Available online at www.sciencedirect.com
ScienceDirect
International Journal of Industrial Organization
24 (2006) 1299 – 1332
International Journal of Industrial Organization
www.elsevier.com/locate/econbase

Competition policy, collusion, and tacit collusion ☆

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Available online 6 June 2006

Abstract

In this paper, I pursue three goals. The first is to model collusion in a way that is distinct from noncooperative collusion. The second and third are to develop a particular specification of a standard model of noncooperative collusion that permits explicit solution for equilibrium outputs and reversion thresholds and to extend this analysis to allow for a deterrence-based competition policy that investigates conduct based on observed high prices (investigation thresholds).
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JEL classification: L40; L41

Keywords: Competition policy; Antitrust policy; Collusion; Tacit collusion

1. Introduction

From Friedman (1971a) onward, industrial economists have modelled collusion as the noncooperative equilibrium of a repeated game. Careful work in this tradition refers to the behavior that is studied as "tacit collusion" or "noncooperative collusion." But models of imperfectly competitive markets, and models of collusion are no exception, often have as a primary purpose the forming of advice for the conduct of antitrust policy, and there is a fundamental disconnect between treating collusion as the outcome of a noncooperative game

☆ I am grateful for comments received from Brigham Young University, BiGSEM (University of Bielefeld), Università degli Studi di Lecce, the Portuguese Competition Authority, Washington University, St. Louis, and from Jeroen Hinloopen. Responsibility for errors is my own.
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0167-7187/$ - see front matter © 2006 Published by Elsevier B.V.
doi:10.1016/j.ijindorg.2006.04.007

and the antitrust concept of collusion. The legal offense of collusion typically requires that firms agree. This is true of U.S. antitrust (*Theatre Enterprises*, 346 U.S. 537 at 540–541, 1954):[1]

> The crucial question is whether respondents' conduct toward petitioner stemmed from independent decision or from an agreement, tacit or express. To be sure, business behavior is admissible circumstantial evidence from which the fact finder may infer agreement. ... But [the U.S. Supreme] Court has never held that proof of parallel business behavior conclusively establishes agreement, or, phrased differently, that such behavior itself constitutes a Sherman Act offense.

The same seems to be true for European Union competition policy toward collusion. In the *Woodpulp decision*,[2] the European Court of Justice set aside a Commission decision motivated largely by demonstrated parallel behavior on the ground that ([1993] 4 CMLR 407 at 478) "if there is a plausible explanation for the conduct found to exist which is consistent with an independent choice by the undertakings concerned, concertation remains unproven."

It is the essence of the noncooperative behavior that characterizes equilibrium in repeated games that agreement does not take place (Baker, 1993; Martin, 1993). Models of tacit collusion do describe a phenomenon that takes place in the real world, but that phenomenon does not involve agreement (Friedman, 1971b, p. 106):

> A noncooperative approach will . . . involve each firm in isolated decision making. This is not to imply that each firm ignores the effects of its rivals' decisions on its own profit or of its own decision on its rivals behavior (and hence its own profits). Noncooperative formulations will generally assume the firms not to make decisions jointly.

To this extent, models of tacit collusion are fundamentally unsuited for the analysis of collusion.

Many of the common heuristic expositions of the concept of noncooperative equilibrium lend themselves to the interpretation that interactions of some type among players underlie equilibrium behavior (Johansen, 1982). To be precise, however, the absence of communication is an essential aspect of noncooperative equilibrium, a theory that (Nash, 1951, p. 286) "is based on the absence of coalitions in that it is assumed that each participant acts independently, without collaboration or communication with any of the others."

The absence of communication condition applies to one-shot games, in which (Johansen, 1982, pp. 429–430) "A player does not know the actions to be taken by other players until they all reveal their decisions; he can only analyze the situation on the basis of his information about the action possibility sets and utility functions, and on this basis make his own decision." It applies equally to noncooperative equilibrium strategies in repeated games. In the class of Green–Porter models of tacit collusion to which the models considered in this paper belong, the action each firm takes if a sufficiently low price is observed in the immediately preceding period is an action that is taken without communication with any other firm.

Two approaches to the use of models of noncooperative behavior to inform public policy toward collusion seem possible. One approach might be called heuristic: recognizing that a model implies genuinely independent (and therefore noncollusive) behavior, we might nonetheless

[1] Judge Posner's decision in *In re High Fructose Corn Syrup* (295 F.3d 651 7th Cir. 2002) is not as much a departure from this standard as it is sometimes made out to be; it can be viewed as an elaboration of the factors from which existence of an agreement may be inferred.

[2] A. Ahlström OY and others *v.* E.C. Commission [1988] 4 CMLR 901; [1993] 4 CMLR 407.

consider that in application, firms could not coordinate on the kinds of noncooperative equilibrium behavior indicated by the model without some kind of communication taking place. Observed outcomes of the kind indicated by models of noncooperative behavior in "sufficiently complex" environments might then be seen as supporting the conclusion that agreement, collusion in a legal sense, had taken place.

This approach may well be valid in some circumstances. But there may well be other circumstances in which real firms in real markets are able to reach what look like cooperative outcomes based on genuinely independent decisions. This is the "oligopoly problem" that long antedates the application of game theory to questions of industrial economics (Fellner, 1950, p. 54):

> it should be realized that oligopolistic co-operation may stem largely from the spontaneous co-ordination of business policies, and that it does not presuppose direct contacts, or collusion in the sense proper.

In this paper, I pursue three goals. The first is to model collusion in a way that is distinct from noncooperative collusion. The second and third are to develop a particular specification of a standard model of noncooperative collusion that permits explicit solution for equilibrium outputs and reversion thresholds, and to extend this analysis to allow for a deterrence-based competition policy that investigates conduct based on observed high prices (investigation thresholds).

Section 2 presents the demand and cost specifications that are used in the rest of the paper. Section 3 considers the case of tacit collusion supported by a grim trigger strategy and compares tacit collusion with overt collusion, without a competition policy. Section 4 introduces a competition policy. Section 5 concludes. Proofs are given in Appendix A.

2. Demand and cost

Let realized price p be

$$p = P(Q) + \varepsilon. \tag{1}$$

The random part of demand ε is assumed to have zero mean. $P(Q)$ is thus expected price. Assume that $P' < 0$ and that ε has a well-behaved density function $f(\varepsilon)$, defined on the interval

$$\underline{\varepsilon} \leq \varepsilon \leq \overline{\varepsilon} \leq \infty. \tag{2}$$

Assume also that

$$\begin{matrix} f'(\varepsilon) > 0 & \varepsilon < 0 \\ f'(\varepsilon) < 0 & \varepsilon > 0 \end{matrix}. \tag{3}$$

These assumptions are satisfied by the case of linear inverse expected demand

$$P = a - bQ \tag{4}$$

and a random part of demand distributed according to the symmetric triangular distribution. The equation of the density function of the symmetric triangular distribution is

$$f(\varepsilon) = \begin{cases} \dfrac{\beta + \varepsilon}{\beta^2} & -\beta \leq \varepsilon \leq 0 \\ \dfrac{\beta - \varepsilon}{\beta^2} & 0 \leq \varepsilon \leq \beta \\ 0 & \text{otherwise} \end{cases}. \tag{5}$$

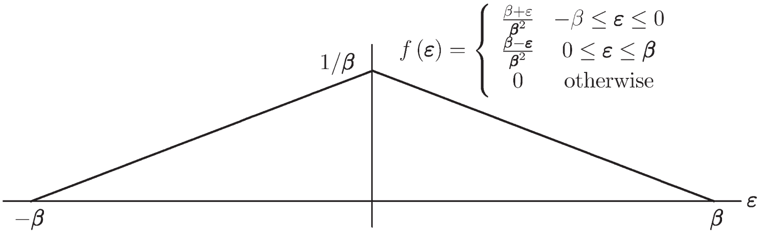

Fig. 1. Triangle density function.

Fig. 1 shows the density function (5). The triangular distribution has appealing properties – more probability mass is centrally located than in the tails, variance ($\beta^2/6$) rises with β – and is tractable, in contrast to other distributions (such as the truncated normal) with similar properties.[3] The specific results of the paper will be presented for the case of linear inverse expected demand and triangular density for the random part of demand.

Finally, assume that firms produce at constant average and marginal cost c per unit of output. For simplicity in evaluation of expected profit, assume also that if firms produce the noncooperative equilibrium output of a one-shot game, the least possible realized price is not less than marginal cost:

$$P(Q_{\mathrm{N}}) + \underline{\varepsilon} \geq c. \tag{6}$$

3. No competition policy

3.1. One-shot game (equivalent) outcome

If all other firms produce output Q_{-i}, the value of firm i satisfies

$$V_i = \frac{\pi_i}{1+r} + \frac{1}{1+r} V_i, \tag{7}$$

where at the end of the period, the firm receives expected payoff

$$\pi_i = [P(Q_{-i} + q_i) - c] q_i. \tag{8}$$

Combining terms,

$$V_i = \frac{[P(Q_{-i} + q_i) - c] q_i}{r}. \tag{9}$$

In noncooperative quantity-setting oligopoly, equilibrium output for firm i must maximize V_i, given the equilibrium outputs of all other firms. The first-order condition to maximize V_i is

$$P(Q_{-i} + q_i) - c + q_i P'(Q_{-i} + q_i) \equiv 0. \tag{10}$$

[3] On the triangular distribution, see Johnson *et al.* (1999, p. 297–298). A random variable with the triangle distribution is, among other characterizations, the sum of two independently distributed uniform random variables (Freeman, 1963, pp. 176–177).

In symmetric equilibrium, all firms produce the same output. The condensed first-order condition, which defines equilibrium output, is

$$P(nq_{\mathrm{N}}) - c + q_{\mathrm{N}}P'(nq_{\mathrm{N}}) \equiv 0. \tag{11}$$

For equilibrium firm value, use (11) to write

$$V_{\mathrm{N}} = \frac{[P(Q_{\mathrm{N}}) - c]q_{\mathrm{N}}}{r} = \frac{P'(nq_{\mathrm{N}})q_{\mathrm{N}}^2}{r} = \frac{\pi_{\mathrm{N}}}{r}, \tag{12}$$

where

$$Q_{\mathrm{N}} = nq_{\mathrm{N}}. \tag{13}$$

For linear inverse expected demand (4), (10) and (12) reduce to the familiar

$$q_{\mathrm{N}} = \frac{1}{n+1}\frac{a-c}{b} \quad \pi_{\mathrm{N}} = bq_{\mathrm{N}}^2 = b\left(\frac{1}{n+1}\frac{a-c}{b}\right)^2. \tag{14}$$

3.2. Tacit collusion supported by a trigger strategy

When demand has a random part, the trigger strategy model of noncooperative collusion specifies that if price falls below a threshold price L,[4] firms revert to forever playing the noncooperative equilibrium strategy of a one-shot game.[5] In what follows, I characterize the one-shot game, trigger strategy, and collusive equilibria, without and with a deterrence-based competition policy.

Suppose that by use of a trigger strategy, firms can noncooperatively restrict total output to a level Q_{ts}, output per firm to a level q_{ts}, with

$$Q_{\mathrm{ts}} = nq_{\mathrm{ts}}. \tag{15}$$

At the end of the first period, each firm has expected payoff

$$\pi_{\mathrm{ts}} = [P(Q_{\mathrm{ts}}) - c]q_{\mathrm{ts}}. \tag{16}$$

With probability, ρ_{ts}, realized price is below the threshold price L (which is determined as part of the trigger strategy) and firms revert forever to the Nash equilibrium outputs of a one-shot game.

The probability of reversion with output Q is

$$\begin{aligned}\rho(P-L) &= Pr(p \leq L) = Pr[P(Q) + \varepsilon \leq L] = Pr\{\varepsilon \leq -[P(Q) - L]\} \\ &= \int_{\underline{\varepsilon}}^{-[P(Q)-L]} f(\varepsilon)\mathrm{d}\varepsilon. \end{aligned} \tag{17}$$

See Fig. 2 for the triangular distribution.[6]

[4] Ellison (1994) discusses a variety of signals that might be monitored by outputrestricting firms.

[5] For the case without uncertainty, see Friedman (1971a,b). The specification considered in this section is a special case of the mechanism analyzed by Porter (1983a,b) and Green and Porter (1984). Results for the case in which firms resume output restriction after a fixed number of periods are available on request. In that case, expected firm value rises as the length of the reversion period increases, so that it is the grim trigger strategy that maximizes firm value.

[6] I limit attention to cases that satisfy $\beta \geq P - L > 0$. This assumption simplifies the calculation of ρ by keeping ρ in the range $0 \leq \rho \leq 1/2$.

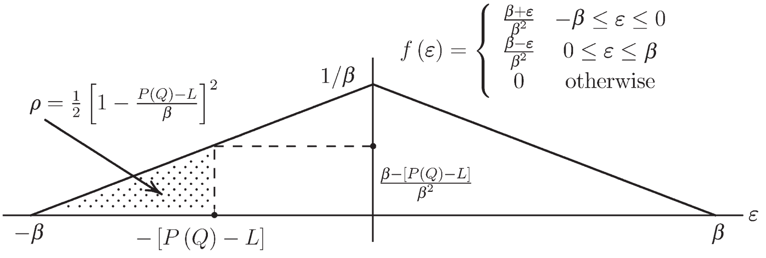

Fig. 2. Probability of reversion, triangle density function; Q=output, L=reversion threshold.

