Competition Policy, Collusion, and Tacit Collusion

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Abstract

In this paper, I pursue three goals. The first two are to develop a particular specification of a standard model of noncooperative collusion that permits explicit solution for equilibrium outputs and reversion thresholds, and to extend this analysis to allow for a deterrence-based competition policy that investigates conduct based on observed high prices (investigation thresholds). The third is to model collusion in a way that is distinct from noncooperative collusion.

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Contents

1 Introduction

From Friedman (1971a) onward, economists have modelled collusion as the noncooperative equilibrium of a repeated game. Porter (1983a, b) and Green and Porter (1984) extend Friedman's analysis to games in which there is a random element to demand and firms observe price but not firm-specific outputs. In this formulation, firms restrict output unless realized price falls below some threshold level, in which case they revert to Nash behavior forever (a grim trigger strategy) or for a specified number of periods.

Careful work in this tradition refers to the behavior that is studied as "tacit collusion" or "noncooperative collusion." But models of imperfectly competitive markets, and models of collusion are no exception, often have as a primary purpose the forming of advice for the conduct of antitrust policy. Yet there is a fundamental disconnect between treating collusion as the outcome of a noncooperative game and the antitrust concept of collusion. The legal offense of collusion requires that firms agree. For example (*Theatre Enterprises*, 346 U.S. 537 at 540–541, 1954):[1]

> The crucial question is whether respondents' conduct toward petitioner stemmed from independent decision or from an agreement, tacit or express. To be sure, business behavior is admissible circumstantial evidence from which the fact finder may infer agreement. ...But [the U.S. Supreme] Court has never held that proof of parallel business behavior conclusively establishes agreement, or, phrased differently, that such behavior itself constitutes a Sherman Act offense.

It is the essence of the noncooperative behavior that characterizes equilibrium in repeated games that agreement does not take place (Baker, 1993; Martin, 1993). Models of tacit collusion do describe a phenomenon that takes place in the real world, but that phenomenon is not collusion (Fellner, 1950, p. 54):[2]

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[1] For the European Union, see the Woodpulp decision (*A. Ahlström OY and others* v. *E.C. Commission* [1988] 4 CMLR 901; [1993] 4 CMLR 407).

[2] See also Friedman (1971b, p. 106):

> A noncooperative approach will, by contrast, involve each firm in isolated decision making. This is not to imply that each firm ignores the effects of its rivals' decisions on its own profit or of its own decision on its rivals behavior

> it should be realized that oligopolistic co-operation may stem largely from the spontaneous co-ordination of business policies, and that it does not presuppose direct contacts, or collusion in the sense proper.

To this extent, models of tacit collusion are fundamentally unsuited for the analysis collusion and of the impact of competition policy on collusion.

In this paper, I pursue three goals. The first two are to develop a particular specification of a standard model of noncooperative collusion that permits explicit solution for equilibrium outputs and reversion thresholds, and to extend this analysis to allow for a deterrence-based competition policy that investigates conduct based on observed high prices (investigation thresholds). The third is to model collusion in a way that is distinct from noncooperative collusion.

Section 2 presents that demand and cost specifications that are used in the rest of the paper. Section 3 considers the case of tacit collusion supported by a grim trigger strategy and compares tacit collusion with overt collusion, with and without competition policy. Consideration of a trigger strategy is a useful starting point to develop the distinction between tacit collusion and collusion and lay out the intuition behind the impact of competition policy on market performance.

2 Demand and cost

Let realized price p be

$$p = P\left(Q\right) + \varepsilon. \tag{1}$$

The random part of demand ε is assumed to have zero mean. $P\left(Q\right)$ is thus expected demand. Assume that $P' < 0$ and that ε has a well-behaved density function $f\left(\varepsilon\right)$, defined on the interval

$$\underline{\varepsilon} \leq \varepsilon \leq \overline{\varepsilon} \leq \infty. \tag{2}$$

Assume also that

$$\begin{array}{ll} f'\left(\varepsilon\right) > 0 & \varepsilon < 0 \\ f'\left(\varepsilon\right) < 0 & \varepsilon > 0 \end{array}. \tag{3}$$

(and hence its own profits). Noncooperative formulations will generally assume the firms not to make decisions jointly.

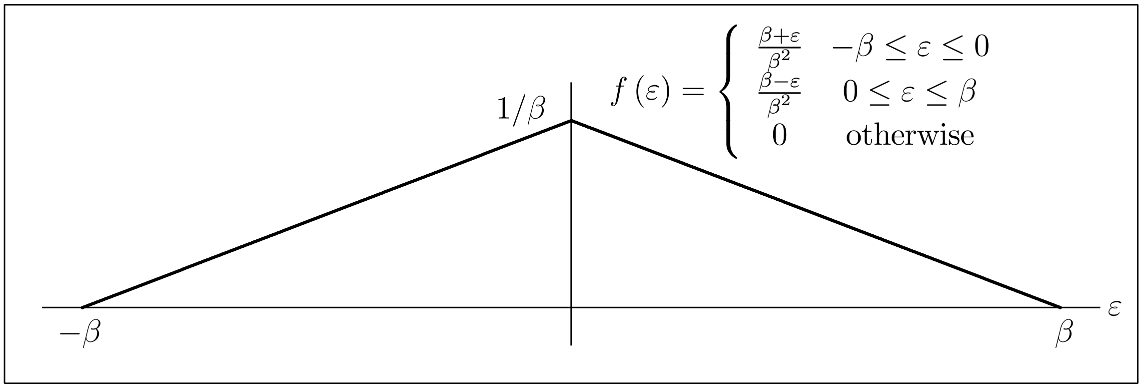

Figure 1: Triangle density function.

These assumptions are satisfied by the case of linear inverse expected demand

$$P = a - bQ \tag{4}$$

and a random part of demand distributed according to the symmetric triangular distribution. The equation of the density function of the symmetric triangular distribution is

$$f(\varepsilon) = \begin{cases} \frac{\beta+\varepsilon}{\beta^2} & -\beta \le \varepsilon \le 0 \\ \frac{\beta-\varepsilon}{\beta^2} & 0 \le \varepsilon \le \beta \\ 0 & \text{otherwise} \end{cases} . \tag{5}$$

Figure 1 shows the density function (5). The triangular distribution has appealing properties — more probability mass is centrally located than in the tails, variance ($\beta^2/6$) rising with β — and is tractable, in contrast to other distributions (such as the truncated normal) with similar properties.[3] The specific results of the paper will be presented for the case of linear inverse expected demand and triangular density for the random part of demand.

Finally, assume that firms produce at constant average and marginal cost c per unit of output. For simplicity in evaluation of expected profit, assume also that if firms produce the noncooperative equilibrium output of a one-shot game, the least possible realized price is not less than marginal cost:

$$P(Q_N) + \underline{\varepsilon} \ge c. \tag{6}$$

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[3] On the triangular distribution, see Johnson *et al.* (1995, p. 297–298). A random variable with the triangle distribution is, among other characterizations, the sum of two independently distributed uniform random variables (Freeman, 1963, pp. 176–177).

3 Trigger strategy

In this context, the trigger strategy model of noncooperative collusion takes the form that if price falls below a threshold price L,[4] firms revert to forever playing the noncooperative equilibrium strategy of a one-shot game.[5] In what follows, I characterize the one-shot game, trigger strategy, and collusive equilibria, without and with a deterrence-based competition policy.

3.1 No competition policy

3.1.1 One-shot game (equivalent) outcome

If all other firms produce output Q_{-i}, the value of firm i satisfies

$$V_i = \frac{\pi_i}{1+r} + \frac{1}{1+r} V_i, \tag{7}$$

where at the end of the period, the firm receives expected payoff

$$\pi_i = \left[P\left(Q_{-i} + q_i\right) - c\right] q_i. \tag{8}$$

Combining terms,

$$V_i = \frac{\left[P\left(Q_{-i} + q_i\right) - c\right] q_i}{r}. \tag{9}$$

Equilibrium output for firm i must maximize V_i, given the output of all other firms. The first-order condition is

$$P\left(Q_{-i} + q_i\right) - c + q_i P'\left(Q_{-i} + q_i\right) \equiv 0. \tag{10}$$

In symmetric equilibrium, all firms produce the same output. The condensed first-order condition, which defines equilibrium output, is

$$P\left(nq_N\right) - c + q_N P'\left(nq_N\right) \equiv 0. \tag{11}$$

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[4] Ellison (1994) discusses a variety of signals that might be monitored by output-restricting firms.

[5] For the case without uncertainty, see Friedman (1971). The specification considered in this section is a special case of the mechanism analyzed by Porter (1983) and Green and Porter (1984). Results for the case in which firms resume output restriction after a fixed number of periods (corresponding to Result 2) are available on request. In the case, expected firm value rises as the length of the reversion period increases, so that it is the grim trigger strategy that maximizes firm value.

For equilibrium firm value, use (11) to write

$$V_N = \frac{[P(Q_N) - c]\, q_N}{r} = \frac{-P'(nq_N)\, q_N^2}{r} = \frac{\pi_N}{r}, \tag{12}$$

where

$$Q_N = nq_N. \tag{13}$$

For linear inverse expected demand (4), (10) and (12) reduce to the familiar

$$q_N = \frac{1}{n+1}\frac{a-c}{b} \qquad \pi_N = bq_N^2 = b\left(\frac{1}{n+1}\frac{a-c}{b}\right)^2. \tag{14}$$

3.1.2 Tacit collusion supported by a trigger strategy

Suppose that by use of a trigger strategy, firms can noncooperatively restrict total output to a level Q_{ts}, output per firm to a level q_{ts}, with

$$Q_{ts} = nq_{ts}. \tag{15}$$

At the end of the first period, each firm has expected payoff

$$\pi_{ts} = [P(Q_{ts}) - c]\, q_{ts}. \tag{16}$$

With probability ρ_{ts}, realized price is below the threshold price L (which is determined as part of the trigger strategy) and firms revert forever to the Nash equilibrium outputs of a one-shot game.

The probability of reversion with output Q is

$$\rho(P-L) = \Pr(p \leq L) = \Pr[P(Q) + \varepsilon \leq L]$$

$$= \Pr\{\varepsilon \leq -[P(Q) - L]\} = \int_{\underline{\varepsilon}}^{-[P(Q)-L]} f(\varepsilon)\, d\varepsilon, \tag{17}$$

where $P(Q) - L > 0$ when expected price exceeds the reversion threshold. See Figure 2 for the triangular distribution.[6]

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[6] I limit attention to cases that satisfy

$$\beta \geq P - L > 0. \tag{18}$$

This assumption simplifies the calculation of ρ by keeping ρ in the range $0 \leq \rho \leq \frac{1}{2}$. If $P - L > \beta$, then $P - \beta > L$, the lowest possible realized price exceeds the reversion threshold, and $\rho = 1$. If $P - L < -\beta$, then $P + \beta < L$, the highest possible price is below the reversion threshold, and $\rho = 0$. For $-\beta \leq P - L \leq 0$, $\rho = 1 - \frac{1}{2}\left(\frac{\beta - L + P}{\beta}\right)^2$.

Increasing output or raising the threshold price increases the probability of reversion:

$$\frac{\partial \rho}{\partial Q} = -f\left[-\left(P-L\right)\right]P' > 0 \qquad \frac{\partial \rho}{\partial L} = f\left[-\left(P-L\right)\right] > 0. \tag{19}$$

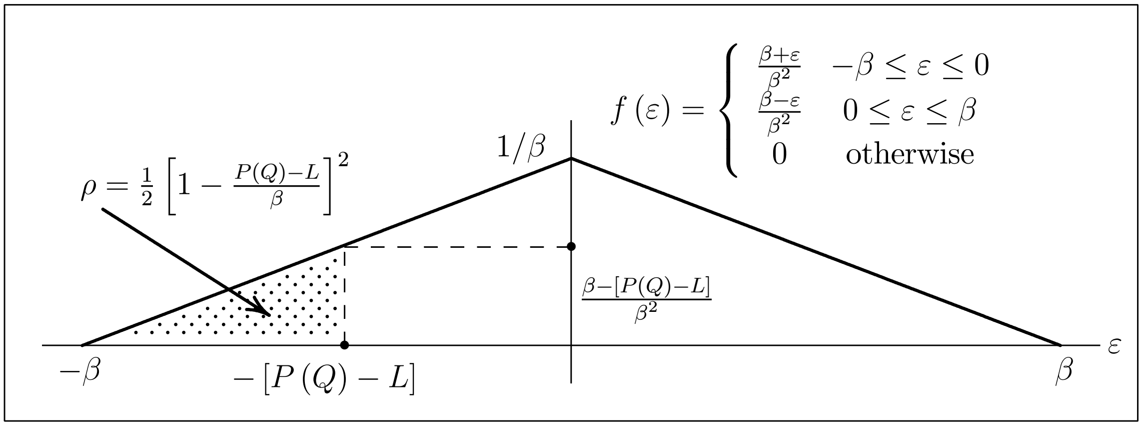

Figure 2: Probability of reversion, triangle density function; Q = output, L = reversion threshold.

If reversion occurs, the present discounted value of income of the firm, discounted to the beginning of the first reversionary period, is V_N. With probability $1-\rho_{ts}$, there is no reversion, and firm value from the end of the first period is again V_{ts}. V_{ts} thus satisfies the recursive relationship

$$V_{ts} = \frac{\pi_{ts}}{1+r} + \frac{\rho_{ts}}{1+r}\frac{\pi_N}{r} + \frac{1-\rho_{ts}}{1+r}V_{ts}, \tag{20}$$

so that

$$V_{ts} = V_N + \frac{\pi_{ts}-\pi_N}{r+\rho_{ts}}. \tag{21}$$

The tacit collusion problem is to maximize V_{ts} by choice of q_{ts} and L, subject to the constraint that the chosen values imply that no firm has an incentive to defect from the trigger strategy.

