Abstract

We study the information asymmetry issues in a decentralized inventory sharing system consisting of a manufacturer and two independent retailers, who privately hold demand information, non-cooperatively place their orders, but cooperatively share inventories with each other. We find that while the manufacturer needs retailers’ mean demand and standard deviation for her wholesale price decision, each retailer only needs to know the other retailer’s demand standard deviation for his order quantity decision. However, an incentive compatibility analysis shows that retailers have incentives to share their demand information untruthfully. Although a truth-inducing scheme can be developed for a system with symmetric retailers who share information between themselves, no such scheme can be developed to ensure truth-telling to the manufacturer. Further, we develop a coordination mechanism (CIS) for the decentralized inventory sharing system, considering information asymmetry. We show that CIS coordinates the manufacturer-retailers system and leads to an all-win situation under complete information. More importantly, CIS minimizes the value of information such that each party can obtain expected profits very close to their first-best profits even under asymmetric information, hence indirectly solves the information asymmetry problem. To our knowledge, this work is the first to study decentralized inventory sharing and its coordination considering asymmetric information.

Key Words: Decentralized inventory sharing, asymmetric information, information sharing, incentive compatibility, coordination.
1 Introduction and Literature Review

Inventory sharing (transshipment) among retailers has drawn increased attention from retailers and manufacturers as they seek to succeed in a highly competitive market. The practice of inventory transshipment from a retailer with surplus inventory to a retailer who stocks out has become prevalent in the automotive and machine tool industries (Narus and Anderson, 1996) and is also routinely performed in fashion industry (Dong and Rudi, 2004) and wholesale/retail industry (e.g., Gallagher 2002).

This work is motivated by our interactions with a leading heavy-machine manufacturer, whose situation is very typical in this industry. Recognizing the importance of helping its many independent dealers provide better services to their customers, the manufacturer has set up a dealer service parts inventory sharing program and even provides the information system and incentives for participation. While dealers share service parts inventory, they are very sensitive about sharing demand information due to various reasons. Thus, several interesting questions arise: What piece(s) of demand information is important to the manufacturer and each dealer/retailer for her/his decision? How should the manufacturer/dealers make their wholesale price/ordering decisions if the dealers do not share demand information with other parties? If the dealers share demand information, will they share it truthfully? Can the decentralized system be coordinated with asymmetric information? What is the impact of information asymmetry under the coordination mechanism?

In this note, we aim to answer these questions by studying a decentralized inventory sharing system consisting of a manufacturer (she) and two independent retailers (he). The manufacturer, considering the inventory sharing opportunities between retailers, determines her wholesale price. The retailers privately hold demand information, non-cooperatively determine their order quantities, but cooperatively share inventories with each other.

The first part of the work is focused on information asymmetry issues in the decentralized system (section 2). We first analyze the full/complete information scenario (FIS) to see what information is necessary for each player’s decision. We then study two scenarios under asymmetric information, one in which retailers do not share information (NIS), hence we
study a Bayesian game, and the other in which retailers reveal/share their information (IRS), hence we conduct an incentive compatibility analysis to investigate whether retailers will share true information and then have a discussion of truth-inducing.

The second focus of this work is to develop a coordination mechanism (CIS) for the decentralized inventory sharing system, \textit{with the consideration of information asymmetry} (section 3). We first analyze CIS under the same three scenarios, complete information (CFIS), asymmetric information with no information sharing (CNIS), and asymmetric information with information revealing/sharing (CIRS), as those analyzed under the decentralized system. We then conduct an extensive numerical study to demonstrate the value of coordination and the value of information in the decentralized system and under CIS (section 3.2).

When taking into consideration inventory transshipment opportunities, retailers will modify their order quantities to gain maximum profit from satisfying their own demand as well as the inventory sharing requests from other retailers. Much literature has focused on inventory stocking and/or transshipment decisions in these systems (e.g., Tagaras (1989), Anupindi, et al. (2001), Rudi, et al. (2001), Zhao, et al. (2005, 2006,2008)). However, few have considered manufacturer’s response to the retailers’ inventory sharing opportunities in her wholesale price decisions (except Dong and Rudi (2004) and Shao, et al. (2009)). And even fewer, if any, have considered the information issues in the system. All the previous work on inventory sharing has assumed retailers’ demand information known to all parties in the network, an assumption that can be far from reality for decentralized supply chains in which each independent retailer has his own objective. As for coordination, there has been limited work on coordination mechanisms in inventory sharing systems and none of which considers asymmetric information. Anupindi, et al. (1999) and Rudi, et al. (2001), considering an exogenous wholesale price, use transfer prices for the shared units to coordinate the retailers. Such coordinating prices do not always exist (Anupindi, et al. (1999) and Hu, et al. (2007)) and such mechanism runs into significant incentive compatibility issues when applied to asymmetric information case. One big differential advantage of the mechanism we develop (CIS) is that it minimizes the impact of information such that each party can obtain
expected profits very close to what they would obtain in a complete information scenario (i.e., their first-best profits) even under asymmetric information, hence indirectly solves the information asymmetry problems.

In addition to the above contributions, compared with the large amount of literature on asymmetric information and information sharing, our work also possesses some other distinguishing characteristics. First, while most of the previous literature studies asymmetric information between vertical parties in the supply chain, e.g., Li (2002), Özer and Wei (2006), we study both vertical (between the retailers and the manufacturer) and horizontal (between the retailers themselves) information asymmetry. Further, while most of the literature on information sharing assumes players reveal true information, we consider information credibility issue which is non-trivial in this inventory sharing case. We also investigate truth-inducing schemes.

2 Decentralized Inventory Sharing System with Asymmetric Information

Consider a decentralized system consisting of a manufacturer (she) and two independent retailers (he), each serving his own customer base at two distinct locations, indexed by $i, j = 1, 2$ ($i \neq j$), without competing for their demands. We study a single-period model with each retailer (say retailer $i$) facing uncertain demand, $D_i$, represented by a cdf $F_i(\cdot)$ with a mean of $\mu_i$ and a standard deviation (SD) of $\sigma_i$, independently drawn from what we refer to as the standardizable distributions, where $Z_i = \frac{D_i - \mu_i}{\sigma_i}$ are i.i.d. with $E[Z_i] = 0$ and $Var[Z_i] = 1$, e.g., normal distribution, uniform distribution, etc. This group of distributions is also used in Zhang (2005). We assume $\mu_i >> \sigma_i$ so that the probability of negative demand is very small. We assume that retailer $i$ has private and accurate information on his true demand distribution parameters, $\{\mu_i, \sigma_i\}$, because of his exclusive proximity to his market, while others only have prior information (detailed later) of what $\mu_i$ and $\sigma_i$ may be. Retailers order from the manufacturer whose production cost is $c$ per unit and who sells to
the retailers at a unit wholesale price, \( w \). Retailer \( i \) obtains revenue \( r_i > w \) for each unit sold and incurs a penalty cost, \( p_i \), for each unit of unsatisfied demand. Unsold inventory has a unit salvage value \( s_i < w \). To focus on demand information, we assume all cost parameters are common knowledge to all players.