Increasing output or raising the threshold price increases the probability of reversion:

$$\frac{\partial\rho}{\partial Q}=-f[-(P-L)]P'>0 \quad \frac{\partial\rho}{\partial L}=f[-(P-L)]>0. \tag{18}$$

If reversion occurs, the present discounted value of income of the firm, discounted to the beginning of the first reversionary period, is $V_N=\pi_N/r$. With probability $1-\rho_{ts}$, there is no reversion, and firm value from the end of the first period is again V_{ts}. V_{ts} thus satisfies the recursive relationship

$$V_{ts}=\frac{\pi_{ts}}{1+r}+\frac{\rho_{ts}}{1+r}\frac{\pi_N}{r}+\frac{1-\rho_{ts}}{1+r}V_{ts}, \tag{19}$$

so that

$$V_{ts}=V_N+\frac{\pi_{ts}-\pi_N}{r+\rho_{ts}}. \tag{20}$$

The tacit collusion problem is to maximize V_{ts} by choice of q_{ts} and L, subject to the constraint that no firm has an incentive to defect from the trigger strategy.

To formalize the no-defection constraint, note that if all other firms produce combined output Q_{-i}, the value of firm i is

$$V_i=V_N+\frac{\pi_i-\pi_N}{r+\rho_i}=V_N+\frac{[P(Q_{-i}+q_i)-c]q_i-\pi_N}{r+\int_{\underline{\varepsilon}}^{-[P(Q_{-i}+q_i-L)]}f(\varepsilon)\mathrm{d}\varepsilon}. \tag{21}$$

For the trigger strategy to be an equilibrium, it must be that if each other firm produces its tacit collusion output q_{ts}, firm i's best response is to produce q_{ts}. That is, it must be that the first-order condition for maximization of (21) with respect to q_i holds for $Q_{-i}=(n-1)q_{ts}$ and $q_i=q_{ts}$:

$$(r+\rho_{ts})(P_{ts}-c+q_{ts}P_{ts}{}')+(\pi_{ts}-\pi_N)f_{ts}P_{ts}{}'\equiv 0 \tag{22}$$

(where the subscript *ts* indicates that a function is evaluated at trigger strategy equilibrium values of its arguments).[7]

[7] I assume that the second-order condition,

$$\frac{\partial^2}{\partial q_i^2}\left(V_i-\frac{\pi_N}{r}\right)\Big|_{ts}=\frac{(r+\rho_i)\frac{\partial^2\pi_i}{\partial q_i^2}-(\pi_i-\pi_N)\frac{\partial^2\rho_i}{\partial q_i^2}}{(r+\rho_i)^2}\Bigg|_{ts} \tag{23}$$

$$=\frac{(r+\rho_{ts})(2P'_{ts}+q_{ts}P''_{ts})-(\pi_{ts}-\pi_N)[f'_{ts}(P'_{ts})^2-f_{ts}P''_{ts}]}{(r+\rho_i)^2}<0 \tag{24}$$

for the defector's maximization problem is satisfied. Considering the numerator on the right in (23), the second-order condition is satisfied if firm profit is concave, and the probability of reversion convex, in the defector's output. Considering the numerator on the right in (24), $2P'+q_{ts}P''<0$ if the Hahn–Novshek condition is satisfied. Making this assumption, if P''_{ts} is small in magnitude, $f'_{ts}>0$ is sufficient to make the second term, which is subtracted, positive.

The first-order condition (22) can be rewritten[8]

$$\frac{\pi_{\mathrm{ts}} - \pi_{\mathrm{N}}}{r + \rho_{\mathrm{ts}}} + \frac{P_{\mathrm{ts}} - c + q_{\mathrm{ts}} P'_{\mathrm{ts}}}{f_{\mathrm{ts}} P'_{\mathrm{ts}}} \equiv 0. \tag{25}$$

The trigger strategy problem can then be formulated as

$$\max_{q_{\mathrm{ts}}, L} V_{\mathrm{ts}} - V_{\mathrm{N}} \quad \ni: \quad \left. \frac{\partial (V_i - V_{\mathrm{N}})}{\partial q_i} \right|_{\mathrm{eq}} \equiv 0. \tag{26}$$

or equivalently, using (25),

$$\max_{q_{\mathrm{ts}}, L} \frac{\pi_{\mathrm{ts}} - \pi_{\mathrm{N}}}{r + \rho_{\mathrm{ts}}} \quad \ni: \quad \frac{\pi_{\mathrm{ts}} - \pi_{\mathrm{N}}}{r + \rho_{\mathrm{ts}}} = -\frac{P_{\mathrm{ts}} + q_{\mathrm{ts}} P_{\mathrm{ts}}{}' - c}{f_{\mathrm{ts}} P_{\mathrm{ts}}{}'} \tag{27}$$

Details of the solution of this constrained optimization problem for the case of linear inverse expected demand and the triangular distribution are given in Appendix A, which contains a proof of

Result 1. For linear inverse expected demand and the triangular distribution, the trigger strategy equilibrium output per firm and reversion threshold are

$$q_{\mathrm{ts}} = \frac{1}{n} \left[\frac{a - c}{2b} + \frac{(n+1)^3}{n-1} \frac{\beta^2}{a - c} \frac{r}{b} \right] \tag{28}$$

and

$$L_{\mathrm{ts}} = c + \frac{a - c}{2} - \beta - (n+1)^2 \frac{\beta^2 r}{a - c}, \tag{29}$$

respectively.

It follows from (28) that the trigger strategy equilibrium expected price is[9]

$$P_{\mathrm{ts}} = c + \frac{a - c}{2} - \frac{(n+1)^3}{n-1} \frac{\beta^2 r}{a - c}. \tag{32}$$

From (28), q_{ts} exceeds the expected-joint-profit-maximizing level, $(a-c)/2nb$, and rises with β and r. Total output nq_{ts} rises as n rises. From (29), the reversion threshold falls as n, β, or r rise.

[8] For $\pi_{\mathrm{ts}} - \pi_{\mathrm{N}} > 0$, (25) implies that marginal revenue is greater than marginal cost, $P_{\mathrm{ts}} + q_{\mathrm{ts}} P'_{\mathrm{ts}} > c$, so that in trigger strategy equilibrium a firm restricts output below the privately profit-maximizing level.

[9] Result 1 is obtained on the assumption that $P_{\mathrm{ts}} - L_{\mathrm{ts}} \geq 0$. From (29) and (32),

$$P_{\mathrm{ts}} - L_{\mathrm{ts}} = \left[1 - \frac{2(n+1)^2}{n-1} \frac{\beta r}{a - c} \right] \beta, \tag{30}$$

and $P_{\mathrm{ts}} - L_{\mathrm{ts}} \geq 0$ requires

$$\frac{2(n+1)^2}{n-1} \frac{\beta r}{a - c} \leq 1. \tag{31}$$

Condition (31) is satisfied for the numerical results presented below.

When inverse demand has a random part, a low realized price might be the result of defection, or it might be the result of a large negative realized value of the random part of demand. To align incentives so that defection does not occur in equilibrium, output is expanded above the monopoly level, reducing the expected price. The reversion threshold is reduced accordingly.

Fig. 3 shows the trigger strategy price and reversion threshold for a numerical example as functions of the number of firms (treated, for simplicity, as a continuous variable) and for two different values of β. When β is small, relative to the price axis intercept of expected inverse demand, the trigger strategy price is very nearly at the monopoly level (60), and it would require a negative ε near the maximum possible magnitude to trigger reversion. For large β, the trigger strategy price is much lower, approaching the one-shot game Nash equilibrium level for $n=7$.

The trigger strategy equilibrium firm value is

$$V_{\rm ts} = V_{\rm N} + \frac{b/n}{r+\rho_{\rm ts}}\left\{\left(\frac{n-1}{n+1}\frac{a-c}{2b}\right)^2 - \left[\frac{(n+1)^3}{n-1}\frac{\beta^2}{a-c}\frac{r}{b}\right]^2\right\}, \tag{33}$$

from which $V_{\rm ts} \geq V_{\rm N} \Leftrightarrow$

$$a-c \geq \sqrt{2r}\frac{(n+1)^2}{n-1}\beta. \tag{34}$$

$V_{\rm ts}-V_{\rm N}$ rises with $a-c$ and falls as β, r, and n (for $n\geq 3$) rise. $V_{\rm ts}\geq V_{\rm N}$ provided the range of the random part of demand is not too great.

Fig. 4 shows trigger strategy equilibrium firm value, for the numerical example of Fig. 3, as the number of firms rises from 2 to 7. For $n=7$ and $\beta=5$, tacit collusion supported by a trigger strategy more than doubles firm value, compared with repeated play of the Nash–Cournot equilibrium of a one-shot game. For $n=7$ and $\beta=20$, the use of a trigger strategy means roughly 11% more value than repeated play of the Nash–Cournot equilibrium.

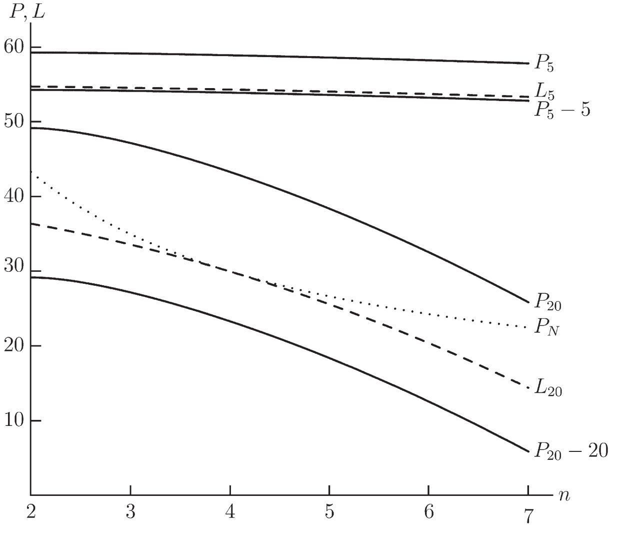

Fig. 3. Trigger strategy price P, reversion threshold L, $P-\beta$, and one-shot-game Nash–Cournot price $P_{\rm N}$, as functions of number of firms; $a=110$, $c=10$, $b=1$, $r=1/10$, $\beta=5$ and $\beta=20$.

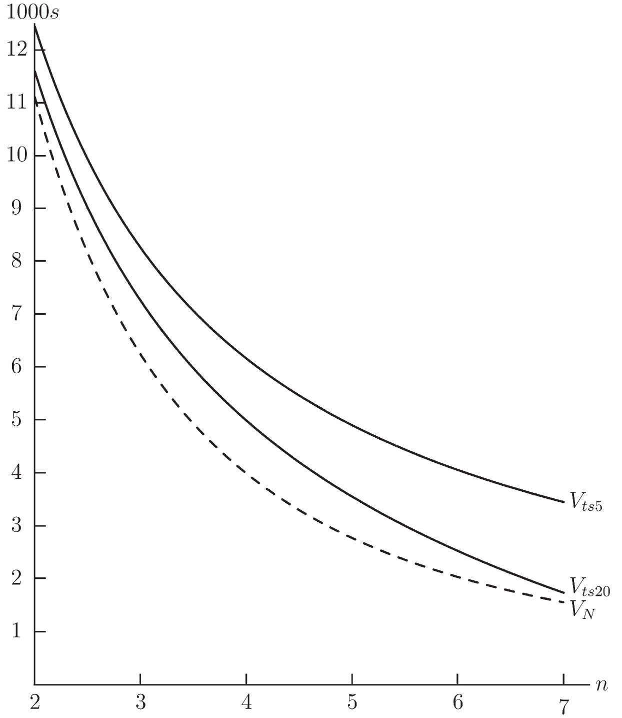

Fig. 4. One-shot game and trigger-strategy equilibrium firm value as functions of number of firms: $a=110$, $c=10$, $b=1$, $r=1/10$, $\beta=5$ and $\beta=20$.

Trigger strategy equilibrium reversion probability is

$$\rho_{\text{ts}} = \frac{1}{2}\left(\frac{\beta - a + nbq_{\text{ts}}}{\beta}\right)^2 = 2\left[\frac{(n+1)^2}{n-1}\frac{\beta r}{a-c}\right]^2. \tag{35}$$

The mean time to reversion is $1/\rho_{\text{ts}}$. ρ_{ts} rises with β and r and falls as $a-c$ rises. ρ_{ts} falls moving from 2 to 3 firms, and rises as n rises thereafter.

Table 1 reports equilibrium reversion probabilities for 2 through 7 firms, $a-c=100$, $b=1$, and $r=1/10$. For low values of β, reversion probabilities are strikingly small. Even with 7 firms and relatively large β – price ranging ±20 when the expected trigger strategy price is 25.87 – the mean time to reversion is almost 11 periods.

3.3. Collusion

A trigger strategy allows firms to restrict output, until reversion occurs, and, provided condition (34) is satisfied, to increase value compared with repeated play of the one-shot game Nash–Cournot equilibrium. It does not allow firms to restrict output to the joint-profit-maximizing level: to make defection unattractive, firms expand total output above that level that would be offered by a monopoly supplier (see (28)).

The result is that explicit agreement to restrict output may be profitable. Even in the absence of an antitrust or competition policy that prohibits such agreements, the approach of the Anglo-

Table 1
Equilibrium reversion probability as a function of the number of firms; $a=110$, $c=10$, $b=1$, $r=1/10$, $\beta=5$, 10, 15, and 20

n	β			
	5	10	15	20
2	0.0041	0.0162	0.0364	0.0648
3	0.0032	0.0128	0.0288	0.0512
4	0.0035	0.0139	0.0312	0.0556
5	0.0041	0.0162	0.0364	0.0648
6	0.0048	0.0192	0.0432	0.0768
7	0.0057	0.0228	0.0512	0.0910

Saxon common law was that collusive agreements, while not affirmatively illegal, could not be enforced in courts of law.[10] In such circumstances, successful collusion requires the private investment of resources to establish mechanisms that sustain collusion. Such mechanisms might involve the establishment of private institutions to monitor the conduct of cartel members, and to adjudicate apparent breakdowns in collusive behavior.[11] Some such mechanisms might well be modeled as elements of an equilibrium strategy in a repeated game. Here I treat private enforcement of a collusive agreement as a black box, and suppose that by paying a cost K per period per firm, firms can successfully restrict output to the joint-profit-maximizing level. K may be thought of as the cost per firm of setting up and maintaining procedures that would instantly detect a unilateral expansion of output above the collusive level should it occur, permitting immediate retaliation and thus rendering such defection unprofitable.