To formalize the no-defection constraint, note that if all other firms produce combine output Q_{-i}, the value of firm i is

$$V_i = V_N + \frac{\pi_i - \pi_N}{r+\rho_i} = V_N + \frac{\left[P\left(Q_{-i}+q_i\right)-c\right]q_i - \pi_N}{r + \int_{\underline{\varepsilon}}^{-\left[P\left(Q_{-i}+q_i\right)-L\right]} f\left(\varepsilon\right)d\varepsilon}. \tag{22}$$

For the trigger strategy to be an equilibrium, it must be that if each other firm produces its tacit collusion output q_{ts}, firm i's best response is to produce q_{ts}. That is, it must be that the first-order condition for maximization of (22) with respect to q_i holds for $Q_{-i} = (n-1)\, q_{ts}$ and $q_i = q_{ts}$:

$$(r + \rho_{ts})\left(P_{ts} - c + q_{ts} P'_{ts}\right) + \left(\pi_{ts} - \pi_N\right) f_{ts} P'_{ts} \equiv 0 \tag{23}$$

(where the subscript ts indicates that a function is evaluated at trigger strategy equilibrium values of its arguments).[7]

The first-order condition (23) can be rewritten[8]

$$\frac{\pi_{ts} - \pi_N}{r + \rho_{ts}} + \frac{P_{ts} - c + q_{ts} P'_{ts}}{f_{ts} P'_{ts}} \equiv 0. \tag{26}$$

The trigger strategy problem can then be formulated as

$$\max_{q_{ts}, L} V_{ts} - V_N \quad \ni \quad \left.\frac{\partial\left(V_i - V_N\right)}{\partial q_i}\right|_{eq} \equiv 0, \tag{27}$$

or equivalently, using (26),

$$\max_{q_{ts}, L} \frac{\pi_{ts} - \pi_N}{r + \rho_{ts}} \quad \ni \quad \frac{\pi_{ts} - \pi_N}{r + \rho_{ts}} = -\frac{P_{ts} + q_{ts} P'_{ts} - c}{f_{ts} P'_{ts}}. \tag{28}$$

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[7] I assume that the second-order condition,

$$\frac{\partial^2}{\partial q_i^2}\left(V_i - \frac{\pi_N}{r}\right)\Bigg|_{ts} = \left.\frac{\left(r + \rho_i\right)\frac{\partial^2 \pi_i}{\partial q_i^2} - \left(\pi_i - \pi_N\right)\frac{\partial^2 \rho_i}{\partial q_i^2}}{\left(r + \rho_i\right)^2}\right|_{ts} \tag{24}$$

$$= \frac{\left(r + \rho_{ts}\right)\left(2P'_{ts} + q_{ts} P''_{ts}\right) - \left(\pi_{ts} - \pi_N\right)\left[f'_{ts}\left(P'_{ts}\right)^2 - f_{ts} P''_{ts}\right]}{\left(r + \rho_i\right)^2} < 0 \tag{25}$$

for the defector's maximization problem is satisfied. Considering the numerator on the right in (24), the second-order condition is satisfied if firm profit is concave, and the probability of reversion convex, in the defector's output. Considering the numerator on the right in (25), $2P' + q_{ts} P'' < 0$ if the Hahn-Novshek condition is satisfied. Making this assumption, if P''_{ts} is small in magnitude, $f'_{ts} > 0$ is sufficient to make the second term, which is subtracted, positive.

[8] For $\pi_{ts} - \pi_N > 0$, (26) implies that marginal revenue is greater than marginal cost, $P_{ts} + q_{ts} P'_{ts} > c$, so that in trigger strategy equilibrium a firm restricts output below the privately profit-maximizing level.

Details of the solution of this constrained optimization problem for the case of linear inverse expected demand and the triangular distribution are given in Appendix I, which contains a proof of

Result 1: for linear inverse expected demand and the triangular distribution, the trigger strategy equilibrium output per firm and reversion threshold are

$$q_{ts} = \frac{1}{n}\left[\frac{a-c}{2b} + \frac{(n+1)^3}{n-1}\frac{\beta^2}{a-c}\frac{r}{b}\right] \tag{29}$$

and

$$L_{ts} = c + \frac{a-c}{2} - \beta - (n+1)^2\frac{\beta^2 r}{a-c}, \tag{30}$$

respectively.

From (29), q_{ts} exceeds the expected-joint-profit-maximizing level, $(a-c)/2nb$, and rises with β and r. Total output nq_{ts} rises as n rises. From (30), the reversion threshold falls as n, β, or r rises.

Other characteristics of the trigger strategy tacit collusion equilibrium follow from Result 1. The trigger strategy equilibrium price is[9]

$$P_{ts} = c + \frac{a-c}{2} - \frac{(n+1)^3}{n-1}\frac{\beta^2 r}{a-c}. \tag{33}$$

When the inverse demand curve has a random part, a low realized price might be the result of defection, or it might be the result of a negative realized value of the random part of demand. To align incentives so that defection does not occur in equilibrium, output is expanded above the monopoly level, reducing the expected price. The reversion threshold is reduced accordingly.

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[9] From (30) and (33),

$$P_{ts} - L_{ts} = \left[1 - \frac{2(n+1)^2}{n-1}\frac{\beta r}{a-c}\right]\beta. \tag{31}$$

Result 1 is obtained on the assumption that $P_{ts} - L_{ts} \geq 0$. This requires

$$\frac{2(n+1)^2}{n-1}\frac{\beta r}{a-c} \leq 1, \tag{32}$$

and (32) is satisfied for numerical results presented below.

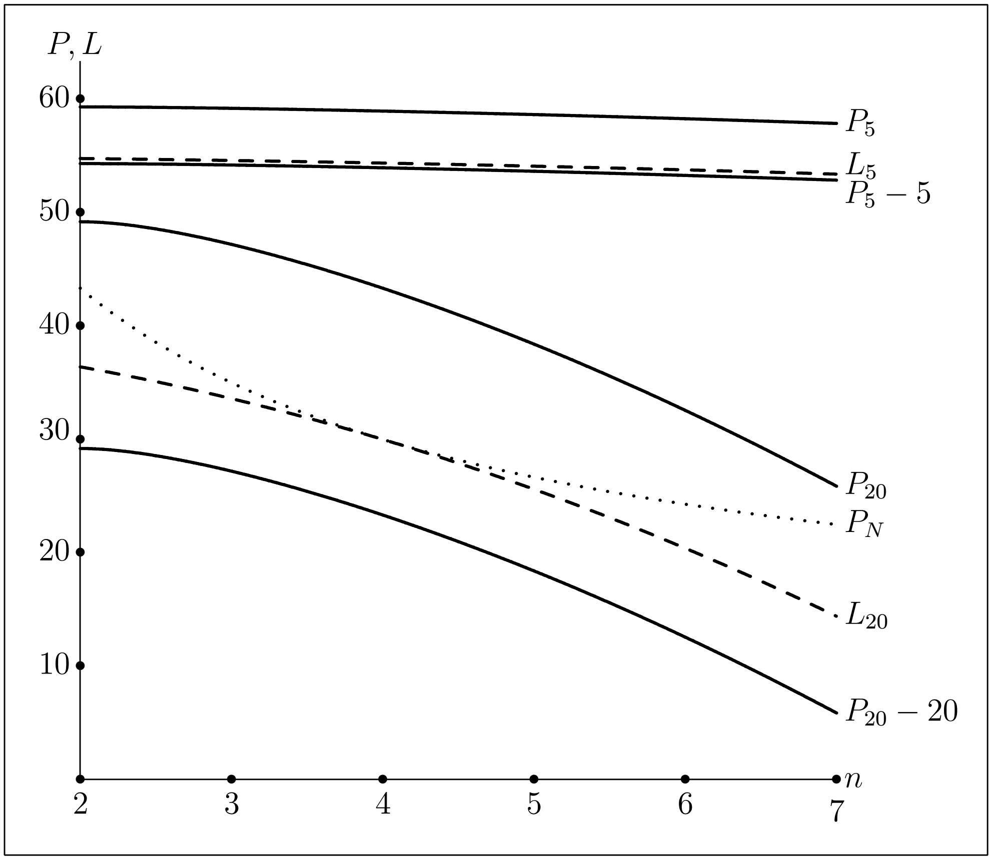

Figure 3: Tacit collusion price P, reversion threshold L, $P - \beta$, and Nash-Cournot price P_N, as functions of number of firms; $a = 110$, $c = 10$, $b = 1$, $r = 1/10$, $\beta = 5$ and $\beta = 20$.

Figure 3 shows the trigger strategy price and reversion threshold for a numerical example as functions of the number of firms (treated, for simplicity, as a continuous variable) and for two different values of β. When β is small, relative to the price axis intercept of expected inverse demand, the trigger strategy price is very nearly at the monopoly level (60), and it would require a negative ε near its maximum magnitude to trigger reversion. For large β, the trigger strategy price is much lower, approaching the one-shot-game Nash equilibrium level for $n = 7$.

Trigger strategy equilibrium firm value is

$$V_{ts} = V_N + \frac{b/n}{r + \rho_{ts}} \left\{ \left(\frac{n-1}{n+1} \frac{a-c}{2b} \right)^2 - \left[\frac{(n+1)^3}{n-1} \frac{\beta^2}{a-c} \frac{r}{b} \right]^2 \right\}, \tag{34}$$

from which $V_{ts} \geq V_N \Leftrightarrow$

$$a - c \geq \sqrt{2r} \frac{(n+1)^2}{n-1} \beta. \tag{35}$$

$V_{ts} - V_N$ rises with $a - c$ and falls as β, r, and n (for $n \geq 3$) rise. To interpret (35), note that for the present specification, $a - c$ is the difference between the price-axis intercept of the expected inverse demand curve, while $(-\beta, \beta)$ is the vertical range of the random part of demand. Thus $V_{ts} \geq V_N$ provided the range of the random part of demand is not too great.

Figure 4 shows trigger strategy equilibrium firm value, for the numerical example of Figure 3, as the number of firms rises from 2 to 7. For $n = 7$ and $\beta = 5$, tacit collusion supported by a trigger strategy more than doubles firm value, compared with repeated play of the Nash-Cournot equilibrium of a one-shot game. For $n = 7$ and $\beta = 20$, the use of a trigger strategy means roughly 11 per cent more value than repeated play of the Nash-Cournot equilibrium.

Trigger strategy equilibrium reversion probability is

$$\rho_{ts} = \frac{1}{2} \left(\frac{\beta - a + L + nbq_{ts}}{\beta} \right)^2 = 2 \left[\frac{(n+1)^2}{n-1} \frac{\beta r}{a-c} \right]^2. \tag{36}$$

The mean time to reversion is $1/\rho_{ts}$. ρ_{ts} rises with β and r and falls as $a - c$ rises. It falls moving from 2 to 3 firms, and rises as n rises thereafter.

Table 1 reports equilibrium reversion probabilities for 2 through 7 firms, $a - c = 100$, $b = 1$, and $r = 1/10$. For low values of β, reversion probabilities

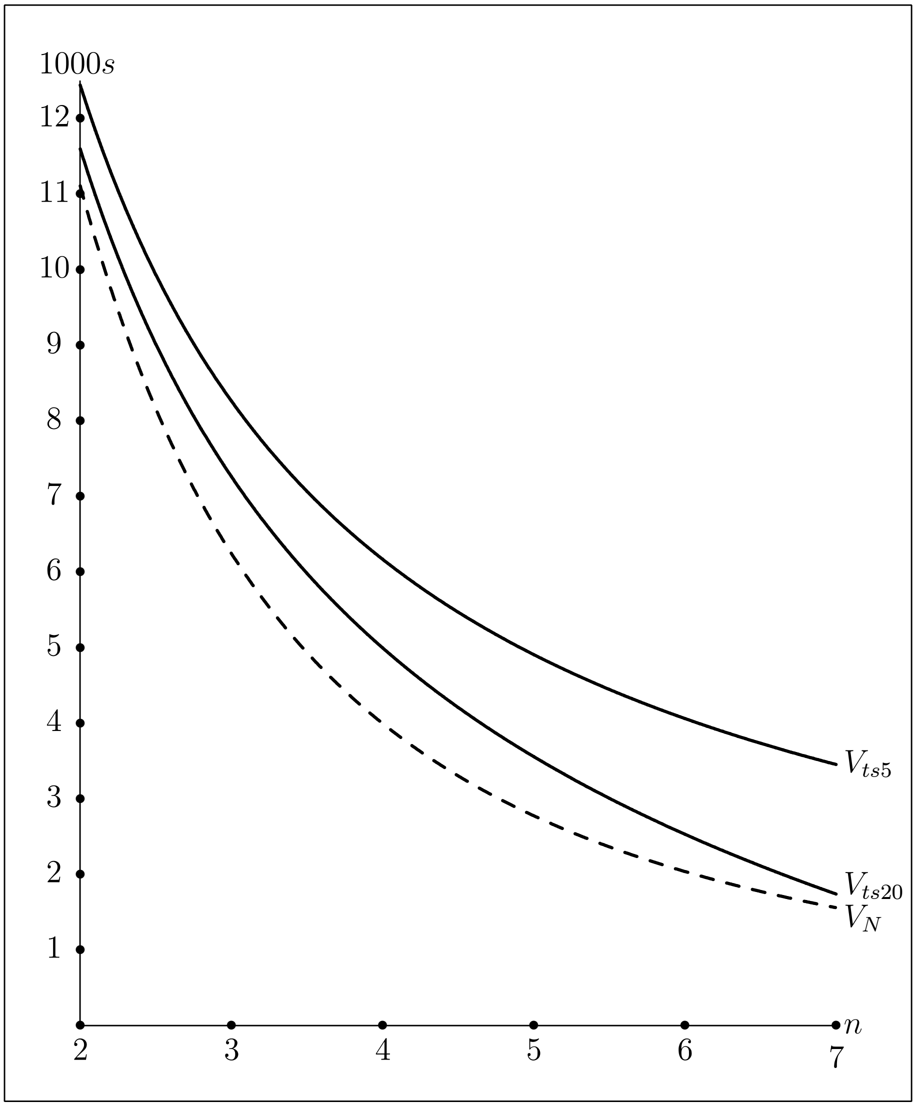

Figure 4: One-shot game and trigger strategy equilibrium firm value as functions of number of firms; $a = 110$, $c = 10$, $b = 1$, $r = 1/10$, for $\beta = 5$ and $\beta = 20$.

n	β			
$\downarrow$	5	10	15	20
2	0.0041	0.0162	0.0364	0.0648
3	0.0032	0.0128	0.0288	0.0512
4	0.0035	0.0139	0.0312	0.0556
5	0.0041	0.0162	0.0364	0.0648
6	0.0048	0.0192	0.0432	0.0768
7	0.0057	0.0228	0.0512	0.0910

Table 1: Equilibrium reversion probability as a function of the number of firms; $a = 110$, $c = 10$, $b = 1$, $r = 1/10$, $\beta = 5$, 10, 15, and 20.

are strikingly small. Even with 7 firms and relatively large β — price ranging plus or minus 20 when the expected tacit collusion price is 25.87 — the mean time to reversion is almost 11 periods.

3.1.3 Collusion

A trigger strategy allows firms to restrict output, until reversion occurs, and, provided (35) is satisfied, to increase value compared with repeated play of the one-shot game Cournot-Nash equilibrium. It does not allow firms to restrict output to the joint-profit-maximizing level: to make defection unattractive, firms expand total output above that level that would be offered by a monopoly supplier (see 29).

