\right) + \frac{\chi_A}{\chi_B}\theta_{\phi A}$ connects the points $\left(0, \left(1 - \frac{\chi_A}{\chi_B}\right)\right)$ and (θ_L, θ_L).

When would firm B sell nothing?

$$\theta\chi_B - p_B \leq 0 \;\forall\; \theta$$

$$\theta\chi_B \leq p_B \;\forall\; \theta$$

$$\theta \leq \frac{p_B}{\chi_B} \;\forall\; \theta$$

$$\theta_H \leq \frac{p_B}{\chi_B}$$

$$\theta_H \leq \theta_{\phi B} \tag{23}$$

q_B also equals zero if $\theta_{AB} \geq \theta_H$:

$$\frac{p_B - p_A}{\chi_B - \chi_A} \geq \theta_H$$

$$\frac{p_B - p_A}{\chi_B - \chi_A} \geq \theta_H$$

$$p_B \geq p_A + \theta_H(\chi_B - \chi_A)$$

$$\frac{p_B}{\chi_B} \geq \frac{\chi_A}{\chi_B}\frac{p_A}{\chi_A} + \theta_H\left(1 - \frac{\chi_A}{\chi_B}\right)$$

$$\theta_{\phi B} \geq \frac{\chi_A}{\chi_B}\theta_{\phi A} + \theta_H\left(1 - \frac{\chi_A}{\chi_B}\right) \tag{24}$$

As before, the line $\theta_{\phi A} = \frac{\chi_B}{\chi_A}\theta_{\phi B} - \theta_H\left(\frac{\chi_B}{\chi_A} - 1\right)$ connects the points $\left(0, \theta_H\left(1 - \frac{\chi_A}{\chi_B}\right)\right)$ and (θ_H, θ_H) (now thinking in terms of $(\theta_{\phi A}, \theta_{\phi B})$-space).

What about $\theta_L \leq \theta_{\phi B} \leq \theta_{\phi A} \leq \theta_H$? $\theta_{\phi B} \leq \theta_{\phi A} \Longrightarrow \theta_{AB} \leq \theta_{\phi B} \leq \theta_{\phi A}$.; then

$$q_A = \frac{1}{\theta_H - \theta_L}\left(\theta_H - \max(\theta_{\phi B}, \theta_L)\right) = \frac{\theta_H - \theta_{\phi B}}{\theta_H - \theta_L} \tag{25}$$

What about $\theta_L \leq \theta_{\phi B}$, $\theta_{\phi A} \leq \theta_{\phi B} \leq \frac{\chi_A}{\chi_B}\theta_{\phi A} + \theta_H\left(1 - \frac{\chi_A}{\chi_B}\right)$?

$\theta_{\phi A} \leq \theta_{\phi B} \Longrightarrow \theta_{\phi A} \leq \theta_{\phi B} \leq \theta_{AB}$; by its derivation, for $\theta_{\phi B} \leq \frac{\chi_A}{\chi_B}\theta_{\phi A} + \theta_H\left(1 - \frac{\chi_A}{\chi_B}\right)$, $\theta_{AB} \leq \theta_H$; hence

$$q_A = \frac{N}{\theta_H - \theta_L}\left(\theta_H - \theta_{AB}\right) \tag{26}$$

Final region: $\theta_L\left(1 - \frac{\chi_A}{\chi_B}\right) + \frac{\chi_A}{\chi_B}\theta_{\phi A} \leq \theta_{\phi B} \leq \theta_L \leq \frac{\chi_A}{\chi_B}\theta_{\phi A} + \theta_H\left(1 - \frac{\chi_A}{\chi_B}\right)$, $0 \leq \theta_{\phi A} \leq \theta_L$.

By its derivation, $\theta_L\left(1 - \frac{\chi_A}{\chi_B}\right) + \frac{\chi_A}{\chi_B}\theta_{\phi A} \leq \theta_{\phi B} \Longrightarrow \theta_{AB} \geq \theta_L$; so $\theta_{\phi B} \leq \theta_L \leq \theta_{AB}$.

Hence also

$$q_A = \frac{\theta_H - \theta_{AB}}{\theta_H - \theta_L}. \tag{27}$$

2 Firm B's best response function

Firm B's payoff must analyzed for two ranges of $\theta_{\phi A}$, $0 \leq \theta_{\phi A} \leq \theta_L$ and $\theta_L \leq \theta_{\phi A} \leq \theta_H$.

(B1) $0 \leq \theta_{\phi A} \leq \theta_L$. The quantity demanded of firm B is

$$q_B \begin{cases} 1 & 0 \leq \theta_{\phi B} \leq \frac{\chi_A}{\chi_B}\theta_{\phi A} + \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_L \\ \frac{\theta_H - \theta_{AB}}{\theta_H - \theta_L} & \frac{\chi_A}{\chi_B}\theta_{\phi A} + \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_L \leq \theta_{\phi B} \leq \frac{\chi_A}{\chi_B}\theta_{\phi A} + \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H \\ 0 & \frac{\chi_A}{\chi_B}\theta_{\phi A} + \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H \leq \theta_{\phi B} \leq \theta_H \end{cases} \tag{28}$$

(B1.1) $q_B = 1$, $\pi_B = p_B q_B = \chi_B \theta_{\phi B} q_B = \chi_B \theta_{\phi B}$. The local maximum of B's payoff function occurs when B makes $\theta_{\phi B}$ as large as possible consistent with being within the limits that define region B1.1, that is,

$$\theta^*_{\phi B} = \frac{\chi_A}{\chi_B}\theta_{\phi A} + \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_L. \tag{29}$$

(B1.2) $q_B = \frac{\theta_H - \theta_{AB}}{\theta_H - \theta_L}$, $\pi_B = p_B q_B = \chi_B \theta_{\phi B} \frac{\theta_H - \theta_{AB}}{\theta_H - \theta_L} = \frac{\chi_B}{\theta_H - \theta_L}\theta_{\phi B}\left(\theta_H - \frac{\chi_B \theta_{\phi B} - \chi_A \theta_{\phi A}}{\chi_B - \chi_A}\right)$.
The global maximum of the region B.1.2 payoff function occurs at

$$\theta_H - \frac{2\chi_B \theta_{\phi B} - \chi_A \theta_{\phi A}}{\chi_B - \chi_A} \equiv 0, \tag{30}$$

and for $\theta_{\phi B}$ defined by this first-order condition

$$\pi_B = \frac{\chi_B}{(\theta_H - \theta_L)(\chi_B - \chi_A)}\theta^2_{\phi B}. \tag{31}$$

Solve the first-order condition for

$$\theta^*_{\phi B} = \frac{1}{2}\left[\frac{\chi_A}{\chi_B}\theta_{\phi A} + \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H\right]. \tag{32}$$

$\theta^*_{\phi B}$ lies within region B.1.2 if

$$\frac{\chi_A}{\chi_B}\theta_{\phi A} + \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_L \leq \theta^*_{\phi B} \leq \frac{\chi_A}{\chi_B}\theta_{\phi A} + \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H. \tag{33}$$

If this inequality is satisfied, the global maximum of the B.1.2 payoff function lies within region B.1.2 and, because the B.1.2 payoff function is a parabola, $\theta^*_{\phi B}$ is firm B's best response price.

The right-hand inequality is always satisfied. The condition for the left-hand inequality to be satisfied is

$$\frac{\chi_A}{\chi_B}\theta_{\phi A} + \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_L \leq \frac{1}{2}\left[\frac{\chi_A}{\chi_B}\theta_{\phi A} + \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H\right] \tag{34}$$

$$2\frac{\chi_A}{\chi_B}\theta_{\phi A} + 2\left(1 - \frac{\chi_A}{\chi_B}\right)\theta_L \leq \frac{\chi_A}{\chi_B}\theta_{\phi A} + \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H$$

$$\frac{\chi_A}{\chi_B}\theta_{\phi A} + 2\left(1 - \frac{\chi_A}{\chi_B}\right)\theta_L \leq \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H$$

$$\frac{\chi_A}{\chi_B}\theta_{\phi A} \leq \left(1 - \frac{\chi_A}{\chi_B}\right)(\theta_H - 2\theta_L)$$

$$\theta_{\phi A} \leq \left(\frac{\chi_B}{\chi_A} - 1\right)(\theta_H - 2\theta_L) \tag{35}$$

Remark 1: if $\theta_H < 2\theta_L$, the right-hand side of (35) is negative and the global maximum of the B1.2 region payoff function lies in region B1.1; the local maximum of the B1.2 region payoff function in region B1.2 occurs at the boundary between region B1.1 and region B1.2, and this is firm B's best response price:

$$\theta_{\phi B}^{br}(\theta_{\phi A}) = \frac{\chi_A}{\chi_B}\theta_{\phi A} + \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_L. \tag{36}$$

Remark 2: we are considering the case $0 \leq \theta_{\phi A} \leq \theta_L$. If

$$\theta_L \leq \left(\frac{\chi_B}{\chi_A} - 1\right)(\theta_H - 2\theta_L), \tag{37}$$

then the left-hand inequality in (33) is always satisfied for case B1, the global maximum of the region B1.2 payoff function lies in region B1.2 and is, by the shape of the payoff function, firm B's best response price.

(37) can be rewritten

$$1 \leq \left(\frac{\chi_B}{\chi_A} - 1\right)\left(\frac{\theta_H}{\theta_L} - 2\right). \tag{38}$$

(38) is satisfied if χ_B/χ_A and θ_H/θ_L are both sufficiently large — if qualities are sufficiently different and preferences (marginal utilities of income) sufficiently diverse.

If

$$0 \leq \left(\frac{\chi_B}{\chi_A} - 1\right)\left(\frac{\theta_H}{\theta_L} - 2\right) \leq 1, \tag{39}$$

then (34) is satisfied for

$$0 \leq \theta_{\phi A} \leq \left(\frac{\chi_B}{\chi_A} - 1\right)(\theta_H - 2\theta_L), \tag{40}$$

and over this range firm B's best-response price is (32)

$$\theta_{\phi B}^{br}(\theta_{\phi A}) = \frac{1}{2}\left[\frac{\chi_A}{\chi_B}\theta_{\phi A} + \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H\right];$$

while for

$$\left(\frac{\chi_B}{\chi_A}-1\right)(\theta_H-2\theta_L)\le\theta_{\phi A}\le\theta_L, \tag{41}$$

(34) is violated and firm B's best response price is given by (36)

$$\theta_{\phi B}^{br}(\theta_{\phi A})=\frac{\chi_A}{\chi_B}\theta_{\phi A}+\left(1-\frac{\chi_A}{\chi_B}\right)\theta_L.$$

(B1.3) $q_B=0$, $\pi_B=0$; B's best response price never lies in this range.
For $\theta_L=\chi_A=1$, $\theta_H=\chi_B=5$,

$$\left(\frac{\chi_B}{\chi_A}-1\right)\left(\frac{\theta_H}{\theta_L}-2\right)=\left(\frac{5}{1}-1\right)\left(\frac{5}{1}-2\right)=12>1$$

For these parameter values$\theta_{\phi B}^{*}=\frac{1}{2}\left[\frac{\chi_A}{\chi_B}\theta_{\phi A}+\left(1-\frac{\chi_A}{\chi_B}\right)\theta_H\right]$

$$0\le\theta_{\phi A}\le 1\Longrightarrow\theta_{\phi B}^{br}(\theta_{\phi A})=\frac{1}{2}\left[\frac{1}{5}\theta_{\phi A}+\left(1-\frac{1}{5}\right)5\right]=2+\frac{1}{10}\theta_{\phi A}.$$

(B2) $\theta_L\le\theta_{\phi A}\le\theta_H$. The quantity demanded of firm B is

$$q_B\begin{cases}1 & 0\le\theta_{\phi B}\le\theta_L\\ \frac{\theta_H-\theta_{\phi A}}{\theta_H-\theta_L} & \theta_L\le\theta_{\phi B}\le\theta_{\phi A}\\ \frac{\theta_H-\theta_{AB}}{\theta_H-\theta_L} & \theta_{\phi A}\le\theta_{\phi B}\le\frac{\chi_A}{\chi_B}\theta_{\phi A}+\left(1-\frac{\chi_A}{\chi_B}\right)\theta_H\\ 0 & \frac{\chi_A}{\chi_B}\theta_{\phi A}+\left(1-\frac{\chi_A}{\chi_B}\right)\theta_H\le\theta_{\phi B}\le\theta_H\end{cases}. \tag{42}$$

(B2.1) $0\le\theta_{\phi B}\le\theta_L$

$q_B=1$, $\pi_B=p_Bq_B=\chi_B\theta_{\phi B}q_B=\chi_B\theta_{\phi B}$. The local maximum of B's payoff function occurs when B makes $\theta_{\phi B}$ as large as possible consistent with being within the limits that define region B2.1, that is,

$$\theta_{\phi B}^{*}=\theta_L. \tag{43}$$

For $\theta_L=\chi_A=1$, $\theta_H=\chi_B=5$, the local maximum payoff on region B2.1 is

$$\pi_B^{*}=\chi_B\theta_{\phi B}^{*}q_B=5(1)(1)=5.$$

(B2.2) $\theta_L\le\theta_{\phi B}\le\theta_{\phi A}$

$q_B = \frac{\theta_H - \theta_{\phi A}}{\theta_H - \theta_L}$, $\pi_B = p_B q_B = \chi_B \theta_{\phi B} \frac{\theta_H - \theta_{\phi B}}{\theta_H - \theta_L} = \frac{\chi_B}{\theta_H - \theta_L} \theta_{\phi B} \left(\theta_H - \theta_{\phi B}\right)$.

$$\pi_B = \frac{\chi_B}{\theta_H - \theta_L} \theta_{\phi B} \left(\theta_H - \theta_{\phi B}\right)$$

is a parabola with value 0 at $\theta_{\phi B} = 0, \theta_H$ and global maximum of the region B.2.2 payoff function occurs at

$$\theta^*_{\phi B} = \frac{1}{2}\theta_H, \tag{44}$$

and for $\theta_{\phi B}$ defined by this first-order condition firm B's payoff is

$$\pi_B = \frac{1}{4} \frac{\chi_B}{\theta_H - \theta_L} \theta_H^2. \tag{45}$$

(44) lies in the range that defines region B2.2 if

$$\theta_L \leq \frac{1}{2}\theta_H \leq \theta_{\phi A}. \tag{46}$$

If

$$\theta_H \leq 2\theta_L, \tag{47}$$

the left-hand inequality in (46) fails, the global maximum of the region B2.2 payoff function lies in region B2.1 and the local maximum of the region B2.2 payoff function on region B2.2 is the boundary between region B2.2 and region B2.1, $\theta^*_{\phi B} = \theta_L$, (43).