For linear inverse expected demand, joint-profit-maximizing output per firm is

$$q_{\mathrm{m}}=\frac{1}{n}\frac{a-c}{2b}, \tag{36}$$

so that the value of a colluding firm is

$$V_{\mathrm{m}}=\frac{(a-c-bnq_{\mathrm{m}})q_{\mathrm{m}}-K}{r}=\frac{1}{r}\left[\frac{b}{n}\left(\frac{a-c}{2b}\right)^2-K\right]. \tag{37}$$

Comparing (37) and trigger strategy equilibrium firm value, (33), some manipulation shows that $V_{\mathrm{m}}\geq V_{\mathrm{ts}}$ (that is, collusion yields at least as great a payoff as tacit collusion) provided the cost of collusion does not exceed a critical level:

$$K\leq K^*=\frac{(n+1)^2}{2n}\frac{\beta^2 r}{b}. \tag{38}$$

As should be expected, K^* rises with n: the greater the number of firms, the less is V_{ts}, the greater the incremental value from collusion, and the greater the cost of collusion that can be supported without making collusion less profitable than tacit collusion.

[10] See, for example, *Mogul Steamship Co.* v. *McGregor, Gow and Co. et al.* 54 L.J.Q.B. 540 (1884/1885); 57 L.J.K.B. 541 (1887/1888); 23 Q.B.D. 598 (C.A.)(1889); [1992] A.C. 25, or Judge Taft's discussion in U.S. v. Addyston Pipe and Steel 85 Fed 271 (1898).

[11] For discussions of the enforcement mechanisms employed by specific cartels, see Hudson (1891), Porter (1983b), Ellison (1994), Levinstein (1997), Grossman (1996), Genesove and Mullin (1998, 1999, 2001), Clyde and Reitzes (1998), and Connor (2001). See also Hay and Kelley (1974).

4. Competition policy

A Competition Authority with limited resources[12] will not be able to continuously monitor all firms in all industries. It will have to evaluate signals of various kinds to determine which industries invite close examination. I suppose here that the Competition Authority has responsibility for enforcing an anticollusion policy[13] and that it relies on price signals to determine industries that will be investigated.[14]

Specifically, I suppose that for each industry, the Competition Authority sets an investigation threshold U and investigates the industry if realized price equals or exceeds U. Investigation is required because the observation of a high price, in and of itself, is insufficient to determine whether or not collusion has occurred. A high realized price may reflect a large positive disturbance to the expected inverse demand curve; it may be the result of tacit collusion rather than collusion.

The probability of an investigation with output Q is

$$\tau = Pr(p \geq U) = Pr[P(Q) + \varepsilon \geq U] = Pr[\varepsilon \geq U - P(Q)] = \int_{U-P(Q)}^{\bar{\varepsilon}} f(\varepsilon)\mathrm{d}\varepsilon. \tag{39}$$

For the triangular distribution, the probability of investigation is (see Fig. 5)[15]

$$\tau = \frac{1}{2}\left[1 - \frac{U - P(Q)}{\beta}\right]^2 \tag{41}$$

There is inherent uncertainty in the enforcement process, and this uncertainty encompasses not only the types of behavior that may trigger an investigation but also the nature of the result that will follow if an investigation takes place.

Depending on the details of the enforcement regime, the Competition Authority may have the power to make an initial finding in its own right, in which case that finding is typically open to appeal to the courts. Alternatively, the Competition Authority's options after an investigation may be to drop the matter or to initiate action in the courts, where the first finding would be made. In either case, firms cannot predict with certainty what conclusion the Competition Authority will reach based on a particular set of facts. Neither firms nor the Competition Authority can predict with certainty what conclusions a court will reach if faced with a particular set of arguments.[16]

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[12] The Wall Street Journal reports (26 June 2003, internet edition, "Endless EU Antitrust Investigations Leave Enough Time for Frustration") that "the U.S. Federal Trade Commission and Department of Justice's Antitrust Division in 2002 shared an annual budget of \$206 million... EU enforcers in 2002 had an annual budget of €=60 million...."

[13] Thus, I do not deal with competition policy toward single-firm conduct (monopolization under U.S. antitrust law, abuse of a dominant position in the European Union), toward changes in market structure, or toward state aid to business (an element of EU competition policy).

[14] See Souam (2001) for an analysis of this type of antitrust regime when the antitrust authority is uncertain of the level of constant marginal cost.

[15] I limit attention to the case

$$0 \leq U - P(Q) \leq \beta, \tag{40}$$

which keeps τ in the range $0 \leq \tau \leq 1/2$.

[16] In the *Woodpulp* case, the European Commission found that collusion had taken place in violation of Article 81 of the EC Treaty. This decision was overturned by the European Court of Justice ([1988] 4 CMLR 901) on the ground (as described by an economist) that the Commission had not shown that observed market behavior was not the result of tacit collusion rather than collusion. In *U.S.* v. *Pfizer* (367 F. Supp. 91 (S.D.N.Y. 1973), the Court of Appeals was unwilling to find collusion in violation of the Sherman Act, despite a record showing meetings by the presidents of the firms alleged to have colluded at which the matters alleged to be the subject of a collusive agreement were discussed.

$$f(\varepsilon) = \begin{cases} \frac{\beta+\varepsilon}{\beta^2} & -\beta \le \varepsilon \le 0 \\ \frac{\beta-\varepsilon}{\beta^2} & 0 \le \varepsilon \le \beta \\ 0 & \text{otherwise} \end{cases}$$

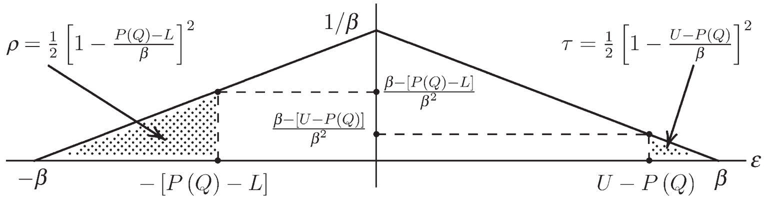

Fig. 5. Triangle density function, probability of reversion, probability of investigation: Q=output, L=reversion threshold, U=investigation threshold.

Here I meld this uncertainty into a single parameter, ω, the probability competition authority/courts make a mistake, if there is an investigation.[17] Let F denote the total fines in the event firms that are found to have colluded.[18]

4.1. One-shot game (equivalent) outcome

Firms that behave noncooperatively (whether in a one-shot or a repeated game) do not, in principle, risk being penalized for having colluded unless the enforcement process makes a mistake. If mistakes are possible, however, firms that behave noncooperatively may be fined, and the possibility of such a fine affects equilibrium behavior. Assume that any fine that is assessed is divided equally among firms found to have colluded.

If in any period all other firms produce output Q_{-i}, firm i's value is

$$V_i = \frac{\pi_i - \omega\tau_{\mathrm{N}}\dfrac{F}{n}}{r} = \frac{[P(Q_{-i}+q_i)-c]q_i - \dfrac{\omega F}{n}\displaystyle\int_{U-P(Q_{-i}+q_i)}^{\overline{\varepsilon}} f(\varepsilon)\mathrm{d}\varepsilon}{r}. \tag{42}$$

The equilibrium first-order condition to maximize V_i is

$$P(nq_{\mathrm{N}}) - c + \left\{q_{\mathrm{N}} - \frac{\omega F}{n} f[U - P(nq_{\mathrm{N}})]\right\}P'(nq_{\mathrm{N}}) \equiv 0, \tag{43}$$

and this implicitly defines Nash–Cournot equilibrium output per firm if firms play noncooperatively with a one-period time horizon.

For linear inverse demand (4) and the triangular distribution (5), the output implied by (43) is

$$q_{\mathrm{N}} = \frac{a - c + \frac{\omega F b}{n\beta^2}(a + \beta - U)}{\left(n + 1 + \frac{\omega F b}{\beta^2}\right)b}, \tag{44}$$

which reduces to (14) if ω=0.

[17] It would be possible to distinguish type I error (finding that a colluding firm had not colluded) and type II error (convicting a firm that had not colluded), or to allow for distinct probabilities of making a mistake on the part of the Competition Authority and the courts. In the interest of simplicity, these refinements are not pursued here.

[18] In a broader sense, F may be thought of as including expenses incurred by firms during the legal process. In a generalized model, F might be a function of a firm's turnover (as in the European Union) or of collusive profits. The latter introduces a dynamic element to the analysis, as considered by Harrington (2001).

Equilibrium firm value is

$$V_{\rm N} = \frac{1}{r}\left[(a-c-nbq_{\rm N})q_{\rm N} - \frac{\omega F}{2n\beta^2}(\beta - U + a - nbq_{\rm N})^2\right]. \tag{45}$$

4.2. Trigger strategy

Consider a trigger strategy of the form analyzed in Section 3.2. If all other firms produce combined output Q_{-i}, the value of firm i satisfies

$$V_i - V_{\rm N} = \frac{\pi_i - rV_{\rm N} - \frac{\omega F}{n}\tau_i}{r+\rho_i} = \frac{[P(Q_{-i}+q_i)-c]q_i - \frac{\omega F}{n}\int_{U-P(Q_{-i}+q_i)}^{\overline{\varepsilon}} f(\varepsilon)\mathrm{d}\varepsilon - rV_{\rm N}}{r + \int_{\underline{\varepsilon}}^{-[P(Q_{-i}+q_i)-L]} f(\varepsilon)\mathrm{d}\varepsilon} \tag{46}$$

The first-order condition to maximize (46) is

$$\frac{(r+\rho_i)\left(\frac{\partial \pi_i}{\partial q_i} - \frac{\omega F}{n}\frac{\partial \tau_i}{\partial q_i}\right) - \left(\pi_i - \frac{\omega F}{n}\tau_i - rV_{\rm N}\right)\frac{\partial \rho_i}{\partial q_i}}{(r+\rho_i)^2} \equiv 0. \tag{47}$$

For the trigger strategy to be stable, (47) must hold in equilibrium; this requires

$$\frac{\pi_{\rm ts} - \frac{\omega F}{n}\tau_{\rm ts} - rV_{\rm N}}{r+\rho_{\rm ts}} = \left.\frac{\frac{\partial \pi_i}{\partial q_i} - \frac{\omega F}{n}\frac{\partial \tau_i}{\partial q_i}}{\frac{\partial \rho_i}{\partial q_i}}\right|_{\rm ts}. \tag{48}$$

The trigger strategy problem is to

$$\max_{q_{\rm ts},L} \frac{\pi_{\rm ts} - \frac{\omega F}{n}\tau_{\rm ts} - rV_{\rm N}}{r+\rho_{\rm ts}} \tag{49}$$

subject to the constraint (48).

The solution to this problem is reached in the same way as is Result 1, and is given in Appendix A.

For notational compactness, write

$$k_1 = a + \beta - U$$

$$k_2 = U - c - \beta$$

$$k_3 = n + 1 + \frac{\omega F b}{\beta^2}$$

Result 2. For linear inverse expected demand and the triangular distribution, the trigger strategy output per firm and reversion threshold with competition policy are

$$q_{\rm ts} = \frac{1}{n}\frac{1}{2+\frac{\omega F b}{\beta^2}}\left(\frac{a-c}{b} + \frac{\omega F}{\beta^2}k_1 + \frac{k_3^3}{n-1}\frac{2\beta^2}{a-c+\frac{\omega F b}{\beta^2}k_2}\frac{r}{b}\right) \tag{50}$$

and

$$L_{\mathrm{ts}} = c - \beta + \frac{1}{2 + \frac{\omega F b}{\beta^2}}\left(a - c + \frac{\omega F b}{\beta^2}k_2 - k_3^3 \frac{2\beta^2 r}{a - c + \frac{\omega F b}{\beta^2}k_2}\right), \tag{51}$$

respectively.

From Result 2, it follows that the trigger strategy equilibrium price is

$$\rho_{\mathrm{ts}} = c + \frac{1}{2 + \frac{\omega F b}{\beta^2}}\left(a - c + \frac{\omega F b}{\beta^2}k_2 - \frac{k_3^2}{n-1}\frac{2\beta^2 r}{a - c + \frac{\omega F b}{\beta^2}k_2}\right). \tag{52}$$

(52) and (51) imply

$$\frac{\partial P_{\mathrm{ts}}}{\partial U} \geq 0 \quad \text{and} \quad \frac{\partial L}{\partial U} \geq 0, \tag{53}$$

where each derivative is equal to 0 if and only if $\omega=0$. If there is some possibility that noncooperative output restriction supported by a trigger strategy will lead to a penalty, then a lower investigation threshold (a tougher competition policy) means a lower equilibrium price and a lower reversion threshold, all else equal.

Fig. 6 illustrates Result 2 for particular parameter values. Expected price is below, and the maximum possible realized price above, the investigation threshold. Expected price is above, and the lowest possible realized price below, the reversion threshold. As the investigation threshold falls – as competition policy becomes tougher – the gap between U and P generally becomes smaller, and the probability of investigation rises.