The result is that explicit agreement to restrict output may be profitable. Even in the absence of an antitrust or competition policy that prohibits such agreements, the approach of the Anglo-Saxon common law was that collusive agreements, while not affirmatively illegal, could not be enforced in courts of law.[10] In such circumstances, successful collusion requires the private investment of resources to establish mechanisms that sustain collusion.[11] Here I examine a stylized version of private enforcement, and suppose that by pay-

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[10] See, for example, *Mogul Steamship Co.* v. *McGregor, Gow & Co. et. al.* 54 L.J.Q.B. 540 (1884/1885); 57 L.J.K.B. 541 (1887/1888); 23 Q.B.D. 598 (C.A.)(1889); [1992] A.C. 25, or Judge Taft's discussion in U.S. *v.* Addyston Pipe and Steel 85 Fed 271 (1898).

[11] For discussions of the enforcement mechanisms employed by specific cartels, see Hudson (1891), Porter (1983b), Ellison (1994), Levinstein (1997), Grossman (1996), Genosove and Mullin (1998, 1999), Clyde, Paul S. and James D. Reitzes (1998), and Connor (2001). See also Hay and Kelly (1974).

ing a cost K per period per firm, firms can successfully restrict output to the joint-profit-maximizing level. K may be thought of as the cost per firm of setting up and maintaining procedures that would instantly detect a unilateral expansion of output above the collusive level should it occur, permitting immediate retaliation and thus rendering such defection unprofitable.

For linear inverse expected demand, joint-profit-maximizing output per firm is

$$q_m = \frac{1}{n}\frac{a-c}{2b}, \tag{37}$$

so that the value of a colluding firm is

$$V_m = \frac{(a-c-bnq_m)\,q_m - K}{r} = \frac{1}{r}\left[\frac{b}{n}\left(\frac{a-c}{2b}\right)^2 - K\right]. \tag{38}$$

Comparing (38) and trigger strategy equilibrium firm value, (34), some manipulation shows that $V_m \geq V_{ts}$ (that is, collusion yields at least as great a payoff as tacit collusion) provided the cost of collusion does not exceed a critical level:

$$K \leq K^* = \frac{(n+1)^2}{2n}\frac{\beta^2 r}{b}. \tag{39}$$

As should be expected, K^* rises with n: the greater the number of firms, the less is V_{ts}, the greater the incremental value from collusion, and the greater the cost of collusion that can be supported without making collusion less profitable than tacit collusion.

3.2 Competition policy

A Competition Authority with limited resources[12] will not be able to continuously monitor all firms in all industries. It will have to evaluate signals of various kinds to determine which industries invite close examination. I suppose here that the Competition Authority has responsibility for enforcing an anticollusion policy[13] and (abstracting from such sources of information

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[12] The Wall Street Journal reports (26 June 2003, internet edition, "Endless EU Antitrust Investigations Leave Enough Time for Frustration") that "the U.S. Federal Trade Commission and Department of Justice's Antitrust Division in 2002 shared an annual budget of $206 million ... EU enforcers in 2002 had an annual budget of €60 million...".

[13] Thus, I do not deal with competition policy toward single-firm conduct (monopolization under U.S. antitrust law, abuse of a dominant position in the European Union), toward changes in market structure, or toward state aid to business (an element of European Union competition policy).

as complaints from consumers or from rivals) that it relies on price signals to determine industries that will be investigated.[14]

Specifically, I suppose that for each industry, the Competition Authority sets an investigation threshold U and investigates the industry if realized price equals or exceeds U. Investigation is required because the observation of a high price, in and of itself, is insufficient to determine whether or not collusion has occurred. A high realized price may reflect a large positive disturbance to the expected inverse demand curve; it may be the result of tacit collusion rather than collusion.

The probability of an investigation with output Q is

$$\tau = \Pr\left(p \geq U\right) = \Pr\left[P\left(Q\right) + \varepsilon \geq U\right]$$

$$= \Pr\left[\varepsilon \geq U - P\left(Q\right)\right] = \int_{U-P(Q)}^{\overline{\varepsilon}} f\left(\varepsilon\right) d\varepsilon. \tag{40}$$

For the triangular distribution,[15]

$$\tau = \frac{1}{2}\left[1 - \frac{U - P\left(Q\right)}{\beta}\right]^2 \tag{42}$$

(see Figure 5).

There is inherent uncertainty in the enforcement process, and this uncertainty encompasses not only the types of behavior that may trigger an investigation but also the nature of the result that will follow if an investigation takes place.

Depending on the details of the enforcement regime, the Competition Authority may have the power to make an initial finding in its own right, in which case that finding is typically open to appeal the courts. Alternatively, the Competition Authority's options after an investigation may be to drop the matter or to initiate action in the courts, where the first finding would

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[14] See Souam (2001) for an analysis of this type of antitrust regime when the antitrust authority is uncertain of the level of constant marginal cost.

[15] I limit attention to the case

$$0 \leq U - P\left(Q\right) \leq \beta, \tag{41}$$

which simplifies the calculation of τ and keeps τ in the range $0 \leq \tau \leq \frac{1}{2}$. If $U - P > \beta$, then $U > P + \beta$ and investigation never occurs. If $U - P < -\beta$, then $U < P - \beta$ and investigation always occurs. For $-\beta \leq U - P\left(Q\right) \leq 0$, $\tau = 1 - \frac{1}{2}\left(\frac{\beta+U-P}{\beta}\right)^2$.

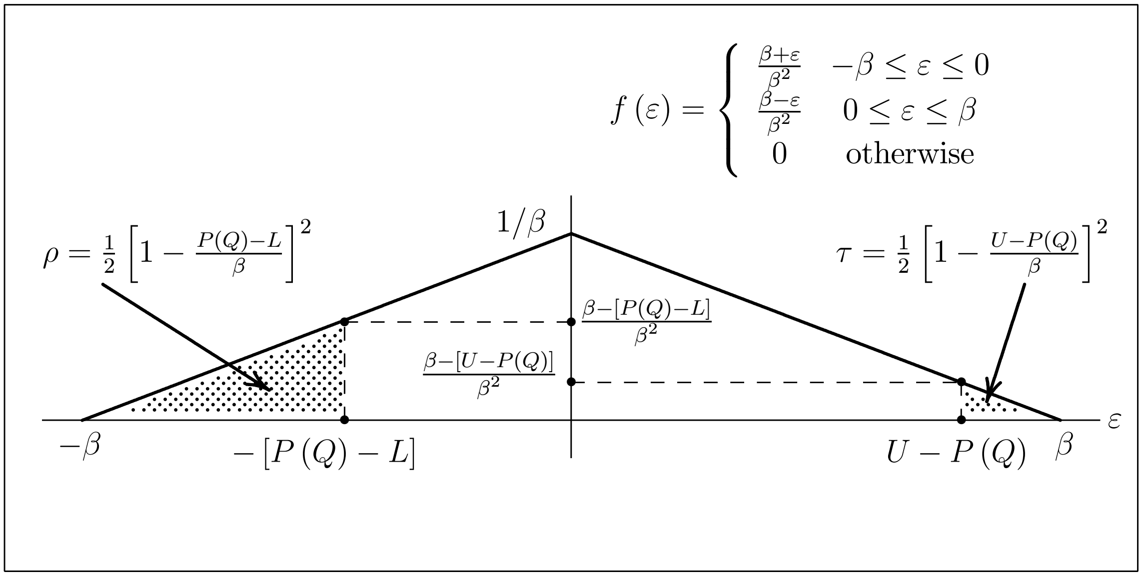

Figure 5: Triangle density function, probability of reversion, probability of investigation; $Q =$ output, $L =$ reversion threshold, $U =$ investigation threshold.

be made. Firms cannot predict with certainty what conclusion the Competition Authority will reach based on a particular set of facts. Neither firms nor the Competition Authority can predict with certainty what conclusions a court will reach if faced with a particular set of arguments.[16] Here I meld this uncertainty into a single parameter, ω, the probability competition authority/courts make a mistake, if there is an investigation.[17] Let F denote

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[16] In the *Woodpulp* case, the European Commission found that collusion had taken place in violation of Article 81 of the EC Treaty. This decision was overturned by the European Court of Justice ([1988] 4 CMLR 901) on the ground (as described by an economist) that the Commission had not shown that observed market behavior was not the result of tacit collusion rather than collusion. In *U.S.* v. *Pfizer* (367 F. Supp. 91 (S.D.N.Y. 1973), the Court of Appeals was unwilling to find collusion in violation of the Sherman Act, despite a record showing meetings by the presidents of the firms alleged to have colluded at which the matters alleged to be the subject of a collusive agreement were discussed.

[17] It would be possible to distinguish type I (finding that a colluding firm had not colluded) and type II error (convicting a firm that had not colluded), or to allow for distinct probabilities of making a mistake on the part of the Competition Authority and the courts. In the interest of simplicity, these extensions are not pursued here.

the total fine in the event firms are found to have colluded.[18]

3.2.1 One-shot game (equivalent) outcome

Firms that behave noncooperatively (whether in a one-shot or a repeated game) do not, in principle, risk being penalized for having colluded unless the enforcement process makes a mistakes. If mistakes are possible, however, firms that behave noncooperatively may be fined, and the possibility of such a fine affects equilibrium behavior.

If in any period all other firms produce output Q_{-i}, firm i's value is

$$V_i = \frac{\pi_i - \omega \tau_N \frac{F}{n}}{r} = \frac{\left[P\left(Q_{-i}+q_i\right)-c\right] q_i - \frac{\omega F}{n} \int_{U-P\left(Q_{-i}+q_i\right)}^{\bar{\varepsilon}} f\left(\varepsilon\right) d\varepsilon}{r}, \qquad (43)$$

which assumes that firms divide a fine equally if they are mistakenly found to have colluded.

The equilibrium first-order condition to maximize V_i is

$$P\left(nq_N\right) - c + \left\{ q_N - \frac{\omega F}{n} f\left[U - P\left(nq_N\right)\right] \right\} P'\left(nq_N\right) \equiv 0, \qquad (44)$$

and this implicitly defines Nash-Cournot equilibrium output per firm if firms play noncooperatively with a one-period time horizon.

For linear inverse demand (4) and the triangular distribution (5), the output implied by (44) is

$$q_N = \frac{a - c + \frac{\omega F b}{n\beta^2}\left(a + \beta - U\right)}{\left(n + 1 + \frac{\omega F b}{\beta^2}\right) b}, \qquad (45)$$

which reduces to (14) if $\omega = 0$.

Equilibrium firm value is

$$V_N = \frac{1}{r}\left[\left(a - c - nbq_N\right) q_N - \frac{\omega F}{2n\beta^2}\left(\beta - U + a - nbq_N\right)^2\right]. \qquad (46)$$

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[18] In a broader sense, F may be thought of as including expenses incurred by firms during the legal process. In a generalized model, F might be a function of a firm's turnover (as in the European Union) or of collusive profits. The latter introduces a dynamic element to the analysis, as considered by Harrington (2001).

3.2.2 Trigger strategy

Consider a trigger strategy of the form analyzed in Section 3.1.2. If all other firms produce combined output Q_{-i}, the value of firm i satisfies

$$V_i - V_N = \frac{\pi_i - rV_N - \frac{\omega F}{n}\tau_i}{r + \rho_i}$$

$$= \frac{\left[P\left(Q_{-i}+q_i\right) - c\right] q_i - \frac{\omega F}{n}\int_{U-P(Q_{-i}+q_i)}^{\overline{\varepsilon}} f\left(\varepsilon\right) d\varepsilon - rV_N}{r + \int_{\underline{\varepsilon}}^{-[P(Q_{-i}+q_i)-L]} f\left(\varepsilon\right) d\varepsilon}. \tag{47}$$

The first-order condition to maximize (47) is

$$\frac{\left(r+\rho_i\right)\left(\frac{\partial \pi_i}{\partial q_i} - \frac{\omega F}{n}\frac{\partial \tau_i}{\partial q_i}\right) - \left(\pi_i - \frac{\omega F}{n}\tau_i - rV_N\right)\frac{\partial \rho_i}{\partial q_i}}{\left(r+\rho_i\right)^2} \equiv 0. \tag{48}$$

For the trigger strategy to be stable, (48) must hold in equilibrium; this requires

$$\frac{\pi_{ts} - \frac{\omega F}{n}\tau_{ts} - rV_N}{r + \rho_{ts}} = \left.\frac{\frac{\partial \pi_i}{\partial q_i} - \frac{\omega F}{n}\frac{\partial \tau_i}{\partial q_i}}{\frac{\partial \rho_i}{\partial q_i}}\right|_{ts}. \tag{49}$$

The trigger strategy problem is to

$$\max_{q_{ts},L} \frac{\pi_{ts} - \frac{\omega F}{n}\tau_{ts} - rV_N}{r + \rho_{ts}} \tag{50}$$

subject to the constraint (49).

The solution to this problem is reached in the same way as is Result 1, and is given in Appendix I.

Result 2: for linear inverse expected demand and the triangular distribution, the trigger strategy output per firm and reversion threshold with competition policy are

$$q_{ts} =$$

$$\frac{1}{n}\frac{1}{2+\frac{\omega F b}{\beta^2}}\left[\frac{a-c}{b} + \frac{\omega F}{\beta^2}\left(a+\beta-U\right) + \frac{\left(n+1+\frac{\omega F b}{\beta^2}\right)^3}{n-1}\frac{2\beta^2}{a-c+\frac{\omega F b}{\beta^2}\left(U-c-\beta\right)}\frac{r}{b}\right] \tag{51}$$

and

$$L_{ts} = c - \beta +$$

$$\frac{1}{2+\frac{\omega F b}{\beta^2}}\left[a-c-\frac{\omega F b}{\beta^2}\left(c+\beta-U\right)-\left(n+1+\frac{\omega F b}{\beta^2}\right)^2\frac{2\beta^2 r}{a-c+\frac{\omega F b}{\beta^2}\left(U-c-\beta\right)}\right], \tag{52}$$

respectively.

From Result 2, it follows that the trigger strategy equilibrium price is

$$P_{ts} = c +$$

$$\frac{1}{2+\frac{\omega F b}{\beta^2}}\left[a-c-\frac{\omega F b}{\beta^2}\left(c+\beta-U\right)-\frac{\left(n+1+\frac{\omega F b}{\beta^2}\right)^3}{n-1}\frac{2\beta^2 r}{a-c+\frac{\omega F b}{\beta^2}\left(U-c-\beta\right)}\right]. \tag{53}$$

(53) and (52) imply

$$\frac{\partial P_{ts}}{\partial U} \geq 0 \text{ and } \frac{\partial L}{\partial U} \geq 0, \tag{54}$$

where each derivative is equal to 0 if and only if $\omega = 0$. If there is some possibility that noncooperative output restriction supported by a trigger strategy will lead to a penalty, then a lower investigation threshold (a tougher competition policy) means a lower equilibrium price and a lower reversion threshold, all else equal.