If $\theta_H \geq 2\theta_L$, the local maximum of the region B2.2 payoff function on region B2.2 is

$$\theta^*_{\phi B} = \begin{cases} \theta_L & \theta_L \leq \theta_{\phi A} \leq \frac{1}{2}\theta_H \\ \frac{1}{2}\theta_H & \frac{1}{2}\theta_H \leq \theta_{\phi A} \leq \theta_H \end{cases} \tag{48}$$

with corresponding payoff

$$\pi^*_B = \begin{cases} \chi_B \theta_L & \theta_L \leq \theta_{\phi A} \leq \frac{1}{2}\theta_H \\ \frac{1}{4}\frac{\chi_B}{\theta_H - \theta_L}\theta_H^2 & \frac{1}{2}\theta_H \leq \theta_{\phi A} \leq \theta_H \end{cases} \tag{49}$$

For $\theta_L = \chi_A = 1$, $\theta_H = \chi_B = 5$, local maxima and local maximum payoffs on region B2.2 are

$$\theta^*_{\phi B} = \begin{cases} 1 & 1 \leq \theta_{\phi A} \leq \frac{5}{2} \\ \frac{5}{2} & \frac{5}{2} \leq \theta_{\phi A} \leq 5 \end{cases}$$

$$\pi_B^* = \left\{ \begin{array}{cc} 5 & 1 \le \theta_{\phi A} \le \frac{5}{2} \\ \frac{1}{4}\frac{5}{5-4}(5)^2 = \frac{125}{4} & \frac{5}{2} \le \theta_{\phi A} \le 5 \end{array} \right. .$$

(B2.3) $\theta_{\phi A} \le \theta_{\phi B} \le \frac{\chi_A}{\chi_B}\theta_{\phi A} + \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H$
$q_B = \frac{\theta_H - \theta_{AB}}{\theta_H - \theta_L}$, $\pi_B = p_B q_B = \chi_B \theta_{\phi B} \frac{\theta_H - \theta_{AB}}{\theta_H - \theta_L} = \frac{\chi_B}{\theta_H - \theta_L}\theta_{\phi B}\left(\theta_H - \frac{\chi_B\theta_{\phi B} - \chi_A\theta_{\phi A}}{\chi_B - \chi_A}\right)$.
The global maximum of the region B.2.3 payoff function occurs at

$$\theta_H - \frac{2\chi_B\theta_{\phi B} - \chi_A\theta_{\phi A}}{\chi_B - \chi_A} \equiv 0, \qquad (50)$$

and for $\theta_{\phi B}$ defined by this first-order condition

$$\pi_B = \frac{\chi_B^2}{(\theta_H - \theta_L)(\chi_B - \chi_A)}\theta_{\phi B}^2. \qquad (51)$$

Solve the first-order condition for

$$\theta_{\phi B}^* = \frac{1}{2}\left[\frac{\chi_A}{\chi_B}\theta_{\phi A} + \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H\right]. \qquad (52)$$

$\theta_{\phi B}^*$ lies within region B.2.3 if

$$\theta_{\phi A} \le \frac{1}{2}\left[\frac{\chi_A}{\chi_B}\theta_{\phi A} + \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H\right] \le \frac{\chi_A}{\chi_B}\theta_{\phi A} + \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H \qquad (53)$$

$$2\theta_{\phi A} \le \frac{\chi_A}{\chi_B}\theta_{\phi A} + \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H \le \left[\frac{\chi_A}{\chi_B}\theta_{\phi A} + \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H\right]. \qquad (54)$$

The right-hand inequality is always satisfied. The left-hand inequality is satisfied for

$$2\theta_{\phi A} \le \frac{\chi_A}{\chi_B}\theta_{\phi A} + \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H$$

$$\left(2 - \frac{\chi_A}{\chi_B}\right)\theta_{\phi A} \le \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H$$

$$\theta_{\phi A} \le \frac{1 - \frac{\chi_A}{\chi_B}}{2 - \frac{\chi_A}{\chi_B}}\theta_H$$

We now consider the case $\theta_L \leq \theta_{\phi A} \leq \theta_H$. If

$$\frac{1 - \frac{\chi_A}{\chi_B}}{2 - \frac{\chi_A}{\chi_B}} \theta_H \leq \theta_L, \tag{55}$$

the left-hand inequality in (54) is never satisfied.

(55) can be rewritten

$$\frac{2 - \frac{\chi_A}{\chi_B} - 1}{2 - \frac{\chi_A}{\chi_B}} \leq \frac{\theta_L}{\theta_H}$$

$$1 - \frac{1}{2 - \frac{\chi_A}{\chi_B}} \leq \frac{\theta_L}{\theta_H}$$

$$1 - \frac{\theta_L}{\theta_H} \leq \frac{1}{2 - \frac{\chi_A}{\chi_B}} \tag{56}$$

$$\left(1 - \frac{\theta_L}{\theta_H}\right)\left(2 - \frac{\chi_A}{\chi_B}\right) \leq 1 \tag{57}$$

(Compare with (38) and (39)).

If θ_L/θ_H and χ_A/χ_B satisfy (57), the global maximum of the region B2.3 payoff function occurs in region B2.2 and the local maximum of the B2.3 payoff function on region B2.3 is the boundary between region B2.3 and region B2.2,

$$\theta^*_{\phi B} = \theta_{\phi A}. \tag{58}$$

If

$$1 \leq \left(1 - \frac{\theta_L}{\theta_H}\right)\left(2 - \frac{\chi_A}{\chi_B}\right), \tag{59}$$

then

$$\theta_L \leq \frac{1 - \frac{\chi_A}{\chi_B}}{2 - \frac{\chi_A}{\chi_B}} \theta_H \leq \theta_H. \tag{60}$$

Best response prices and payoffs are

$$\theta^*_{\phi B} = \begin{cases} \frac{1}{2}\left[\frac{\chi_A}{\chi_B}\theta_{\phi A} + \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H\right] & \theta_L \leq \theta_{\phi A} \leq \frac{1-\frac{\chi_A}{\chi_B}}{2-\frac{\chi_A}{\chi_B}}\theta_H \\ \theta_{\phi A} & \frac{1-\frac{\chi_A}{\chi_B}}{2-\frac{\chi_A}{\chi_B}}\theta_H \leq \theta_{\phi A} \leq \theta_H \end{cases} \tag{61}$$

and

$$\theta^*_{\phi B} = \frac{1}{2}\left[\frac{\chi_A}{\chi_B}\theta_{\phi A} + \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H\right] \qquad \frac{1}{4}\frac{\chi_B^2}{(\theta_H - \theta_L)(\chi_B - \chi_A)}\left[\frac{\chi_A}{\chi_B}\theta_{\phi A} + \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H\right]^2$$

$$\pi^*_B = \begin{cases} \frac{1}{4}\frac{\chi_B^2}{(\theta_H-\theta_L)(\chi_B-\chi_A)}\left[\frac{\chi_A}{\chi_B}\theta_{\phi A} + \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H\right]^2 & \theta_L \le \theta_{\phi A} \le \frac{1-\frac{\chi_A}{\chi_B}}{2-\frac{\chi_A}{\chi_B}}\theta_H \\ \frac{\chi_B}{\theta_H-\theta_L}\theta_{\phi A}\left(\theta_H - \frac{\chi_B\theta_{\phi A}-\chi_A\theta_{\phi A}}{\chi_B-\chi_A}\right) = \chi_B\theta_{\phi A}\left(\frac{\theta_H-\theta_{\phi A}}{\theta_H-\theta_L}\right) & \frac{1-\frac{\chi_A}{\chi_B}}{2-\frac{\chi_A}{\chi_B}}\theta_H \le \theta_{\phi A} \le \theta_H \end{cases} \tag{62}$$

respectively.

For $\theta_L = \chi_A = 1$, $\theta_H = \chi_B = 5$, the local maximum best response function and payoff on region B2.3 are

$$\theta^*_{\phi B} = \begin{cases} \frac{1}{2}\left[\frac{1}{5}\theta_{\phi A} + \left(1 - \frac{1}{5}\right)(5)\right] = 2 + \frac{1}{10}\theta_{\phi A} & 1 \le \theta_{\phi A} \le \frac{1-\frac{1}{5}}{2-\frac{1}{5}}(5) = \frac{20}{9} \\ \theta_{\phi A} & \frac{20}{9} \le \theta_{\phi A} \le 5 \end{cases}$$

$$\pi^*_B = \begin{cases} \frac{1}{4}\frac{25}{(4)(4)}\left(4 + \frac{1}{5}\theta_{\phi A}\right)^2 = \frac{25}{16}\left(2 + \frac{1}{10}\theta_{\phi A}\right)^2 & 1 \le \theta_{\phi A} \le \frac{20}{9} = 2\frac{2}{9} \\ \chi_B\theta_{\phi A}\left(\frac{\theta_H-\theta_{\phi A}}{\theta_H-\theta_L}\right) = \frac{5}{4}\theta_{\phi A}(5 - \theta_{\phi A}) & \frac{20}{9} \le \theta_{\phi A} \le 5 \end{cases}$$

(B2.4) $\frac{\chi_A}{\chi_B}\theta_{\phi A} + \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H \le \theta_{\phi B} \le \theta_H$
$q_B = 0$, $\pi_B = 0$; B's best response price never lies in this range.
Summarize results for the four ranges that make up case B2:
(B2.1) $0 \le \theta_{\phi B} \le 1$: $\theta^*_{\phi B} = \theta_L = 1$, $\pi^*_B = 5(1) = 5$.
(B2.2) $\theta_L \le \theta_{\phi B} \le \theta_{\phi A}$:

$$\theta^*_{\phi B} = \begin{cases} 1 & 1 \le \theta_{\phi A} \le \frac{5}{2} \\ \frac{5}{2} & \frac{5}{2} \le \theta_{\phi A} \le 5 \end{cases}$$

$$\pi^*_B = \begin{cases} 5 & 1 \le \theta_{\phi A} \le \frac{5}{2} \\ \frac{1}{4}\frac{5}{5-1}\left(\frac{5}{2}\right)^2 = \frac{125}{64} = 1\frac{61}{64} & \frac{5}{2} \le \theta_{\phi A} \le 5 \end{cases}.$$

(B2.3) $\theta_{\phi A} \le \theta_{\phi B} \le 4 + \frac{1}{5}\theta_{\phi A}$

$$\theta^*_{\phi B} = \begin{cases} 2 + \frac{1}{10}\theta_{\phi A} & 1 \le \theta_{\phi A} \le \frac{20}{9} \\ \theta_{\phi A} & \frac{20}{9} \le \theta_{\phi A} \le 5 \end{cases}$$

$$\pi_B^* = \begin{cases} \frac{1}{4}\frac{5}{(4)(4)}\left(4+\frac{1}{5}\theta_{\phi A}\right)^2 = \frac{25}{16}\left(2+\frac{1}{10}\theta_{\phi A}\right)^2 & \theta_L \leq \theta_{\phi A} \leq \frac{20}{9} = 2\frac{2}{9} \\ \chi_B \theta_{\phi A}\left(\frac{\theta_H - \theta_{\phi A}}{\theta_H - \theta_L}\right) = \frac{5}{4}\theta_{\phi A}\left(5-\theta_{\phi A}\right) & \frac{20}{9} \leq \theta_{\phi A} \leq 5 \end{cases}$$

(B2.4) $4+\frac{1}{5}\theta_{\phi A} \leq \theta_{\phi A} \leq 5$: $q_B = 0$, $\pi_B = 0$.

Comparing regions B2.2 and B2.3, we can assemble the table

	$\theta_{\phi B}^*$		$\pi_{\phi B}^*$	
	B2.2	B2.3	B2.2	B2.3
$1 \leq \theta_{\phi A} \leq 2\frac{2}{9}$	1	$2+\frac{1}{10}\theta_{\phi A}$	5	$\frac{25}{16}\left(2+\frac{1}{10}\theta_{\phi A}\right)^2$
$2\frac{2}{9} \leq \theta_{\phi A} \leq 2\frac{1}{2}$	1	$\theta_{\phi A}$	5	$\frac{5}{4}\theta_{\phi A}\left(5-\theta_{\phi A}\right)$
$2\frac{1}{2} \leq \theta_{\phi A} \leq 5$	$\frac{5}{2}$	$\theta_{\phi A}$	$\frac{125}{64}$	$\frac{5}{4}\theta_{\phi A}\left(5-\theta_{\phi A}\right)$

Compare payoffs at the two local maxima in each of these three regions.

$1 \leq \theta_{\phi A} \leq 2\frac{2}{9}$: B's payoff in region B2.2 at $\theta_{\phi B} = 1$ is 5. As $\theta_{\phi A}$ varies from θ_L to θ_H, B's payoff in region B2.3 increases from $\frac{25}{16}\left(2+\frac{1}{10}(1)\right)^2 = \frac{441}{64} = 6\frac{57}{64}$ to $\frac{25}{16}\left(2+\frac{1}{10}(5)\right)^2 = \frac{625}{64} = 9\frac{49}{64}$ and is always greater than B's payoff in region B2.2 at $\theta_{\phi B} = 1$. For $1 \leq \theta_{\phi A} \leq 2\frac{2}{9}$, B's best-response quality-adjusted price is $\theta_{\phi B}^{br}(\theta_{\phi A}) = 2+\frac{1}{10}\theta_{\phi A}$.