The sequence of events is as follows. First, the manufacturer announces the wholesale price \( w \) to the two retailers, who separately place their orders \( (Q_i) \). Then, demand realizes at both retailers. After each retailer satisfies his demand with his own inventory, the retailers share (transship) inventory if one faces shortage (referred to as the buyer) and the other has surplus inventory (referred to as the seller). With \( \tau_{ij} \) being the cost of transshipping one unit from retailer \( i \) to retailer \( j \), \( u_{ij} = r_j - \tau_{ij} - s_i + p_j \) represents the net profit from transshipping one unit from retailer \( i \) to retailer \( j \). We assume \( u_{ij} > 0 \) such that transshipment is mutually profitable to the retailers. In addition, we also assume \( s_i < s_j + \tau_{ji} \) and \( r_i + p_i < r_j + p_j + \tau_{ji} \), which ensure transshipment occur only when one retailer has surplus and the other has shortage (Tagaras (1989)). The two retailers cooperatively divide the transshipment profits according to pre-specified proportions, \((\alpha_i, \alpha_j)\), with \( \alpha_i \in [0, 1] \) representing the proportion of profits allocated to retailer \( i \) for each unit transshipped from \( i \) to \( j \). There is a one-to-one correspondence between \((\alpha_i, \alpha_j)\) and \((C_{ij}, C_{ji})\) used in Rudi, et al. (2001) where \( C_{ij} \) is the transfer price paid by retailer \( j \) to retailer \( i \) for each unit of inventory transshipped from retailer \( i \) to \( j \).  

Since each retailer \((i)\) receives a constant profit, \( \alpha_i u_{ij} \), for each unit shared from \( i \) to \( j \), retailers are sure to truthfully report their surplus or shortage, the retailers have asymmetric information regarding ex-ante demand and ex-post visibility of excess demand/stock that is automatically guaranteed.

**Inventory Sharing Game Under Full/Complete Information Scenario (FIS)**

This benchmark scenario, where retailers’ demand distribution parameters are known to all parties, corresponds to a two-stage stackelberg game which can be solved backwards. Specifically, given the manufacturer’s wholesale price, \( w \), each retailer, \( i \), chooses his order

\[ C_{ij} - s_i - \tau_{ij} = \alpha_i u_{ij}. \]
quantity, $Q_i(w)$, expecting the inventory sharing opportunities at the end of the period. This is the same as the inventory sharing game analyzed in Rudi, et al. (2001). Using our notation, retailer $i$’s expected profit, $\Pi^F_i(Q_i, Q_j)$, can be written as

$$\Pi^F_i(Q_i, Q_j) = E \left[ r_i \min(D_i, Q_i) + s_i(Q_i - D_i)^+ - p_i(D_i - Q_i)^+ \right. $$

$$+ \alpha_i u_{ij} T_{ij} + (1 - \alpha_j) u_{ji} T_{ji} \left. - wQ_i, \right]$$

where $T_{ij} = \min[(Q_i - D_i)^+, (D_j - Q_j)^+]$ and the retailers’ best response order quantities, $(Q^F_i(w), Q^F_j(w))$, can be solved from the following two first order equations:

$$F_{D_i}(Q_i) = \frac{r_i + p_i - w}{r_i + p_i - s_i} + \frac{\alpha_i u_{ij}}{r_i + p_i - s_i} P\{Q_i - (D_j - Q_j)^+ < D_i < Q_i\} $$

$$- \frac{(1 - \alpha_j) u_{ji}}{r_i + p_i - s_i} P\{Q_i < D_i < Q_i + (Q_j - D_j)^+\}, \quad i, j = 1, 2. \quad (1)$$

Then, given $(Q^F_i(w), Q^F_j(w))$, the manufacturer’s optimal wholesale price, $w^F$, can be solved from maximizing her expected profit, $\Pi^F_M(w) = (w - c) (Q^F_i(w) + Q^F_j(w))$.

Although there do not exist closed-form solutions to $(Q^F_i(w), Q^F_j(w))$ or $w^F$, we can show (proof in appendix) that there exists the manufacturer’s optimal wholesale price $w^F$ and a unique pair of retailers’ corresponding equilibrium order quantities, $(Q^F_i(w^F), Q^F_j(w^F))$.

The following theorem helps us understand the impact of retailers’ demand parameters on the manufacturer’s optimal wholesale price.

**Theorem 1** As retailer $i$’s mean demand $\mu_i$ increases, the manufacturer’s wholesale price $w^F$ increases. As retailer $i$’s demand SD $\sigma_i$ increases, the manufacturer’s wholesale price $w^F$ can either increase or decrease.

Theorem 1 indicates that the manufacturer will choose a higher wholesale price for a higher mean demand at a retailer, because with the higher mean demand she can offset the lower order quantity caused by the higher price. On the other hand, the influence of $\sigma_i$ is less predictable, depending on the transshipment benefit allocation proportion $\alpha_i$ and the magnitudes of retailers’ demand parameters.
Although the manufacturer’s decision, \( w^F \), depends on both retailers’ \( \{\mu_i, \sigma_i\} \) (Theorem 1), we show next that given \( w \), a retailer only needs to know the other retailer’s SD in determining his order quantity.