(53) and (52) also imply that the equilibrium reversion probability is

$$\rho_{ts} = 2\left[\frac{\left(n+1+\frac{\omega F b}{\beta^2}\right)^2}{n-1}\frac{\beta r}{a-c+\frac{\omega F b}{\beta^2}\left(U-c-\beta\right)}\right]^2, \tag{55}$$

from which

$$\frac{\partial \rho_{ts}}{\partial U} \leq 0 \tag{56}$$

(see Figure 8). Once again, the derivative equals zero if and only if $\omega = 0$. Otherwise, a lower investigation threshold increases the probability of reversion.

Figure 6 illustrates Result 2 for particular parameter values. Expected price is below, and the maximum possible realized price above, the investigation threshold. Expected price is above, and the lowest possible realized

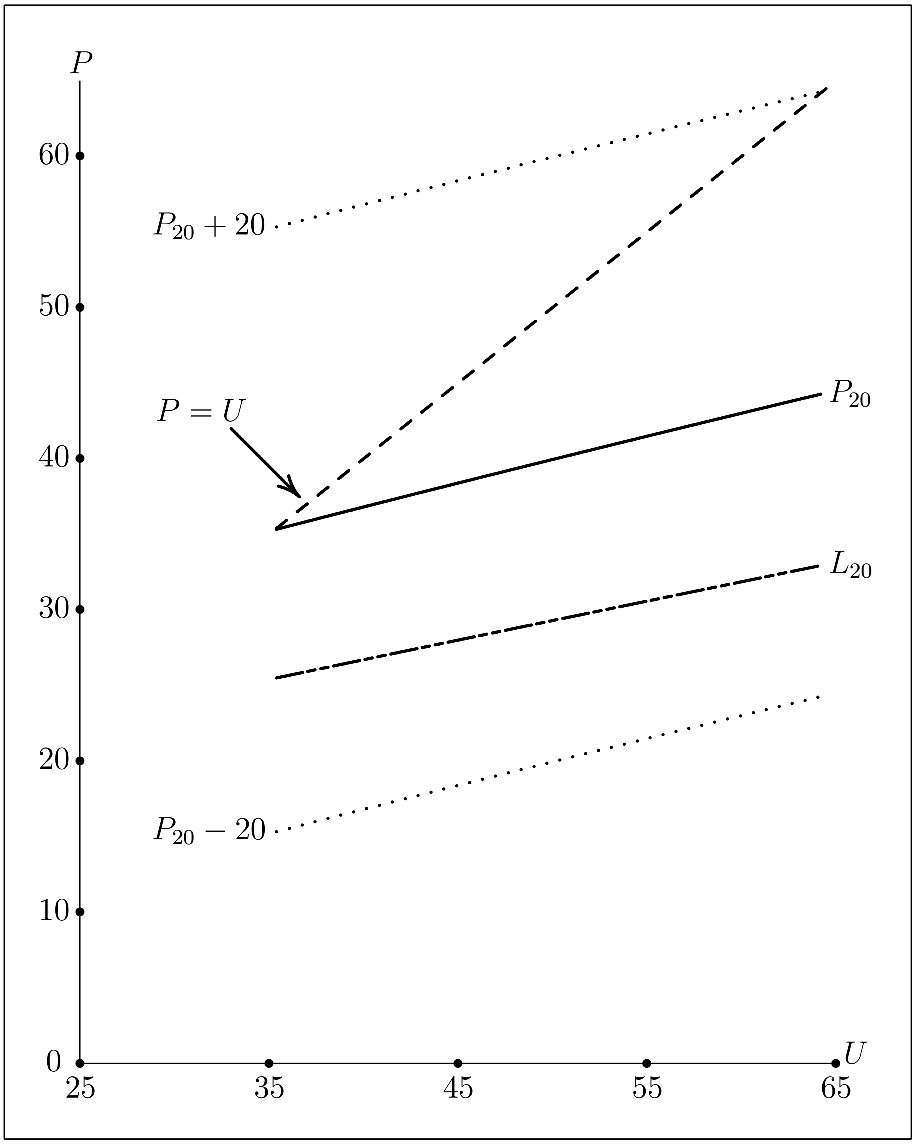

Figure 6: Tacit collusion price P, and reversion threshold L as function of investigation threshold U. $a = 110$, $c = 10$, $b = 1$, $r = 1/10$, $\beta = 20$, $n = 2$, $\omega = 1/5$, $F = 1250$.

price below, the reversion threshold. As the investigation threshold falls — as competition policy becomes tougher — the gap between U and P generally becomes smaller, and the probability of investigation rises.

This can also be seen in Figure 7, which shows the relationship between ρ and U for three value of β.[19] While the probability of investigation tends generally to rise as U falls, for very low U and small β, τ rises as U rises.

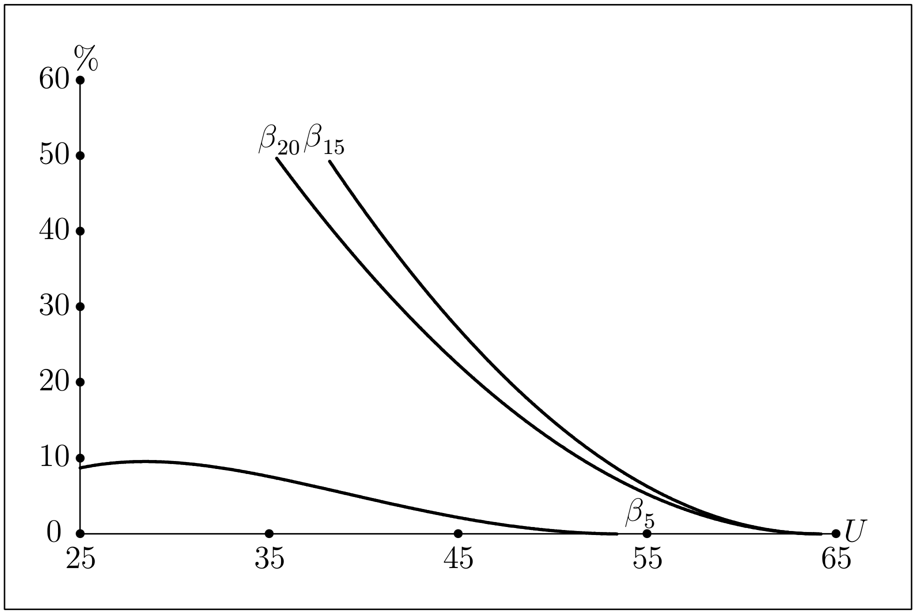

Figure 7: Probability of investigation τ as a function of investigation threshold U. $a = 110$, $c = 10$, $b = 1$, $r = 1/10$, $\beta = 5, 15, 20$, $n = 2$, $\omega = 1/5$, $F = 1250$.

As U falls, the gap between P and L also becomes smaller. The impact of changes in U on ρ is generally smaller in magnitude than the impact of changes in U on τ; this can also be seen in Figure 8.

For a fixed value of U, the impact of changes in β on both τ and ρ is nonlinear, as shown in Figures 9 and 10.

For parameter values that yield typical results, lower values of U — tougher competition policy — reduce trigger strategy equilibrium value (Ta-

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[19] The U–τ curve for $\beta = 10$ very nearly coincides with that for $\beta = 20$, and is omitted from Figure 7 to avoid visual clutter.

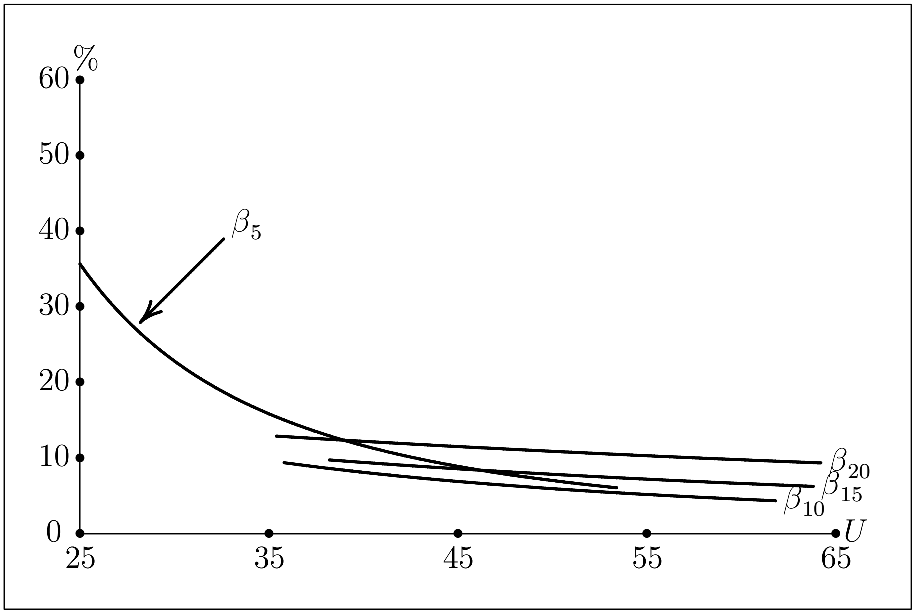

Figure 8: Probability of reversion ρ as a function of investigation threshold U. $a = 110$, $c = 10$, $b = 1$, $r = 1/10$, $\beta = 5, 20$, $n = 2$, $\omega = 1/5$, $F = 1250$.

ble 2).[20] For low values of β, the trigger strategy yields greater equilibrium firm value than repeated play of the equilibrium output of a one-shot game. For high values of β it is the one-shot game output that yields greatest value: when uncertainty is large, the trigger strategy requires a sufficiently great output expansion to nullify the temptation to defect that firms are better off without it. For a narrow intermediate range of β, the trigger strategy yields greater value for high U, the one-shot game strategy for low U.

3.2.3 Collusion

If firms collude, reversion is not an issue: the per-firm cost K per puts mechanisms in place that render defection unprofitable.[21] Then in the presence

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[20] For calculations underlying Table 2, U is less than 60 (the monopoly price) and otherwise limited to a range that keeps reversion and investigation probabilities between 0 and 1/2. See Footnotes 6 and 15.

[21] Ellison (1994, p. 52) notes that there are many possible explanations for apparent secret price cuts. One explanation, suggested by the results of this section, is that changes in market conditions mean a change from an environment in which $V_m \geq \max(V_{ts}, V_N)$

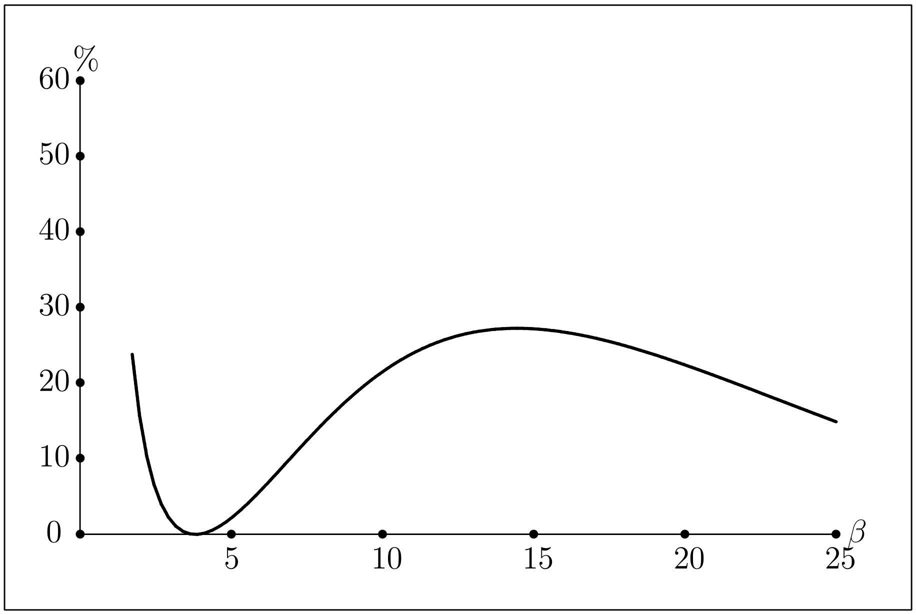

Figure 9: Probability of investigation τ as a function of β. $a = 110$, $c = 10$, $b = 1$, $r = 1/10$, $U = 45$, $\omega = 1/5$, $F = 1250$.

β	(U, V_N, V_{ts})	$V_{ts} = V_N$ for $U \approx$	(U, V_N, V_{ts})
6	$(44.7, 10625.90, 10922.84)$	N/A	$(49.3, 11108.13, 11477.41)$
7	$(46.3, 10754.70, 11133.19)$	N/A	$(50.3, 11108.55, 11552.47)$
10	$(48.1, 10800.08, 11299.79)$	N/A	$(53.3, 11109.47, 11685.79)$
15	$(46.9, 10668.16, 11085.21)$	N/A	$(58.3, 11110.24, 11640.34)$
20	$(47.5, 10690.36, 10837.33)$	N/A	$(60, 11051.09, 11280.74)$
21	$(48.1, 10711.85, 10787.13)$	N/A	$(60, 11036.96, 11185.23)$
22	$(48.7, 10731.24, 10728.25)$	49.25	$(60, 11024.19, 11085.58)$
23	$(49.2, 10745.78, 10657.65)$	N/A	$(60, 11012.61, 10981.74)$
24	$(49.6, 10756.21, 10576.43)$	N/A	$(60, 11002.07, 10873.68)$
25	$(50, 10765.87, 10488.64)$	N/A	$(60, 10992.43, 10761.36)$
30	$(51.7, 10798.55, 9950.847)$	N/A	$(60, 10954.66, 10134.90)$

Table 2: Investigation threshold and firm value, trigger strategy vs. repeated one-shot Nash values; $a = 110$, $c = 10$, $b = 1$, $r = 1/10$, $F = 625$, $\omega = 1/5$, $n = 2$.