$2\frac{2}{9} \leq \theta_{\phi A} \leq 2\frac{1}{2}$: B's payoff in region B2.2 at $\theta_{\phi B} = 1$ is 5.

$$5 \geq \frac{5}{4}\theta_{\phi A}\left(5-\theta_{\phi A}\right)$$

$$4 \geq 5\theta_{\phi A} - \theta_{\phi A}^2$$

$$\theta_{\phi A}^2 - 5\theta_{\phi A} + 4 \geq 0$$

$$\left(\theta_{\phi A} - 1\right)\left(\theta_{\phi A} - 4\right) \geq 0$$

This fails for $1 \leq \theta_{\phi A} < 4$, is satisfied for $4 \leq \theta_{\phi A} \leq 5$. Since we consider the range $2\frac{2}{9} \leq \theta_{\phi A} \leq 2\frac{1}{2}$, the global maximum of B's payoff is $\frac{5}{4}\theta_{\phi A}\left(5-\theta_{\phi A}\right) = \frac{5}{4}\theta_{\phi A}\left(5-\theta_{\phi A}\right)$ for the best-response quality-adjusted price $\theta_{\phi B}^{br}(\theta_{\phi A}) = \theta_{\phi A}$.

$2\frac{1}{2} \leq \theta_{\phi A} \leq 5$: B's payoff in region B2.2 at $\theta_{\phi B} = 2.5$ is $125/64$. B's payoff in region B2.3 at $\theta_{\phi B} = \theta_{\phi A}$ is $\frac{5}{4}\theta_{\phi A}\left(5-\theta_{\phi A}\right)$. This takes its maximum value $\frac{5}{4}\frac{5}{2}\left(5-\frac{5}{2}\right) = \frac{125}{16}$ for $\theta_{\phi A} = 5/2$ and falls to 0 for $\theta_{\phi A} = 5$. The global maximum of B's payoff is at $\theta_{\phi B}^{br}(\theta_{\phi A}) = 2.5$.

Firm B's best-response function is

$$\theta_{\phi B}^{br}(\theta_{\phi A}) = \begin{cases} 2+\frac{1}{10}\theta_{\phi A} & 0 \leq \theta_{\phi A} \leq 1 \\ 2+\frac{1}{10}\theta_{\phi A} & 1 \leq \theta_{\phi A} \leq \frac{20}{9} \\ \theta_{\phi A} & \frac{20}{9} \leq \theta_{\phi A} \leq \frac{5}{2} \\ \frac{1}{2}\theta_H & \frac{5}{2} \leq \theta_{\phi A} \leq 5 \end{cases}$$

3 Firm A's best response function

Firm A's payoff must be analyzed over the following ranges:

(A1) $0 \leq \theta_{\phi B} \leq \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_L$

(A2) $\left(1 - \frac{\chi_A}{\chi_B}\right)\theta_L \leq \theta_{\phi B} \leq \theta_L$

(A3) $\theta_L \leq \theta_{\phi B} \leq \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H$

(A4) $\left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H \leq \theta_{\phi B} \leq \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H + \frac{\chi_A}{\chi_B}\theta_L$

(A5) $\left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H + \frac{\chi_A}{\chi_B}\theta_L \leq \theta_{\phi B} \leq \theta_H$

(A1) $0 \leq \theta_{\phi B} \leq \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_L$

$q_A = 0$, $\pi_A = 0$. Firm A's payoff is 0 no matter what price it sets.

(A2) $\left(1 - \frac{\chi_A}{\chi_B}\right)\theta_L \leq \theta_{\phi B} \leq \theta_L$

(A2.1) $0 \leq \theta_{\phi A} \leq \frac{\chi_B}{\chi_A}\theta_{\phi B} - \left(\frac{\chi_B}{\chi_A} - 1\right)\theta_L$.

The right-hand boundary is the line along which $\theta_{AB} = \theta_L$.

$q_A = \frac{\theta_{AB} - \theta_L}{\theta_H - \theta_L}$, $\pi_A = \chi_A \theta_{\phi A} \frac{\theta_{AB} - \theta_L}{\theta_H - \theta_L}$.

$$\pi_A = \frac{\chi_A}{\theta_H - \theta_L}\theta_{\phi A}\left(\frac{\chi_B\theta_{\phi B} - \chi_A\theta_{\phi A}}{\chi_B - \chi_A} - \theta_L\right). \tag{63}$$

The first-order condition is

$$\frac{\chi_B\theta_{\phi B} - 2\chi_A\theta_{\phi A}}{\chi_B - \chi_A} - \theta_L \equiv 0, \tag{64}$$

from which firm A's payoff along the first-order condition is

$$\pi_A^* = \frac{(\chi_A\theta_{\phi A}^*)^2}{(\theta_H - \theta_L)(\chi_B - \chi_A)}. \tag{65}$$

Solve (64) for

$$\chi_B\theta_{\phi B} - 2\chi_A\theta_{\phi A} = (\chi_B - \chi_A)\theta_L$$

$$2\chi_A\theta_{\phi A} = \chi_B\theta_{\phi B} - (\chi_B - \chi_A)\theta_L$$

$$\theta_{\phi A}^* = \frac{1}{2}\left[\frac{\chi_B}{\chi_A}\theta_{\phi B} - \left(\frac{\chi_B}{\chi_A} - 1\right)\theta_L\right]. \tag{66}$$

$\theta^*_{\phi A}$ given by (66) lies in region A2.1 if

$$0 \leq \frac{1}{2}\left[\frac{\chi_B}{\chi_A}\theta_{\phi B} - \left(\frac{\chi_B}{\chi_A} - 1\right)\theta_L\right] \leq \frac{\chi_B}{\chi_A}\theta_{\phi B} - \left(\frac{\chi_B}{\chi_A} - 1\right)\theta_L. \tag{67}$$

Both inequalities in (67) are satisfied for

$$\frac{\chi_B}{\chi_A}\theta_{\phi B} - \left(\frac{\chi_B}{\chi_A} - 1\right)\theta_L \geq 0$$

$$\theta_{\phi B} \geq \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_L, \tag{68}$$

a condition which is met by the definition of region A2.

Hence the global maximum of firm A's region A2.1 payoff function lies in region A2.1, and the local maximum of firm A's payoff function in region A2.1 is given by (66).

(A2.2) $\frac{\chi_B}{\chi_A}\theta_{\phi B} - \left(\frac{\chi_B}{\chi_A} - 1\right)\theta_L \leq \theta_{\phi A} \leq \theta_H$: $q_A = 0$, $\pi_A = 0$.

Firm A's payoff is 0 no matter what price it sets. Hence firm A's best-response quality-adjusted price in region A2 is given by (66):

$$\theta^{br}_{\phi A}(\theta_{\phi B}) = \frac{1}{2}\left[\frac{\chi_B}{\chi_A}\theta_{\phi B} - \left(\frac{\chi_B}{\chi_A} - 1\right)\theta_L\right] \text{ for } \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_L \leq \theta_{\phi B} \leq \theta_L. \tag{69}$$

For $\theta_L = \chi_A = 1$, $\theta_H = \chi_B = 5$, (69) is

$$\theta^{br}_{\phi A}(\theta_{\phi B}) = \frac{1}{2}\left[\frac{5}{1}\theta_{\phi B} - \left(\frac{5}{1} - 1\right)(1)\right] \text{ for } (1)\left(1 - \frac{1}{5}\right) \leq \theta_{\phi B} \leq 1.$$

$$\theta^{br}_{\phi A}(\theta_{\phi B}) = \frac{5}{2}\theta_{\phi B} - 2 \text{ for } \frac{4}{5} \leq \theta_{\phi B} \leq 1.$$

This is the equation of the straight line connecting $(0, \frac{4}{5})$ and $(\frac{1}{2}, 1)$ in $(\theta_{\phi A}, \theta_{\phi B})$-space.

(A3) $\theta_L \leq \theta_{\phi B} \leq \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H$

(A3.1) $0 \leq \theta_{\phi A} \leq \theta_L$: $q_A = \frac{\theta_{AB} - \theta_L}{\theta_H - \theta_L}$, $\pi_A = \chi_A\theta_{\phi A}\frac{\theta_{AB} - \theta_L}{\theta_H - \theta_L}$.

$$\pi_A = \chi_A\theta_{\phi A}\frac{\theta_{AB} - \theta_L}{\theta_H - \theta_L} = \frac{\chi_A}{\theta_H - \theta_L}\theta_{\phi A}\left(\frac{\chi_B\theta_{\phi B} - \chi_A\theta_{\phi A}}{\chi_B - \chi_A} - \theta_L\right). \tag{70}$$

From the discussion of case A2.1, the global maximum of firm A's region A3.1 payoff function occurs at (66)

$$\theta_{\phi A}^{*} = \frac{1}{2}\left[\frac{\chi_B}{\chi_A}\theta_{\phi B} - \left(\frac{\chi_B}{\chi_A} - 1\right)\theta_L\right].$$

This lies within region A3.1 for

$$0 \leq \frac{1}{2}\left[\frac{\chi_B}{\chi_A}\theta_{\phi B} - \left(\frac{\chi_B}{\chi_A} - 1\right)\theta_L\right] \leq \theta_L. \tag{71}$$

The condition for the left-hand inequality to be satisfied is

$$\frac{\chi_B}{\chi_A}\theta_{\phi B} - \left(\frac{\chi_B}{\chi_A} - 1\right)\theta_L \geq 0$$

$$\frac{\chi_B}{\chi_A}\theta_{\phi B} \geq \left(\frac{\chi_B}{\chi_A} - 1\right)\theta_L$$

$$\theta_{\phi B} \geq \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_L, \tag{72}$$

which is met by definition of region A3.

The condition for the right-hand inequality to be satisfied is

$$\frac{\chi_B}{\chi_A}\theta_{\phi B} - \left(\frac{\chi_B}{\chi_A} - 1\right)\theta_L \leq 2\theta_L$$

$$\frac{\chi_B}{\chi_A}\theta_{\phi B} \leq \left(\frac{\chi_B}{\chi_A} - 1\right)\theta_L + 2\theta_L$$

$$\frac{\chi_B}{\chi_A}\theta_{\phi B} \leq \left(\frac{\chi_B}{\chi_A} + 1\right)\theta_L$$

$$\theta_{\phi B} \leq \left(1 + \frac{\chi_A}{\chi_B}\right)\theta_L. \tag{73}$$

Case A3 is defined for values of $\theta_{\phi B}$ in the range

$$\theta_L \leq \theta_{\phi B} \leq \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H.$$

If

$$\left(1-\frac{\chi_A}{\chi_B}\right)\theta_H \leq \left(1+\frac{\chi_A}{\chi_B}\right)\theta_L, \tag{74}$$

then (73) is satisfied for all of case A3.1, the global maximum of the region A3.1 profit function occurs in region A3.1, and is given by (66).

(74) can be rewritten

$$\frac{1-\frac{\chi_A}{\chi_B}}{1+\frac{\chi_A}{\chi_B}} \leq \frac{\theta_L}{\theta_H}$$

$$\frac{1+\frac{\chi_A}{\chi_B}-2\frac{\chi_A}{\chi_B}}{1+\frac{\chi_A}{\chi_B}} \leq \frac{\theta_L}{\theta_H}$$

$$1-\frac{2\frac{\chi_A}{\chi_B}}{1+\frac{\chi_A}{\chi_B}} \leq \frac{\theta_L}{\theta_H}$$

$$1-\frac{\theta_L}{\theta_H} \leq \frac{2}{1+\frac{\chi_B}{\chi_A}}$$

$$\left(1-\frac{\theta_L}{\theta_H}\right)\left(1+\frac{\chi_B}{\chi_A}\right) \leq 2. \tag{75}$$

This will be satisfied if θ_L/θ_H is sufficiently large (marginal utilities of quality slightly dispersed) and χ_B/χ_A sufficiently small (qualities relatively close together).

If (75) is not met, then

$$\theta_L \leq \left(1+\frac{\chi_A}{\chi_B}\right)\theta_L \leq \left(1-\frac{\chi_A}{\chi_B}\right)\theta_H \tag{76}$$

For $\theta_L \leq \theta_{\phi B} \leq \left(1+\frac{\chi_A}{\chi_B}\right)\theta_L$, the right-hand inequality in (71) is satisfied, the global maximum of the region A3.1 profit function occurs in region A3.1, and is given by (66). For $\left(1+\frac{\chi_A}{\chi_B}\right)\theta_L \leq \theta_{\phi B} \leq \left(1-\frac{\chi_A}{\chi_B}\right)\theta_H$, the right-hand inequality in (71) is violated and the global maximum of the region A3.1 profit lies to the right of region A3.1. The local maximum of the region

A3.1 profit function in region A3.1 is the boundary between region A3.1 and A3.2, the line $\theta_{\phi A} = \theta_L$.

The local maximum of the region A3.1 profit function in region A3.1 occurs at

$$\theta^*_{\phi A} = \begin{cases} \frac{1}{2}\left[\frac{\chi_B}{\chi_A}\theta_{\phi B} - \left(\frac{\chi_B}{\chi_A} - 1\right)\theta_L\right] & \theta_L \le \theta_{\phi B} \le \left(1 + \frac{\chi_A}{\chi_B}\right)\theta_L \\ \theta_L & \left(1 + \frac{\chi_A}{\chi_B}\right)\theta_L \le \theta_{\phi B} \le \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H \end{cases}. \tag{77}$$

If $\theta_{\phi B} = \left(1 + \frac{\chi_A}{\chi_B}\right)\theta_L$, the value of $\theta^*_{\phi A}$ along the first segment is

$$\frac{1}{2}\left(\frac{\chi_B}{\chi_A}\left(1 + \frac{\chi_A}{\chi_B}\right)\theta_L - \left(\frac{\chi_B}{\chi_A} - 1\right)\theta_L\right) = \theta_L.$$

Payoffs are

$$\pi^*_{\phi A} = \begin{cases} \frac{1}{4}\frac{1}{(\theta_H-\theta_L)(\chi_B-\chi_A)}\left[\chi_B\theta_{\phi B} - (\chi_B - \chi_A)\theta_L\right]^2 & \theta_L \le \theta_{\phi B} \le \left(1 + \frac{\chi_A}{\chi_B}\right)\theta_L \\ \left(\frac{\chi_A\chi_B}{\chi_B-\chi_A}\right)\left(\frac{\theta_L}{\theta_H-\theta_L}\right)(\theta_{\phi B} - \theta_L) & \left(1 + \frac{\chi_A}{\chi_B}\right)\theta_L \le \theta_{\phi B} \le \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H \end{cases} \tag{78}$$

For $\theta_L = \chi_A = 1$, $\theta_H = \chi_B = 5$, (77) is

$$\theta^*_{\phi A} = \begin{cases} \frac{1}{2}\left[\frac{5}{1}\theta_{\phi B} - \left(\frac{5}{1} - 1\right)(1)\right] = \frac{5}{2}\theta_{\phi B} - 2 & 1 \le \theta_{\phi B} \le \frac{6}{5} \\ 1 & \frac{6}{5} \le \theta_{\phi B} \le 4 \end{cases}.$$

The first segment connects the points $(\frac{1}{2}, 1)$ and $(1, \frac{6}{5})$ in $(\theta_{\phi A}, \theta_{\phi B})$-space. The second segment connects the points $(1, \frac{6}{5})$ and $(1, 4)$ in $(\theta_{\phi A}, \theta_{\phi B})$-space.