To see this, we use a transformation of variables by standardizing the demand variable \( D_i \) with \( Z_i = \frac{D_i - \mu_i}{\sigma_i} \) and \( Q_i \) with \( z_i = \frac{Q_i - \mu_i}{\sigma_i} \). Thus, equation (1) can be rewritten as

\[
F_{Z_i}(z_i) = \frac{r_i + p_i - w}{r_i + p_i - s_i} + \frac{\alpha_i u_{ij}}{r_i + p_i - s_i} P\{z_i - \frac{\sigma_j}{\sigma_i} (Z_j - z_j)^+ < Z_i < z_i\} - \frac{(1 - \alpha_j) u_{ji}}{r_i + p_i - s_i} P\{z_i < Z_i < z_i + \frac{\sigma_j}{\sigma_i} (z_j - Z_j)^+\}, i, j = 1, 2. \tag{2}
\]

Therefore, solving for \((Q_i^F(w), Q_j^F(w))\) in equations (1) is equivalent to solving for \((z_i^F(w), z_j^F(w))\) in equations (2) due to the following theorem.

**Theorem 2** \( Q_i^F(w) = \mu_i + \sigma_i z_i^F(w) \), where, of the retailers’ demand parameters, \( z_i^F(w) \) is a function of only \( \frac{\sigma_i}{\sigma_j} \), i.e., \( z_i^F(w) = z_i^F(\frac{\sigma_i}{\sigma_j}, w) \), \( i, j = 1, 2 \).

Theorem 2 reveals a rather interesting fact: Given a wholesale price, \( w \), each retailer’s equilibrium order quantity (say, \( Q_i^F \)) is not influenced by the other retailer’s mean demand \( (\mu_j) \), but influenced only by the other retailer’s SD \( (\sigma_j) \). In other words, a retailer only needs the other retailer’s demand SD for his order quantity decision. This result can be explained by the fact that retailers’ equilibrium order quantities, with the consideration of the inventory sharing opportunities at the end of the period, depend on the distribution of the shortage or surplus at each retailer, which is influenced only by the variance of the demand (and not the mean demand). This result provides us important insights into the information asymmetry problem in decentralized inventory sharing systems.

Next, we consider the asymmetric information case in which each retailer, \( i \), holds private information of his demand distribution parameters, \( \{\mu_i, \sigma_i\} \). Specifically, we assume \( \mu_i \) takes one of \( m_i \) possible values, and \( \sigma_i \) takes one of \( n_i \) possible values, i.e., \( \mu_i \in O_i = \{\mu_{i,k}, k = 1, 2, ... m_i\} \), and \( \sigma_i \in S_i = \{\sigma_{i,l}, l = 1, 2, ... n_i\} \), with \( P_{i,\mu}(\mu_i) \) and \( P_{i,\sigma}(\sigma_i) \) being the probabilities that retailer \( i \)'s mean demand and demand SD take the value of \( \mu_i \) and \( \sigma_i \), respectively. We assume \( \{O_i, S_i\} \), and the probability distributions, \( \{P_{i,\mu}(\mu_i)\} \), and \( \{P_{i,\sigma}(\sigma_i)\} \), \( i = 1, 2 \), are common knowledge to all parties.
Inventory Sharing Game Under No Information Sharing (NIS)

For this scenario, we solve each player’s equilibrium decision when each retailer, \(i\), holds private information, \(\{\mu_i, \sigma_i\}\), and does not share this information. Hence, the decentralized inventory sharing system under this scenario is a Bayesian Stackelberg game.

Given \(w\), the two retailers calculate their Bayesian Nash equilibrium order quantities based on their prior information of the other’s private information. Define \(\{Q^B_i(\mu_i, \sigma_i, w)\}\), \(\mu_i \in O_i, \sigma_i \in S_i\) as retailer \(i\)’s Bayesian Nash equilibrium order quantities for each pair of his demand parameter \((\mu_i, \sigma_i)\), given \(w\). Following the same transformation in Section 2 and omitting all the technical details, we can show that

\[
Q^B_i(\mu_i, \sigma_i, w) = \mu_i + \sigma_i z_i^B(\sigma_i, w),
\]

where \(z_i^B(\sigma_i, w)\), solved from the \(n_i + n_j\) equations, (3), only depends on all possible ratios of the two retailers’ standard deviations, \(\{\sigma_i, \sigma_i \in S_i, \sigma_j \in S_j\}\).

Then, based on her prior information of the retailers’ demand parameters, the manufacturer can solve for her optimal wholesale price \(w^B\) that maximizes her expected profit

\[
E[\Pi^B_M(w)] = (w - c) \sum_{\mu_i, \sigma_i, \mu_j, \sigma_j} P_{i, \mu}(\mu_i) P_{i, \sigma}(\sigma_i) P_{j, \mu}(\mu_j) P_{j, \sigma}(\sigma_j) \left( Q^B_i(\mu_i, \sigma_i, w) + Q^B_j(\mu_j, \sigma_j, w) \right).
\]

We can show (details omitted) that there exists the manufacturer’s optimal wholesale price, \(w^B\), and a unique set of corresponding Bayesian Nash equilibrium order quantities for retailer \(i\), \(Q^B_i(\mu_i, \sigma_i, w^B)\), for each pair of his demand distribution parameters, \(\{\mu_i, \sigma_i\}\), \(\mu_i \in O_i, \sigma_i \in S_i\).

Information Revealing Scenario (IRS) and Information Credibility
In this scenario, the retailers who have private demand information reveal/share this information to other parties. The question is: Will the retailers reveal true information?

We assume each retailer will share only the necessary information, i.e., retailer $i$ will share $\{\mu_i, \sigma_i\}$ with the manufacturer before she determines her wholesale price and will share $\sigma_i$ with retailer $j$ upon receiving the wholesale price before they each make the order quantity decisions.