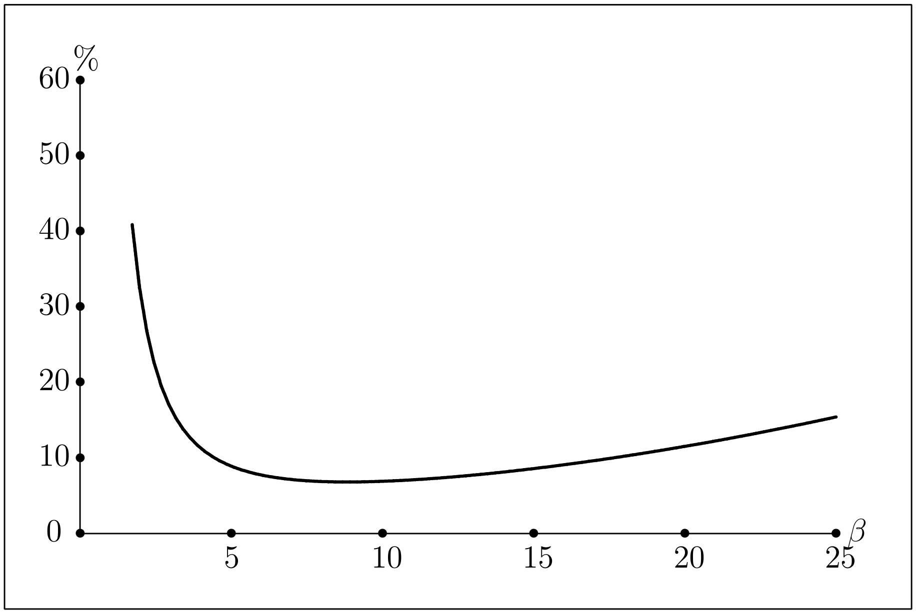

Figure 10: Probability of reversion ρ as a function of β. $a = 110$, $c = 10$, $b = 1$, $r = 1/10$, $U = 45$, $\omega = 1/5$, $F = 1250$.

of a competition policy, the value of a colluding firm is

$$V_m = \frac{1}{r}\left[\pi_m - K - \frac{(1-\omega)F}{n}\tau_m\right]. \tag{57}$$

A fine is levied if and only if there is an investigation and the legal system does not make a mistake.

One might well suspect that the cost of collusion, K, would be greater if there is an anticollusion policy, since successful collusion have to be kept secret. Leaving this aside from the formal model, the existence of an anticollusion policy affects collusive value in two ways.

First, to reduce the probability of investigation, firms will expand output above the no-competition policy-level (37). This output expansion reduces π_m.

Second, the expected fine $\frac{(1-\omega)F}{n}\tau_m$ reduces the margin $V_m - \max(V_{ts}, V_N)$ that is available to cover the cost of collusion.

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to one in which $V_m < \max(V_{ts}, V_N)$.

For linear inverse expected demand and the triangular distribution, (57) is

$$V_m = \frac{\left(a-c-bnq_m\right)q_m - K - \frac{(1-\omega)F}{2n\beta^2}\left(a+\beta-U-bnq_m\right)^2}{r}. \tag{58}$$

The first-order condition to maximize (58) is

$$a-c-2bnq_m + \frac{(1-\omega)\,bF}{\beta^2}\left(a+\beta-U-bnq_m\right) \equiv 0, \tag{59}$$

from which collusive output per firm is

$$q_m = \frac{1}{n}\frac{a-c+\frac{(1-\omega)bF}{\beta^2}\left(a+\beta-U\right)}{b\left[2+\frac{(1-\omega)bF}{\beta^2}\right]}. \tag{60}$$

Substituting the first-order condition into (58), equilibrium collusive firm value is.

$$V_m = \frac{1}{r}\left\{\left[1+\frac{(1-\omega)\,bF}{2\beta^2}\right]bnq_m^2 - \frac{(1-\omega)\,F}{2n\beta^2}\left(a+\beta-U\right)^2 - K\right\}. \tag{61}$$

If $V_{ts} \geq \frac{\pi_N}{r}$, the relevant comparison is between collusion and tacit collusion; collusion yields a greater equilibrium firm value than tacit collusion if

$$\left[1+\frac{(1-\omega)\,bF}{2\beta^2}\right]bnq_m^2 - \frac{(1-\omega)\,F}{2n\beta^2}\left(a+\beta-U\right)^2 - K \geq$$

$$\frac{\left(a-c-nbq_{ts}\right)q_{ts} - \frac{\omega F}{2n\beta^2}\left(a+\beta-U-nbq_{ts}\right)^2 - rV_N}{r+\frac{1}{2\beta^2}\left(\beta-a+L+nbq_{ts}\right)^2}. \tag{62}$$

Otherwise, repeated play of one-shot game equilibrium output yields a greater value than tacit collusion, and collusion yields at least as great a value as the one-shot game equilibrium if

$$\left[1+\frac{(1-\omega)\,bF}{2\beta^2}\right]bnq_m^2 - \frac{(1-\omega)\,F}{2n\beta^2}\left(a+\beta-U\right)^2 - K \geq$$

$$\left(a-c-nbq_N\right)q_N - \frac{\omega F}{2n\beta^2}\left(a+\beta-U-nbq_N\right)^2. \tag{63}$$

3.3 Comparison

Suppose $\omega = 0$, so that the legal system does not make mistakes. Then equilibrium outputs (45), (51), and (60) simplify to

$$q_N = \frac{1}{n+1} \frac{a-c}{b} \tag{64}$$

$$q_{ts} = \frac{1}{n} \left[\frac{a-c}{2b} + \frac{(n+1)^3}{n-1} \frac{\beta^2}{a-c} \frac{r}{b} \right] \tag{65}$$

and

$$q_m = \frac{1}{n} \frac{a - c + \frac{bF}{\beta^2} (a + \beta - U)}{b \left(2 + \frac{bF}{\beta^2} \right)}, \tag{66}$$

respectively.

For the no-mistakes case, competition policy has no impact on the one-shot-game equivalent or trigger strategy outcomes. If these regimes should be investigated, the enforcement regime would correctly conclude that collusion had not taken place, and no fine would be levied.

For the no-mistakes case, q_N is unaffected by β, while q_{ts} rises as β rises (see the discussion of (29)). (64) and (65) imply that $q_N \geq q_{ts}$, trigger strategy equilibrium output is weakly less than "one-shot game" equilibrium output, if and only if β is not too great relative to $a - c$,

$$a - c \geq \sqrt{2r} \left(n + 3 + \frac{4}{n-1} \right) \beta. \tag{67}$$

(See the discussion of (35).)

From (66) it follows that q_m may rise or fall as β rises, all else equal, and in fact both directions of change are realized for the parameters of Table 3.

From (65) and (66), $q_{ts} \geq q_m$ if and only if

$$\frac{(n+1)^3}{n-1} \frac{2\beta^2 r}{a-c} + \frac{\frac{bF}{\beta^2}}{2 + \frac{bF}{\beta^2}} \left[2 (U - c - \beta) - (a - c) \right] \geq 0.$$

While this condition is not as informative as one might like, it is met if $\frac{bF}{\beta^2}$ is sufficiently small (bF small implies that competition policy has little effect on equilibrium behavior) or if n is sufficiently large, all else equal.

	(q_N, V_N) CS_N	(q_{ts}, V_{ts}) CS_{ts}	(U, q_m, V_m) (CS_m, τ_m)	(U, q_m, V_m) (CS_m, τ_m)
$\beta = 5$	$(33.33, 11111)$ 22222	$(25.34, 12444)$ 13205	$(43, 35.19, 10259)$ $(24760, 0.05)$	$(60, 27.31, 12384)$ $(14922, 0.003)$
$\beta = 10$	$(33.33, 11111)$ 22222	$(26.35, 12275)$ 15049	$(43, 35.23, 9739)$ $(24819, 0.21)$	$(60, 28.79, 12121)$ $(16575, 0.03)$
$\beta = 20$	$(33.33, 11111)$ 22222	$(30.40, 11600)$ 19953	$(44.4, 32.81, 9721)$ $(21526, 0.50)$	$(60, 29.39, 11623)$ $(17271, 0.16)$
$\beta = 30$	$(33.33, 11111)$ 22222	$(37.15, 10475)$ 24411	$(49.6, 30.21, 10397)$ $(18248, 0.50)$	$(60, 28.87, 11340)$ $(16665, 0.28)$

Table 3: Equilibrium values, alternative β and modes of strategic interaction. $a = 110$, $c = 10$, $b = 1$, $r = 1/10$, $n = 2$, $F = 625$, $\omega = 0$.

Table 3 reports equilibrium output, firm value, and expected present discounted value of consumer surplus for the three different strategic regimes and (for collusion) for a range of values of the investigation threshold.[22] Values under collusion are measured before allowing for the cost of collusion. Thus, for $\beta = 30$ and $U = 60$, the difference $11340 - 10475 = 865$ gives the maximum cost of collusion that would make net firm value under collusion weakly greater than firm value under tacit collusion supported by a grim trigger strategy; the difference $11340 - 11111 = 229$ gives the maximum cost of collusion that would make net firm value under collusion weakly greater than firm value with repeated play of the one-shot game equilibrium output.

Assuming the legal system does not make mistakes, and for a fine upon conviction that is one-quarter of single-period monopoly profit, then holding β constant, a lower investigation threshold (a tougher competition policy) increases collusive output. Tacit collusion yields the greatest value for low values of β. Tacit collusion equilibrium value falls as β rises. For higher values of β, collusion yields the greatest value if the investigation threshold is set sufficiently high, the one-shot game strategy yields the greatest value if the investigation threshold is set sufficiently low.

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[22] Recall that for the parameters of Table 3 marginal cost is 10 and the unconstrained monopoly price 60. For Table 3 I have used 60 as the upper limit of the range of investigation thresholds considered, while the lower limit is the smallest value of U that keeps the probability of investigation below one-half.

4 Conclusion

The model of collusion developed here identifies collusion with the private investment that reduces (in the present idealized version, eliminates) the incentive to defect from an output-restricting equilibrium. It thus directs anticollusion enforcement attention toward market institutions and patterns of firm conduct that alter incentives to defect.

Greater expected antitrust penalties generally increase consumer welfare, making it more likely that tacit collusion will yield greater firm value than collusion and more likely that one-shot-game behavior will yield greater firm value than tacit collusion. These effects depend on expected antitrust fines be sufficiently large, and may be reversed if the enforcement system makes mistakes.

5 Appendix

5.1 Proof of Result 1

A Lagrangian for (28) is

$$\mathcal{L} = \frac{\pi_{ts} - \pi_N}{r + \rho_{ts}} + \lambda \left[\frac{\pi_{ts} - \pi_N}{r + \rho_{ts}} + \frac{P_{ts} + q_{ts} P'_{ts} - c}{f_{ts} P'_{ts}} \right] \tag{68}$$

$$= (1 + \lambda) \frac{\left[P\left(nq_{ts}\right) - c \right] q_{ts} - \pi_N}{r + \int_{-\beta}^{L - P(nq_{ts})} f\left(\varepsilon\right) d\varepsilon} + \lambda \frac{P\left(nq_{ts}\right) + q_{ts} P'\left(nq_{ts}\right) - c}{f\left[L - P\left(nq_{ts}\right)\right] P'\left(nq_{ts}\right)}. \tag{69}$$

5.1.1 Kuhn-Tucker necessary conditions

q_{ts}:

$$(1 + \lambda) \frac{\left(r + \rho_{ts}\right)\left(P_{ts} - c + nq_{ts} P'_{ts}\right) + n\left(\pi_{ts} - \pi_N\right) f_{ts} P'_{ts}}{\left(r + \rho_{ts}\right)^2}$$

$$+\lambda \frac{f_{ts} P'_{ts} \left[\left(n + 1\right) P'_{ts} + nq_{ts} P''_{ts}\right] - n\left(P_{ts} + q_{ts} P'_{ts} - c\right)\left[f_{ts} P''_{ts} - \left(P'_{ts}\right)^2 f'_{ts}\right]}{\left(f_{ts} P'_{ts}\right)^2} \equiv 0. \tag{70}$$

L:

$$-\left(1 + \lambda\right) \frac{\pi_{ts} - \pi_N}{\left(r + \rho_{ts}\right)^2} f_{ts} - \frac{\lambda}{P'_{ts}} \frac{P\left(nq_{ts}\right) + q_{ts} P'\left(nq_{ts}\right) - c}{\left(f_{ts}\right)^2} f'_{ts} \equiv 0. \tag{71}$$

λ: (this simply gives back the no-defection condition)

$$\frac{\pi_{ts}-\pi_N}{r+\rho_{ts}}+\frac{P_{ts}+q_{ts}P'_{ts}-c}{f_{ts}P'_{ts}}\equiv 0. \tag{72}$$

(72) implies

$$\left(r+\rho_{ts}\right)\left(P_{ts}+q_{ts}P'_{ts}-c\right)+\left(\pi_{ts}-\pi_N\right)f_{ts}P'_{ts}=0. \tag{73}$$

The numerator of the coefficient of $1+\lambda$ in (70) is

$$\left(r+\rho_{ts}\right)\left(P_{ts}-c+nq_{ts}P'_{ts}\right)+n\left(\pi_{ts}-\pi_N\right)f_{ts}P'_{ts}=$$

$$\left(r+\rho_{ts}\right)\left[P_{ts}+q_{ts}P'_{ts}-c+\left(n-1\right)q_{ts}P'_{ts}\right]+\left(\pi_{ts}-\pi_N\right)f_{ts}P'_{ts}+\left(n-1\right)\left(\pi_{ts}-\pi_N\right)f_{ts}P'_{ts}=$$

and using (73),

$$\left(n-1\right)\left[\left(r+\rho_{ts}\right)q_{ts}+\left(\pi_{ts}-\pi_N\right)f_{ts}\right]P'_{ts}. \tag{74}$$

Once again from (72)

$$\left(\pi_{ts}-\pi_N\right)f_{ts}=-\left(r+\rho_{ts}\right)\frac{P_{ts}+q_{ts}P'_{ts}-c}{P'_{ts}}. \tag{75}$$

Substitute in (74) and rearrange terms to obtain

$$\left(n-1\right)\left[\left(r+\rho_{ts}\right)q_{ts}-\left(r+\rho_{ts}\right)\frac{P_{ts}+q_{ts}P'_{ts}-c}{P'_{ts}}\right]P'_{ts}=-\left(n-1\right)\left(r+\rho_{ts}\right)\left(P_{ts}-c\right). \tag{76}$$

Hence the Kuhn-Tucker condition for q_{ts} can be written

$$-\left(n-1\right)\left(1+\lambda\right)\frac{P_{ts}-c}{r+\rho_{ts}}$$

$$+\lambda\frac{f_{ts}P'_{ts}\left[\left(n+1\right)P'_{ts}+nq_{ts}P''_{ts}\right]-n\left(P_{ts}+q_{ts}P'_{ts}-c\right)\left[f_{ts}P''_{ts}-\left(P'_{ts}\right)^2f'_{ts}\right]}{\left(f_{ts}P'_{ts}\right)^2}\equiv 0. \tag{77}$$