Payoffs are

$$\pi^*_{\phi A} = \begin{cases} \frac{1}{4}\frac{1}{(\theta_H-\theta_L)(\chi_B-\chi_A)}\left[\chi_B\theta_{\phi B} - (\chi_B - \chi_A)\theta_L\right]^2 & 1 \le \theta_{\phi B} \le \frac{6}{5} \\ \frac{5}{16}(\theta_{\phi B} - 1) & \frac{6}{5} \le \theta_{\phi B} \le 4 \end{cases}$$

(A3.2) $\theta_L \le \theta_{\phi A} \le \theta_{\phi B}$: $q_A = \frac{\theta_{AB}-\theta_{\phi A}}{\theta_H-\theta_L}$, $\pi_A = \chi_A\theta_{\phi A}\frac{\theta_{AB}-\theta_{\phi A}}{\theta_H-\theta_L}$
Rewrite the expression for $\theta_{AB} - \theta_{\phi A}$ as

$$\theta_{AB} - \theta_{\phi A} = \frac{\chi_B\theta_{\phi B} - \chi_A\theta_{\phi A}}{\chi_B - \chi_A} - \theta_{\phi A} = \frac{\chi_B}{\chi_B - \chi_A}(\theta_{\phi B} - \theta_{\phi A}) \tag{79}$$

$$\pi_A = \frac{\chi_A\chi_B}{(\theta_H - \theta_L)(\chi_B - \chi_A)}\theta_{\phi A}(\theta_{\phi B} - \theta_{\phi A}). \tag{80}$$

The first-order condition to maximize (80) is

$$\theta_{\phi B} - 2\theta_{\phi A} \equiv 0, \tag{81}$$

from which firm A's payoff on the first-order condition is

$$\pi_A^* = \frac{\chi_A \chi_B}{(\theta_H - \theta_L)(\chi_B - \chi_A)} \left(\theta_{\phi A}^*\right)^2. \tag{82}$$

Solve (81) for

$$\theta_{\phi A}^* = \frac{1}{2}\theta_{\phi B}. \tag{83}$$

$\theta_{\phi A}^* = \frac{1}{2}\theta_{\phi B}$ lies within region A3.2 for

$$\theta_L \leq \frac{1}{2}\theta_{\phi B} \leq \theta_{\phi B} \tag{84}$$

The right-hand inequality is always satisfied. The left-hand inequality is satisfied for

$$2\theta_L \leq \theta_{\phi B}. \tag{85}$$

Case A3 is defined by the inequalities $\theta_L \leq \theta_{\phi B} \leq \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H$. If

$$\left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H \leq 2\theta_L, \tag{86}$$

then (85) is violated for all $\theta_{\phi B}$ in Case A3.2, the global maximum of the region A3.2 payoff function lies in region A3.1, and the local maximum of the A3.2 payoff function on region A3.2 is the boundary between regions A3.1 and A3.2, $\theta_{\phi A} = \theta_L$.

(86) can be rewritten

$$1 - \frac{\chi_A}{\chi_B} \leq 2\frac{\theta_L}{\theta_H}$$

$$1 \leq 2\frac{\theta_L}{\theta_H} + \frac{\chi_A}{\chi_B}. \tag{87}$$

If (87) is violated, then $\theta_L \leq 2\theta_L \leq \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H$ and

$$\theta_{\phi A}^* = \begin{cases} \theta_L & \theta_L \leq \theta_{\phi B} \leq 2\theta_L \\ \frac{1}{2}\theta_{\phi B} & 2\theta_L \leq \theta_{\phi B} \leq \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H \end{cases} \tag{88}$$

For $\theta_L = \chi_A = 1$, $\theta_H = \chi_B = 5$, (88) is

$$\theta^*_{\phi A} = \begin{cases} 1 & 1 \leq \theta_{\phi B} \leq 2 \\ \frac{1}{2}\theta_{\phi B} & 2 \leq \theta_{\phi B} \leq 4 \end{cases}.$$

The first segment is a line connecting $(1, 1)$ and $(1, 2)$ in $(\theta_{\phi A}, \theta_{\phi B})$-space. The second segment is a line connecting $(1, 2)$ and $(2, 4)$ in $(\theta_{\phi A}, \theta_{\phi B})$-space.

(A3.3) $\theta_{\phi B} \leq \theta_{\phi A} \leq \theta_H$: $q_A = 0$, $\pi_A = 0$.

Collect results for Case A3:

A3.1:

$$\theta^*_{\phi A} = \begin{cases} \frac{5}{2}\theta_{\phi B} - 2 & 1 \leq \theta_{\phi B} \leq \frac{6}{5} \\ 1 & \frac{6}{5} \leq \theta_{\phi B} \leq 4 \end{cases}.$$

with payoffs

$$\pi^*_{\phi A} = \begin{cases} \frac{(1)^2}{(5-1)(5-1)}\left(\frac{5}{2}\theta_{\phi B} - 2\right)^2 = \frac{1}{16}\left(\frac{5}{2}\theta_{\phi B} - 2\right)^2 & 1 \leq \theta_{\phi B} \leq \frac{6}{5} \\ \frac{1}{5-1}(1)\left(\frac{5\theta_{\phi B}-(1)(1)}{5-1} - 1\right) = \frac{5}{16}\left(\theta_{\phi B} - 1\right) & \frac{6}{5} \leq \theta_{\phi B} \leq 4 \end{cases}.$$

(A3.2)

$$\theta^*_{\phi A} = \begin{cases} 1 & 1 \leq \theta_{\phi B} \leq 2 \\ \frac{1}{2}\theta_{\phi B} & 2 \leq \theta_{\phi B} \leq 4 \end{cases}.$$

with payoffs

$$\pi^*_{\phi A} = \begin{cases} \frac{1}{5-1}(1)\left(\frac{5\theta_{\phi B}-(1)(1)}{5-1} - 1\right) = \frac{5}{16}\left(\theta_{\phi B} - 1\right) & 1 \leq \theta_{\phi B} \leq 2 \\ \frac{(1)(5)}{(5-1)(5-1)}\left(\frac{1}{2}\theta_{\phi B}\right)^2 = \frac{5}{64}\theta^2_{\phi B} & 2 \leq \theta_{\phi B} \leq 4 \end{cases}$$

$$\frac{(1)(5)}{(5-1)(5-1)}\left(\frac{1}{2}\theta_{\phi B}\right)^2$$

- $1 \leq \theta_{\phi B} \leq \frac{6}{5}$: compare payoffs at the two local maxima, $\frac{1}{16}\left(\frac{5}{2}\theta_{\phi B} - 2\right)^2$ versus $\frac{5}{16}\left(\theta_{\phi B} - 1\right)$

$$\frac{1}{16}\left(\frac{5}{2}\theta_{\phi B} - 2\right)^2 - \frac{5}{16}\left(\theta_{\phi B} - 1\right) =$$

$$\frac{25}{64}\left(\frac{6}{5} - \theta_{\phi B}\right)^2 \geq 0.$$

Hence setting $\theta_{\phi A} = \frac{5}{2}\theta_{\phi B} - 2$ gives firm A the greatest payoff for $1 \leq \theta_{\phi B} \leq \frac{6}{5}$;

$$\theta^{br}_{\phi A}(\theta_{\phi B}) = \frac{5}{2}\theta_{\phi B} - 2 \text{ for } 1 \leq \theta_{\phi B} \leq \frac{6}{5}.$$

This is a straight line connecting $(\frac{1}{2}, 1)$ and $(1, \frac{6}{5})$ in $(\theta_{\phi A}, \theta_{\phi B})$-space.

- $\frac{6}{5} \leq \theta_{\phi B} \leq 2$: $\theta^{br}_{\phi A}(\theta_{\phi B}) = 1$.

This is a straight line connecting $(1, \frac{6}{5})$ and $(1, 2)$ in $(\theta_{\phi A}, \theta_{\phi B})$-space.

- $2 \leq \theta_{\phi B} \leq 4$: compare payoffs at the two local maxima, $\frac{5}{16}(\theta_{\phi B} - 1)$ versus $\frac{5}{64}\theta^2_{\phi B}$.

$$\frac{5}{64}\theta^2_{\phi B} - \frac{5}{16}(\theta_{\phi B} - 1) = \frac{5}{64}(\theta_{\phi B} - 2)^2 > 0.$$

Hence

$$\theta^{br}_{\phi A}(\theta_{\phi B}) = \frac{1}{2}\theta_{\phi B} \text{ for } 2 \leq \theta_{\phi B} \leq 4.$$

This is a straight line connecting $(1, 2)$ and $(2, 4)$ in $(\theta_{\phi A}, \theta_{\phi B})$-space.

Firm A's best response function in region A3 is

$$\theta^{br}_{\phi A}(\theta_{\phi B}) = \begin{cases} \frac{5}{2}\theta_{\phi B} - 2 & 1 \leq \theta_{\phi B} \leq \frac{6}{5} \\ 1 & \frac{6}{5} \leq \theta_{\phi B} \leq 2 \\ \frac{1}{2}\theta_{\phi B} & 2 \leq \theta_{\phi B} \leq 4 \end{cases}$$

(A4) $\left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H \leq \theta_{\phi B} \leq \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H + \frac{\chi_A}{\chi_B}\theta_L$

(A4.1) $0 \leq \theta_{\phi A} \leq \frac{\chi_B}{\chi_A}\theta_{\phi B} - \left(\frac{\chi_B}{\chi_A} - 1\right)\theta_H$: $q_A = 1$, $\pi_A = \chi_A\theta_{\phi A}$

The right boundary is the line along which $\theta_{AB} = \theta_H$.

A maximizes its profit in this region by making $\theta_{\phi A}$ as large as possible consistent with remaining in the region,

$$\theta^*_{\phi A} = \frac{\chi_B}{\chi_A}\theta_{\phi B} - \left(\frac{\chi_B}{\chi_A} - 1\right)\theta_H, \tag{89}$$

and A's payoff is

$$\pi^*_A = \chi_A\left[\frac{\chi_B}{\chi_A}\theta_{\phi B} - \left(\frac{\chi_B}{\chi_A} - 1\right)\theta_H\right] = \chi_B\theta_{\phi B} - \theta_H(\chi_B - \chi_A). \tag{90}$$

For $\theta_L = \chi_A = 1$, $\theta_H = \chi_B = 5$,

$$\theta^*_{\phi A} = 5\left(\theta_{\phi B} - 4\right), \; \pi^*_A = 5\left(\theta_{\phi B} - 4\right).$$

(A4.2) $\frac{\chi_B}{\chi_A}\theta_{\phi B} - \left(\frac{\chi_B}{\chi_A} - 1\right)\theta_H \leq \theta_{\phi A} \leq \theta_L$: $q_A = \frac{\theta_{AB}-\theta_L}{\theta_H-\theta_L}$, $\pi_A = \chi_A\theta_{\phi A}\frac{\theta_{AB}-\theta_L}{\theta_H-\theta_L}$

$$\pi_A = \frac{\chi_A}{\theta_H - \theta_L}\theta_{\phi A}\left(\frac{\chi_B\theta_{\phi B} - \chi_A\theta_{\phi A}}{\chi_B - \chi_A} - \theta_L\right). \tag{91}$$

The first-order condition to maximize (91) is

$$\frac{\chi_B\theta_{\phi B} - 2\chi_A\theta_{\phi A}}{\chi_B - \chi_A} - \theta_L \equiv 0, \tag{92}$$

from which firm A's profit along the first-order condition is

$$\pi^*_A = \frac{(\chi_A\theta^*_{\phi A})^2}{(\theta_H - \theta_L)(\chi_B - \chi_A)}. \tag{93}$$

Solve (92) for

$$\theta^*_{\phi A} = \frac{1}{2}\left[\frac{\chi_B}{\chi_A}\theta_{\phi B} - \left(\frac{\chi_B}{\chi_A} - 1\right)\theta_L\right]. \tag{94}$$

$\theta^*_{\phi A}$ defined by (94) lies within region A4.2 for

$$\frac{\chi_B}{\chi_A}\theta_{\phi B} - \left(\frac{\chi_B}{\chi_A} - 1\right)\theta_H \leq \frac{1}{2}\left[\frac{\chi_B}{\chi_A}\theta_{\phi B} - \left(\frac{\chi_B}{\chi_A} - 1\right)\theta_L\right] \leq \theta_L. \tag{95}$$

The left-hand inequality is satisfied for

$$2\frac{\chi_B}{\chi_A}\theta_{\phi B} - 2\left(\frac{\chi_B}{\chi_A} - 1\right)\theta_H \leq \frac{\chi_B}{\chi_A}\theta_{\phi B} - \left(\frac{\chi_B}{\chi_A} - 1\right)\theta_L$$

$$\frac{\chi_B}{\chi_A}\theta_{\phi B} \leq \left(\frac{\chi_B}{\chi_A} - 1\right)(2\theta_H - \theta_L).$$

$$\theta_{\phi B} \leq \left(1 - \frac{\chi_A}{\chi_B}\right)(2\theta_H - \theta_L). \tag{96}$$

Case A4 is defined by the inequalities

$$\left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H \leq \theta_{\phi B} \leq \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H + \frac{\chi_A}{\chi_B}\theta_L. \tag{97}$$

If

$$\left(1-\frac{\chi_A}{\chi_B}\right)\theta_H+\frac{\chi_A}{\chi_B}\theta_L\leq\left(1-\frac{\chi_A}{\chi_B}\right)\left(2\theta_H-\theta_L\right), \tag{98}$$

$$\frac{\chi_A}{\chi_B}\theta_L\leq\left(1-\frac{\chi_A}{\chi_B}\right)\left(\theta_H-\theta_L\right)$$

$$1\leq\left(\frac{\chi_B}{\chi_A}-1\right)\left(\frac{\theta_H}{\theta_L}-1\right), \tag{99}$$

then (96) and the left-hand inequality in (95) are satisfied for all $\theta_{\phi B}$ in case A4; the global maximum of the region A4.2 payoff function does not lie in region A4.1.