(52) and (51) also imply that the equilibrium reversion probability is

$$\rho_{\mathrm{ts}} = 2\left(\frac{k_3^2}{n-1}\frac{\beta r}{a - c + \frac{\omega F b}{\beta^2}k_2}\right)^2, \tag{54}$$

from which

$$\frac{\partial \rho_{\mathrm{ts}}}{\partial U} \leq 0. \tag{55}$$

Once again, the derivative equals zero if and only if $\omega=0$. Otherwise, a lower investigation threshold increases the probability of reversion. This can be seen in Fig. 7, which shows the relationship between ρ and U for four values of β. The impact of reductions in U on the reversion probability is small, but reductions in U imply an increase in ρ.

As U falls, the gap between P and L also becomes smaller. Lower values of U increase the probability of investigation. The impact of changes in U on ρ is generally smaller in magnitude than the impact of changes in U on τ; this can also be seen comparing Figs. 7 and 8.

For a fixed value of U, the impact of changes in β on both τ and ρ is nonlinear, as shown in Fig. 9. Increases in β, unless from very low levels, increase the probability of reversion (which, however, remains low for the parameter values considered).

Table 2 shows firm values following different strategies for different levels of β and for a range of investigation thresholds.[19] Lower values of U – tougher competition policy –

[19] For calculations underlying Table 2, U does not exceed 60 (the monopoly price) and otherwise limited to a range that keeps reversion and investigation probabilities between 0 and 1/2. See Footnotes 6 and 15.

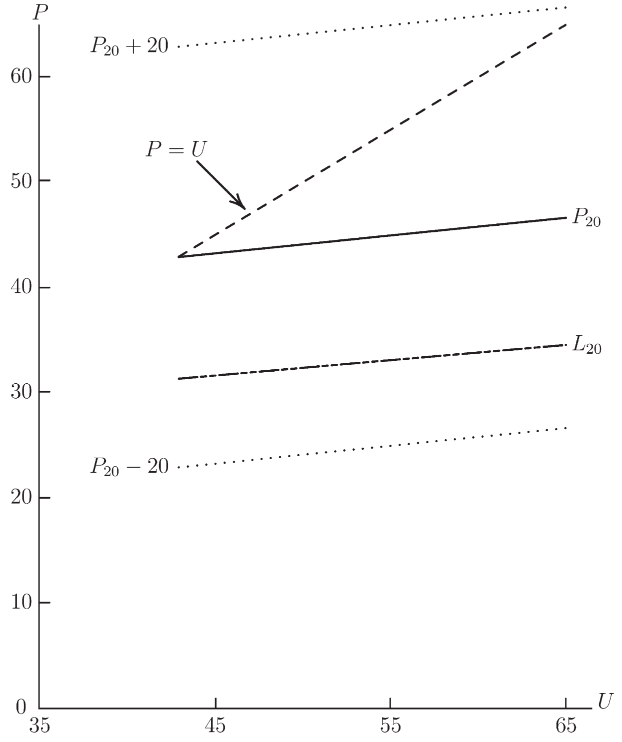

Fig. 6. Tacit collusion price P, and reversion threshold L as functions of the investigation threshold U. a=110, c=10, b=1, r=1/10, β=20, n=2, ω=1/5, F=625.

reduce both noncooperative and trigger strategy equilibrium value. For low values of β, the trigger strategy yields greater equilibrium firm value than repeated play of the equilibrium output of a one-shot game. For high values of β it is the one-shot game output that yields greatest value: when uncertainty is large, the trigger strategy requires a sufficiently great output expansion to nullify the temptation to defect that firms are better off without it. For a narrow intermediate range of β, the trigger strategy yields greater value for high U, the one-shot game strategy for low U. Overall, Table 2 suggests that it is uncertainty more than competition policy that determines which strategy yields the greatest equilibrium payoffs.

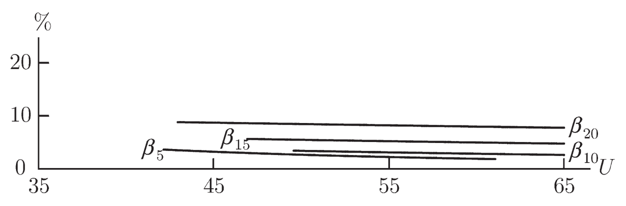

Fig. 7. Probability of reversion ρ as a function of the investigation threshold U. a=110, c=10, b=1, r=1/10, n=2, ω=1/5, F=625.

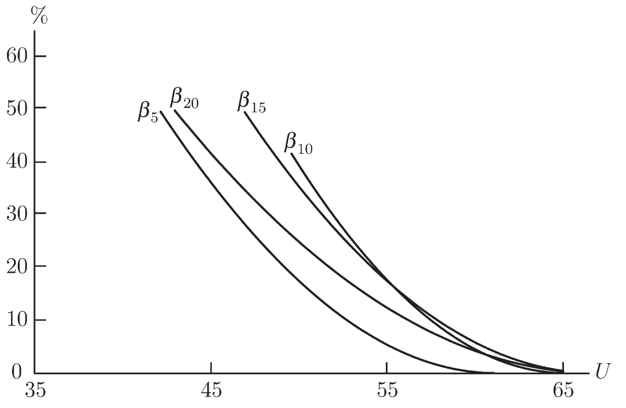

Fig. 8. Probability of investigation τ as a function of the investigation threshold U. $a=110$, $c=10$, $b=1$, $r=1/10$, $\beta=5$, 10, 15, 20, $n=2$, $\omega=1/5$, $F=625$.

4.3. Collusion

If firms collude, reversion is not an issue: the per-firm cost K puts mechanisms in place that render defection unprofitable. In the presence of a competition policy, the value of a colluding firm is

$$V_{\mathrm{m}} = \frac{1}{r}\left[\pi_{\mathrm{m}} - K - \frac{(1-\omega)F}{n}\tau_{\mathrm{m}}\right]. \tag{56}$$

A fine is levied if and only if there is an investigation and the legal system does not make a mistake.

One might well suspect that the cost of collusion, K, would be greater if there is an anticollusion policy, since successful collusion have to be kept secret. Leaving this aside from the formal model, the existence of an anticollusion policy affects collusive value in two ways.

First, to reduce the probability of investigation, firms will expand output above the no-competition policy-level (36). This output expansion reduces π_{m}.

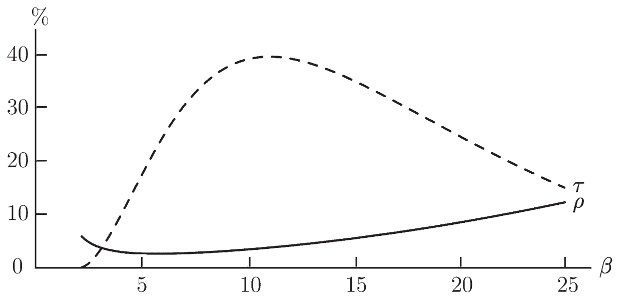

Fig. 9. Probability of reversion ρ and of investigation τ as functions of β. $a=110$, $c=10$, $b=1$, $r=1/10$, $U=50$, $\omega=1/5$, $F=625$.

Table 2
Investigation threshold and firm value, trigger strategy vs. repeated one-shot Nash values; $a=110$, $c=10$, $b=1$, $r=1/10$, $F=625$, $\omega=1/5$, $n=2$

β	(U, V_N, V_{ts})	$V_{ts}=V_N$ for $U\approx$	(U, V_N, V_{ts})
6	(44.7, 10625.90, 10922.84)	N/A	(49.3, 11108.13, 11477.41)
7	(46.3, 10754.70, 11133.19)	N/A	(50.3, 11108.55, 11552.47)
10	(48.1, 10800.08, 11299.79)	N/A	(53.3, 11109.47, 11685.79)
15	(46.9, 10668.16, 11085.21)	N/A	(58.3, 11110.24, 11640.34)
20	(47.5, 10690.36, 10837.33)	N/A	(60, 11051.09, 11280.74)
21	(48.1, 10711.85, 10787.13)	N/A	(60, 11036.96, 11185.23)
22	(48.7, 10731.24, 10728.25)	49.25	(60, 11024.19, 11085.58)
23	(49.2, 10745.78, 10657.65)	N/A	(60, 11012.61, 10981.74)
24	(49.6, 10756.21, 10576.43)	N/A	(60, 11002.07, 10873.68)
25	(50, 10765.87, 10488.64)	N/A	(60, 10992.43, 10761.36)
30	(51.7, 10798.55, 9950.847)	N/A	(60, 10954.66, 10134.90)

Second, the expected fine $\frac{(1-\omega)F}{n}\tau_m$ reduces the margin $V_m - \max(V_{ts}, V_N)$ that is available to cover the cost of collusion.

For linear inverse expected demand and the triangular distribution, (56) is

$$V_m = \frac{(a-c-bnq_m)q_m - K - \frac{(1-\omega)F}{2n\beta^2}(k_1 - bnq_m)^2}{r}. \tag{57}$$

The first-order condition to maximize (57) is

$$a - c - 2bnq_m + \frac{(1-\omega)bF}{\beta^2}(k_1 - bnq_m) \equiv 0, \tag{58}$$

from which collusive output per firm is

$$q_m = \frac{1}{n}\frac{a-c+\frac{(1-\omega)bF}{\beta^2}k_1}{b\left[2+\frac{(1-\omega)bF}{\beta^2}\right]}. \tag{59}$$

Substituting the first-order condition into (57), equilibrium collusive firm value is.

$$V_m = \frac{1}{r}\left\{\left[1+\frac{(1-\omega bF)}{2\beta^2}\right]bnq_m^2 - \frac{(1-\omega)F}{2n\beta^2}k_1^2 - K\right\}. \tag{60}$$

If $V_{ts} \geq \frac{\pi_N}{r}$, the relevant comparison is between collusion and tacit collusion. Collusion yields a greater equilibrium firm value than tacit collusion if

$$\left[1+\frac{(1-\omega)bF}{2\beta^2}\right]bnq_m^2 - \frac{(1-\omega)F}{2n\beta^2}k_1^2 \\ - K \geq \frac{(a-c-nbq_{ts})q_{ts} - \frac{\omega F}{2n\beta^2}(k_1 - nbq_{ts})^2 - rV_N}{r + \frac{1}{2\beta^2}(\beta - a + L + nbq_{ts})^2}. \tag{61}$$

Otherwise, repeated play of one-shot game equilibrium output yields a greater value than tacit collusion, and collusion yields at least as great a value as the one-shot game equilibrium outcome if

$$\left[1+\frac{(1-\omega)bF}{2\beta^2}\right]bnq_{\mathrm{m}}^2-\frac{(1-\omega)F}{2n\beta^2}k_1^2-K\geq(a-c-nbq_{\mathrm{N}})q_{\mathrm{N}} \\ -\frac{\omega F}{2n\beta^2}(k_1-nbq_{\mathrm{N}})^2. \tag{62}$$

4.4. Comparison

Suppose $\omega=0$, so that the legal system does not make mistakes. Then equilibrium outputs (44), (50), and (59) simplify to

$$q_{\mathrm{N}}=\frac{1}{n+1}\frac{a-c}{b} \tag{63}$$

$$q_{\mathrm{ts}}=\frac{1}{n}\left[\frac{a-c}{2b}+\frac{(n+1)^3}{n-1}\frac{\beta^2}{a-c}\frac{r}{b}\right] \tag{64}$$

and

$$q_{\mathrm{m}}=\frac{1}{n}\frac{a-c+\frac{bF}{\beta^2}(a+\beta-U)}{b\left(2+\frac{bF}{\beta^2}\right)}, \tag{65}$$

respectively.

For the no-mistakes case, competition policy has no impact on the one-shot-game equivalent or trigger strategy outcomes. If these regimes should be investigated, the enforcement regime would correctly conclude that collusion had not taken place, and no fine would be levied.

For the no-mistakes case, q_{N} is unaffected by β, while q_{ts} rises as β rises (see the discussion of (28)). (63) and (64) imply that $q_{\mathrm{N}}\geq q_{\mathrm{ts}}$, trigger strategy equilibrium output is weakly less than “one-shot game” equilibrium output, if and only if β is not too great relative to $a-c$,

$$a-c\leq\sqrt{2r}\left(n+3+\frac{4}{n-1}\right)\beta. \tag{66}$$

(See the discussion of (34).)

From (65) it follows that q_{m} may rise or fall as β rises, all else equal, and in fact both directions of change are realized for the parameters of Table 3.

From (64) and (65), $q_{\mathrm{ts}}\geq q_{\mathrm{m}}$ if and only if

$$\frac{(n+1)^3}{n-1}\frac{2\beta^2r}{a-c}+\frac{\frac{bF}{\beta^2}}{2+\frac{bF}{\beta^2}}[2(U-c-\beta)-(a-c)]\geq0.$$

While this condition is not as informative as one might like, it is met if $\frac{bF}{\beta^2}$ is sufficiently small (bF small implies that competition policy has little effect on equilibrium behavior) or if n is sufficiently large, all else equal.

Table 3 reports equilibrium output, firm value, and expected present discounted value of consumer surplus for the three different strategic regimes and (for collusion) for a range of values

Table 3
Equilibrium values, alternative β and modes of strategic interaction. $a=110$, $c=10$, $b=1$, $r=1/10$, $n=2$, $F=625$, $\omega=0$

	(q_N, V_N) CS_N	(q_{ts}, V_{ts}) CS_{ts}	(U, q_m, V_m) (CS_m, τ_m)	(U, q_m, V_m) (CS_m, τ_m)
$\beta=5$	(33.33, 11111)	(25.34, 12444)	(43, 35.19, 10259)	(60, 27.31, 12384)
	22222	13205	(24760, 0.05)	(14922, 0.003)
$\beta=10$	(33.33, 11111)	(26.35, 12275)	(43, 35.23, 9739)	(60, 28.79, 12121)
	22222	15049	(24819, 0.21)	(16575, 0.03)
$\beta=20$	(33.33, 11111)	(30.40, 11600)	(44.4, 32.81, 9721)	(60, 29.39, 11623)
	22222	19953	(21526, 0.50)	(17271, 0.16)
$\beta=30$	(33.33, 11111)	(37.15, 10475)	(49.6, 30.21, 10397)	(60, 28.87, 11340)
	22222	24411	(18248, 0.50)	(16665, 0.28)

of the investigation threshold.[20] Values under collusion are measured before allowing for the cost of collusion. Thus, for $\beta=30$ and $U=60$, the difference $11340-10475=865$ gives the maximum cost of collusion that would make net firm value under collusion weakly greater than firm value under tacit collusion supported by a grim trigger strategy; the difference $11340-11111=229$ gives the maximum cost of collusion that would make net firm value under collusion weakly greater than firm value with repeated play of the one-shot game equilibrium output.