Simplify the numerator of the second term to obtain

$$-\left(n-1\right)\left(1+\lambda\right)\frac{P_{ts}-c}{r+\rho_{ts}}$$

$$+\lambda\frac{\left[(n+1)f_{ts}+n\left(P_{ts}+q_{ts}P'_{ts}-c\right)f'_{ts}\right]\left(P'_{ts}\right)^2-n\left(P_{ts}-c\right)f_{ts}P''_{ts}}{\left(f_{ts}P'_{ts}\right)^2}\equiv 0. \quad (78)$$

Now return to the Kuhn-Tucker condition for L:

$$(1+\lambda)\frac{\pi_{ts}-\pi_N}{\left(r+\rho_{ts}\right)^2}f_{ts}+\frac{\lambda}{P'_{ts}}\frac{P_{ts}+q_{ts}P'_{ts}-c}{\left(f_{ts}\right)^2}f'_{ts}\equiv 0$$

From the no-defection condition

$$\frac{\pi_{ts}-\pi_N}{r+\rho_{ts}}f_{ts}\equiv-\frac{P_{ts}+q_{ts}P'_{ts}-c}{P'_{ts}}. \quad (79)$$

Substitute in the Kuhn-Tucker condition for L:

$$-\frac{1+\lambda}{r+\rho_{ts}}\frac{P_{ts}+q_{ts}P'_{ts}-c}{P'_{ts}}+\frac{\lambda}{P'_{ts}}\frac{P_{ts}+q_{ts}P'_{ts}-c}{\left(f_{ts}\right)^2}f'_{ts}\equiv 0$$

$$\left[-\frac{1+\lambda}{r+\rho_{ts}}+\frac{\lambda}{\left(f_{ts}\right)^2}f'_{ts}\right]\frac{P_{ts}+q_{ts}P'_{ts}-c}{P'_{ts}}\equiv 0$$

Hence

$$\frac{1+\lambda}{r+\rho_{ts}}=\frac{\lambda}{\left(f_{ts}\right)^2}f'_{ts}$$

$$\frac{1+\lambda}{\lambda}=\frac{r+\rho_{ts}}{\left(f_{ts}\right)^2}f'_{ts}. \quad (80)$$

Substitute in (78) to obtain one equation in q_{ts} and L.

$$-(n-1)f'_{ts}\left(P_{ts}-c\right)+$$

$$\frac{\left[(n+1)f_{ts}+n\left(P_{ts}+q_{ts}P'_{ts}-c\right)f'_{ts}\right]\left(P'_{ts}\right)^2-n\left(P_{ts}-c\right)f_{ts}P''_{ts}}{\left(P'_{ts}\right)^2}\equiv 0 \quad (81)$$

or equivalently

$$\frac{f\left[L-P\left(nq_{ts}\right)\right]}{f'\left[L-P\left(nq_{ts}\right)\right]}=-\frac{P\left(nq_{ts}\right)-c+nq_{ts}P'\left(nq_{ts}\right)}{n+1-n\frac{P\left(nq_{ts}\right)-c}{\left[P'\left(nq_{ts}\right)\right]^2}P''\left(nq_{ts}\right)}, \quad (82)$$

The equation other is the no-defection condition, (72), which implies

$$\left[\left(P_{ts}-c\right)q_{ts}-\pi_N\right]f_{ts}P'_{ts}+\left(P_{ts}+q_{ts}P'_{ts}-c\right)\int_{-\beta}^{L-P\left(nq_{ts}\right)}f\left(\varepsilon\right)d\varepsilon=$$

$$-\left(P_{ts} + q_{ts}P'_{ts} - c\right) r. \tag{83}$$

Go back to (82):

$$\frac{f\left[L - P\left(nq_{ts}\right)\right]}{f'\left[L - P\left(nq_{ts}\right)\right]} = -\frac{P\left(nq_{ts}\right) - c + nq_{ts}P'\left(nq_{ts}\right)}{n + 1 - n\frac{P(nq_{ts}) - c}{\left[P'(nq_{ts})\right]^2}P''\left(nq_{ts}\right)} \tag{84}$$

Suppose inverse demand is linear, (4). Then (84) becomes

$$\frac{f\left(L - a + nbq_{ts}\right)}{f'\left(L - a + nbq_{ts}\right)} = -\frac{a - c - 2nbq_{ts}}{n + 1}. \tag{85}$$

Now suppose that the random part of demand has the triangular distribution and that expected price exceeds the reversion threshold, $-\beta \leq \varepsilon \leq 0$, so that

$$f\left[-\left(a - nbq_{ts} - L\right)\right] = \frac{\beta + L - a + nbq_{ts}}{\beta^2} \qquad f'\left[-\left(a - nbq_{ts} - L\right)\right] = \frac{1}{\beta^2}. \tag{86}$$

Then (85) becomes

$$\beta + L - a + nbq_{ts} = -\frac{a - c - 2nbq_{ts}}{n + 1} \tag{87}$$

For linear inverse demand and the triangular distribution, the no-defection condition is

$$\left\{\left[\left(a - bnq_{ts} - c\right)q_{ts} - b\left(\frac{1}{n+1}\frac{a - c}{b}\right)^2\right]b - \frac{1}{2}\left[a - c - \left(n + 1\right)bq_{ts}\right]\left(\beta - a + nbq_{ts} + L\right)\right\} \times$$

$$\frac{\beta - a + nbq_{ts} + L}{\beta^2} = \left[a - \left(n + 1\right)bq_{ts} - c\right]r \tag{88}$$

Substitute (87) in the expression in braces and simplify to obtain

$$\left[\left(a - bnq_{ts} - c\right)q_{ts} - b\left(\frac{1}{n+1}\frac{a - c}{b}\right)^2\right]b + \frac{1}{2}\left[a - c - \left(n + 1\right)bq_{ts}\right]\frac{a - c - 2nbq_{ts}}{n + 1}$$

$$= \left(n - 1\right)\frac{a - c - \left(n + 1\right)bq_{ts}}{2\left(n + 1\right)^2}\left(a - c\right). \tag{89}$$

Substituting (89) and again (87), (88) becomes

$$\left[\frac{n - 1}{2\left(n + 1\right)^2}\frac{\left(a - c\right)}{\beta^2}\frac{a - c - 2nbq_{ts}}{n + 1} + r\right]\left[a - c\left(n + 1\right)bq_{ts}\right] = 0. \tag{90}$$

Solving the expression in brackets for q_{ts} gives (29). This and (87) imply (30).

5.2 Proof of Result 2[23]

A Lagrangian for the maximization of (50) subject to the constraint (49) is

$$\mathcal{L} = \frac{\pi_{ts} - \frac{\omega F}{n}\tau_{ts} - rV_N}{r + \rho_{ts}} +$$

$$\lambda \left\{ \frac{\pi_{ts} - \frac{\omega F}{n}\tau_{ts} - rV_N}{r + \rho_{ts}} - \frac{a - c - (n+1)\, bq_{ts} + \frac{\omega F b}{n\beta^2}\left(a + \beta - U - nbq_{ts}\right)}{\frac{b}{\beta^2}\left(\beta - a + L + nbq_{ts}\right)} \right\} \tag{91}$$

$$= (1+\lambda)\, \frac{\left(a - c - nbq_{ts}\right) q_{ts} - \frac{\omega F}{2n\beta^2}\left(a + \beta - U - nbq_{ts}\right)^2 - rV_N}{r + \frac{1}{2\beta^2}\left(\beta - a + L + nbq_{ts}\right)^2}$$

$$-\lambda \frac{\beta^2}{b} \frac{a - c - (n+1)\, bq_{ts} + \frac{\omega F b}{n\beta^2}\left(a + \beta - U - nbq_{ts}\right)}{\beta - a + L + nbq_{ts}}. \tag{92}$$

5.2.1 Kuhn-Tucker necessary conditions

q_{ts}: The Kuhn-Tucker for q_{ts} condition is

$$(1+\lambda)\, \frac{\left\{ \begin{array}{c} \left(r + \rho_{ts}\right) \left[a - c - 2nbq_{ts} - \frac{\omega F}{n\beta^2}\left(a + \beta - U - nbq_{ts}\right)(-nb)\right] \\ - \left(\pi_{ts} - \frac{\omega F}{n}\tau_{ts} - rV_N\right) \frac{nb}{\beta^2}\left(\beta - a + L + nbq_{ts}\right) \end{array} \right\}}{\left(r + \rho_{ts}\right)^2}$$

$$-\lambda \frac{\beta^2}{b} \frac{\left\{ \begin{array}{c} \left(\beta - a + L + nbq_{ts}\right) \left[-(n+1)\, b - \frac{\omega F b}{n\beta^2} nb\right] \\ - \left[a - c - (n+1)\, bq_{ts} + \frac{\omega F b}{n\beta^2}\left(a + \beta - U - nbq_{ts}\right)\right] nb \end{array} \right\}}{\left(\beta - a + L + nbq_{ts}\right)^2} \equiv 0.$$

Cancel b in the term that is the coefficient of λ, take β^2 back into denominator:

$$(1+\lambda)\, \frac{\left\{ \begin{array}{c} \left(r + \rho_{ts}\right) \left[a - c - 2nbq_{ts} + \frac{\omega F b}{\beta^2}\left(a + \beta - U - nbq_{ts}\right)\right] \\ - \left(\pi_{ts} - \frac{\omega F}{n}\tau_{ts} - rV_N\right) \frac{nb}{\beta^2}\left(\beta - a + L + nbq_{ts}\right) \end{array} \right\}}{\left(r + \rho_{ts}\right)^2}$$

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[23] See the document *Tri4.tex* for a version of this proof that includes all intermediate calculations.

$$+\lambda \frac{\left\{ \begin{array}{c} \left(\beta - a + L + nbq_{ts}\right) \left(n+1+\frac{\omega F b}{\beta^2}\right) \\ +\left[a-c-\left(n+1\right)bq_{ts} + \frac{\omega F b}{n\beta^2}\left(a+\beta-U-nbq_{ts}\right)\right] n \end{array} \right\}}{\left(\frac{\beta-a+L+nbq_{ts}}{\beta}\right)^2} \equiv 0$$

Distribute n in second term, rewrite the denominator in terms of ρ_{ts}:

$$\left(1+\lambda\right) \frac{\left\{ \begin{array}{c} \left(r+\rho_{ts}\right)\left[a-c-2nbq_{ts}+\frac{\omega F b}{\beta^2}\left(a+\beta-U-nbq_{ts}\right)\right] \\ -\left(\pi_{ts}-\frac{\omega F}{n}\tau_{ts}-rV_N\right)\frac{nb}{\beta^2}\left(\beta-a+L+nbq_{ts}\right) \end{array} \right\}}{\left(r+\rho_{ts}\right)^2}$$

$$+\lambda \frac{\left\{ \begin{array}{c} \left(\beta - a + L + nbq_{ts}\right) \left(n+1+\frac{\omega F b}{\beta^2}\right) \\ +n\left[a-c-\left(n+1\right)bq_{ts}\right] + \frac{\omega F b}{\beta^2}\left(a+\beta-U-nbq_{ts}\right) \end{array} \right\}}{2\rho_{ts}} \equiv 0 \tag{93}$$

This will be further simplified below using the other Kuhn-Tucker conditions.

L: The Kuhn-Tucker for L condition is

$$-\left(1+\lambda\right)\frac{\pi_{ts}-\frac{\omega F}{n}\tau_{ts}-rV_N}{\left(r+\rho_{ts}\right)^2}\frac{\beta-a+L+nbq_{ts}}{\beta^2}$$

$$+\lambda\frac{\beta^2}{b}\frac{a-c-\left(n+1\right)bq_{ts}+\frac{\omega F b}{n\beta^2}\left(a+\beta-U-nbq_{ts}\right)}{\left(\beta-a+L+nbq_{ts}\right)^2} \equiv 0.$$

Take $\frac{\beta^2}{b}$ back into the denominator of the second term:

$$-\left(1+\lambda\right)\frac{\pi_{ts}-\frac{\omega F}{n}\tau_{ts}-rV_N}{\left(r+\rho_{ts}\right)^2}\frac{\beta-a+L+nbq_{ts}}{\beta^2}$$

$$+\lambda\frac{a-c-\left(n+1\right)bq_{ts}+\frac{\omega F b}{n\beta^2}\left(a+\beta-U-nbq_{ts}\right)}{b\left(\frac{\beta-a+L+nbq_{ts}}{\beta}\right)^2} \equiv 0.$$

Rewrite denominator of second term in terms of ρ_{ts}:

$$-\left(1+\lambda\right)\frac{\pi_{ts}-\frac{\omega F}{n}\tau_{ts}-rV_N}{\left(r+\rho_{ts}\right)^2}\frac{\beta-a+L+nbq_{ts}}{\beta^2}$$

$$+\lambda \frac{a-c-(n+1)\,bq_{ts}+\frac{\omega Fb}{n\beta^2}\left(a+\beta-U-nbq_{ts}\right)}{2b\rho_{ts}} \equiv 0. \tag{94}$$

λ: The Kuhn-Tucker condition for λ gives back the no-defection condition:

$$\frac{\pi_{ts}-\frac{\omega F}{n}\tau_{ts}-rV_N}{r+\rho_{ts}} - \frac{a-c-(n+1)\,bq_{ts}+\frac{\omega Fb}{n\beta^2}\left(a+\beta-U-nbq_{ts}\right)}{\frac{b}{\beta^2}\left(\beta-a+L+nbq_{ts}\right)} \geq 0 \tag{95}$$

$$\lambda\left[\frac{\pi_{ts}-\frac{\omega F}{n}\tau_{ts}-rV_N}{r+\rho_{ts}} - \frac{a-c-(n+1)\,bq_{ts}+\frac{\omega Fb}{n\beta^2}\left(a+\beta-U-nbq_{ts}\right)}{\frac{b}{\beta^2}\left(\beta-a+L+nbq_{ts}\right)}\right] \equiv 0. \tag{96}$$

$$\lambda \geq 0 \tag{97}$$

Assume that the no-defection constraint is binding, so that

$$\frac{\pi_{ts}-\frac{\omega F}{n}\tau_{ts}-rV_N}{r+\rho_{ts}}$$
$$-\frac{a-c-(n+1)\,bq_{ts}+\frac{\omega Fb}{n\beta^2}\left(a+\beta-U-nbq_{ts}\right)}{\frac{b}{\beta^2}\left(\beta-a+L+nbq_{ts}\right)} \equiv 0. \tag{98}$$