If (99) is not satisfied, then

$$\left(1-\frac{\chi_A}{\chi_B}\right)\left(2\theta_H-\theta_L\right)\leq\left(1-\frac{\chi_A}{\chi_B}\right)\theta_H+\frac{\chi_A}{\chi_B}\theta_L$$

and for

$$\left(2\theta_H-\theta_L\right)\left(1-\frac{\chi_A}{\chi_B}\right)\leq\theta_{\phi B}\leq\left(1-\frac{\chi_A}{\chi_B}\right)\theta_H+\frac{\chi_A}{\chi_B}\theta_L \tag{100}$$

the left-hand inequality in (95) is violated; the global maximum of the region A4.2 payoff function lies in region A4.1, and the local maximum of the region A4.2 payoff function on region A4.2 is at the boundary between region A4.1 and A4.2.

The right-hand inequality in (95) is satisfied for

$$\frac{\chi_B}{\chi_A}\theta_{\phi B}-\left(\frac{\chi_B}{\chi_A}-1\right)\theta_L\leq 2\theta_L$$

$$\frac{\chi_B}{\chi_A}\theta_{\phi B}\leq 2\theta_L+\left(\frac{\chi_B}{\chi_A}-1\right)\theta_L$$

$$\frac{\chi_B}{\chi_A}\theta_{\phi B}\leq\left(\frac{\chi_B}{\chi_A}+1\right)\theta_L$$

$$\theta_{\phi B}\leq\left(1+\frac{\chi_A}{\chi_B}\right)\theta_L \tag{101}$$

From (97), if

$$\left(1-\frac{\chi_A}{\chi_B}\right)\theta_H+\frac{\chi_A}{\chi_B}\theta_L\le\left(1+\frac{\chi_A}{\chi_B}\right)\theta_L, \tag{102}$$

then (101) is satisfied for all $\theta_{\phi B}$ in case A4 and the global maximum of the region A4.2 payoff function does not lie to the right of region A4.2.

Rewrite (102) as

$$\left(1-\frac{\chi_A}{\chi_B}\right)\theta_H\le\theta_L$$

$$1\le\frac{\theta_L}{\theta_H}+\frac{\chi_A}{\chi_B}. \tag{103}$$

If

$$1\le\min\left[\left(\frac{\chi_B}{\chi_A}-1\right)\left(\frac{\theta_H}{\theta_L}-1\right),\frac{\theta_L}{\theta_H}+\frac{\chi_A}{\chi_B}\right] \tag{104}$$

then (99) and (103) are both satisfied and the global maximum of the region A4.2 payoff function lies within region A4.2 for all $\theta_{\phi B}$ in region A4. If (104),

$$\theta^*_{\phi A}=\frac{1}{2}\left[\frac{\chi_B}{\chi_A}\theta_{\phi B}-\left(\frac{\chi_B}{\chi_A}-1\right)\theta_L\right]\text{ for }\left(1-\frac{\chi_A}{\chi_B}\right)\theta_H\le\theta_{\phi B}\le\left(1-\frac{\chi_A}{\chi_B}\right)\theta_H+\frac{\chi_A}{\chi_B}\theta_L.$$

If

$$\left(\frac{\chi_B}{\chi_A}-1\right)\left(\frac{\theta_H}{\theta_L}-1\right)\le1\le\frac{\theta_L}{\theta_H}+\frac{\chi_A}{\chi_B}, \tag{105}$$

then the right-hand inequality is satisfied for all $\theta_{\phi B}$ in region A4, the left-hand inequality is not satisfied for some $\theta_{\phi B}$ or all in region A4.

Noting that

$$\left(1-\frac{\chi_A}{\chi_B}\right)\theta_H\le\left(1-\frac{\chi_A}{\chi_B}\right)(2\theta_H-\theta_L), \tag{106}$$

we have for this case

$$\left(1-\frac{\chi_A}{\chi_B}\right)\theta_H\le\left(1-\frac{\chi_A}{\chi_B}\right)(2\theta_H-\theta_L)\le\left(1-\frac{\chi_A}{\chi_B}\right)\theta_H+\frac{\chi_A}{\chi_B}\theta_L\le\left(1+\frac{\chi_A}{\chi_B}\right)\theta_L \tag{107}$$

If (107), both inequalities in (95) are satisfied in the lower range of $\theta_{\phi B}$, $\left(1-\frac{\chi_A}{\chi_B}\right)\theta_H \le \theta_{\phi B} \le \left(1-\frac{\chi_A}{\chi_B}\right)(2\theta_H-\theta_L)$, while the left-hand inequality is violated for the upper range of $\theta_{\phi B}$, $\left(1-\frac{\chi_A}{\chi_B}\right)(2\theta_H-\theta_L) \le \theta_{\phi B} \le \left(1-\frac{\chi_A}{\chi_B}\right)\theta_H+\frac{\chi_A}{\chi_B}\theta_L$.

$$\theta^*_{\phi A} =$$

$$\begin{cases} \frac{1}{2}\left[\frac{\chi_B}{\chi_A}\theta_{\phi B}-\left(\frac{\chi_B}{\chi_A}-1\right)\theta_L\right] & \left(1-\frac{\chi_A}{\chi_B}\right)\theta_H \le \theta_{\phi B} \le \left(1-\frac{\chi_A}{\chi_B}\right)(2\theta_H-\theta_L) \\ \frac{\chi_B}{\chi_A}\theta_{\phi B}-\left(\frac{\chi_B}{\chi_A}-1\right)\theta_H & \left(1-\frac{\chi_A}{\chi_B}\right)(2\theta_H-\theta_L) \le \theta_{\phi B} \le \left(1-\frac{\chi_A}{\chi_B}\right)\theta_H+\frac{\chi_A}{\chi_B}\theta_L \end{cases} \tag{108}$$

If

$$\frac{\theta_L}{\theta_H}+\frac{\chi_A}{\chi_B} \le 1 \le \left(\frac{\chi_B}{\chi_A}-1\right)\left(\frac{\theta_H}{\theta_L}-1\right), \tag{109}$$

then the left-hand inequality in (95) is satisfied for all $\theta_{\phi B}$ in region A4,

$$\left(1-\frac{\chi_A}{\chi_B}\right)\theta_H+\frac{\chi_A}{\chi_B}\theta_L \le \left(1-\frac{\chi_A}{\chi_B}\right)(2\theta_H-\theta_L)$$

the right-hand inequality is violated for some $\theta_{\phi B}$ in region A4,

$$\left(1+\frac{\chi_A}{\chi_B}\right)\theta_L \le \left(1-\frac{\chi_A}{\chi_B}\right)\theta_H+\frac{\chi_A}{\chi_B}\theta_L.$$

Hence if (109)

$$\left(1+\frac{\chi_A}{\chi_B}\right)\theta_L \le \left(1-\frac{\chi_A}{\chi_B}\right)\theta_H+\frac{\chi_A}{\chi_B}\theta_L \le \left(1-\frac{\chi_A}{\chi_B}\right)(2\theta_H-\theta_L). \tag{110}$$

There are two subcases to consider,

$$\left(1-\frac{\chi_A}{\chi_B}\right)\theta_H \le \left(1+\frac{\chi_A}{\chi_B}\right)\theta_L \le \left(1-\frac{\chi_A}{\chi_B}\right)\theta_H+\frac{\chi_A}{\chi_B}\theta_L \le \left(1-\frac{\chi_A}{\chi_B}\right)(2\theta_H-\theta_L). \tag{111}$$

and

$$\left(1+\frac{\chi_A}{\chi_B}\right)\theta_L \le \left(1-\frac{\chi_A}{\chi_B}\right)\theta_H \le \left(1-\frac{\chi_A}{\chi_B}\right)\theta_H+\frac{\chi_A}{\chi_B}\theta_L \le \left(1-\frac{\chi_A}{\chi_B}\right)(2\theta_H-\theta_L). \tag{112}$$

The condition for (111) is

$$\left(1-\frac{\chi_A}{\chi_B}\right)\theta_H \le \left(1+\frac{\chi_A}{\chi_B}\right)\theta_L$$

$$\frac{1-\frac{\chi_A}{\chi_B}}{1+\frac{\chi_A}{\chi_B}} \le \frac{\theta_L}{\theta_H}$$

$$\frac{2-\left(1+\frac{\chi_A}{\chi_B}\right)}{1+\frac{\chi_A}{\chi_B}} \le \frac{\theta_L}{\theta_H}$$

$$\frac{2}{1+\frac{\chi_A}{\chi_B}}-1 \le \frac{\theta_L}{\theta_H}$$

$$2 \le \left(1+\frac{\theta_L}{\theta_H}\right)\left(1+\frac{\chi_A}{\chi_B}\right). \tag{113}$$

If (113), then

$$\theta^*_{\phi A} =$$

$$\begin{cases} \frac{1}{2}\left[\frac{\chi_B}{\chi_A}\theta_{\phi B}-\left(\frac{\chi_B}{\chi_A}-1\right)\theta_L\right] & \left(1-\frac{\chi_A}{\chi_B}\right)\theta_H \le \theta_{\phi B} \le \left(1+\frac{\chi_A}{\chi_B}\right)\theta_L \\ \theta_L & \left(1+\frac{\chi_A}{\chi_B}\right)\theta_L \le \theta_{\phi B} \le \left(1-\frac{\chi_A}{\chi_B}\right)\theta_H+\frac{\chi_A}{\chi_B}\theta_L \end{cases}, \tag{114}$$

with payoffs

$$\pi^*_{\phi A} =$$

$$\begin{cases} \frac{\left(\chi_B\theta_{\phi B}-(\chi_B-\chi_A)\theta_L\right)^2}{4(\theta_H-\theta_L)(\chi_B-\chi_A)} & \left(1-\frac{\chi_A}{\chi_B}\right)\theta_H \le \theta_{\phi B} \le \left(1+\frac{\chi_A}{\chi_B}\right)\theta_L \\ \frac{\chi_A\chi_B\theta_L}{(\theta_H-\theta_L)(\chi_B-\chi_A)}\left(\theta_{\phi B}-\theta_L\right) & \left(1+\frac{\chi_A}{\chi_B}\right)\theta_L \le \theta_{\phi B} \le \left(1-\frac{\chi_A}{\chi_B}\right)\theta_H+\frac{\chi_A}{\chi_B}\theta_L \end{cases}. \tag{115}$$

If in contrast

$$2 \geq \left(1 + \frac{\theta_L}{\theta_H}\right)\left(1 + \frac{\chi_A}{\chi_B}\right), \tag{116}$$

then in

$$\left(1 + \frac{\chi_A}{\chi_B}\right)\theta_L \leq \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H \leq \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H + \frac{\chi_A}{\chi_B}\theta_L \leq \left(1 - \frac{\chi_A}{\chi_B}\right)(2\theta_H - \theta_L),$$

$\left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H \leq \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H + \frac{\chi_A}{\chi_B}\theta_L \leq \left(1 - \frac{\chi_A}{\chi_B}\right)(2\theta_H - \theta_L)$ implies that the left-hand inequality in (95) is satisfied for all $\theta_{\phi B}$ in case A4, while $\left(1 + \frac{\chi_A}{\chi_B}\right)\theta_L \leq \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H \leq \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H + \frac{\chi_A}{\chi_B}\theta_L$ implies that the right-hand inequality is violated for all $\theta_{\phi B}$ in case A4. The global maximum of the region A4.2 payoff function lies to the right of region A4.2 throughout, and the local maximum of the region A4.2 payoff function on region A4.2 is the boundary between region A4.2 and region A4.3,

$$\theta^*_{\phi A} = \theta_L \text{ for } \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H \leq \theta_{\phi B} \leq \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H + \frac{\chi_A}{\chi_B}\theta_L \tag{117}$$

with payoff

$$\pi^*_{\phi A} = \frac{\chi_A \chi_B \theta_L}{(\theta_H - \theta_L)(\chi_B - \chi_A)}(\theta_{\phi B} - \theta_L) \text{ for } \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H \leq \theta_{\phi B} \leq \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H + \frac{\chi_A}{\chi_B}\theta_L. \tag{118}$$

For $\theta_L = \chi_A = 1$, $\theta_H = \chi_B = 5$, region A4 is defined by the inequalities

$$5\left(1 - \frac{1}{5}\right) \leq \theta_{\phi B} \leq \frac{1}{5}(1) + (5)\left(1 - \frac{1}{5}\right)$$

$$4 \leq \theta_{\phi B} \leq 4\frac{1}{5}.$$

(112) is satisfied

$$\left(1 + \frac{\chi_A}{\chi_B}\right)\theta_L \leq \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H \leq \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H + \frac{\chi_A}{\chi_B}\theta_L \leq \left(1 - \frac{\chi_A}{\chi_B}\right)(2\theta_H - \theta_L)$$

$$1\frac{1}{5} \leq 4 \leq 4\frac{1}{5} \leq 7\frac{1}{5},$$

$$\theta^*_{\phi A} = 1 \text{ for } 4 \leq \theta_{\phi B} \leq 4\frac{1}{5}$$

with payoff

$$\pi^*_{\phi A} = \frac{5}{16}\left(\theta_{\phi B} - 1\right) \text{ for } 4 \leq \theta_{\phi B} \leq 4\frac{1}{5}.$$

(A4.3) $\theta_L \leq \theta_{\phi A} \leq \theta_{\phi B}$: $q_A = \frac{\theta_{AB}-\theta_{\phi A}}{\theta_H - \theta_L}$, $\pi_A = \chi_A \theta_{\phi A} \frac{\theta_{AB}-\theta_{\phi A}}{\theta_H-\theta_L}$.

$$\pi_A = \frac{\chi_A \chi_B}{\left(\theta_H - \theta_L\right)\left(\chi_B - \chi_A\right)} \theta_{\phi A}\left(\theta_{\phi B} - \theta_{\phi A}\right). \tag{119}$$

From (82), firm A's payoff along the first-order condition is

$$\pi^*_A = \frac{\chi_A \chi_B}{\left(\theta_H - \theta_L\right)\left(\chi_B - \chi_A\right)} \left(\theta^*_{\phi A}\right)^2,$$

and the global maximum of (119) occurs at (83)

$$\theta^*_{\phi A} = \frac{1}{2}\theta_{\phi B}.$$

$\theta^*_{\phi A}$ defined by (83) lies within region A4.3 for

$$\theta_L \leq \frac{1}{2}\theta_{\phi B} \leq \theta_{\phi B}. \tag{120}$$

The right-hand inequality is always satisfied. The left-hand inequality is satisfied for

$$2\theta_L \leq \theta_{\phi B}$$

Region A4 is defined by the inequalities

$$\left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H \leq \theta_{\phi B} \leq \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H + \frac{\chi_A}{\chi_B}\theta_L$$

If

$$2\theta_L \leq \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H, \tag{121}$$

then the left-hand inequality in (120) is satisfied for all $\theta_{\phi B}$ in case A4.