Assuming the legal system does not make mistakes, and for a fine upon conviction that is one-quarter of single-period monopoly profit, then holding β constant, a lower investigation threshold (a tougher competition policy) increases collusive output. Tacit collusion yields the greatest value for low values of β. Tacit collusion equilibrium value falls as β rises. For higher values of β, collusion yields the greatest value if the investigation threshold is set sufficiently high, the one-shot game strategy yields the greatest value if the investigation threshold is set sufficiently low.

Ellison (1994, p. 52) notes that there are many possible reasons for apparent breakdowns in collusion. One explanation, suggested by the results of this section, is that the existence of an antitrust policy that punishes collusion may mean a change from an environment in which the payoff from collusion is greater than the payoff from either tacit collusion or one-shot game equilibrium behavior to an environment in which it is less. Given the asymmetry that characterizes real-world markets, the way different firms rank the payoffs associated with different types of conduct may differ, rendering coordination more difficult.[21]

5. Conclusion

The model of collusion developed here identifies collusion with private investment of resources that reduces (in the extreme version considered here, eliminates) the incentive to defect from an output-restricting equilibrium. The model thus directs the attention of antitrust enforcers toward market institutions and patterns of firm conduct that reduce uncertainty and alter incentives to defect.

Greater expected antitrust penalties generally increase consumer welfare, making it more likely that tacit collusion will yield greater firm value than collusion and more likely that one-

[20] Recall that for the parameters of Table 3 marginal cost is 10 and the unconstrained monopoly price 60. For Table 3 I have used 60 as the upper limit of the range of investigation thresholds considered, while the lower limit is the smallest value of U that keeps the probability of investigation below one-half.

[21] The question how firms settle on one of multiple equilibria poses problems of its own (Johansen, 1982, p. 437).

shot-game behavior will yield greater firm value than tacit collusion. These effects depend on expected antitrust fines being sufficiently large, and may be reversed if the enforcement system makes mistakes.

Appendix A

A.1. Proof of Result 1

A Lagrangian for (27) is

$$\mathcal{L} = \frac{\pi_{\mathrm{ts}} - \pi_{\mathrm{N}}}{r + \rho_{\mathrm{ts}}} + \lambda\left[\frac{\pi_{\mathrm{ts}} - \pi_{\mathrm{N}}}{r + \rho_{\mathrm{ts}}} + \frac{P_{\mathrm{ts}} + q_{\mathrm{ts}}P'_{\mathrm{ts}} - c}{f_{\mathrm{ts}}P'_{\mathrm{ts}}}\right] \tag{67}$$

$$= (1+\lambda)\frac{[P(nq_{\mathrm{ts}} - c)]q_{\mathrm{ts}} - \pi_{\mathrm{N}}}{r + \int_{-\beta}^{L-P(nq_{\mathrm{ts}})} f(\varepsilon)\mathrm{d}\varepsilon} + \lambda\frac{P(nq_{\mathrm{ts}}) + q_{\mathrm{ts}}P'(nq_{\mathrm{ts}}) - c}{f[L - P(nq_{\mathrm{ts}})]P'(nq_{\mathrm{ts}})}. \tag{68}$$

A.1.1. Kuhn–Tucker necessary conditions

q_{ts}:

$$(1+\lambda)\frac{(r+\rho_{\mathrm{ts}})(P_{\mathrm{ts}} - c + nq_{\mathrm{ts}}P'_{\mathrm{ts}}) + n(\pi_{\mathrm{ts}} - \pi_{\mathrm{N}})f_{\mathrm{ts}}P'_{\mathrm{ts}}}{(r+\rho_{\mathrm{ts}})^2} \\ + \lambda\frac{f_{\mathrm{ts}}P'_{\mathrm{ts}}[(n+1)P'_{\mathrm{ts}} + nq_{\mathrm{ts}}P''_{\mathrm{ts}}] - n(P_{\mathrm{ts}} + q_{\mathrm{ts}}P'_{\mathrm{ts}} - c)[f_{\mathrm{ts}}P''_{\mathrm{ts}} - (P'_{\mathrm{ts}})^2 f'_{\mathrm{ts}}]}{(f_{\mathrm{ts}}P'_{\mathrm{ts}})^2} \equiv 0. \tag{69}$$

L:

$$-(1+\lambda)\frac{\pi_{\mathrm{ts}} - \pi_{\mathrm{N}}}{(r+\rho_{\mathrm{ts}})^2}f_{\mathrm{ts}} - \frac{\lambda}{P'_{\mathrm{ts}}}\frac{P(nq_{\mathrm{ts}}) + q_{\mathrm{ts}}P'(nq_{\mathrm{ts}}) - c}{(f_{\mathrm{ts}})^2}f'_{\mathrm{ts}} \equiv 0. \tag{70}$$

λ: (this simply gives back the no-defection condition)

$$\frac{\pi_{\mathrm{ts}} - \pi_{\mathrm{N}}}{r + \rho_{\mathrm{ts}}} + \frac{P_{\mathrm{ts}} + q_{\mathrm{ts}}P'_{\mathrm{ts}} - c}{f_{\mathrm{ts}}P'_{\mathrm{ts}}} \equiv 0. \tag{71}$$

(71) implies

$$(r+\rho_{\mathrm{ts}})(P_{\mathrm{ts}} + q_{\mathrm{ts}}P'_{\mathrm{ts}} - c) + (\pi_{\mathrm{ts}} - \pi_{\mathrm{N}})f_{\mathrm{ts}}P'_{\mathrm{ts}} = 0. \tag{72}$$

The numerator of the coefficient of $1+\lambda$ in (69) is

$$(r+\rho_{\mathrm{ts}})(P_{\mathrm{ts}} - c + nq_{\mathrm{ts}}P'_{\mathrm{ts}}) + n(\pi_{\mathrm{ts}} - \pi_{\mathrm{N}})f_{\mathrm{ts}}P'_{\mathrm{ts}} \\ = (r+\rho_{\mathrm{ts}})[P_{\mathrm{ts}} + q_{\mathrm{ts}}P'_{\mathrm{ts}} - c + (n-1)q_{\mathrm{ts}}P'_{\mathrm{ts}}] + (\pi_{\mathrm{ts}} - \pi_{\mathrm{N}})f_{\mathrm{ts}}P'_{\mathrm{ts}} \\ + (n-1)(\pi_{\mathrm{ts}} - \pi_{\mathrm{N}})f_{\mathrm{ts}}P'_{\mathrm{ts}} =$$

(using (72),)

$$(n-1)[(r+\rho_{\mathrm{ts}})q_{\mathrm{ts}} + (\pi_{\mathrm{ts}} - \pi_{\mathrm{N}})f_{\mathrm{ts}}]P'_{\mathrm{ts}}. \tag{73}$$

Once again from (71)

$$(\pi_{\rm ts}-\pi_{\rm N})f_{\rm ts}=-(r+\rho_{\rm ts})\frac{P_{\rm ts}+q_{\rm ts}P'_{\rm ts}-c}{P_{\rm ts}{}'}. \tag{74}$$

Substitute in (73) and rearrange terms to obtain

$$(n-1)\left[(r+\rho_{\rm ts})q_{\rm ts}-(r+\rho_{\rm ts})\frac{P_{\rm ts}+q_{\rm ts}P'_{\rm ts}-c}{P'_{\rm ts}}\right]P'_{\rm ts} = -(n-1)(r+\rho_{\rm ts})(P_{\rm ts}-c). \tag{75}$$

Hence the Kuhn–Tucker condition for $q_{\rm ts}$ can be written

$$-(n-1)(1+\lambda)\frac{P_{\rm ts}-c}{r+\rho_{\rm ts}} + \lambda\frac{f_{\rm ts}P'_{\rm ts}[(n+1)P'_{\rm ts}+nq_{\rm ts}P''_{\rm ts}]-n(P_{\rm ts}+q_{\rm ts}P'_{\rm ts}-c)[f_{\rm ts}P''_{\rm ts}-(P'_{\rm ts})^2f'_{\rm ts}]}{(f_{\rm ts}P'_{\rm ts})^2}\equiv 0. \tag{76}$$

Simplify the numerator of the second term to obtain

$$-(n-1)(1+\lambda)\frac{P_{\rm ts}-c}{r+\rho_{\rm ts}} + \lambda\frac{[(n+1)f_{\rm ts}+n(P_{\rm ts}+q_{\rm ts}P'_{\rm ts}-c)f'_{\rm ts}](P'_{\rm ts})^2-n(P_{\rm ts}-c)f_{\rm ts}P''_{\rm ts}}{(f_{\rm ts}P'_{\rm ts})^2}\equiv 0. \tag{77}$$

Now return to the Kuhn–Tucker condition for L:

$$(1+\lambda)\frac{\pi_{\rm ts}-\pi_{\rm N}}{(r+\rho_{\rm ts})^2}f_{\rm ts}+\frac{\lambda}{P'_{\rm ts}}\frac{P_{\rm ts}+q_{\rm ts}P'_{\rm ts}-c}{(f_{\rm ts})^2}f'_{\rm ts}\equiv 0$$

From the no-defection condition

$$\frac{\pi_{\rm ts}-\pi_{\rm N}}{r+\rho_{\rm ts}}f_{\rm ts}\equiv -\frac{P_{\rm ts}+q_{\rm ts}P'_{\rm ts}-c}{P'_{\rm ts}}. \tag{78}$$

Substitute in the Kuhn–Tucker condition for L:

$$-\frac{1+\lambda}{r+\rho_{\rm ts}}\frac{P_{\rm ts}+q_{\rm ts}P'_{\rm ts}-c}{P'_{\rm ts}}+\frac{\lambda}{P'_{\rm ts}}\frac{P_{\rm ts}+q_{\rm ts}P'_{\rm ts}-c}{P'_{\rm ts}}\equiv 0$$

$$\left[-\frac{1+\lambda}{r+\rho_{\rm ts}}+\frac{\lambda}{(f_{\rm ts})^2}f'_{\rm ts}\right]\frac{P_{\rm ts}+q_{\rm ts}P'_{\rm ts}-c}{P'_{\rm ts}}\equiv 0$$

Hence

$$\frac{1+\lambda}{r+\rho_{\rm ts}}=\frac{\lambda}{(f_{\rm ts})^2}f'_{\rm ts}$$

$$\frac{1+\lambda}{\lambda}=\frac{r+\rho_{\rm ts}}{(f_{\rm ts})^2}f'_{\rm ts}. \tag{79}$$

Substitute in (77) to obtain one equation in q_{ts} and L.

$$-(n-1)f'_{\mathrm{ts}}(P_{\mathrm{ts}}-c) + \frac{[(n+1)f_{\mathrm{ts}}+n(P_{\mathrm{ts}}+q_{\mathrm{ts}}P'_{\mathrm{ts}}-c)f'_{\mathrm{ts}}](P'_{\mathrm{ts}})^2 - n(P_{\mathrm{ts}}-c)f_{\mathrm{ts}}P''_{\mathrm{ts}}}{(P'_{\mathrm{ts}})^2} \equiv 0 \tag{80}$$

or equivalently

$$\frac{f[L-P(nq_{\mathrm{ts}})]}{f'[L-P(nq_{\mathrm{ts}})]} = -\frac{P(nq_{\mathrm{ts}})-c+nq_{\mathrm{ts}}P'(nq_{\mathrm{ts}})}{n+1-n\frac{P(nq_{\mathrm{ts}})-c}{[P'(nq_{\mathrm{ts}})]^2}P''(nq_{\mathrm{ts}})}, \tag{81}$$

The o is the no-defection condition, (71), which implies

$$[(P_{\mathrm{ts}}-c)q_{\mathrm{ts}}-\pi_{\mathrm{N}}]f_{\mathrm{ts}}P'_{\mathrm{ts}} + (P_{\mathrm{ts}}+q_{\mathrm{ts}}P'_{\mathrm{ts}}-c)\int_{-\beta}^{L-P(nq_{\mathrm{ts}})} f(\varepsilon)\mathrm{d}\varepsilon = -(P_{\mathrm{ts}}+q_{\mathrm{ts}}P'_{\mathrm{ts}}-c)r. \tag{82}$$

Go back to (81):

$$\frac{f[L-P(nq_{\mathrm{ts}})]}{f'[L-P(nq_{\mathrm{ts}})]} = -\frac{P(nq_{\mathrm{ts}})-c+nq_{\mathrm{ts}}P'(nq_{\mathrm{ts}})}{n+1-n\frac{P(nq_{\mathrm{ts}})-c}{[P'(nq_{\mathrm{ts}})]^2}P''(nq_{\mathrm{ts}})} \tag{83}$$

Suppose inverse demand is linear, (4). Then (83) becomes

$$\frac{f(L-a+nbq_{\mathrm{ts}})}{f'(L-a+nbq_{\mathrm{ts}})} = -\frac{a-c-2nbq_{\mathrm{ts}}}{n+1}. \tag{84}$$