Following Hammond (1979) and Mas-Colell and Vives (1993), we conduct the IC analysis by checking whether under IRS, retailer $i$ will report his information truthfully, given that others trust what retailer $i$ reports as true information and that retailer $j$ reports his demand distribution parameters truthfully. If not, then truth-telling is not an equilibrium strategy. Let $\{\mu_{ri}, \sigma_{ri}\}$ and $\{\mu_{ri}, \sigma_{ri}\}$ denote retailer $i$’s true and reported demand distribution parameters, respectively. Since retailer $i$’s order quantity (observable to the manufacturer) should be consistent with $\{\mu_{ri}, \sigma_{ri}\}$, retailers’ order quantities can be calculated as $Q_{ri}(\mu_{ri}, \sigma_{ri}, w) = \mu_{ri} + \sigma_{ri} z_{ri}(w)$ and $Q_{rj}(\mu_{ri}, \sigma_{ri}, w) = \mu_{rj} + \sigma_{rj} z_{rj}(w)$, where $\{z_{ri}(w), z_{rj}(w)\}$ are solved from equations (2) with $\sigma_i = \sigma_{ri}$ and $\sigma_j = \sigma_{rj}$. Given $\{Q_{ri}(\mu_{ri}, \sigma_{ri}, w), Q_{rj}(\mu_{ri}, \sigma_{ri}, w)\}$, we can calculate the manufacturer’s optimal $w$ given retailer $i$ reports $\{\mu_{ri}, \sigma_{ri}\}$, $w^*(\mu_{ri}, \sigma_{ri})$. Therefore, IC is guaranteed if and only if reporting true demand information maximizes a retailer’s expected profit. Define $\{\mu_{ri}^*, \sigma_{ri}^*\} \triangleq \arg\max_{\{\mu_{ri}, \sigma_{ri}\}} \Pi_{ri}(Q_{ri}^*(\mu_{ri}^*, \sigma_{ri}^*, w^*(\mu_{ri}^*, \sigma_{ri}^*)), Q_{rj}^*(\mu_{ri}^*, \sigma_{ri}^*, w^*(\mu_{ri}^*, \sigma_{ri}^*)))$. The following theorem presents the results of the incentive compatibility analysis.

**Theorem 3** Truth-telling is not retailers’ equilibrium reporting strategy when they share demand information with the manufacturer and the other retailer, i.e., $\{\mu_{ri}^*, \sigma_{ri}^*\} \neq \{\mu_{ri}, \sigma_{ri}\}$.

Theorem 3 demonstrates that, under IRS, a retailer has incentives to report demand information untruthfully to distort the manufacturer’s wholesale price and the other retailer’s order quantity such that he can obtain a higher profit. Next, is it possible to develop truth-inducing schemes? The answer is yes, to some extent. Using a reward/penalty contingent

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3We assume each retailer (say $i$) reports the same demand SD to the manufacturer and retailer $j$ because the manufacturer can easily find out any inconsistency in the information by observing the retailers’ order quantities and can always obtain higher profits if the retailers shared the same information with them (by adjusting $w$).
upon the information reported by the retailers, we can develop a truth-inducing scheme for a system with symmetric retailers (identical cost parameters and $\alpha = 0.5$) who share information between themselves (details in Yan and Zhao (2008)). However, due to the complex interactions among the manufacturer’s wholesale price, retailers’ order quantities, and the reported information, no such schemes can be developed to ensure truth-telling to the manufacturer.

3 A Coordinated Inventory Sharing (CIS) Mechanism With Asymmetric Demand Information

In this section, we develop and analyze a coordination mechanism of the inventory sharing system, with the consideration of asymmetric information. This mechanism is developed through the involvement of the manufacturer and can coordinate just the two retailers or the manufacturer-retailers system and can apply to a system with either complete information or asymmetric information.

The coordination mechanism we propose (CIS) specifies how to operate the inventory sharing system, with a set of certain payments. The sequence of events under CIS is as follows:

1. At the beginning of the game (selling season), a fixed fee (premium) to join the inventory sharing program for this season, $\delta_i$, is collected from each retailer by the manufacturer.
2. According to her best information of the retailers’ demand, the manufacturer determines the wholesale price and announces it to the retailers.
3. Upon receiving the wholesale price, each retailer separately places his order according to his best information of the other retailer’s demand.
4. Demand realizes at each retailer’s location. After satisfying demand with his own inventory, each retailer reports to the manufacturer how much surplus/shortage he has.
5. If there is surplus at one retailer and shortage at the other, inventory is shared and the
payments are arranged as follows: Without loss of generality, assuming the manufacturer pays the transportation cost, $\tau_{ij}$, the manufacturer will charge the buyer $s_i + \tau_{ij}$ and pay the seller $r_j - \tau_{ij} + p_j$ for each unit shipped from $i$ to $j$.

There are a few points that will help us understand the rationale behind CIS. First, CIS described above is similar to a two-part tariff (a fixed premium and a per-unit wholesale price paid to the manufacturer from each retailer) plus a set of transshipment payments handled through the manufacturer, which is key to the coordination. Second, depending on who pays for the transportation cost, we may have different sets of payments for the transshipped units. The bottom line is to ensure each retailer obtain $u_{ij}$ as the net profit for each unit shared from $i$ to $j$, as accomplished by the set of payments described in step 5. Third, recall that the system gains a net of $u_{ij}$ for each unit shared. By allowing each retailer to have a net profit of $u_{ij}$, the manufacturer needs to pay a net $u_{ij}$ for each unit shared from $i$ to $j$. This amount is provided from the premium charged at the beginning of the game. Finally, under CIS, since each retailer obtains a constant net profit of $u_{ij}$ for each unit shared from $i$ to $j$, we can be sure that each retailer will truthfully report his surplus or shortage to the manufacturer.

3.1 Analytical Investigation of CIS

In this section, we analyze CIS under three scenarios, complete information (CFIS), private information with no information sharing (CNIS), and private information with information revealing/sharing (CIRS), corresponding to FIS, NIS, and IRS in the decentralized system, respectively.

**Theorem 4** Under complete information, given any $w$, CIS coordinates the two retailers. Further, when setting $\hat{w} = c$, CIS coordinates the two-level manufacturer-retailers system. In either case, (1) there exists a unique equilibrium of the retailers’ order quantities that equal to the respective centralized order quantities, and (2) there exists at least one pair of $\{\delta_i, \delta_j\}$

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4Note that the physical inventory does not have to be routed through the manufacturer, but the money needs to in order to ensure that each retailer obtains $u_{ij}$ for each unit shared from $i$ to $j$. 

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such that each player can gain higher expected profit compared to a decentralized system (e.g., FIS, NIS).

Theorem 4 shows that under complete information, CIS maximizes the two retailers’ total profit for any given \( w \) and further, it maximizes the total profit of the manufacturer-retailers system if the manufacturer sets \( w = c \). With proper allocation of the maximized total profit, CIS can lead to an all-win situation for all the players.

Under asymmetric demand information, CIS can still be implemented following the steps listed earlier in the section. Notice that the manufacturer does not need the retailers’ information under CIS, leaving only the demand SD the useful information for each other.