Rewrite (121) as

$$2\frac{\theta_L}{\theta_H} \leq 1 - \frac{\chi_A}{\chi_B}$$

$$2\frac{\theta_L}{\theta_H} + \frac{\chi_A}{\chi_B} \leq 1. \tag{122}$$

If (122) is satisfied, then so is the left-hand inequality in (120), and the global maximum of the region A4.3 payoff function lies in region A4.3.

If (122) is not satisfied, then

$$\left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H \leq 2\theta_L;$$

for

$$\left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H \leq \theta_{\phi B} \leq \min\left[2\theta_L, \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H + \frac{\chi_A}{\chi_B}\theta_L\right] \tag{123}$$

the local maximum of the region A4.3 payoff function on region A4.3 is the boundary between region A4.2 and A4.3.

For $\theta_L = \chi_A = 1$, $\theta_H = \chi_B = 5$, region A4 is defined by the inequalities

$$4 \leq \theta_{\phi B} \leq 4\frac{1}{5}.$$

(122) is satisfied:

$$2\left(\frac{1}{5}\right) + \frac{1}{5} \leq 1.$$

The global maximum of the payoff function lies in region A4.3 for

$$1 \leq \frac{1}{2}\theta_{\phi B} \leq \theta_{\phi B}$$

$$2 \leq \theta_{\phi B} \leq 2\theta_{\phi B},$$

and this is satisfied for $\theta_{\phi B}$ in the range $4 \leq \theta_{\phi B} \leq 4\frac{1}{5}$.

A's payoff is

$$\pi_A^* = \frac{(1)(5)}{(5-1)(5-1)}\left(\frac{1}{2}\theta_{\phi B}\right)^2 = \frac{5}{64}\theta_{\phi B}^2$$

(A4.4) $\theta_{\phi B} \leq \theta_{\phi A} \leq \theta_H$: $q_A = 0, \pi_A = 0$.

Firm A's best-response price would never be found in this region.

Collect results for case A4, $4 \leq \theta_{\phi B} \leq 4\frac{1}{5}$.

(A4.1) $0 \leq \theta_{\phi A} \leq 5(\theta_{\phi B} - 4)$: $\theta_{\phi A}^* = 5(\theta_{\phi B} - 4)$, $\pi_A^* = 5(\theta_{\phi B} - 4)$.

(A4.2) $5\theta_{\phi B} - 20 \leq \theta_{\phi A} \leq 1$: $\theta_{\phi A}^* = \theta_L = 1$, $\pi_A^* = \frac{5}{16}(\theta_{\phi B} - 1)$

(A4.3) $1 \leq \theta_{\phi A} \leq \theta_{\phi B}$; $\theta^*_{\phi A} = \frac{1}{2}\theta_{\phi B}$; $\pi^*_A = \frac{5}{64}\theta^2_{\phi B}$

The region A4.2 payoff is never less than the region A4.1 payoff:

$$\frac{5}{16}\left(\theta_{\phi B} - 1\right) - 5\left(\theta_{\phi B} - 4\right) = \frac{5}{16}(63 - 15\theta_{\phi B}).$$

This difference is positive for $\theta_{\phi B} = 4$ and declines to zero for $\theta_{\phi B} = 4\frac{1}{5}$.

For these parameters, the global optimum never occurs in region A4.1.

Now compare the region A4.3 payoff and the region A4.2 payoff:

$$\frac{5}{64}\theta^2_{\phi B} - \frac{5}{16}\left(\theta_{\phi B} - 1\right) = \frac{5}{64}\left(\theta_{\phi B} - 2\right)^2,$$

which is positive.

A earns the greatest profit at the local maximum of its payoff function on region A4.3. A's best-response quality-adjusted price is therefore

$$\theta^{br}_{\phi A}(\theta_{\phi B}) = \frac{1}{2}\theta_{\phi B} \text{ for } 4 \leq \theta_{\phi B} \leq 4\frac{1}{5}.$$

This is a straight line connecting $(2, 4)$ and $(2.1, 4.2)$ in $(\theta_{\phi A}, \theta_{\phi B})$-space.

(A5) $\left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H + \frac{\chi_A}{\chi_B}\theta_L \leq \theta_{\phi B} \leq \theta_H$

(A5.1) $0 \leq \theta_{\phi A} \leq \theta_L$: $q_A = 1$, $\pi_A = \chi_A\theta_{\phi A}$

Firm A maximizes its payoff charging the highest price consistent with being in region A5.1, $\theta^*_{\phi A} = \theta_L$. A's payoff is $\pi^*_A = \chi_A\theta_L$.

(A5.2) $\theta_L \leq \theta_{\phi A} \leq \frac{\chi_B}{\chi_A}\theta_{\phi B} - \left(\frac{\chi_B}{\chi_A} - 1\right)\theta_H$: $q_A = \frac{\theta_H - \theta_{\phi A}}{\chi_B - \chi_A}$, $\pi_A = \chi_A\theta_{\phi A}\frac{\theta_H - \theta_{\phi A}}{\chi_B - \chi_A}$

The right boundary is the line along which $\theta_{AB} = \theta_H$.

$$\pi_A = \frac{\chi_A}{\chi_B - \chi_A}\theta_{\phi A}\left(\theta_H - \theta_{\phi A}\right). \tag{124}$$

The first-order condition to maximize (124) is

$$\theta_H - 2\theta_{\phi A} \equiv 0, \tag{125}$$

from which A's payoff along the first-order condition is

$$\pi^*_A = \frac{\chi_A}{\chi_B - \chi_A}\left(\theta^*_{\phi A}\right)^2. \tag{126}$$

Solve (125) for

$$\theta^*_{\phi A} = \frac{1}{2}\theta_H. \tag{127}$$

$\theta^*_{\phi A} = \frac{1}{2}\theta_H$ lies in region A5.2 for

$$\theta_L \leq \frac{1}{2}\theta_H \leq \frac{\chi_B}{\chi_A}\theta_{\phi B} - \left(\frac{\chi_B}{\chi_A} - 1\right)\theta_H. \tag{128}$$

The left-hand inequality is satisfied for

$$\theta_H \geq 2\theta_L. \tag{129}$$

If (129) is satisfied, the global maximum of the region A5.2 payoff function does not lie in region A5.1. If (129) is violated, the global maximum of the region A5.2 payoff function lies in region A5.1, and the local maximum of the region A5.2 payoff function on region A5.2 is the boundary between regions A5.1 and A5.2.

The right-hand inequality in (128) is satisfied for

$$\frac{1}{2}\theta_H \leq \frac{\chi_B}{\chi_A}\theta_{\phi B} - \left(\frac{\chi_B}{\chi_A} - 1\right)\theta_H$$

$$\frac{1}{2}\theta_H + \left(\frac{\chi_B}{\chi_A} - 1\right)\theta_H \leq \frac{\chi_B}{\chi_A}\theta_{\phi B}$$

$$\theta_H\left(\frac{\chi_B}{\chi_A} - \frac{1}{2}\right) \leq \frac{\chi_B}{\chi_A}\theta_{\phi B}$$

$$\theta_H\left(1 - \frac{1}{2}\frac{\chi_A}{\chi_B}\right) \leq \theta_{\phi B}. \tag{130}$$

Region A5 is defined by the inequalities

$$\left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H + \frac{\chi_A}{\chi_B}\theta_L \leq \theta_{\phi B} \leq \theta_H.$$

If

$$\theta_H\left(1 - \frac{1}{2}\frac{\chi_A}{\chi_B}\right) \leq \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H + \frac{\chi_A}{\chi_B}\theta_L, \tag{131}$$

then (130) and the right-hand inequality in (128) are satisfied for all $\theta_{\phi B}$ in case A5.2; the global maximum of the region A5.2 payoff function does not lie to the right of region A5.2.

Rewrite (131) as

$$\theta_H \left(1 - \frac{1}{2}\frac{\chi_A}{\chi_B}\right) - \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H \leq \frac{\chi_A}{\chi_B}\theta_L$$

$$\frac{1}{2}\frac{\chi_A}{\chi_B}\theta_H \leq \frac{\chi_A}{\chi_B}\theta_L$$

$$\theta_H \leq 2\theta_L. \tag{132}$$

(132) is the contrary of (129). Except in the borderline case $\theta_H = 2\theta_L$, if the left-hand inequality in (128) is satisfied, the right-hand inequality is not, and vice versa.

If (132) is satisfied, the global maximum of firm A's region A5.2 payoff function does not lie to the right of region A5.2. If (132) is satisfied, (129) is violated; the global maximum of firm A's region A5.2 payoff function lies in region A5.1 and the local maximum of firm A's region A5.2 payoff function is the boundary between region A5.1 and region A5.2, $\theta^*_{\phi A} = \theta_L$.

If (132) is violated, then

$$\left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H + \frac{\chi_A}{\chi_B}\theta_L \leq \theta_H \left(1 - \frac{1}{2}\frac{\chi_A}{\chi_B}\right) \leq \theta_H. \tag{133}$$

For

$$\left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H + \frac{\chi_A}{\chi_B}\theta_L \leq \theta_{\phi B} \leq \theta_H \left(1 - \frac{1}{2}\frac{\chi_A}{\chi_B}\right), \tag{134}$$

(130) and the right-hand inequality in (128) are violated. The local maximum of firm A's region A5.2 payoff function on region A5.2 is the boundary between region A5.2 and A5.3,

$$\theta^*_{\phi A} = \frac{\chi_B}{\chi_A}\theta_{\phi B} - \left(\frac{\chi_B}{\chi_A} - 1\right)\theta_H. \tag{135}$$

Firm A's payoff is

$$\pi^*_A =$$

$$\frac{\chi_A}{\chi_B - \chi_A}\left(\frac{\chi_B}{\chi_A}\theta_{\phi B} - \left(\frac{\chi_B}{\chi_A} - 1\right)\theta_H\right)\left(\theta_H - \frac{\chi_B}{\chi_A}\theta_{\phi B} + \left(\frac{\chi_B}{\chi_A} - 1\right)\theta_H\right) = \tag{136}$$

$$\frac{\chi_A}{\chi_B-\chi_A}\left(\theta_H-\frac{\chi_B}{\chi_A}(\theta_H-\theta_{\phi B})\right)\left(\frac{\chi_B}{\chi_A}(\theta_H-\theta_{\phi B})\right)=$$

$$\frac{\chi_B}{\chi_B-\chi_A}\left(\theta_H-\frac{\chi_B}{\chi_A}(\theta_H-\theta_{\phi B})\right)(\theta_H-\theta_{\phi B}) \tag{137}$$

For

$$\theta_H\left(1-\frac{1}{2}\frac{\chi_A}{\chi_B}\right)\leq\theta_{\phi B}\leq\theta_H, \tag{138}$$

(130) and the right-hand inequality in (128) are satisfied. The global maximum of firm A's region A5.2 payoff function lies within region A5.2.

Hence if (132) is violated

$$\theta^*_{\phi A}=\begin{cases}\frac{\chi_B}{\chi_A}\theta_{\phi B}-\left(\frac{\chi_B}{\chi_A}-1\right)\theta_H & \frac{\chi_B}{\chi_A}\theta_{\phi B}-\left(\frac{\chi_B}{\chi_A}-1\right)\theta_H\leq\theta_{\phi B}\leq\theta_H\left(1-\frac{1}{2}\frac{\chi_A}{\chi_B}\right)\\ \frac{1}{2}\theta_H & \theta_H\left(1-\frac{1}{2}\frac{\chi_A}{\chi_B}\right)\leq\theta_{\phi B}\leq\theta_H\end{cases} \tag{139}$$

with payoffs

$$\pi^*_{\phi A}=\begin{cases}\frac{\chi_B}{\chi_B-\chi_A}\left(\theta_H-\frac{\chi_B}{\chi_A}(\theta_H-\theta_{\phi B})\right)(\theta_H-\theta_{\phi B}) & \frac{\chi_B}{\chi_A}\theta_{\phi B}-\left(\frac{\chi_B}{\chi_A}-1\right)\theta_H\leq\theta_{\phi B}\leq\theta_H\left(1-\frac{1}{2}\right.\\ \frac{1}{4}\frac{\chi_A}{\chi_B-\chi_A}\theta_H^2 & \theta_H\left(1-\frac{1}{2}\frac{\chi_A}{\chi_B}\right)\leq\theta_{\phi B}\leq\theta_H\end{cases} \tag{140}$$

For $\theta_L=\chi_A=1$, $\theta_H=\chi_B=5$, the global maximum of A's region A5.2 payoff function is

$$\theta^*_{\phi A}=\frac{1}{2}\theta_H=\frac{5}{2}.$$

Region A5.2 is defined by the inequalities

$$1=\theta_L\leq\theta_{\phi A}\leq\frac{\chi_B}{\chi_A}\theta_{\phi B}-\left(\frac{\chi_B}{\chi_A}-1\right)\theta_H=\frac{5}{1}\theta_{\phi B}-5\left(\frac{5}{1}-1\right)=5\left(\theta_{\phi B}-4\right)$$

for $\theta_{\phi B}$ in the range $\frac{21}{5}\leq\theta_{\phi B}\leq 5$. $5\left(\frac{21}{5}\right)-20=1$; $5\left(5\right)-20=5$.