Now suppose that the random part of demand has the triangular distribution and that expected price exceeds the reversion threshold, $-\beta \le \varepsilon \le 0$, so that

$$f[-(a-nbq_{\mathrm{ts}}-L)] = \frac{\beta+L-a+nbq_{\mathrm{ts}}}{\beta^2} \quad f'[-(a-nbq_{\mathrm{ts}}-L)] = \frac{1}{\beta^2}. \tag{85}$$

Then (84) becomes

$$\beta+L-a+nbq_{\mathrm{ts}} = -\frac{a-c-2nbq_{\mathrm{ts}}}{n+1} \tag{86}$$

For linear inverse demand and the triangular distribution, the no-defection condition is

$$\left\{\left[(a-bnq_{\mathrm{ts}}-c)q_{\mathrm{ts}} - b\left(\frac{1}{n+1}\frac{a-c}{b}\right)^2\right]b - \frac{1}{2}[a-c-(n+1)bq_{\mathrm{ts}}](\beta-a+nbq_{\mathrm{ts}}+L)\right\} \times \frac{\beta-a+nbq_{\mathrm{ts}}+L}{\beta^2} = [a-(n+1)bq_{\mathrm{ts}}-c]r \tag{87}$$

Substitute (86) in the expression in braces and simplify to obtain

$$\left[(a-bnq_{\mathrm{ts}}-c)q_{\mathrm{ts}} - b\left(\frac{1}{n+1}\frac{a-c}{b}\right)^2\right]b + \frac{1}{2}[a-c-(n+1)bq_{\mathrm{ts}}]\frac{a-c-2nbq_{\mathrm{ts}}}{n+1} = (n-1)\frac{a-c-(n+1)bq_{\mathrm{ts}}}{2(n+1)^2}(a-c). \tag{88}$$

Substituting (88) and again (86), (87) becomes

$$\left[\frac{n-1}{2(n+1)^2}\frac{(a-c)}{\beta^2}\frac{a-c-2nbq_{\mathrm{ts}}}{n+1}+r\right][a-c(n+1)bq_{\mathrm{ts}}]=0. \tag{89}$$

Solving the expression in brackets for q_{ts} gives (28). (28) and (86) imply (29).

A.2. Proof of Result 2[22]

A Lagrangian for the maximization of (49) subject to the constraint (48) is

$$\begin{aligned}\mathcal{L}=&\frac{\pi_{\mathrm{ts}}-\frac{\omega F}{n}\tau_{\mathrm{ts}}-rV_{\mathrm{N}}}{r+\rho_{\mathrm{ts}}}\\&+\lambda\left\{\frac{\pi_{\mathrm{ts}}-\frac{\omega F}{n}\tau_{ts}-rV_{\mathrm{N}}}{r+\rho_{\mathrm{ts}}}-\frac{a-c-(n+1)bq_{\mathrm{ts}}+\frac{\omega Fb}{n\beta^2}(a+\beta-U-nbq_{\mathrm{ts}})}{\frac{b}{\beta^2}(\beta-a+L+nbq_{\mathrm{ts}})}\right\}\end{aligned} \tag{90}$$

$$\begin{aligned}=&(1+\lambda)\frac{(a-c-nbq_{\mathrm{ts}})q_{\mathrm{ts}}-\frac{\omega F}{2n\beta^2}(a+\beta-U-nbq_{\mathrm{ts}})^2-rV_{\mathrm{N}}}{r+\frac{1}{2\beta^2}(\beta-a+L+nbq_{\mathrm{ts}})^2}\\&-\lambda\frac{\beta^2}{b}\frac{a-c-(n+1)bq_{\mathrm{ts}}+\frac{\omega Fb}{n\beta^2}(a+\beta-U-nbq_{\mathrm{ts}})}{\beta-a+L+nbq_{\mathrm{ts}}}.\end{aligned} \tag{91}$$

A.2.1. Kuhn–Tucker necessary conditions

q_{ts}: The Kuhn–Tucker for q_{ts} condition is

$$\begin{aligned}&(1+\lambda)\frac{\left\{(r+\rho_{\mathrm{ts}})\left[a-c-2nbq_{\mathrm{ts}}-\frac{\omega F}{n\beta^2}(a+\beta-U-nbq_{\mathrm{ts}})(-nb)\right]-\left(\pi_{\mathrm{ts}}-\frac{\omega F}{n}\tau_{\mathrm{ts}}-rV_{\mathrm{N}}\right)\frac{nb}{\beta^2}(\beta-a+L+nbq_{\mathrm{ts}})\right\}}{(r+\rho_{\mathrm{ts}})^2}\\&-\lambda\frac{\beta^2}{b}\frac{\left\{(\beta-a+L+nbq_{\mathrm{ts}})\left[-(n+1)b-\frac{\omega Fb}{n\beta^2}nb\right]-\left[a-c-(n+1)bq_{\mathrm{ts}}+\frac{\omega Fb}{n\beta^2}(a+\beta-U-nbq_{\mathrm{ts}})\right]nb\right\}}{(\beta-a+L+nbq_{\mathrm{ts}})^2}\equiv 0\end{aligned}$$

Cancel b in the term that is the coefficient of λ, take β^2 back into denominator:

$$\begin{aligned}&(1+\lambda)\frac{\left\{\left(r+\rho_{\mathrm{ts}}\left[a-c-2nbq_{\mathrm{ts}}+\frac{\omega Fb}{\beta^2}+(a+\beta-U-nbq_{\mathrm{ts}})\right]-\left(\pi_{\mathrm{ts}}-\frac{\omega F}{n}\tau_{\mathrm{ts}}-rV_{\mathrm{N}}\right)\frac{nb}{\beta^2}(\beta-a+L+nbq_{\mathrm{ts}})\right\}}{(r+\rho_{\mathrm{ts}})^2}\\&+\lambda\frac{\left\{(\beta-a+L+nbq_{\mathrm{ts}})\left(n+1+\frac{\omega Fb}{\beta^2}\right)+\left[a-c-(n+1)bq_{\mathrm{ts}}+\frac{\omega Fb}{n\beta^2}+(a+\beta-U-nbq_{\mathrm{ts}})\right]n\right\}}{\left(\frac{\beta-a+L+nbq_{\mathrm{ts}}}{\beta}\right)^2}\equiv 0\end{aligned}$$

Distribute n in second term, rewrite the denominator in terms of ρ_{ts}:

$$\begin{aligned}&(1+\lambda)\frac{\left\{(r+\rho_{\mathrm{ts}})\left[a-c-2nbq_{\mathrm{ts}}+\frac{\omega Fb}{\beta^2}(a+\beta-U-nbq_{\mathrm{ts}})\right]-\left(\pi_{\mathrm{ts}}\frac{\omega F}{n}\tau_{\mathrm{ts}}-rV_{\mathrm{N}}\right)\frac{nb}{\beta^2}(\beta-a+L+nbq_{\mathrm{ts}})\right\}}{(r+\rho_{\mathrm{ts}})^2}\\&+\lambda\frac{\left\{(\beta-a+L+nbq_{\mathrm{ts}})\left(n+1+\frac{\omega Fb}{\beta^2}\right)+n[a-c-(n+1)bq_{\mathrm{ts}}]+\frac{\omega Fb}{\beta^2}(a+\beta-U-nbq_{\mathrm{ts}})\right\}}{2\rho_{\mathrm{ts}}}\equiv 0\end{aligned} \tag{92}$$

---

[22] A statement of the Proof of Result 2 that includes all intermediate steps is available on request from the author.

This will be further simplified below using the other Kuhn–Tucker conditions.

L: The Kuhn–Tucker for L condition is

$$-(1+\lambda)\frac{\pi_{\mathrm{ts}}-\frac{\omega F}{n}\tau_{\mathrm{ts}}-rV_{\mathrm{N}}}{(r+\rho_{\mathrm{ts}})^2}\frac{\beta-a+L+nbq_{\mathrm{ts}}}{\beta^2} + \lambda\frac{\beta^2}{b}\frac{a-c-(n+1)bq_{\mathrm{ts}}+\frac{\omega Fb}{n\beta^2}(a+\beta-U-nbq_{\mathrm{ts}})}{(\beta-a+L+nbq_{\mathrm{ts}})^2}\equiv 0.$$

Take $\frac{\beta^2}{b}$ back into the denominator of the second term:

$$-(1+\lambda)\frac{\pi_{\mathrm{ts}}-\frac{\omega F}{n}\tau_{\mathrm{ts}}-rV_{\mathrm{N}}}{(r+\rho_{\mathrm{ts}})^2}\frac{\beta-a+L+nbq_{\mathrm{ts}}}{\beta^2} + \lambda\frac{a-c-(n+1)bq_{\mathrm{ts}}+\frac{\omega Fb}{n\beta^2}(a+\beta-U-nbq_{\mathrm{ts}})}{b\left(\frac{\beta-a+L+nbq_{\mathrm{ts}}}{\beta}\right)^2}\equiv 0.$$

Rewrite denominator of second term in terms of ρ_{ts}:

$$-(1+\lambda)\frac{\pi_{\mathrm{ts}}-\frac{\omega F}{n}\tau_{\mathrm{ts}}-rV_{\mathrm{N}}}{(r+\rho_{\mathrm{ts}})^2}\frac{\beta-a+L+nbq_{V\mathrm{ts}}}{\beta^2} + \lambda\frac{a-c-(n+1)bq_{\mathrm{ts}}+\frac{\omega Fb}{n\beta^2}(a+\beta-U-nbq_{\mathrm{ts}})}{2b\rho_{\mathrm{ts}}}\equiv 0. \tag{93}$$

λ: The Kuhn–Tucker condition for λ gives back the no-defection condition:

$$\frac{\pi_{\mathrm{ts}}-\frac{\omega F}{n}\tau_{\mathrm{ts}}-rV_{\mathrm{N}}}{r+\rho_{\mathrm{ts}}}-\frac{a-c-(n+1)bq_{\mathrm{ts}}+\frac{\omega Fb}{n\beta^2}(a+\beta-U-nbq_{\mathrm{ts}})}{\frac{b}{\beta^2}(\beta-a+L+nbq_{\mathrm{ts}})}\geq 0 \tag{94}$$

$$\lambda\left[\frac{\pi_{\mathrm{ts}}-\frac{\omega F}{n}-rV_{\mathrm{N}}}{r+\rho_{\mathrm{ts}}}-\frac{a-c-(n+1)bq_{\mathrm{ts}}+\frac{\omega Fb}{n\beta^2}(a+\beta-U-nbq_{\mathrm{ts}})}{\frac{b}{\beta^2}(\beta-a)+L+nbq_{\mathrm{ts}}}\right]\equiv 0. \tag{95}$$

$$\lambda\geq 0 \tag{96}$$

Assume that the no-defection constraint is binding, so that

$$\frac{\pi_{\mathrm{ts}}-\frac{\omega F}{n}\tau_{\mathrm{ts}}-rV_{\mathrm{N}}}{r+\rho_{\mathrm{ts}}}-\frac{a-c-(n+1)bq_{\mathrm{ts}}+\frac{\omega Fb}{n\beta^2}(a+\beta-U-nbq_{\mathrm{ts}})}{\frac{b}{\beta^2}(\beta-a+L+nbq_{\mathrm{ts}})}\equiv 0. \tag{97}$$

Return to the Kuhn–Tucker condition for q_{ts}, (92), and use the Kuhn–Tucker condition for λ, which implies

$$(r+\rho_{\mathrm{ts}})\left[a-c-(n+1)bq_{\mathrm{ts}}+\frac{\omega Fb}{n\beta^2}(a+\beta-U-nbq_{\mathrm{ts}})\right] - \left(\pi_{\mathrm{ts}}-\frac{\omega F}{n}\tau_{\mathrm{ts}}-rV_{\mathrm{N}}\right)\frac{b}{\beta^2}(\beta-a+L+nbq_{\mathrm{ts}})\equiv 0 \tag{98}$$

to simplify (92). Rewrite the numerator of the fraction that is the coefficient of $1+\lambda$:

$$(r+\rho_{\mathrm{ts}})\left[a-c-(n+1)bq_{\mathrm{ts}}+\frac{(n-1+1)\omega Fb}{n\beta^2}(a+\beta-U-nbq_{\mathrm{ts}})-(n-1)bq_{\mathrm{ts}}\right]$$
$$-\left(\pi_{\mathrm{ts}}-\frac{\omega F}{n}\tau_{\mathrm{ts}}-rV_N\right)\frac{(n-1)b+b}{\beta^2}(\beta-a+L+nbq_{\mathrm{ts}});$$

substituting (98), this becomes

$$(r+\rho_{\mathrm{ts}})\left[\frac{(n-1)\omega Fb}{n\beta^2}(a+\beta-U-nbq_{\mathrm{ts}})-(n-1)bq_{\mathrm{ts}}\right]$$
$$-\left(\pi_{\mathrm{ts}}-\frac{\omega F}{n}\tau_{\mathrm{ts}}-rV_{\mathrm{N}}\right)\frac{(n-1)b}{\beta^2}(\beta-a+L+nbq_{\mathrm{ts}}).$$

Then (92) becomes

$$-(1+\lambda)(n-1)b\frac{\left\{(r+\rho_{\mathrm{ts}})\left[q_{\mathrm{ts}}-\frac{\omega F}{n\beta^2}(a+\beta-U-nbq_{\mathrm{ts}})\right]+\left(\pi_{\mathrm{ts}}-\frac{\omega F}{n}\tau_{\mathrm{ts}}-rV_{\mathrm{N}}\right)\frac{(\beta-a+L+nbq_{\mathrm{ts}})}{\beta^2}\right\}}{(r+\rho_{\mathrm{ts}})^2}$$
$$+\lambda\frac{\left\{(\beta-a+L+nbq_{\mathrm{ts}})\left(n+1+\frac{\omega Fb}{\beta^2}\right)+n[a-c-(n+1)bq_{\mathrm{ts}}]+\frac{\omega Fb}{\beta^2}(a+\beta-U-nbq_{\mathrm{ts}})\right\}}{2\rho_{\mathrm{ts}}}\equiv 0. \quad (99)$$