Return to the Kuhn-Tucker condition for q_{ts}, (93), and use the defector's first-order condition (**??**), or equivalently, the Kuhn-Tucker condition for λ, which implies

$$\left(r+\rho_{ts}\right)\left[a-c-(n+1)\,bq_{ts}+\frac{\omega Fb}{n\beta^2}\left(a+\beta-U-nbq_{ts}\right)\right]$$
$$-\left(\pi_{ts}-\frac{\omega F}{n}\tau_{ts}-rV_N\right)\frac{b}{\beta^2}\left(\beta-a+L+nbq_{ts}\right) \equiv 0 \tag{99}$$

to simplify (93). Rewrite the numerator of the fraction that is the coefficient of $1+\lambda$:

$$\left(r+\rho_{ts}\right)\left[a-c-(n+1)\,bq_{ts}+\frac{(n-1+1)\,\omega Fb}{n\beta^2}\left(a+\beta-U-nbq_{ts}\right)-(n-1)\,bq_{ts}\right]$$
$$-\left(\pi_{ts}-\frac{\omega F}{n}\tau_{ts}-rV_N\right)\frac{(n-1)\,b+b}{\beta^2}\left(\beta-a+L+nbq_{ts}\right);$$

substituting (99), this becomes

$$\left(r+\rho_{ts}\right)\left[\frac{\left(n-1\right)\omega F b}{n\beta^{2}}\left(a+\beta-U-nbq_{ts}\right)-\left(n-1\right)bq_{ts}\right]$$

$$-\left(\pi_{ts}-\frac{\omega F}{n}\tau_{ts}-rV_{N}\right)\frac{\left(n-1\right)b}{\beta^{2}}\left(\beta-a+L+nbq_{ts}\right).$$

Then (93) becomes

$$-\left(1+\lambda\right)\left(n-1\right)b\frac{\left\{\begin{array}{c}\left(r+\rho_{ts}\right)\left[q_{ts}-\frac{\omega F}{n\beta^{2}}\left(a+\beta-U-nbq_{ts}\right)\right]\\+\left(\pi_{ts}-\frac{\omega F}{n}\tau_{ts}-rV_{N}\right)\frac{\left(\beta-a+L+nbq_{ts}\right)}{\beta^{2}}\end{array}\right\}}{\left(r+\rho_{ts}\right)^{2}}$$

$$+\lambda\frac{\left\{\begin{array}{c}\left(\beta-a+L+nbq_{ts}\right)\left(n+1+\frac{\omega F b}{\beta^{2}}\right)\\+n\left[a-c-\left(n+1\right)bq_{ts}\right]+\frac{\omega F b}{\beta^{2}}\left(a+\beta-U-nbq_{ts}\right)\end{array}\right\}}{2\rho_{ts}}\equiv 0. \qquad (100)$$

Once again from the Kuhn-Tucker condition for λ:

$$\left(\pi_{ts}-\frac{\omega F}{n}\tau_{ts}-rV_{N}\right)\frac{\left(\beta-a+L+nbq_{ts}\right)}{\beta^{2}}$$

$$=\left(r+\rho_{ts}\right)\left[\frac{a-c}{b}-\left(n+1\right)q_{ts}+\frac{\omega F}{n\beta^{2}}\left(a+\beta-U-nbq_{ts}\right)\right].$$

Substitute in the second term of the numerator of the fraction that is the coefficient of $\left(1+\lambda\right)\left(n-1\right)b$ in (100) to obtain

$$-\left(1+\lambda\right)\left(n-1\right)b\frac{\left\{\begin{array}{c}\left(r+\rho_{ts}\right)\left[q_{ts}-\frac{\omega F}{n\beta^{2}}\left(a+\beta-U-nbq_{ts}\right)\right]\\+\left(r+\rho_{ts}\right)\left[\frac{a-c}{b}-\left(n+1\right)q_{ts}+\frac{\omega F}{n\beta^{2}}\left(a+\beta-U-nbq_{ts}\right)\right]\end{array}\right\}}{\left(r+\rho_{ts}\right)^{2}}$$

$$+\lambda\frac{\left\{\begin{array}{c}\left(\beta-a+L+nbq_{ts}\right)\left(n+1+\frac{\omega F b}{\beta^{2}}\right)\\+n\left[a-c-\left(n+1\right)bq_{ts}\right]+\frac{\omega F b}{\beta^{2}}\left(a+\beta-U-nbq_{ts}\right)\end{array}\right\}}{2\rho_{ts}}=0.$$

In the first term, cancel $r+\rho_{ts}$ and multiply through the first term by b:

$$-\left(1+\lambda\right)\left(n-1\right)\frac{\left\{\begin{array}{c} bq_{ts}-\frac{\omega Fb}{n\beta^2}\left(a+\beta-U-nbq_{ts}\right) \\ +a-c-\left(n+1\right)bq_{ts}+\frac{\omega Fb}{n\beta^2}\left(a+\beta-U-nbq_{ts}\right)\end{array}\right\}}{r+\rho_{ts}}$$

$$+\lambda\frac{\left\{\begin{array}{c}\left(\beta-a+L+nbq_{ts}\right)\left(n+1+\frac{\omega Fb}{\beta^2}\right) \\ +n\left[a-c-\left(n+1\right)bq_{ts}\right]+\frac{\omega Fb}{\beta^2}\left(a+\beta-U-nbq_{ts}\right)\end{array}\right\}}{2\rho_{ts}}=0.$$

Simplify the numerator of the first term to obtain:

$$-\left(1+\lambda\right)\left(n-1\right)\frac{a-c-nbq_{ts}}{r+\rho_{ts}}$$

$$+\lambda\frac{\left\{\begin{array}{c}\left(\beta-a+L+nbq_{ts}\right)\left(n+1+\frac{\omega Fb}{\beta^2}\right) \\ +n\left[a-c-\left(n+1\right)bq_{ts}\right]+\frac{\omega Fb}{\beta^2}\left(a+\beta-U-nbq_{ts}\right)\end{array}\right\}}{2\rho_{ts}}=0.$$

Rewrite the numerator of the second term as

$$\left(\beta-a+L+nbq_{ts}\right)\left(n+1+\frac{\omega Fb}{\beta^2}\right)$$

$$+n\left[a-c-\left(n+1\right)bq_{ts}\right]+\frac{\omega Fb}{\beta^2}\left(a+\beta-U-nbq_{ts}\right)=$$

$$\left(n+1+\frac{\omega Fb}{\beta^2}\right)\left(\beta-c+L\right)+\frac{\omega Fb}{\beta^2}\left(\beta+c-U\right)-\left(a-c\right).$$

Hence the Kuhn-Tucker condition for q_{ts} can be written

$$-\left(1+\lambda\right)\left(n-1\right)\frac{a-c-nbq_{ts}}{r+\rho_{ts}}$$

$$+\lambda\frac{\left(n+1+\frac{\omega Fb}{\beta^2}\right)\left(\beta-c+L\right)+\frac{\omega Fb}{\beta^2}\left(\beta+c-U\right)-\left(a-c\right)}{2\rho_{ts}}=0. \qquad (101)$$

Now turn to the Kuhn-Tucker condition for L: , (94):

$$-\left(1+\lambda\right)\frac{\pi_{ts}-\frac{\omega F}{n}\tau_{ts}-rV_N}{\left(r+\rho_{ts}\right)^2}\frac{\beta-a+L+nbq_{ts}}{\beta^2}$$

$$+\lambda \frac{a-c-(n+1)\,bq_{ts}+\frac{\omega F b}{n\beta^2}\left(a+\beta-U-nbq_{ts}\right)}{2b\rho_{ts}} \equiv 0$$

From the no-defection condition

$$\frac{\pi_{ts}-\frac{\omega F}{n}\tau_{ts}-rV_N}{r+\rho_{ts}} \frac{\left(\beta-a+L+nbq_{ts}\right)}{\beta^2}.$$

$$= \frac{a-c-(n+1)\,bq_{ts}+\frac{\omega F b}{n\beta^2}\left(a+\beta-U-nbq_{ts}\right)}{b}.$$

Hence the Kuhn-Tucker condition for L can be rewritten

$$-\frac{1+\lambda}{r+\rho_{ts}} \frac{a-c-(n+1)\,bq_{ts}+\frac{\omega F b}{n\beta^2}\left(a+\beta-U-nbq_{ts}\right)}{b}$$

$$+\lambda \frac{a-c-(n+1)\,bq_{ts}+\frac{\omega F b}{n\beta^2}\left(a+\beta-U-nbq_{ts}\right)}{2b\rho_{ts}} \equiv 0.$$

Factor:

$$\frac{a-c-(n+1)\,bq_{ts}+\frac{\omega F b}{n\beta^2}\left(a+\beta-U-nbq_{ts}\right)}{b}\left(-\frac{1+\lambda}{r+\rho_{ts}}+\frac{\lambda}{2\rho_{ts}}\right) \equiv 0.$$

The numerator of the first term is known to be nonzero (from the equilibrium no-defection condition); hence the Kuhn-Tucker condition for L implies

$$\frac{1+\lambda}{r+\rho_{ts}} = \frac{\lambda}{2\rho_{ts}}. \tag{102}$$

Now return to the Kuhn-Tucker condition for q_{ts}, (101):

$$-\frac{1+\lambda}{\lambda}(n-1)\frac{a-c-nbq_{ts}}{r+\rho_{ts}}+$$

$$\frac{\left(n+1+\frac{\omega F b}{\beta^2}\right)(\beta-c+L)+\frac{\omega F b}{\beta^2}(\beta+c-U)-(a-c)}{2\rho_{ts}}=0$$

Substitute (102) to obtain

$$-\frac{r+\rho_{ts}}{2\rho_{ts}}(n-1)\frac{a-c-nbq_{ts}}{r+\rho_{ts}}+$$

$$\frac{\left(n+1+\frac{\omega F b}{\beta^2}\right)(\beta-c+L)+\frac{\omega F b}{\beta^2}(\beta+c-U)-(a-c)}{2\rho_{ts}}=0.$$

Multiply through by $2\rho_{ts}$:

$$-(n-1)(a-c-nbq_{ts})+$$

$$\left(n+1+\frac{\omega F b}{\beta^2}\right)(\beta-c+L)+\frac{\omega F b}{\beta^2}(\beta+c-U)-(a-c)=0. \tag{103}$$

(103) is a linear equation in q_{ts} and L.

Rewrite (103) so that it is in terms of $\beta-a+L$ and $a+\beta-U$: considering the last three terms,

$$\left(n+1+\frac{\omega F b}{\beta^2}\right)(\beta-a+L+a-c)+\frac{\omega F b}{\beta^2}\left[a+\beta-U-(a-c)\right]-(a-c)=$$

$$\left(n+1+\frac{\omega F b}{\beta^2}\right)(\beta-a+L)+\frac{\omega F b}{\beta^2}(a+\beta-U)+n(a-c).$$

Hence (103) can be rewritten

$$-(n-1)(a-c-nbq_{ts})$$

$$+\left(n+1+\frac{\omega F b}{\beta^2}\right)(\beta-a+L)+\frac{\omega F b}{\beta^2}(a+\beta-U)+n(a-c)=0. \tag{104}$$

Solve (104) for $\beta-a+L+nbq_{ts}$, which will be needed later:

$$\beta-a+L+nbq_{ts}=-\frac{a-c-n\left(2+\frac{\omega F b}{\beta^2}\right)bq_{ts}+\frac{\omega F b}{\beta^2}(a+\beta-U)}{n+1+\frac{\omega F b}{\beta^2}}. \tag{105}$$

For notational compactness, write

$$x=a-c$$

$$y=bq_{ts}$$

$$u=a+\beta-U$$

$$z=\frac{\omega F b}{n\beta^2}.$$

Then (105) becomes

$$\beta - a + L + nbq_{ts} = -\frac{x - n\left(2 + nz\right)y + nzu}{n + 1 + nz}. \tag{106}$$

The other equation in q_{ts} and L is the no-defection condition

$$\frac{\left(a - c - nbq_{ts}\right)q_{ts} - \frac{\omega F}{2n\beta^2}\left(a + \beta - U - nbq_{ts}\right)^2 - rV_N}{r + \frac{1}{2\beta^2}\left(\beta - a + L + nbq_{ts}\right)^2}$$

$$= \frac{a - c - \left(n+1\right)bq_{ts} + \frac{\omega Fb}{n\beta^2}\left(a + \beta - U - nbq_{ts}\right)}{\frac{b}{\beta^2}\left(\beta - a + L + nbq_{ts}\right)}.$$

Multiply through by b and substitute notation:

$$\frac{\left(x - nbq_{ts}\right)y - \frac{z}{2}\left(u - ny\right)^2 - rbV_N}{r + \frac{1}{2\beta^2}\left(\beta - a + L + nbq_{ts}\right)^2} = \frac{x - \left(n+1\right)y + z\left(u - ny\right)}{\frac{1}{\beta^2}\left(\beta - a + L + nbq_{ts}\right)}.$$

Cross-multiply:

$$\frac{1}{\beta^2}\left(\beta - a + L + nbq_{ts}\right)\left[\left(x - nbq_{ts}\right)y - \frac{z}{2}\left(u - ny\right)^2 - rbV_N\right] =$$

$$\left[r + \frac{1}{2\beta^2}\left(\beta - a + L + nbq_{ts}\right)^2\right]\left[x - \left(n+1\right)y + z\left(u - ny\right)\right].$$

Multiply through by β^2, distribute terms on right-hand side:

$$\left(\beta - a + L + nbq_{ts}\right)\left[\left(x - nbq_{ts}\right)y - \frac{z}{2}\left(u - ny\right)^2 - rbV_N\right] =$$

$$\beta^2 r\left[x - \left(n+1\right)y + z\left(u - nbq_{ts}\right)\right] + \frac{1}{2}\left(\beta - a + L + nbq_{ts}\right)^2\left[x - \left(n+1\right)y + z\left(u - ny\right)\right].$$

Leave terms in $\beta^2 r$ on the right, collect all other terms on the left:

$$\left(\beta - a + L + nbq_{ts}\right)\times$$

$$\left\{\left(x - nbq_{ts}\right)y - \frac{z}{2}\left(u - ny\right)^2 - \frac{1}{2}\left(\beta - a + L + nbq_{ts}\right)\left[x - \left(n+1\right)y + z\left(u - nbq_{ts}\right)\right] - rbV_N\right\}$$

$$= \beta^2 r\left[x - \left(n+1\right)y + z\left(u - ny\right)\right].$$

Now substitute (106) in the first three terms in the expression in braces on the left:

$$(x-ny)\,y-\frac{z}{2}\,(u-ny)^2-\frac{1}{2}\left(-\frac{x-n\,(2+nz)\,y+nzu}{n+1+nz}\right)[x-(n+1)\,y+z\,(u-ny)]=$$

$$(x-ny)\,y-\frac{z}{2}\,(u-ny)^2+\frac{1}{2}\,[x-(n+1)\,y+z\,(u-ny)]\,\frac{x-n\,(2+nz)\,y+nzu}{n+1+nz}.$$