Under CNIS, in step 2 of CIS, each retailer determines his order quantity, denoted as \( \hat{Q}_B \), based on his prior information of the other retailer’s demand SD, \( \sigma_j \), just as in the Bayesian game. Specifically,

\[
\hat{Q}_B \left( \sigma_i \right) = \mu_i + \sigma_i \hat{z}_B \left( \sigma_i \right),
\]

where \( \hat{z}_B \left( \sigma_i, w \right) \) is solved from the following equations:

\[
F_{Z_i} (z_i(\sigma_i)) = \frac{r_i + p_i - w}{r_i + p_i - s_i} + \frac{u_{ij}}{r_i + p_i - s_i} \sum_{\sigma_j \in S_j} P_{j, \sigma} (\sigma_j) \{ z_i(\sigma_i) - \frac{\sigma_j}{\sigma_i} (Z_j - z_j(\sigma_j))^+ < Z_i < z_i(\sigma_i) \} + \frac{u_{ji}}{r_i + p_i - s_i} \sum_{\sigma_j \in S_j} P_{j, \sigma} (\sigma_j) \{ z_i(\sigma_i) < Z_i < z_i(\sigma_i) + \frac{\sigma_j}{\sigma_i} (z_j(\sigma_j) - Z_j)^+ \},
\]

\( i \neq j = 1, 2, \sigma_i \in S_i. \) (4)

Under CIRS, each retailer shares/reveals his SD to the other retailer. We conduct an IC analysis again to see whether retailers will share true information. Since information is only shared between the retailers and one retailer’s order quantity is not observable to the other, the IC analysis (a double moral hazard version with details in Appendix ??) is very different from the one in Section 2. The following theorem presents the results.

**Theorem 5** Under CIRS, truth-telling is not retailers’ equilibrium strategy, i.e., \( \tilde{\sigma}_i^* \neq \sigma_i^* \).

Theorem 5 demonstrates that under CIS, truth-telling is still not retailers’ equilibrium reporting strategy. However, in the next section, we will see that CIS minimizes the impact
of information, hence, the system can enjoy the coordination benefits without having to worry about information sharing or truth-inducing.

3.2 Numerical Investigation of CIS

In this section, we conduct an extensive numerical study to investigate the value of coordination and the impact of information asymmetry. Specifically, by comparing NIS and CNIS, we see the value of coordination (CIS) in the decentralized inventory sharing system under asymmetric demand information. Further, by comparing NIS with FIS and CNIS with CFIS, we see the value of information in the decentralized system and under CIS, respectively. As shown, retailers have incentives to untruthfully share their demand parameter information. Hence, the value of information in a decentralized system and under CIS also provides an indicator of the benefits of untruthful sharing in the decentralized system and under CIS, respectively.

To focus on the impact of demand information, we assume identical cost parameters at the two retailers. We use values in Rudi, et al. (2001) for these parameters, i.e., \( r = 40, \tau = 2, s = 10, p = 0 \) and set \( c = 15 \). We assume retailers’ demand follow normal distribution with mean and SD each taking two levels of values, i.e., \( \{ \mu_H, \mu_L \} \) and \( \{ \sigma_H, \sigma_L \} \), with a 50−50 chance of being high or low. To see the impact of the mean demand, we choose three levels of the expected mean demand, \( \bar{\mu} = 80, 100, 120 \), and for each level, we test three pairs of \( \{ \mu_H, \mu_L \} \) with the ratio of \( \frac{\mu_H}{\mu_L} = 1.2, 1.5, 1.8 \). For example, in the case of \( \bar{\mu} = 100 \), the three pairs of \( \{ \mu_H, \mu_L \} \) are \{109, 91\}, \{120, 80\}, and \{129, 71\}. Similarly, we choose two levels for the expected SD, \( \bar{\sigma} = 30, 18 \), with \( \frac{\sigma_H}{\sigma_L} = 1.0, 2.0, 3.0 \) for each level of \( \bar{\sigma} \). For NIS whose results are affected by \( \alpha \), we test \( \alpha = 1.0, 0.5, 0.0 \). For the total of \( 3^4 \times 2 = 162 \) cases, we calculate the manufacturer’s optimal wholesale price, the retailers’ expected order quantities, system profits and individual profits when appropriate.

We first investigate the impact of information asymmetry in the decentralized system. This will also serve as a benchmark to compare with the impact of information asymmetry under CIS. Table 1 shows the manufacturer’s and retailers’ expected profits under FIS.
and NIS, as well as the value of information measured by the percentage difference between them. Since the change of $\frac{\mu_H}{\mu_L}$ does not bring additional insights, we only show the results with $\frac{\mu_H}{\mu_L} = 1.8$. Several important observations stand out: (1) While knowing retailers’ demand information always benefits the manufacturer, it does not always benefit the retailers. (2) Information asymmetry has a significant impact in the decentralized system and it has a bigger impact on the retailers than on the manufacturer (the percentage difference in retailers’ profit in the absolute value is higher (up to 17.59%) than that of the manufacturer (up to 4.83%)). This is because the manufacturer can adjust her wholesale price which offsets her disadvantage of information. (3) While $\alpha$ (how the transshipment benefit is divided between the retailers) has a significant impact on the impact of information asymmetry, its impact is quite uncertain. Therefore, using transfer prices between retailers (equivalent to $\alpha$) to coordinate the inventory sharing system will lead to considerable incentive compatibility issues under asymmetric information and there do not seem to exist truth-inducing mechanisms for this mechanism (Yan and Zhao (2008)).

Next, we explore the value of the coordination mechanism proposed in this work under asymmetric information. Table 2 compares CNIS and NIS to obtain the value of coordination measured by the percentage increase in the system profit under CNIS over NIS. We make a few interesting observations. First, the value of coordination is significant. For the cases we tested, CIS increases the system profit by 13.61% to 42.07%, depending on the values of different parameters. Second, while the value of coordination increases as $\bar{\mu}$ increases (because $w^*$ under NIS increases), it does not depend on the difference between $\mu_H$ and $\mu_L$, i.e., $\frac{\mu_H}{\mu_L}$ (hence results are only listed for different values of $\bar{\mu}$). This is because retailers’ expected order quantities (hence expected profits) and correspondingly $w^*$ remain the same for fixed $\bar{\mu}$ (recall a retailer’s equilibrium order quantity depends only on his own mean demand and both retailers’ demand SD). On the other hand, the value of coordination depends on both $\bar{\sigma}$ and $\frac{\sigma_H}{\sigma_L}$. As $\bar{\sigma}$ increases, the value of coordination increases because the difference between the retailers’ expected order quantities under NIS and CNIS increases. However, as $\frac{\sigma_H}{\sigma_L}$ increases, it may increase or decrease the value of coordination due to its
uncertain impact on the retailers’ expected order quantities and the manufacturer’s \( w^* \).