Once again from the Kuhn–Tucker condition for λ:

$$\left(\pi_{\mathrm{ts}}-\frac{\omega F}{n}\tau_{\mathrm{ts}}-rV_{\mathrm{N}}\right)\frac{(\beta-a+L+nbq_{\mathrm{ts}})}{\beta^2}=(r+\rho_{\mathrm{ts}})$$
$$\times\left[\frac{a-c}{b}-(n+1)q_{\mathrm{ts}}+\frac{\omega F}{n\beta^2}(a+\beta-U-nbq_{\mathrm{ts}})\right].$$

Substitute in the second term of the numerator of the fraction that is the coefficient of $(1+\lambda)(n-1)b$ in (99) to obtain

$$-(1+\lambda)(n-1)$$
$$\times b\frac{\left\{(r+\rho_{\mathrm{ts}})\left[q_{\mathrm{ts}}-\frac{\omega F}{n\beta^2}(a+\beta-U-nbq_{\mathrm{ts}})\right]+(r+\rho_{\mathrm{ts}})\left[\frac{a-c}{b}-(n+1)q_{\mathrm{ts}}+\frac{\omega F}{n\beta^2}(a+\beta-U-nbq_{\mathrm{ts}})\right]\right\}}{(r+\rho_{\mathrm{ts}})^2}$$
$$+\lambda\frac{\left\{(\beta-a+L+nbq_{\mathrm{ts}})\left(n+1+\frac{\omega Fb}{\beta^2}\right)+n[a-c-(n+1)bq_{\mathrm{ts}}]+\frac{\omega Fb}{\beta^2}(a+\beta-U-nbq_{\mathrm{ts}})\right\}}{2\rho_{\mathrm{ts}}}=0.$$

In the first term, cancel $r+\rho_{\mathrm{ts}}$ and multiply through the first term by b:

$$-(1+\lambda)(n-1)\frac{\left\{bq_{\mathrm{ts}}-\frac{\omega Fb}{n\beta^2}(a+\beta-U-nbq_{\mathrm{ts}})+a-c-(n+1)bq_{\mathrm{ts}}+\frac{\omega Fb}{n\beta^2}(a+\beta-U-nbq_{\mathrm{ts}})\right\}}{r+\rho_{\mathrm{ts}}}$$
$$+\lambda\frac{\left\{(\beta-a+L+nbq_{\mathrm{ts}})\left(n+1+\frac{\omega Fb}{\beta^2}\right)+n[a-c-(n+1)bq_{\mathrm{ts}}]+\frac{\omega Fb}{\beta^2}(a+\beta-U-nbq_{\mathrm{ts}})\right\}}{2\rho_{\mathrm{ts}}}$$
$$=0.$$

Simplify the numerator of the first term to obtain:

$$
\begin{aligned}
&-(1+\lambda)(n-1)\frac{a-c-nbq_{\mathrm{ts}}}{r+\rho_{\mathrm{ts}}}\\
&\quad+\lambda\frac{\left\{(\beta-a+L+nbq_{\mathrm{ts}})\left(n+1+\frac{\omega Fb}{\beta^2}\right)+n[a-c-(n+1)bq_{\mathrm{ts}}]+\frac{\omega Fb}{\beta^2}(a+\beta-U-nbq_{\mathrm{ts}})\right\}}{2\rho_{\mathrm{ts}}}\\
&\quad=0.
\end{aligned}
$$

Rewrite the numerator of the second term as

$$
\begin{aligned}
&(\beta-a+L+nbq_{\mathrm{ts}})\left(n+1+\frac{\omega Fb}{\beta^2}\right)+n[a-c-(n+1)bq_{\mathrm{ts}}]+\frac{\omega Fb}{\beta^2}(a+\beta-U\\
&\quad-nbq_{\mathrm{ts}})\\
&\quad=\left(n+1+\frac{\omega Fb}{\beta^2}\right)(\beta-c+L)+\frac{\omega Fb}{\beta^2}(\beta+c-U)-(a-c).
\end{aligned}
$$

Hence the Kuhn–Tucker condition for q_{ts} can be written

$$
\begin{aligned}
&-(1+\lambda)(n-1)\frac{a-c-nbq_{\mathrm{ts}}}{r+\rho_{\mathrm{ts}}}\\
&\quad+\lambda\frac{\left(n+1+\frac{\omega Fb}{\beta^2}\right)(\beta-c+L)+\frac{\omega Fb}{\beta^2}(\beta+c-U)-(a-c)}{2\rho_{\mathrm{ts}}}\equiv 0.
\end{aligned}
\tag{100}
$$

Now turn to the Kuhn–Tucker condition for L, (93):

$$
\begin{aligned}
&-(1+\lambda)\frac{\pi_{\mathrm{ts}}-\frac{\omega F}{n}\tau_{\mathrm{ts}}-rV_{\mathrm{N}}}{(r+\rho_{\mathrm{ts}})^2}\frac{\beta-a+L+nbq_{\mathrm{ts}}}{\beta^2}\\
&\quad+\lambda\frac{a-c-(n+1)bq_{\mathrm{ts}}+\frac{\omega Fb}{n\beta^2}(a+\beta-U-nbq_{\mathrm{ts}})}{2b\rho_{\mathrm{ts}}}\equiv 0.
\end{aligned}
$$

From the no-defection condition

$$
\frac{\pi_{\mathrm{ts}}-\frac{\omega F}{n}\tau_{\mathrm{ts}}-rV_{\mathrm{N}}}{r+\rho_{\mathrm{ts}}}\frac{\beta-a+L+nbq_{\mathrm{ts}}}{\beta^2}=\frac{a-c-(n+1)bq_{\mathrm{ts}}+\frac{\omega Fb}{n\beta^2}(a+\beta-U-nbq_{\mathrm{ts}})}{b}.
$$

Hence the Kuhn–Tucker condition for L can be rewritten

$$
\begin{aligned}
&-\frac{1+\lambda}{r+\rho_{\mathrm{ts}}}\frac{a-c-(n+1)bq_{\mathrm{ts}}+\frac{\omega Fb}{n\beta^2}(a+\beta-U-nbq_{\mathrm{ts}})}{b}\\
&\quad+\lambda\frac{a-c-(n+1)bq_{\mathrm{ts}}+\frac{\omega Fb}{n\beta^2}(a+\beta-U-nbq_{\mathrm{ts}})}{2b\rho_{\mathrm{ts}}}\equiv 0.
\end{aligned}
$$

Factor:

$$
\frac{a-c-(n+1)bq_{\mathrm{ts}}+\frac{\omega Fb}{n\beta^2}(a+\beta-U-nbq_{\mathrm{ts}})}{b}\left(-\frac{1+\lambda}{r+\rho_{\mathrm{ts}}}+\frac{\lambda}{2\rho_{\mathrm{ts}}}\right)\equiv 0.
$$

The numerator of the first term is known to be nonzero (from the equilibrium no-defection condition); hence the Kuhn–Tucker condition for L implies

$$\frac{1+\lambda}{r+\rho_{\mathrm{ts}}} = \frac{\lambda}{2\rho_{\mathrm{ts}}}. \tag{101}$$

Now return to the Kuhn–Tucker condition for q_{ts}, (100):

$$-\frac{1+\lambda}{\lambda}(n-1)\frac{a-c-nbq_{\mathrm{ts}}}{r+\rho_{\mathrm{ts}}} + \frac{\left(n+1+\frac{\omega Fb}{\beta^2}\right)(\beta-c+L)+\frac{\omega Fb}{\beta^2}(\beta+c-U)-(a-c)}{2\rho_{\mathrm{ts}}} = 0.$$

Substitute (101) to obtain

$$-\frac{r+\rho_{\mathrm{ts}}}{2\rho_{\mathrm{ts}}}(n-1)\frac{a-c-nbq_{\mathrm{ts}}}{r+\rho_{\mathrm{ts}}} + \frac{\left(n+1+\frac{\omega Fb}{\beta^2}\right)(\beta-c+L)+\frac{\omega Fb}{\beta^2}(\beta+c-U)-(a-c)}{2\rho_{\mathrm{ts}}} = 0.$$

Multiply through by $2\rho_{\mathrm{ts}}$:

$$-(n-1)(a-c-nbq_{\mathrm{ts}}) + \left(n+1+\frac{\omega Fb}{\beta^2}\right)(\beta-c+L) + \frac{\omega Fb}{\beta^2}(\beta+c-U)-(a-c) = 0. \tag{102}$$

(102) is a linear equation in q_{ts} and L.

Rewrite (102) so that it is in terms of $\beta-a+L$ and $a+\beta-U$: considering the last three terms,

$$\begin{aligned}&\left(n+1+\frac{\omega Fb}{\beta^2}\right)(\beta-a+L+a-c)+\frac{\omega Fb}{\beta^2}[a+\beta-U-(a-c)]-(a-c)\\ &\quad=\left(n+1+\frac{\omega Fb}{\beta^2}\right)(\beta-a+L)+\frac{\omega Fb}{\beta^2}(a+\beta-U)+n(a-c).\end{aligned}$$

Hence (102) can be rewritten

$$-(n-1)(a-c-nbq_{\mathrm{ts}}) + \left(n+1+\frac{\omega Fb}{\beta^2}\right)(\beta-a+L) + \frac{\omega Fb}{\beta^2}(a+\beta-U)+n(a-c) = 0. \tag{103}$$

Solve (103) for $\beta-a+L+nbq_{\mathrm{ts}}$, which will be needed later:

$$\beta-a+L+nbq_{\mathrm{ts}} = -\frac{a-c-n\left(2+\frac{\omega Fb}{\beta^2}\right)bq_{\mathrm{ts}}+\frac{\omega Fb}{\beta^2}(a+\beta-U)}{n+1+\frac{\omega Fb}{\beta^2}}. \tag{104}$$

For notational compactness, write

$$x = a - c$$

$$y = bq_{\mathrm{ts}}$$

$$u = a + \beta - U$$

$$z = \frac{\omega F b}{n\beta^2}.$$

Then (104) becomes

$$\beta - a + L + nbq_{\mathrm{ts}} = -\frac{x - n(2 + nz)y + nzu}{n + 1 + nz}. \tag{105}$$

The other equation in q_{ts} and L is the no-defection condition

$$\frac{(a - c - nbq_{\mathrm{ts}})q_{\mathrm{ts}} - \frac{\omega F b}{2n\beta^2}(a + \beta - U - nbq_{\mathrm{ts}})^2 - rV_{\mathrm{N}}}{r + \frac{1}{2\beta^2}(\beta - a + L + nbq_{\mathrm{ts}})^2} = \frac{a - c - (n+1)bq_{\mathrm{ts}} + \frac{\omega F b}{n\beta^2}(a + \beta - U - nbq_{\mathrm{ts}})}{\frac{b}{\beta^2}(\beta - a + L + nbq_{\mathrm{ts}})}.$$

Multiply through by b and substitute notation:

$$\frac{(x - nbq_{\mathrm{ts}})y - \frac{z}{2}(u - y)^2 - rbV_{\mathrm{N}}}{r + \frac{1}{2\beta^2}(\beta - a + L + nbq_{\mathrm{ts}})^2} = \frac{x - (n+1)y + z(u - ny)}{\frac{1}{\beta^2}(\beta - a + L + nbq_{\mathrm{ts}})}.$$

Cross-multiply:

$$\frac{1}{\beta^2}(\beta - a + L + nbq_{\mathrm{ts}})\left[(x - nbq_{\mathrm{ts}})y - \frac{z}{2}(u - ny)^2 - rbV_{\mathrm{N}}\right] = \left[r + \frac{1}{2\beta^2}(\beta - a + L + nbq_{\mathrm{ts}})^2\right][x - (n+1)y + z(u - ny)].$$

Multiply through by β^2, distribute terms on right-hand side:

$$(\beta - a + L + nbq_{\mathrm{ts}})\left[(x - nbq_{\mathrm{ts}})y - \frac{z}{2}(u - ny)^2 - rbV_{\mathrm{N}}\right] = \beta^2 r[x - (n+1)y + z(u - nbq_{\mathrm{ts}})] + \frac{1}{2}(\beta - a + L + nbq_{\mathrm{ts}})^2[x - (n+1)y + z(u - ny)].$$

Leave terms in $\beta^2 r$ on the right, collect all other terms on the left:

$$(\beta - a + L + nbq_{\mathrm{ts}}) \times \left\{(x - nbq_{\mathrm{ts}})y - \frac{z}{2}(u - ny)^2 - \frac{1}{2}(\beta - a + L + nbq_{\mathrm{ts}})[x - (n+1)y + z(u - nbq_{\mathrm{ts}})] - rbV_{\mathrm{N}}\right\} = \beta^2 r[x - (n+1)y + z(u - ny)].$$

Now substitute (105) in the first three terms in the expression in braces on the left:

$$(x-ny)y-\frac{z}{2}(u-ny)^2-\frac{1}{2}\left(-\frac{x-n(2+nz)y+nzu}{n+1+nz}\right)[x-(n+1)y+z(u-ny)]$$
$$=(x-ny)y-\frac{z}{2}(u-ny)^2+\frac{1}{2}[x-(n+1)y+z(u-ny)]\frac{x-n(2+nz)y+nzu}{n+1+nz}.$$