The best response and payoff functions are

$$\theta^*_{\phi A}=\begin{cases}5\left(\theta_{\phi B}-4\right) & 4\frac{1}{5}\leq\theta_{\phi B}\leq 4\frac{1}{2}\\ \frac{5}{2} & 4\frac{1}{2}\leq\theta_{\phi B}\leq 5\end{cases}$$

and

$$\pi^*_{\phi A} = \begin{cases} \frac{25}{4}(5-\theta_{\phi B})(\theta_{\phi B}-4) & 4\frac{1}{5} \le \theta_{\phi B} \le 4\frac{1}{2} \\ \frac{25}{16} & 4\frac{1}{2} \le \theta_{\phi B} \le 5 \end{cases} ..$$

respectively.

(A5.3) $\frac{\chi_B}{\chi_A}\theta_{\phi B} - \left(\frac{\chi_B}{\chi_A} - 1\right)\theta_H \le \theta_{\phi A} \le \theta_{\phi B}$: $q_A = \frac{\theta_{AB}-\theta_{\phi A}}{\theta_H-\theta_L}$, $\pi_A = \chi_A \theta_{\phi A} \frac{\theta_{AB}-\theta_{\phi A}}{\theta_H-\theta_L}$

As in the discussion of region (A4.3),

$$\pi_A = \frac{\chi_A \chi_B}{(\theta_H - \theta_L)(\chi_B - \chi_A)} \theta_{\phi A} (\theta_{\phi B} - \theta_{\phi A}). \tag{141}$$

The first-order condition to maximize (141) is

$$\theta_{\phi B} - 2_{\phi A} \equiv 0, \tag{142}$$

from which firm A's profit along the first-order condition is

$$\pi^*_A = \frac{\chi_A \chi_B}{(\theta_H - \theta_L)(\chi_B - \chi_A)} \left(\theta^*_{\phi A}\right)^2. \tag{143}$$

Solve (**??**) for

$$\theta^*_{\phi A} = \frac{1}{2}\theta_{\phi B}. \tag{144}$$

$\theta^*_{\phi A}$ defined by (144) lies within region A5.3 for

$$\frac{\chi_B}{\chi_A}\theta_{\phi B} - \left(\frac{\chi_B}{\chi_A} - 1\right)\theta_H \le \frac{1}{2}\theta_{\phi B} \le \theta_{\phi B}. \tag{145}$$

The right-hand inequality is always satisfied. The left-hand inequality is satisfied for

$$2\frac{\chi_B}{\chi_A}\theta_{\phi B} - 2\left(\frac{\chi_B}{\chi_A} - 1\right)\theta_H \le \theta_{\phi B} \tag{146}$$

$$\left(2\frac{\chi_B}{\chi_A} - 1\right)\theta_{\phi B} \le 2\left(\frac{\chi_B}{\chi_A} - 1\right)\theta_H$$

$$\theta_{\phi B} \le 2\theta_H \frac{\frac{\chi_B}{\chi_A} - 1}{2\frac{\chi_B}{\chi_A} - 1} \tag{147}$$

Region A5 is defined by the inequalities

$$\left(1-\frac{\chi_A}{\chi_B}\right)\theta_H+\frac{\chi_A}{\chi_B}\theta_L\le\theta_{\phi B}\le\theta_H.$$

The right-hand side of (147) is less than θ_H:

$$\theta_H-2\theta_H\frac{\frac{\chi_B}{\chi_A}-1}{2\frac{\chi_B}{\chi_A}-1}=$$

$$\left(1-2\frac{\frac{\chi_B}{\chi_A}-1}{2\frac{\chi_B}{\chi_A}-1}\right)\theta_H=$$

$$\left(\frac{2\frac{\chi_B}{\chi_A}-1-2\frac{\chi_B}{\chi_A}+2}{2\frac{\chi_B}{\chi_A}-1}\right)\theta_H=$$

$$\frac{1}{2\frac{\chi_B}{\chi_A}-1}\theta_H>0. \tag{148}$$

If

$$2\theta_H\frac{\frac{\chi_B}{\chi_A}-1}{2\frac{\chi_B}{\chi_A}-1}\le\left(1-\frac{\chi_A}{\chi_B}\right)\theta_H+\frac{\chi_A}{\chi_B}\theta_L \tag{149}$$

the left-hand inequality in (146) is violated for all $\theta_{\phi B}$ in $\left(1-\frac{\chi_A}{\chi_B}\right)\theta_H+\frac{\chi_A}{\chi_B}\theta_L\le\theta_{\phi B}\le\theta_H$.

Rewrite (149) as

$$2\theta_H\frac{1-\frac{\chi_A}{\chi_B}}{2-\frac{\chi_A}{\chi_B}}\le\left(1-\frac{\chi_A}{\chi_B}\right)\theta_H+\frac{\chi_A}{\chi_B}\theta_L$$

$$2\left(1-\frac{\chi_A}{\chi_B}\right)\theta_H\le\frac{\chi_A}{\chi_B}\left(2-\frac{\chi_A}{\chi_B}\right)\theta_L+\left(1-\frac{\chi_A}{\chi_B}\right)\theta_H\left(2-\frac{\chi_A}{\chi_B}\right)$$

$$\left(2-\left(2-\frac{\chi_A}{\chi_B}\right)\right)\left(1-\frac{\chi_A}{\chi_B}\right)\theta_H\le\frac{\chi_A}{\chi_B}\left(2-\frac{\chi_A}{\chi_B}\right)\theta_L$$

$$\frac{\chi_A}{\chi_B}\left(1-\frac{\chi_A}{\chi_B}\right)\theta_H \leq \frac{\chi_A}{\chi_B}\left(2-\frac{\chi_A}{\chi_B}\right)\theta_L$$

$$\frac{1-\frac{\chi_A}{\chi_B}}{2-\frac{\chi_A}{\chi_B}} \leq \frac{\theta_L}{\theta_H}$$

$$\frac{2-\frac{\chi_A}{\chi_B}-1}{2-\frac{\chi_A}{\chi_B}} \leq \frac{\theta_L}{\theta_H}$$

$$1-\frac{1}{2-\frac{\chi_A}{\chi_B}} \leq \frac{\theta_L}{\theta_H}$$

$$1-\frac{\theta_L}{\theta_H} \leq \frac{1}{2-\frac{\chi_A}{\chi_B}}$$

$$\left(1-\frac{\theta_L}{\theta_H}\right)\left(2-\frac{\chi_A}{\chi_B}\right) \leq 1. \tag{150}$$

If (150) is satisfied, the left-hand inequality in (146) is violated for all $\theta_{\phi B}$ in region A5 and the local maximum of the region A5.3 payoff function on region A5.3 is the boundary between region A5.2 and region A5.3,

$$\theta^*_{\phi A} = \frac{\chi_B}{\chi_A}\theta_{\phi B} - \left(\frac{\chi_B}{\chi_A}-1\right)\theta_H \tag{151}$$

with payoff

$$\pi^*_{\phi A} = \frac{\chi_A\chi_B}{(\theta_H-\theta_L)(\chi_B-\chi_A)}\left(\frac{\chi_B}{\chi_A}\theta_{\phi B} - \left(\frac{\chi_B}{\chi_A}-1\right)\theta_H\right)\left(\theta_{\phi B} - \left(\frac{\chi_B}{\chi_A}\theta_{\phi B} - \left(\frac{\chi_B}{\chi_A}-1\right)\theta_H\right)\right)$$

$$= \frac{\chi_A\chi_B}{(\theta_H-\theta_L)(\chi_B-\chi_A)}\left(\frac{\chi_B}{\chi_A}\theta_{\phi B} - \frac{\chi_B}{\chi_A}\theta_H + \theta_H\right)(\chi_B-\chi_A)\frac{(\theta_H-\theta_{\phi B})}{\chi_A}$$

$$= \chi_B\left[\theta_H - \frac{\chi_B}{\chi_A}(\theta_H-\theta_{\phi B})\right]. \tag{152}$$

If (150) is violated, then

$$\left(1-\frac{\chi_A}{\chi_B}\right)\theta_H+\frac{\chi_A}{\chi_B}\theta_L\leq 2\theta_H\frac{\frac{\chi_B}{\chi_A}-1}{2\frac{\chi_B}{\chi_A}-1}\leq\theta_H. \tag{153}$$

For

$$\left(1-\frac{\chi_A}{\chi_B}\right)\theta_H+\frac{\chi_A}{\chi_B}\theta_L\leq\theta_{\phi B}\leq 2\theta_H\frac{\frac{\chi_B}{\chi_A}-1}{2\frac{\chi_B}{\chi_A}-1}, \tag{154}$$

the left-hand inequality in (145) is satisfied, the local maximum of the region A5.3 payoff function lies in region A5.3, A's best response price is (144) with payoff (143).

For

$$2\theta_H\frac{\frac{\chi_B}{\chi_A}-1}{2\frac{\chi_B}{\chi_A}-1}\leq\theta_{\phi B}\leq\theta_H \tag{155}$$

the left-hand inequality in (145) is violated, the local maximum of the region A5.3 payoff function lies to the left of region A5.3, A's payoff-maximizing price on region A5.3 is the boundary between region A5.3 and region A5.2,

$$\theta^*_{\phi A}=\frac{\chi_B}{\chi_A}\theta_{\phi B}-\left(\frac{\chi_B}{\chi_A}-1\right)\theta_H, \tag{156}$$

with payoff (152).

Overall, if (150) is violated,

$$\theta^*_{\phi A}=$$

$$\begin{cases}\frac{1}{2}\theta_{\phi B} & \left(1-\frac{\chi_A}{\chi_B}\right)\theta_H+\frac{\chi_A}{\chi_B}\theta_L\leq\theta_{\phi B}\leq 2\theta_H\frac{\frac{\chi_B}{\chi_A}-1}{2\frac{\chi_B}{\chi_A}-1}\\ \frac{\chi_B}{\chi_A}\theta_{\phi B}-\left(\frac{\chi_B}{\chi_A}-1\right)\theta_H & 2\theta_H\frac{\frac{\chi_B}{\chi_A}-1}{2\frac{\chi_B}{\chi_A}-1}\leq\theta_{\phi B}\leq\theta_H\end{cases} \tag{157}$$

$$\pi^*_A=$$

$$\begin{cases}\frac{1}{4}\frac{\chi_A\chi_B}{(\theta_H-\theta_L)(\chi_B-\chi_A)}\theta^2_{\phi B} & \left(1-\frac{\chi_A}{\chi_B}\right)\theta_H+\frac{\chi_A}{\chi_B}\theta_L\leq\theta_{\phi B}\leq 2\theta_H\frac{\frac{\chi_B}{\chi_A}-1}{2\frac{\chi_B}{\chi_A}-1}\\ =\chi_B\left[\theta_H-\frac{\chi_B}{\chi_A}(\theta_H-\theta_{\phi B})\right] & 2\theta_H\frac{\frac{\chi_B}{\chi_A}-1}{2\frac{\chi_B}{\chi_A}-1}\leq\theta_{\phi B}\leq\theta_H\end{cases} \tag{158}$$

For $\theta_L = \chi_A = 1$, $\theta_H = \chi_B = 5$, region A5.3 is defined by the inequalities

$$5(\theta_{\phi B} - 4) \le \theta_{\phi A} \le \theta_{\phi B},$$

with $\theta_{\phi B}$ in the range $4\frac{1}{5} \le \theta_{\phi B} \le 5$ and the global maximum of A's region A5.3 payoff function at

$$\theta^*_{\phi A} = \frac{1}{2}\theta_{\phi B}.$$

For the global maximum of the payoff function to lie within the region, we must have

$$5\theta_{\phi B} - 20 \le \frac{1}{2}\theta_{\phi B}$$

$$10\theta_{\phi B} - 40 \le \theta_{\phi B}$$

$$9\theta_{\phi B} \le 40$$

$$\theta_{\phi B} \le \frac{40}{9} = 4\frac{4}{9}.$$

The local maxima and payoff functions are

$$\theta^*_{\phi A} = \begin{cases} \frac{1}{2}\theta_{\phi B} & 4\frac{1}{5} \le \theta_{\phi B} \le 4\frac{4}{9} \\ 5(\theta_{\phi B} - 4) & 4\frac{4}{9} \le \theta_{\phi B} \le 5 \end{cases}$$

$$\pi^*_A = \begin{cases} \frac{5}{64}\theta^2_{\phi B} & 4\frac{1}{5} \le \theta_{\phi B} \le 4\frac{4}{9} \\ 25(\theta_{\phi B} - 4) & 4\frac{4}{9} \le \theta_{\phi B} \le 5 \end{cases}$$

(A5.4) $\theta_{\phi B} \le \theta_{\phi A} \le \theta_H$: $q_A = 0$, $\pi_A = 0$.
Firm A's best-response price would never be found in this region.
Collect results for case A5, $4\frac{1}{5} \le \theta_{\phi B} \le 5$.
A5.1 $0 \le \theta_{\phi A} \le \theta_L$: $\theta^*_{\phi A} = 1$, $\pi^*_A = 1$
A5.2 $\theta_L \le \theta_{\phi A} \le 5(\theta_{\phi B} - 4)$

$$\theta^*_{\phi A} = \begin{cases} 5(\theta_{\phi B} - 4) & 4\frac{1}{5} \le \theta_{\phi B} \le 4\frac{1}{2} \\ \frac{5}{2} & 4\frac{1}{2} \le \theta_{\phi B} \le 5 \end{cases}$$

and

$$\pi^*_{\phi A} = \begin{cases} \frac{25}{4}(5-\theta_{\phi B})(\theta_{\phi B}-4) & 4\frac{1}{5} \le \theta_{\phi B} \le 4\frac{1}{2} \\ \frac{25}{16} & 4\frac{1}{2} \le \theta_{\phi B} \le 5 \end{cases}..$$