Finally, the value of coordination decreases as \( \alpha \) increases. This is because as \( \alpha \) increases, more transshipment profits will be allocated to the retailer with extra units (the seller). Hence, retailers have incentives to order more. In other words, in a vertical context, the increase of \( \alpha \) counter-balances double marginalization, i.e., the value of coordination goes down. This observation is also articulated in Jiang and Anupindi (2010).

After we see the significant value of the coordination mechanism CIS, we investigate the impact of information asymmetry under CIS. Knowing that CIS coordinates the manufacturer-retailers system when retailers’ demand information is common knowledge, we would like to see how CIS behaves under asymmetric information? But before we see this, we first want to take a look at the impact of information asymmetry in the decentralized system for the completeness of our analysis of the decentralized system and as a benchmark to compare with the impact of information asymmetry under the coordination mechanism CIS.

Given the significant value of the coordination mechanism CIS, we next investigate the impact of information asymmetry under CIS. We would like to see how CIS behaves under asymmetric information. Recall from Table 1 that the impact of information may be significant but uncertain, the following results shows the advantage of CIS in dealing with asymmetric information. Table 3 shows the system profits under CFIS and CNIS, as well as the value of information under CIS measured by the percentage difference between them. Since under both CFIS and CNIS, each player’s net profit can be determined as a proportion of the system profit through adjusting the premium, a comparison of the system profits is sufficient.

The results again do not depend on \( \mu_u / \mu_L \), but only depends on \( \bar{\mu} \). Table 3 demonstrates that in all cases we test, the differences between the system profits (and hence each player’s profit) under CNIS and CFIS are very small (close to 0.01%). Therefore, even if retailers do not share their private information, under CIS, each party and the system can obtain profits very close to what they can obtain under complete information (CFIS), i.e., first-best profits. This also indicates that the benefit of untruthful reporting is insignificant under CIS. Therefore,
under CIS, all players can enjoy the first-best profits without requesting the retailers to share their information or truth-inducing! This advantage of CIS may not be expected from other coordination mechanisms. Therefore, it is important to consider information asymmetry when developing coordination mechanisms in a decentralized system.

4 Conclusion and Discussion of Future Work

We consider a supply chain with a manufacturer supplying two independent retailers who may share inventory with each other at the end of a single period. The main contributions of our work include: (1) To the best of our knowledge, it is the first to study inventory sharing systems with asymmetric information. (2) We not only consider the issues of asymmetric information and information sharing, but also information credibility and truth-inducing in the information sharing case. (3) It is also the first to provide a coordination mechanism the decentralized inventory sharing system with the consideration of asymmetric information. It is worth noting that although the analysis was presented for the “standardizable” demand distribution, results in all theorems except Theorem 2 apply to any general distribution. Below we summarize the managerial insights obtained from our work:

1. In an inventory sharing system, different parties need different pieces of demand information for their decisions. Specifically, the manufacturer needs both the retailers’ mean demand and standard deviation to determine the wholesale price, while retailers only need the other retailer’s demand standard deviation for his order quantity decision.

2. Truth-telling cannot be expected voluntarily from independent retailers in an inventory sharing system when they share their private demand information. A truth-inducing scheme can be developed for symmetric retailers (with same cost parameters) who share information with each other. However, no such or similar schemes can be developed to ensure truth-telling to the manufacturer.

3. Information asymmetry has significant impact in the decentralized system. While knowing retailers’ demand information always benefits the manufacturer, it may not always
benefit the retailers. Further, information asymmetry has a bigger impact on the retailers than on the manufacturer.

4. A coordinated inventory sharing mechanism (CIS) can be developed for a decentralized system and achieve significant profit increase for each player and the whole supply chain. One differential advantage of CIS is that it also minimizes the impact of asymmetric information, i.e., each party can obtain an expected profit very close to what they would obtain under complete information scenario (i.e., their first-best profit), even if the retailers do not share information with each other.

5. It is very important to take into consideration the information issues in designing coordination mechanisms. Otherwise, a system may run into significant incentive compatibility issues that are very complicated to solve when the coordination mechanism is implemented under information asymmetry.

Finally, the current research has opened up interesting future research opportunities. A natural extension to this work is to analyze an $n$-retailer inventory sharing system ($n > 2$) with asymmetric information (Zhao and Yan 2010). Many complicated issues arise when there are $n$ decentralized retailers in the inventory sharing game ($n > 2$), e.g., coalition and coordination. Further, this work, as the first mover to study information asymmetry in the inventory sharing area, studies a single-period model. An interesting future work is to study the demand information asymmetry in a multiple-period setting in which retailers may choose to or not to share their inventory, based on their expectation of future demand. As the retailers have more options in sharing their inventory, it would be interesting to see what impact information asymmetry will bring in this even more complicated setting.

**Acknowledgement** The authors thank the Department Editor, Associate Editor and the anonymous referees for their suggestions and comments to improve the paper.

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5 Proofs

Proofs of Theorems 4 and 5 are shown in Online Appendix.

5.1 Proof of Theorem 1 and Theorem 2

If we use the transformation $Z_i = \frac{D_i - \mu_i}{\sigma_i}$, we can see that $z_i^F(w)$ are the solutions to equations (2) if and only if $\mu_i + \sigma_i z_i^F(w)$ are the solutions to equations (1), $i = 1, 2$. From equations (2), we also see that, for given cost parameters and $(\alpha_i, \alpha_j)$, $z_i^F(w)$, $i = 1, 2$, are functions of only $\sigma_i$. Therefore, we have the best responses $Q_i^F(w) = \mu_i + \sigma_i z_i^F(w)$ and $z_i^F(w) = z_i^F(w; \frac{\sigma_i}{\sigma_j})$. This proves the results in Theorem 2.

Now we prove the two bullets in theorem 1.