Note that

$$x-n(2+nz)y+nzu=x-2ny+nz(u-ny).$$

Hence the first three terms can be written

$$(x-ny)y-\frac{z}{2}(u-ny)^2+\frac{1}{2}\frac{x-2ny+nz(u-ny)}{n+1+nz}[x-(n+1)y+z(u-ny)],$$

and the whole expression in braces is

$$(x-ny)y-\frac{z}{2}(u-ny)^2+\frac{1}{2}\frac{x-2ny+nz(u-ny)}{n+1+nz}[x-(n+1)y+z(u-ny)]-rbV_{\mathrm{N}}.$$

Now substitute the expression for V_{N} that is valid for the case $0\le\tau_{\mathrm{N}}\le 1/2$,

$$brV_{\mathrm{N}}=\frac{-\left(1+\frac{z}{2}\right)zu^2+\frac{1}{2}\left\{\frac{2x+[(n^2+1+nz)u-(n-2)nx]z}{n+1+nz}\right\}^2}{2-(n-2)nz},$$

in the expression in braces on the left:

$$(x-ny)y-\frac{z}{2}(u-ny)^2+\frac{1}{2}\frac{x-2ny+nz(u-ny)}{n+1+nz}[x-(n+1)y+z(u-ny)]$$
$$-\frac{-\left(1+\frac{z}{2}\right)zu^2+\frac{1}{2}\left\{\frac{2x+[(n^2+1+nz)u-(n-2)nx]z}{n+1+nz}\right\}^2}{2-(n-2)nz}$$
$$=(n-1)\frac{x+nz(x-u)}{2(n+1+nz)^2}[x-(n+1)y+z(u-ny)].$$

The equation is

$$(\beta-a+L+nbq_{\mathrm{ts}})(n-1)\frac{x+nz(x-u)}{2(n+1+nz)^2}[x-(n+1)y+z(u-ny)]$$
$$=\beta^2r[x-(n+1)y+z(u-ny)],$$

or

$$(\beta-a+L+nbq_{\mathrm{ts}})(n-1)\frac{x+nz(x-u)}{2(n+1+nz)^2}=\beta^2r.$$

Substitute from (105) to eliminate $\beta-a+\mathrm{L}+nbq_{\mathrm{ts}}$:

$$-\frac{x-n(2+nz)y+nzu}{n+1+nz}(n-1)\frac{x+nz(x-u)}{2(n+1+nz)^2}=\beta^2r$$

Solve for y:

$$y = \frac{x + nzu}{n(2 + nz)} + \frac{2(n + 1 + nz)^3}{n(2 + nz)[x + nz(x - u)]} \frac{\beta^2 r}{(n - 1)}.$$

A.2.2. q_{ts}

Now return to the underlying notation ($x = a - c$, $y = bq_{\text{ts}}$, $u = a + \beta - U$, $z = \frac{\omega Fb}{n\beta^2}$):

$$bq_{\text{ts}} = \frac{a - c + n\frac{\omega Fb}{n\beta^2}(a + \beta - U)}{n\left(2 + n\frac{\omega Fb}{n\beta^2}\right)} + \frac{2\left(n + 1 + n\frac{\omega Fb}{n\beta^2}\right)^3}{n\left(2 + n\frac{\omega Fb}{n\beta^2}\right)\left[a - c + n\frac{\omega Fb}{n\beta^2}(a - c - a - \beta + U)\right]} \frac{\beta^2 r}{n - 1}$$

$$q_{\text{ts}} = \frac{1}{n}\left[\frac{\frac{a-c}{b} + \frac{\omega F}{\beta^2}(a + \beta - U)}{2 + \frac{\omega Fb}{\beta^2}} + \frac{2}{2 + \frac{\omega Fb}{\beta^2}} \frac{\left(n + 1 + \frac{\omega Fb}{\beta^2}\right)^3}{n - 1} \frac{\beta^2}{a - c + \frac{\omega Fb}{\beta^2}(U - c - \beta)} \frac{r}{b}\right]. \tag{106}$$

A.2.3. P_{ts}

Expected price:

$$P_{\text{ts}} = c + a - c - nbq_{\text{ts}} = c + \frac{\left(1 + \frac{\omega Fb}{\beta^2}\right)(a - c) - \frac{\omega Fb}{\beta^2}(a + \beta - U)}{2 + \frac{\omega Fb}{\beta^2}} - \frac{2}{2 + \frac{\omega Fb}{\beta^2}} \frac{\left(n + 1 + \frac{\omega Fb}{\beta^2}\right)^3}{n - 1} \frac{\beta^2 r}{a - c + \frac{\omega Fb}{\beta^2}(U - c - \beta)}. \tag{107}$$

Note that by rearranging terms in the numerator of the second term on the right, (107) may be rewritten

$$P_{\text{ts}} = c + \frac{a - c - \frac{\omega Fb}{\beta^2}(c + \beta - U)}{2 + \frac{\omega Fb}{\beta^2}} - \frac{2}{2 + \frac{\omega Fb}{\beta^2}} \frac{\left(n + 1 + \frac{\omega Fb}{\beta^2}\right)^3}{n - 1} \frac{\beta^2 r}{a - c + \frac{\omega Fb}{\beta^2}(U - c - \beta)}. \tag{108}$$

A.2.4. L

Solve (105) for L:

$$\beta + L - (a + nbq_{\text{ts}}) = -\frac{a - c - n\left(2 + \frac{\omega Fb}{\beta^2}\right)bq_{\text{ts}} + \frac{\omega Fb}{\beta^2}(a + \beta - U)}{n + 1 + \frac{\omega Fb}{\beta^2}}.$$

$$\beta + L = P_{\text{ts}} - \frac{a - c - \left(2 + \frac{\omega Fb}{\beta^2}\right)nbq_{\text{ts}} + \frac{\omega Fb}{\beta^2}(a + \beta - U)}{n + 1 + \frac{\omega Fb}{\beta^2}}$$

First simplify the numerator of the fraction on the right:

$$a-c-\left(2+\frac{\omega Fb}{\beta^2}\right)nbq_{\text{ts}}+\frac{\omega Fb}{\beta^2}(a+\beta-U)=\frac{\left(n+1+\frac{\omega Fb}{\beta^2}\right)^3}{n-1}\frac{2\beta^2 r}{a-c+\frac{\omega Fb}{\beta^2}(U-c-\beta)}.$$

Then

$$\beta+L=P_{\text{ts}}-\frac{\frac{\left(n+1+\frac{\omega Fb}{\beta^2}\right)^3}{n-1}\frac{2\beta^2 r}{a-c+\frac{\omega Fb}{\beta^2}(U-c-\beta)}}{n+1+\frac{\omega Fb}{\beta^2}}=c+\frac{a-c-\frac{\omega Fb}{\beta^2}(c+\beta-U)}{2+\frac{\omega Fb}{\beta^2}}$$
$$-\frac{\left(n+1+\frac{\omega Fb}{\beta^2}\right)^2}{2+\frac{\omega Fb}{\beta^2}}\frac{2\beta^2 r}{a-c+\frac{\omega Fb}{\beta^2}(U-c-\beta)}.$$

Hence

$$L=c-\beta+\frac{a-c-\frac{\omega Fb}{\beta^2}(c+\beta-U)}{2+\frac{\omega Fb}{\beta^2}}-\frac{\left(n+1+\frac{\omega Fb}{\beta^2}\right)^2}{2+\frac{\omega Fb}{\beta^2}}\frac{2\beta^2 r}{a-c+\frac{\omega Fb}{\beta^2}(U-c-\beta)}. \tag{109}$$

Remark: go back to (102):

$$-(n-1)(a-c-nbq_{\text{ts}})+\left(n+1+\frac{\omega Fb}{\beta^2}\right)(\beta-c+L)+\frac{\omega Fb}{\beta^2}(\beta+c-U)-(a-c)$$
$$=0.$$

Note that

$$a-c-nbq_{\text{ts}}=P_{\text{ts}}-c$$

Then the equation in q_{ts} and L becomes an equation in P_{ts} and L:

$$-(n-1)(P_{\text{ts}}-c)+\left(n+1+\frac{\omega Fb}{\beta^2}\right)(\beta-c+L)+\frac{\omega Fb}{\beta^2}(\beta+c-U)=(a-c)=0.$$

P_{ts} rises with U (see the discussion of comparative statics below).

The formula for P_{ts} is valid so long as

$$0\leq U-P_{\text{ts}}\leq\beta.$$

If $U-P_{\text{ts}}=\beta$ or $P_{\text{ts}}=U-\beta$, $\tau_{\text{ts}}=0$; for greater values of U,

$$P_{\text{ts}}=U-\beta$$

and L is determined by the equation

$$-(n-1)(P_{\text{ts}}-c)+\left(n+1+\frac{\omega Fb}{\beta^2}\right)(\beta-c+L)+\frac{\omega Fb}{\beta^2}(\beta+c-U)-(a-c)=0,$$

which implies

$$L = \frac{a + c - 2\beta\left(n + \frac{\omega Fb}{\beta^2}\right) + \left(n - 1 + \frac{\omega Fb}{\beta^2}\right)U}{n + 1 + \frac{\omega Fb}{\beta^2}}.$$

A.2.5. $P_{ts}-L$

Rewrite (104):

$$P_{\mathrm{ts}} - L = \beta + \frac{a + c - \left(2 + \frac{\omega Fb}{\beta^2}\right)nbq_{\mathrm{ts}} + \frac{\omega Fb}{\beta^2}(a + \beta - U)}{n + 1 + \frac{\omega Fb}{\beta^2}}.$$

Now substitute

$$nbq_{\mathrm{ts}} = \frac{a - c + \frac{\omega Fb}{\beta^2}(a + \beta - U)}{2 + \frac{\omega Fb}{\beta^2}} + \frac{1}{2 + \frac{\omega Fb}{\beta^2}} \frac{\left(n + 1 + \frac{\omega Fb}{\beta^2}\right)^3}{n - 1} \frac{2\beta^2 r}{a - c + \frac{\omega Fb}{\beta^2}(U - c - \beta)}$$

and rearrange terms to obtain

$$P_{\mathrm{ts}} - L = \beta - \frac{2\left(n + 1 + \frac{\omega Fb}{\beta^2}\right)^2}{n - 1} \frac{\beta^2 r}{a - c + \frac{\omega Fb}{\beta^2}(U - c - \beta)}. \tag{110}$$

Alternatively, using (108) and (109) leads to the same result. In the present model, evidently,

$$P_{\mathrm{ts}} - L - \beta = -\frac{\left(n + 1 + \frac{\omega Fb}{\beta^2}\right)^2}{n - 1} \frac{2\beta^2 r}{a - c + \frac{\omega Fb}{\beta^2}(U - c - \beta)} < 0. \tag{111}$$

A.2.6. ρ_{ts}

Probability of reversion:

$$\rho_{\mathrm{ts}} = \frac{1}{2}\left(\frac{\beta - a + L + nbq_{\mathrm{ts}}}{\beta}\right)^2 = \frac{1}{2}\left[\frac{\beta - (a - nbq_{\mathrm{ts}} + L)}{\beta}\right]^2 = \frac{1}{2}\left(\frac{\beta - P_{\mathrm{ts}} + L}{\beta}\right)^2$$

and from (111)

$$= 2\left[\frac{\left(n + 1 + \frac{\omega Fb}{\beta^2}\right)^2}{n - 1} \frac{\beta r}{a - c + \frac{\omega Fb}{\beta^2}(U - c - \beta)}\right]^2. \tag{112}$$

A.2.7. U−P

We have from (108)

$$P_{\mathrm{ts}} = c + \frac{a - c - \frac{\omega Fb}{\beta^2}(c + \beta - U)}{2 + \frac{\omega Fb}{\beta^2}} - \frac{2}{2 + \frac{\omega Fb}{\beta^2}} \frac{\left(n + 1 + \frac{\omega Fb}{\beta^2}\right)^3}{n - 1} \frac{\beta^2 r}{a - c + \frac{\omega Fb}{\beta^2}(U - c - \beta)}$$

so that

$$U - P_{\mathrm{ts}} = \beta - \frac{a - c + 2(c + \beta - U)}{2 + \frac{\omega Fb}{\beta^2}} + \frac{2}{2 + \frac{\omega Fb}{\beta^2}} \frac{\left(n + 1 + \frac{\omega Fb}{\beta^2}\right)^3}{n - 1} \frac{\beta^2 r}{a - c + \frac{\omega Fb}{\beta^2}(U - c - \beta)}. \tag{113}$$

Then

$$\beta - (U - P_{\mathrm{ts}}) = \frac{a - c + 2(c + \beta - U)}{2 + \frac{\omega Fb}{\beta^2}} - \frac{2}{2 + \frac{\omega Fb}{\beta^2}} \frac{\left(n + 1 + \frac{\omega Fb}{\beta^2}\right)^2}{n - 1} \frac{\beta^2 r}{a - c + \frac{\omega Fb}{\beta^1}(U - c - \beta)}. \tag{114}$$

This must be nonnegative for the solution for P_{ts} to be valid.

A.2.8. τ_{ts}

The probability of investigation is

$$\begin{aligned}\tau_{\mathrm{ts}} &= \frac{1}{2}\left[\frac{\beta - (U - P_{\mathrm{ts}})}{\beta}\right]^2 \\ &= \frac{1}{2}\left[\frac{a - c + 2(c + \beta - U)}{\left(2 + \frac{\omega Fb}{\beta^2}\right)\beta} - \frac{2}{2 + \frac{\omega Fb}{\beta^2}} \frac{\left(n + 1 + \frac{\omega Fb}{\beta^2}\right)^3}{n - 1} \frac{\beta r}{a - c + \frac{\omega Fb}{\beta^2}(U - c - \beta)}\right]^2.\end{aligned} \tag{115}$$

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