A5.3 $5(\theta_{\phi B}-4) \le \theta_{\phi A} \le \theta_{\phi B}$

$$\theta^*_{\phi A} = \begin{cases} \frac{1}{2}\theta_{\phi B} & 4\frac{1}{5} \le \theta_{\phi B} \le 4\frac{4}{9} \\ 5(\theta_{\phi B}-4) & 4\frac{4}{9} \le \theta_{\phi B} \le 5 \end{cases}$$

$$\pi^*_{A} = \begin{cases} \frac{5}{64}\theta^2_{\phi B} & 4\frac{1}{5} \le \theta_{\phi B} \le 4\frac{4}{9} \\ 25(\theta_{\phi B}-4) & 4\frac{4}{9} \le \theta_{\phi B} \le 5 \end{cases}$$

Compare the best payoff in region A5.1 with payoffs in region A5.2:
On the range $4\frac{1}{5} \le \theta_{\phi B} \le 4\frac{1}{2}$:

$$\frac{25}{4}(5-\theta_{\phi B})(\theta_{\phi B}-4) - 1 = \frac{55}{4}\left(\theta_{\phi B} - \frac{21}{5}\right)\left(\frac{24}{5} - \theta_{\phi B}\right) \ge 0$$

On the range $4\frac{1}{2} \le \theta_{\phi B} \le 5$:

$$\frac{25}{16} - 1 > 0.$$

Firm A's best response price is never in region A5.1.
Compare payoffs in regions A5.2 and A5.3:

- $4\frac{1}{5} \le \theta_{\phi B} \le 4\frac{4}{9}$

$$\frac{25}{4}(5-\theta_{\phi B})(\theta_{\phi B}-4) - \frac{5}{64}\theta^2_{\phi B}$$

$$= -\frac{5}{64}81\left(\frac{40}{9} - \theta_{\phi B}\right)^2 \le 0.$$

The payoff in region A5.3 is never less than the payoff in region A5.2, and in general greater:

$$\theta^{br}_{\phi A}(\theta_{\phi B}) = \frac{1}{2}\theta_{\phi B}.$$

This is a straight line connecting $(2.1, 4.2)$ and $(2\frac{2}{9}, 4\frac{4}{9})$ in $(\theta_{\phi A}, \theta_{\phi B})$-space.

- $4\frac{4}{9} \le \theta_{\phi B} \le 4\frac{1}{2}$:

$$\frac{25}{4}\left(5-\theta_{\phi B}\right)\left(\theta_{\phi B}-4\right)-25\left(\theta_{\phi B}-4\right)=$$

$$25\left(\theta_{\phi B}-4\right)\left(\frac{1}{4}\left(5-\theta_{\phi B}\right)-1\right)=$$

$$-\frac{25}{4}\left(\theta_{\phi B}-4\right)\left(\theta_{\phi B}-1\right)<0$$

For $4\frac{4}{9} \le \theta_{\phi B} \le 4\frac{1}{2}$,

$$\theta_{\phi A}^{br}(\theta_{\phi B}) = 5\left(\theta_{\phi B}-4\right).$$

This is a straight line connecting $(2\frac{2}{9}, 4\frac{4}{9})$ and $(2\frac{1}{2}, 4\frac{1}{2})$ in $(\theta_{\phi A}, \theta_{\phi B})$-space.

- $4\frac{1}{2} \le \theta_{\phi B} \le 5$

$$\frac{25}{16}-25\left(\theta_{\phi B}-4\right)$$

$$=25\left(4\frac{1}{16}-\theta_{\phi B}\right),$$

which is negative on the range considered.

For $4\frac{1}{2} \le \theta_{\phi B} \le 5$,

$$\theta_{\phi A}^{br}(\theta_{\phi B}) = 5\left(\theta_{\phi B}-4\right).$$

This is a straight line connecting $(2\frac{1}{2}, 4\frac{1}{2})$ and $(5,5)$ in $(\theta_{\phi A}, \theta_{\phi B})$-space.

Summarize results for case A5:

$$\theta_{\phi A}^{br}(\theta_{\phi B}) = \begin{cases} \frac{1}{2}\theta_{\phi B} & 4\frac{1}{5} \le \theta_{\phi B} \le 4\frac{4}{9} \quad (2\frac{1}{10}, 4\frac{1}{5}) \quad (2\frac{2}{9}, 4\frac{4}{9}) \\ 5\left(\theta_{\phi B}-4\right) & 4\frac{4}{9} \le \theta_{\phi B} \le 5 \quad (2\frac{2}{9}, 4\frac{4}{9}) \quad (5,5) \end{cases}$$

Summarize results for the numerical example for all regions:

A2:

$$\theta_{\phi A}^{br}(\theta_{\phi B}) = \frac{5}{2}\theta_{\phi B} - 2 \text{ for } \frac{4}{5} \le \theta_{\phi B} \le 1.$$

This is a straight line connecting $(0, \frac{4}{5})$ and $(\frac{1}{2}, 1)$ in $(\theta_{\phi A}, \theta_{\phi B})$-space.
A3:

$$\theta^{br}_{\phi A}(\theta_{\phi B}) = \begin{cases} \frac{5}{2}\theta_{\phi B} - 2 & 1 \leq \theta_{\phi B} \leq \frac{6}{5} & (\frac{1}{2}, 1) & (1, \frac{6}{5}) \\ 1 & \frac{6}{5} \leq \theta_{\phi B} \leq 2 & (1, \frac{6}{5}) & (1, 2) \\ \frac{1}{2}\theta_{\phi B} & 2 \leq \theta_{\phi B} \leq 4 & (1, 2) & (2, 4) \end{cases}$$

A4:

$$\theta^{br}_{\phi A}(\theta_{\phi B}) = \frac{1}{2}\theta_{\phi B} \text{ for } 4 \leq \theta_{\phi B} \leq 4\frac{1}{5}.$$

This is a straight line connecting $(2, 4)$ and $(2\frac{1}{10}, 4\frac{1}{5})$ in $(\theta_{\phi A}, \theta_{\phi B})$-space.
A5:

$$\theta^{br}_{\phi A}(\theta_{\phi B}) = \begin{cases} \frac{1}{2}\theta_{\phi B} & 4\frac{1}{5} \leq \theta_{\phi B} \leq 4\frac{4}{9} & (2\frac{1}{10}, 4\frac{1}{5}) & (2\frac{2}{9}, 4\frac{4}{9}) \\ 5\left(\theta_{\phi B} - 4\right) & 4\frac{4}{9} \leq \theta_{\phi B} \leq 5 & (2\frac{2}{9}, 4\frac{4}{9}) & (5, 5) \end{cases}$$

Combining intervals where possible,

$$\theta^{br}_{\phi A}(\theta_{\phi B}) = \begin{cases} \frac{5}{2}\theta_{\phi B} - 2 & \frac{4}{5} \leq \theta_{\phi B} \leq \frac{6}{5} & (0, \frac{4}{5}) & (1, \frac{6}{5}) \\ 1 & \frac{6}{5} \leq \theta_{\phi B} \leq 2 & (1, \frac{6}{5}) & (1, 2) \\ \frac{1}{2}\theta_{\phi B} & 2 \leq \theta_{\phi B} \leq 4\frac{4}{9} & (1, 2) & (2\frac{2}{9}, 4\frac{4}{9}) \\ 5\left(\theta_{\phi B} - 4\right) & 4\frac{4}{9} \leq \theta_{\phi B} \leq 5 & (2\frac{2}{9}, 4\frac{4}{9}) & (\frac{5}{2}, 4\frac{1}{2}) \end{cases}$$

4 Equilibrium values

The two reaction functions intersect on the segments

$$\theta^{br}_{\phi A}(\theta_{\phi B}) = \frac{1}{2}\theta_{\phi B} \qquad \pi^*_A = \frac{1}{4}\frac{\chi_A \chi_B}{\left(\theta_H - \theta_L\right)\left(\chi_B - \chi_A\right)}\theta^2_{\phi B}$$

$$\theta^{br}_{\phi B}(\theta_{\phi A}) = \frac{1}{2}\left[\frac{\chi_A}{\chi_B}\theta_{\phi A} + \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H\right]$$

$$\pi^*_B = \frac{1}{4}\frac{\chi^2_B}{\left(\theta_H - \theta_L\right)\left(\chi_B - \chi_A\right)}\left[\frac{\chi_A}{\chi_B}\theta_{\phi A} + \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H\right]^2$$

or, for the numerical example,

$$\theta^{br}_{\phi A}(\theta_{\phi B}) = \frac{1}{2}\theta_{\phi B} \qquad \pi^*_A = \frac{5}{64}\theta^2_{\phi B}$$

$$\theta^{br}_{\phi B}(\theta_{\phi A}) = 2 + \frac{1}{10}\theta_{\phi A} \qquad \pi^{br}_{\phi B}(\theta_{\phi A}) = \frac{25}{16}\left(2 + \frac{1}{10}\theta_{\phi A}\right)^2$$

Solve the equations of the reaction functions:

$$\theta_{\phi A} = \frac{1}{2}\theta_{\phi B}$$

$$\theta_{\phi B} = \frac{1}{2}\left[\frac{\chi_A}{\chi_B}\theta_{\phi A} + \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H\right]$$

$$\theta_{\phi B} = \frac{1}{2}\left(\frac{\chi_A}{\chi_B}\left(\frac{1}{2}\theta_{\phi B}\right) + \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H\right)$$

$$2\theta_{\phi B} = \frac{\chi_A}{2\chi_B}\theta_{\phi B} + \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H$$

$$\left(2 - \frac{\chi_A}{2\chi_B}\right)\theta_{\phi B} = \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H$$

$$\theta_{\phi B} = \frac{1 - \frac{\chi_A}{\chi_B}}{2 - \frac{\chi_A}{2\chi_B}}\theta_H = 2\frac{1 - \frac{\chi_A}{\chi_B}}{4 - \frac{\chi_A}{\chi_B}}\theta_H$$

$$\theta_{\phi A} = \frac{1 - \frac{\chi_A}{\chi_B}}{4 - \frac{\chi_A}{\chi_B}}\theta_H.$$

Evaluate these for the numerical example:

$$\theta_{\phi B} = 2\frac{1 - \frac{1}{5}}{4 - \frac{1}{5}}(5) = \frac{40}{19}$$

$$\theta_{\phi A} = \frac{20}{19}.$$

Solve the numerical versions:

$$\theta_{\phi B} = 2 + \frac{1}{10}\theta_{\phi A} = 2 + \frac{1}{10}\left(\frac{1}{2}\theta_{\phi B}\right)$$

$$\frac{19}{20}\theta_{\phi B} = 2$$

$$\theta_{\phi B} = \frac{40}{19} = 2\frac{2}{19}$$

$$\theta_{\phi A} = \frac{20}{19} = 1\frac{1}{19}$$

In general terms, payoffs are

$$\pi_A^* = \frac{1}{4}\frac{\chi_A\chi_B}{(\theta_H - \theta_L)(\chi_B - \chi_A)}\left(2\frac{1 - \frac{\chi_A}{\chi_B}}{4 - \frac{\chi_A}{\chi_B}}\theta_H\right)^2 =$$

$$\frac{\theta_H^2\chi_B}{(\theta_H - \theta_L)}\frac{\chi_A}{(\chi_B - \chi_A)}\left(\frac{1 - \frac{\chi_A}{\chi_B}}{4 - \frac{\chi_A}{\chi_B}}\right)^2$$

$$\frac{\partial}{\partial\chi_A}\left(\frac{\chi_A}{(\chi_B - \chi_A)}\left(\frac{1 - \frac{\chi_A}{\chi_B}}{4 - \frac{\chi_A}{\chi_B}}\right)^2\right)$$

$$= (4\chi_B - 7\chi_A)\frac{\chi_B}{(4\chi_B - \chi_A)^3}$$

which is positive for the values of the numerical example.

$$\pi_B^* = \frac{1}{4}\frac{\chi_B^2}{(\theta_H - \theta_L)(\chi_B - \chi_A)}\left[\frac{\chi_A}{\chi_B}\frac{1 - \frac{\chi_A}{\chi_B}}{4 - \frac{\chi_A}{\chi_B}}\theta_H + \left(1 - \frac{\chi_A}{\chi_B}\right)\theta_H\right]^2$$

$$\pi_B^* = \frac{1}{4}\frac{\chi_B^2\theta_H^2}{(\theta_H - \theta_L)(\chi_B - \chi_A)}\left[\frac{\chi_A}{\chi_B}\frac{1 - \frac{\chi_A}{\chi_B}}{4 - \frac{\chi_A}{\chi_B}} + 1 - \frac{\chi_A}{\chi_B}\right]^2$$

$$\pi_B^* = \frac{1}{4}\frac{\chi_B^2\theta_H^2}{(\theta_H - \theta_L)(\chi_B - \chi_A)}\left(\frac{\frac{\chi_A}{\chi_B}}{4 - \frac{\chi_A}{\chi_B}} + 1\right)^2\left(1 - \frac{\chi_A}{\chi_B}\right)^2$$

$$\pi_B^* = \frac{1}{4}\frac{\chi_B^2\theta_H^2}{(\theta_H-\theta_L)(\chi_B-\chi_A)}\left(\frac{4}{4-\frac{\chi_A}{\chi_B}}\right)^2\left(1-\frac{\chi_A}{\chi_B}\right)^2$$

$$\pi_B^* = 4\frac{\theta_H^2}{(\theta_H-\theta_L)}\frac{\chi_B^2}{(\chi_B-\chi_A)}\left(\frac{1-\frac{\chi_A}{\chi_B}}{4-\frac{\chi_A}{\chi_B}}\right)^2$$

$$\frac{\partial}{\partial\chi_B}\left(\frac{\chi_B^2}{(\chi_B-\chi_A)}\left(\frac{1-\frac{\chi_A}{\chi_B}}{4-\frac{\chi_A}{\chi_B}}\right)^2\right)$$

$$= \left(4\chi_B^2 - 3\chi_B\chi_A + 2\chi_A^2\right)\frac{\chi_B}{(4\chi_B-\chi_A)^3}$$

which is positive for the values of the numerical example.