We can rewrite (??) as follows,

$$\frac{d\Pi^F_M(w)}{dw} = \mu_i + \sigma_i z_i^F(w) + \mu_j + \sigma_j z_j^F(w) - (w - c) \frac{A_i}{r_i + p_i - s_i} + \frac{A_i}{r_j + p_j - s_j} \frac{A_j}{A_i A_j + A_i B_j + A_j B_i}.$$

(5)

Suppose we have two different $\mu_i$: $\mu_{i,H}$ and $\mu_{i,L}$ with $\mu_{i,H} > \mu_{i,L}$. We denote the manufacturer’s corresponding optimal wholesale prices are $w^F(\mu_{i,H})$ and $w^F(\mu_{i,L})$, respectively. Therefore, when $\mu_i = \mu_{i,L}$, we must have

$$\frac{d\Pi^F_M(w)}{dw} \bigg|_{w=w^F(\mu_{i,L})} = \mu_{i,L} + \sigma_i z_i^F(\mu_{i,L}) + \mu_j + \sigma_j z_j^F(\mu_{i,L}) - (w^F(\mu_{i,L}) - c) \frac{A_i}{r_i + p_i - s_i} + \frac{A_i}{r_j + p_j - s_j} \frac{A_j}{A_i A_j + A_i B_j + A_j B_i} = 0. \quad (6)$$

Now we consider the case when $\mu_i = \mu_{i,H}$. If we still set $w = w^F(\mu_{i,L})$, then

$$\frac{d\Pi^F_M(w)}{dw} \bigg|_{w=w^F(\mu_{i,L})} = \mu_{i,H} + \sigma_i z_i^F(\mu_{i,L}) + \mu_j + \sigma_j z_j^F(\mu_{i,L}) - (w^F(\mu_{i,L}) - c) \frac{A_i}{r_i + p_i - s_i} + \frac{A_i}{r_j + p_j - s_j} \frac{A_j}{A_i A_j + A_i B_j + A_j B_i}. \quad (7)$$

We prove that $z_i^F(\mu_{i,L})$, $z_j^F(\mu_{i,L})$, and other terms $A_i, B_i, i = 1, 2$ are the same as the ones in the case when $\mu_i = \mu_{i,L}$.
First, from the previous proof, we know that under the same wholesale price \( w^F(\mu_{i,L}) \), \( Z_i^F(w) \) only depends on \( \frac{\sigma_i}{\alpha_i} \). Thus, we know that \( z_i^F(w^F(\mu_{i,L})) \) does not depend on \( \mu_i \), \( i = 1, 2 \) and hence \( z_i^F(w^F(\mu_{i,L})) \) under \( \mu_i = \mu_{i,H} \) are the same as the one under the case when \( \mu_i = \mu_{i,L} \).

Second, we prove that \( A_i \) and \( B_i \) under \( \mu_i = \mu_{i,H} \) are the same as the one under the case when \( \mu_i = \mu_{i,L} \).

We can rewrite \( A_i \) and \( B_i \) as follows (here we omit the arguments of \( z_i^F \) for brevity),

\[
A_i = \frac{1}{\sqrt{2\pi\sigma_i}} \exp\left(-\frac{z_i^F}{2} \left[ 1 - \frac{\alpha_i u_{ij}}{r_i + p_i - s_i} F(z_j^F) - \frac{(1 - \alpha_j) u_{ji}}{r_i + p_i - s_i} F(z_j^F) \right] \right);
\]

\[
B_i = \frac{\alpha_i u_{ij}}{r_i + p_i - s_i} \int_{z_j^F}^{\infty} f(Z_j) \frac{1}{\sqrt{2\pi\sigma_i}} \exp\left(-\frac{(z_i^F + \sigma_i(z_j^F - Z_j))^2}{2} \right) \, dZ_j
+ \frac{(1 - \alpha_j) u_{ji}}{r_i + p_i - s_i} \int_{-\infty}^{z_j^F} f(Z_j) \frac{1}{\sqrt{2\pi\sigma_i}} \exp\left(-\frac{(z_i^F + \sigma_i(z_j^F - Z_j))^2}{2} \right) \, dZ_j,
\]

where \( f(\cdot) \) and \( F(\cdot) \) are pdf and cdf functions of standard normal distribution.

We can see that \( A_i \) and \( B_i \) do not depend on \( \mu_i \) and \( \mu_j \) and hence are the same under \( \mu_i = \mu_{i,H} \) as the ones under the case when \( \mu_i = \mu_{i,L} \).

Therefore, when \( \mu_i = \mu_{i,H} \), \( \frac{d\Pi_M^F(w)}{dw}\big|_{w=w^F(\mu_{i,L})} \) in equation (7) is

\[
\frac{d\Pi_M^F(w)}{dw}\big|_{w=w^F(\mu_{i,L})} = \mu_{i,H} - \mu_{i,L} + \left[ \mu_{i,L} + \sigma_i z_i^F(w^F(\mu_{i,L})) \right] + \sigma_j z_j^F(w^F(\mu_{i,L}))
- (w^F(\mu_{i,L}) - c) \frac{A_i}{r_i + p_i - s_i} + \frac{A_j}{r_j + p_j - s_j}
\]

\[
= \mu_{i,H} - \mu_{i,L} > 0,
\]

where the second equality is true because of equation (6).

Thus, \( \frac{d\Pi_M^F(w)}{dw}\big|_{w=w^F(\mu_{i,L})} > 0 \) when \( \mu_i = \mu_{i,H} \), which means \( w^F(\mu_{i,H}) > w^F(\mu_{i,L}) \), i.e., when \( \mu_i \) increases, \( w^F \) increases.

However, as \( \sigma_i \) increases, it is not clear how \( \frac{d\Pi_M^F(w)}{dw} \) will be changed since \( z_i^F, A_i, B_i, i = 1, 2 \), all have complicated relationships with \( \sigma_i \) and may increase or decrease. Therefore, the manufacturer’s optimal wholesale price may also either increase or decrease.
5.2 Proof of Theorem 3

To show truth-telling is not retailer i’s equilibrium reporting strategy, we only need to show that given retailer j reports his demand information truthfully, i.e., \( \{\mu_j^t, \sigma_j^t = \sigma_j^t \} \), revealing true information of his demand parameters, \( \{\mu_i, \sigma_i\} \), to the other parties, is not i’s best strategy. To simplify the analysis, we first focus on the mean demand information revealing, i.e., assuming \( \sigma_i^t = \sigma_i^t \). We will show that retailer i has incentives to report his mean demand information untruthfully. Similarly, we can show that retailer i also has incentives to report his demand standard deviation untruthfully. This proves that retailers have incentives to report their demand parameters untruthfully.