Note that

$$x-n\,(2+nz)\,y+nzu=x-2ny+nz\,(u-ny)\,.$$

Hence the first three terms can be written

$$(x-ny)\,y-\frac{z}{2}\,(u-ny)^2+\frac{1}{2}\frac{x-2ny+nz\,(u-ny)}{n+1+nz}\,[x-(n+1)\,y+z\,(u-ny)]$$

and the whole expression in braces is

$$(x-ny)\,y-\frac{z}{2}\,(u-ny)^2+\frac{1}{2}\frac{x-2ny+nz\,(u-ny)}{n+1+nz}\,[x-(n+1)\,y+z\,(u-ny)]-rbV_N.$$

Now substitute the expression for V_N that valid for the case $0\le\tau_N\le\frac{1}{2}$,

$$brV_N=\frac{-\left(1+\frac{z}{2}\right)zu^2+\frac{1}{2}\left\{\frac{2x+\left[\left(n^2+1+nz\right)u-(n-2)nx\right]z}{n+1+nz}\right\}^2}{2-(n-2)\,nz},$$

in the expression in braces on the left:

$$(x-ny)\,y-\frac{z}{2}\,(u-ny)^2+\frac{1}{2}\frac{x-2ny+nz\,(u-ny)}{n+1+nz}\,[x-(n+1)\,y+z\,(u-ny)]$$

$$-\frac{-\left(1+\frac{z}{2}\right)zu^2+\frac{1}{2}\left\{\frac{2x+\left[\left(n^2+1+nz\right)u-(n-2)nx\right]z}{n+1+nz}\right\}^2}{2-(n-2)\,nz}=$$

$$(n-1)\,\frac{x+nz\,(x-u)}{2\,(n+1+nz)^2}\,[x-(n+1)\,y+z\,(u-ny)]\,.$$

The equation is

$$(\beta - a + L + nbq_{ts})(n-1)\frac{x + nz(x-u)}{2(n+1+nz)^2}[x-(n+1)y+z(u-ny)]$$

$$= \beta^2 r[x-(n+1)y+z(u-ny)],$$

or

$$(\beta - a + L + nbq_{ts})(n-1)\frac{x + nz(x-u)}{2(n+1+nz)^2} = \beta^2 r.$$

Substitute from (106) to eliminate $\beta - a + L + nbq_{ts}$:

$$-\frac{x - n(2+nz)y + nzu}{n+1+nz}(n-1)\frac{x+nz(x-u)}{2(n+1+nz)^2} = \beta^2 r.$$

Solve for y:

$$y = \frac{x+nzu}{n(2+nz)} + \frac{2(n+1+nz)^3}{n(2+nz)[x+nz(x-u)]}\frac{\beta^2 r}{(n-1)}.$$

Compare with

$$(n-1)\frac{x+nz(x-u)}{2(n+1+nz)^2}[x-(n+1)y+z(u-ny)].$$

Look at the other first terms:

$$(x-ny)y - \frac{z}{2}(u-ny)^2 + \frac{1}{2}\frac{x-2ny+nz(u-ny)}{n+1+nz}(x-(n+1)y+z(u-ny))$$

$$-\frac{(nxy - xy - uxz - nuxz + nuyz - nxyz - x^2 + u^2z + nu^2z - n^2uyz + n^2xyz)}{2(n+nz+1)}$$

5.2.2 q_{ts}

Now return to the underlying notation ($x = a-c$, $y = bq_{ts}$, $u = a+\beta-U$, $z = \frac{\omega F b}{n\beta^2}$):

$$bq_{ts} =$$

$$\frac{a-c+n\frac{\omega F b}{n\beta^2}(a+\beta-U)}{n\left(2+n\frac{\omega F b}{n\beta^2}\right)} + \frac{2\left(n+1+n\frac{\omega F b}{n\beta^2}\right)^3}{n\left(2+n\frac{\omega F b}{n\beta^2}\right)\left[a-c+n\frac{\omega F b}{n\beta^2}(a-c-a-\beta+U)\right]}\frac{\beta^2 r}{n-1}$$

$$q_{ts} =$$

$$\frac{1}{n}\left[\frac{\frac{a-c}{b}+\frac{\omega F}{\beta^2}\left(a+\beta-U\right)}{2+\frac{\omega Fb}{\beta^2}}+\frac{2}{2+\frac{\omega Fb}{\beta^2}}\frac{\left(n+1+\frac{\omega Fb}{\beta^2}\right)^3}{n-1}\frac{\beta^2}{a-c+\frac{\omega Fb}{\beta^2}\left(U-c-\beta\right)}\frac{r}{b}\right]. \tag{107}$$

5.2.3 P_{ts}

Expected price:

$$P_{ts} = c + a - c - nbq_{ts} =$$

$$c+\frac{\left(1+\frac{\omega Fb}{\beta^2}\right)(a-c)-\frac{\omega Fb}{\beta^2}\left(a+\beta-U\right)}{2+\frac{\omega Fb}{\beta^2}}-\frac{2}{2+\frac{\omega Fb}{\beta^2}}\frac{\left(n+1+\frac{\omega Fb}{\beta^2}\right)^3}{n-1}\frac{\beta^2 r}{a-c+\frac{\omega Fb}{\beta^2}\left(U-c-\beta\right)}. \tag{108}$$

Note that by rearranging terms in the numerator of the second term on the right, (108) may be rewritten

$$P_{ts} = c+\frac{a-c-\frac{\omega Fb}{\beta^2}\left(c+\beta-U\right)}{2+\frac{\omega Fb}{\beta^2}}-\frac{2}{2+\frac{\omega Fb}{\beta^2}}\frac{\left(n+1+\frac{\omega Fb}{\beta^2}\right)^3}{n-1}\frac{\beta^2 r}{a-c+\frac{\omega Fb}{\beta^2}\left(U-c-\beta\right)} \tag{109}$$

5.2.4 L

Return to present model, solve (106) for L:

$$\beta + L - (a - nbq_{ts}) = -\frac{a-c-n\left(2+\frac{\omega Fb}{\beta^2}\right)bq_{ts}+\frac{\omega Fb}{\beta^2}\left(a+\beta-U\right)}{n+1+\frac{\omega Fb}{\beta^2}}.$$

$$\beta + L = P_{ts} - \frac{a-c-\left(2+\frac{\omega Fb}{\beta^2}\right)nbq_{ts}+\frac{\omega Fb}{\beta^2}\left(a+\beta-U\right)}{n+1+\frac{\omega Fb}{\beta^2}}$$

First simplify the numerator of the fraction on the right:

$$a-c-\left(2+\frac{\omega Fb}{\beta^2}\right)nbq_{ts}+\frac{\omega Fb}{\beta^2}\left(a+\beta-U\right) =$$

$$\frac{\left(n+1+\frac{\omega F b}{\beta^2}\right)^3}{n-1} \frac{2\beta^2 r}{a-c+\frac{\omega F b}{\beta^2}\left(U-c-\beta\right)}.$$

Then

$$\beta + L = P_{ts} - \frac{\frac{\left(n+1+\frac{\omega F b}{\beta^2}\right)^3}{n-1} \frac{2\beta^2 r}{a-c+\frac{\omega F b}{\beta^2}(U-c-\beta)}}{n+1+\frac{\omega F b}{\beta^2}} =$$

$$c + \frac{a-c-\frac{\omega F b}{\beta^2}\left(c+\beta-U\right)}{2+\frac{\omega F b}{\beta^2}} - \frac{\left(n+1+\frac{\omega F b}{\beta^2}\right)^2}{2+\frac{\omega F b}{\beta^2}} \frac{2\beta^2 r}{a-c+\frac{\omega F b}{\beta^2}\left(U-c-\beta\right)}.$$

Hence

$$L = c - \beta + \frac{a-c-\frac{\omega F b}{\beta^2}\left(c+\beta-U\right)}{2+\frac{\omega F b}{\beta^2}} - \frac{\left(n+1+\frac{\omega F b}{\beta^2}\right)^2}{2+\frac{\omega F b}{\beta^2}} \frac{2\beta^2 r}{a-c+\frac{\omega F b}{\beta^2}\left(U-c-\beta\right)}. \tag{110}$$

Remark: go back to (103):

$$-\left(n-1\right)\left(a-c-nbq_{ts}\right) + \left(n+1+\frac{\omega F b}{\beta^2}\right)\left(\beta-c+L\right) + \frac{\omega F b}{\beta^2}\left(\beta+c-U\right) - \left(a-c\right) = 0.$$

Note that

$$a-c-nbq_{ts} = P_{ts} - c$$

Then the equation in q_{ts} and L becomes an equation in P_{ts} and L:

$$-\left(n-1\right)\left(P_{ts}-c\right) + \left(n+1+\frac{\omega F b}{\beta^2}\right)\left(\beta-c+L\right) + \frac{\omega F b}{\beta^2}\left(\beta+c-U\right) - \left(a-c\right) = 0.$$

P_{ts} rises with U (see the discussion of comparative statics below).
The formula for P_{ts} is valid so long as

$$0 \leq U - P_{ts} \leq \beta.$$

If $U - P_{ts} = \beta$ or $P_{ts} = U - \beta$, $\tau_{ts} = 0$; for greater values of U,

$$P_{ts} = U - \beta$$

and L is determined by the equation

$$-(n-1)(P_{ts}-c)+\left(n+1+\frac{\omega Fb}{\beta^2}\right)(\beta-c+L)+\frac{\omega Fb}{\beta^2}(\beta+c-U)-(a-c)=0,$$

which implies

$$L=\frac{a+c-2\beta\left(n+\frac{\omega Fb}{\beta^2}\right)+\left(n-1+\frac{\omega Fb}{\beta^2}\right)U}{n+1+\frac{\omega Fb}{\beta^2}}.$$

5.2.5 $P_{ts}-L$

Rewrite (105):

$$P_{ts}-L=\beta+\frac{a-c-\left(2+\frac{\omega Fb}{\beta^2}\right)nbq_{ts}+\frac{\omega Fb}{\beta^2}(a+\beta-U)}{n+1+\frac{\omega Fb}{\beta^2}}.$$

Now substitute

$$nbq_{ts}=$$

$$\frac{a-c+\frac{\omega Fb}{\beta^2}(a+\beta-U)}{2+\frac{\omega Fb}{\beta^2}}+\frac{1}{2+\frac{\omega Fb}{\beta^2}}\frac{\left(n+1+\frac{\omega Fb}{\beta^2}\right)^3}{n-1}\frac{2\beta^2 r}{a-c+\frac{\omega Fb}{\beta^2}(U-c-\beta)}$$

and rearrange terms to obtain

$$P_{ts}-L=\beta-\frac{2\left(n+1+\frac{\omega Fb}{\beta^2}\right)^2}{n-1}\frac{\beta^2 r}{a-c+\frac{\omega Fb}{\beta^2}(U-c-\beta)}. \tag{111}$$

Alternatively, using (109) and (110) leads to the same result.

In the present model, evidently,

$$P_{ts}-L-\beta=-\frac{\left(n+1+\frac{\omega Fb}{\beta^2}\right)^2}{n-1}\frac{2\beta^2 r}{a-c+\frac{\omega Fb}{\beta^2}(U-c-\beta)}<0. \tag{112}$$

5.2.6 ρ_{ts}

Probability of reversion:

$$\begin{aligned}\rho_{ts} &= \frac{1}{2}\left(\frac{\beta - a + L + nbq_{ts}}{\beta}\right)^2 \\ &= \frac{1}{2}\left[\frac{\beta - (a - nbq_{ts}) + L}{\beta}\right]^2 \\ &= \frac{1}{2}\left(\frac{\beta - P_{ts} + L}{\beta}\right)^2\end{aligned}$$

and from (112)

$$= 2\left[\frac{\left(n + 1 + \frac{\omega F b}{\beta^2}\right)^2}{n-1}\frac{\beta r}{a - c + \frac{\omega F b}{\beta^2}\left(U - c - \beta\right)}\right]^2. \tag{113}$$

5.2.7 $U - P$

We have from (109)

$$P_{ts} = c + \frac{a - c - \frac{\omega F b}{\beta^2}\left(c + \beta - U\right)}{2 + \frac{\omega F b}{\beta^2}} - \frac{2}{2 + \frac{\omega F b}{\beta^2}}\frac{\left(n + 1 + \frac{\omega F b}{\beta^2}\right)^3}{n-1}\frac{\beta^2 r}{a - c + \frac{\omega F b}{\beta^2}\left(U - c - \beta\right)}$$

so that

$$U - P_{ts} =$$

$$\beta - \frac{a - c + 2\left(c + \beta - U\right)}{2 + \frac{\omega F b}{\beta^2}} + \frac{2}{2 + \frac{\omega F b}{\beta^2}}\frac{\left(n + 1 + \frac{\omega F b}{\beta^2}\right)^3}{n-1}\frac{\beta^2 r}{a - c + \frac{\omega F b}{\beta^2}\left(U - c - \beta\right)}. \tag{114}$$

Then

$$\beta - (U - P_{ts}) =$$

$$\frac{a - c + 2\left(c + \beta - U\right)}{2 + \frac{\omega F b}{\beta^2}} - \frac{2}{2 + \frac{\omega F b}{\beta^2}}\frac{\left(n + 1 + \frac{\omega F b}{\beta^2}\right)^3}{n-1}\frac{\beta^2 r}{a - c + \frac{\omega F b}{\beta^2}\left(U - c - \beta\right)}. \tag{115}$$

This must be nonnegative for the solution for P_{ts} to be valid.

5.2.8 τ_{ts}

The probability of investigation is

$$\tau_{ts} = \frac{1}{2}\left[\frac{\beta - (U - P_{ts})}{\beta}\right]^2 =$$

$$\frac{1}{2}\left[\frac{a - c + 2\left(c + \beta - U\right)}{\left(2 + \frac{\omega F b}{\beta^2}\right)\beta} - \frac{2}{2 + \frac{\omega F b}{\beta^2}} \frac{\left(n + 1 + \frac{\omega F b}{\beta^2}\right)^3}{n - 1} \frac{\beta r}{a - c + \frac{\omega F b}{\beta^2}\left(U - c - \beta\right)}\right]^2 . \tag{116}$$

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