Given \( \sigma_i^t = \sigma_i^t \), denote retailer i’s true profit as \( \Pi_i(Q_i^t, \mu_t^i, w^*(\mu_i^t)), Q_j^t(\mu_i^t, w^*(\mu_i^t))) \). We now check \( \frac{d\Pi_i(Q_i^t, Q_j^*)}{d\mu_i^t} \bigg|_{\mu_i^t = \mu_i^t} \). If \( \mu_i^t = \mu_i^t \) maximizes \( \Pi_i(\cdot, \cdot) \), then we must have \( \frac{d\Pi_i(Q_i^t, Q_j^*)}{d\mu_i^t} \bigg|_{\mu_i^t = \mu_i^t} = 0 \) since it must be a critical point, otherwise \( \mu_i^t = \mu_i^t \) cannot be a maximizer. For brevity, we omit the arguments of \( Q_i^t, Q_j^* \), and \( w^* \) when obvious.

\[
\frac{d\Pi_i(Q_i^t, Q_j^*)}{d\mu_i^t} \bigg|_{\mu_i^t = \mu_i^t} = \frac{\partial\Pi_i}{\partial w^*} \frac{dw^*}{d\mu_i^t} \bigg|_{\mu_i^t = \mu_i^t} + \frac{\partial\Pi_i}{\partial Q_i^t} \frac{dQ_i^t}{d\mu_i^t} \bigg|_{\mu_i^t = \mu_i^t} + \frac{\partial\Pi_j}{\partial Q_j^t} \frac{dQ_j^t}{d\mu_i^t} \bigg|_{\mu_i^t = \mu_i^t} = -Q_i^t \frac{dw^*}{d\mu_i^t} \bigg|_{\mu_i^t = \mu_i^t} + [(1 - \alpha_j)u_j \beta_j(Q_i^t, Q_j^*) - \alpha_i u_i \gamma_j(Q_i^t, Q_j^*)] \frac{dz_i^*}{d\mu_i^t} \bigg|_{\mu_i^t = \mu_i^t},
\]

where \( Q_i^t = \mu_i^t + \sigma_i^t z_i^*, Q_j^* = \mu_j^t + \sigma_j^t z_j^* \), and \( \beta_j(Q_i, Q_j) \) and \( \gamma_j(Q_i, Q_j) \) are defined as,

\[
\beta_j(Q_i, Q_j) = P(Q_j - (D_i - Q_i)^+ < D_j < Q_j),
\]

\[
\gamma_j(Q_i, Q_j) = P(Q_j < D_j < Q_j + (Q_i - D_i)^+),
\]

and \( \{\beta_j(Q_i^t, Q_j^*), \gamma_j(Q_i^t, Q_j^*)\} \) are defined accordingly with \( D_i = D_i^t, D_j = D_j^t, Q_i = Q_i^t \), and \( Q_j = Q_j^* \).

Now we solve \( \frac{dw^*}{d\mu_i^t} \) and \( \frac{dz_i^*}{d\mu_i^t} \). Define \( O = \frac{\sigma_i^t + \sigma_j^t}{(\alpha_i + \alpha_j) + \sigma_i^t + \sigma_j^t} \) in equation (??). Taking the derivative of the right hand side of equation (??) w.r.t. \( \mu_i^t \), we have

\[
\sigma_i^t [1 + (w - c) \frac{\partial O}{\partial Q_i^t}] \frac{dz_i^*}{d\mu_i^t} + \sigma_j^t [1 + (w - c) \frac{\partial O}{\partial Q_j^t}] \frac{dz_i^*}{d\mu_i^t} + O \frac{dw^*}{d\mu_i^t} = -[1 + (w - c) \frac{\partial O}{\partial Q_i^t}],
\]

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Taking the derivative of both sides of equations (2) w.r.t. \( \mu_i^r \), we have

\[
(A_i + B_i) \frac{dz_i^*}{d\mu_i^r} + \frac{\sigma_j}{\sigma_i} B_i \frac{dz_i^*}{d\mu_i^r} = -\frac{1}{r_i + p_i - s_i} \frac{dw^*}{d\mu_i^r}, \quad i = 1, 2.
\]

From the above equations, we can solve

\[
\frac{dz_i^*}{d\mu_i^r} = -\frac{A_j + B_j - \frac{\sigma^t_i}{\sigma_i} B_i}{(A_i A_j + A_i B_j + A_j B_i)(r_i + p_i - s_i)} \frac{dw^*}{d\mu_i^r}, \quad i = 1, 2,
\]

and

\[
\frac{dw^*}{d\mu_i^r} = \frac{1 + (w - c) \frac{\partial O}{\partial Q^*}}{\sigma_i^t (1 + (w - c) \frac{\partial O}{\partial Q^*}) (A_i A_j + A_i B_j + A_j B_i)(r_i + p_i - s_i) + \sigma_j^t (1 + (w - c) \frac{\partial O}{\partial Q^*}) (A_i A_j + A_i B_j + A_j B_i)(r_i + p_i - s_i) + O}
\]

Therefore,

\[
\frac{d\Pi_i^r(Q_i^*, Q_j^*)}{d\mu_i^r} |_{\mu_i^r = \mu_i^r} = -[Q_i^* + (\alpha_i u_{ij} \gamma_j^t(Q_i^*, Q_j^*)) - (1 - \alpha_j) u_{ji} \beta_j^t(Q_i^*, Q_j^*)] \frac{\sigma^t_i (A_i + B_i) - \sigma^t_j B_j}{(A_i A_j + A_i B_j + A_j B_i)(r_i + p_i - s_i)} \frac{dw^*}{d\mu_i^r} |_{\mu_i^r = \mu_i^r}
\]

Since in general, \( \frac{dw^*}{d\mu_i^r} \neq 0 \) and \( Q_i^* + (\alpha_i u_{ij} \gamma_j^t(Q_i^*, Q_j^*)) - (1 - \alpha_j) u_{ji} \beta_j^t(Q_i^*, Q_j^*) \neq 0 \) at \( \mu_i^r = \mu_i^r \), we have \( \frac{d\Pi_i^r(Q_i^*, Q_j^*)}{d\mu_i^r} |_{\mu_i^r = \mu_i^r} \neq 0 \), i.e., truth telling is not retailer i’s equilibrium reporting strategy. \( \blacksquare \)