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Caps on Bidding in All-Pay Auctions: Comments on the Experiments of A. Rapoport and W. Amaldoss

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Abstract

In an article published in this journal, Rapoport and Amaldoss (2000, *Journal of Economic Behavior and Organization*, 42, 483-521) analyze symmetric and asymmetric investment games similar to two-player all-pay auctions with bid caps. In this note, we correct an error in their characterization of the set of Nash equilibria of their symmetric investment game. In particular, we find equilibria that Rapoport and Amaldoss (2000) fail to identify. Taking these equilibria into account has important implications for the analysis of data from Rapoport and Amaldoss's experiments.

JEL classification: C72, C92, D44

Keywords: All-pay auction, Mixed strategies, Discrete strategy space, Bid caps, Experiments

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1 Introduction

In a recent article in this journal, Rapoport and Amaldoss (2000) analyze an investment game of complete information in which two firms compete for a patent or the monopoly rights to a product. The firms compete by simultaneously investing in R&D, the level of which may be chosen from a common finite grid of feasible R&D levels. Both firms face budget constraints (endowments) that are common knowledge and that place upper bounds on each firm's feasible amount invested. The firm that invests the higher amount wins the patent or monopoly right (henceforth the "prize") and earns a payoff equal to the value of the prize plus its endowment minus its investment. The firm that invests the lower amount obtains a payoff equal to its endowment minus its investment. If firms set the same level of investment neither wins the prize and both lose their investment. It is assumed that the value of the prize is identical to the two firms, but two separate assumptions on budget constraints are examined: in Case 1 firms have identical budget constraints and in Case 2 one firm has a marginally larger budget than the other firm (\$1 larger with a grid of mesh \$1).

The two games examined by Rapoport and Amaldoss (henceforth R-A) are specific examples of a type of game that is known in the rent-seeking literature as an all-pay auction with bid caps. Except for the fact that R-A examine an all-pay auction with discrete strategy space and for the assumption made by R-A that in the event of a tie in expenditure both firms lose the prize, the game analyzed is very similar to the treatment of the all-pay auction with bid caps analyzed by Che and Gale (1998).¹

In R-A (2000) the authors undertake an experimental analysis of the two cases of all-pay auctions indicated above. This note focuses on the conclusions of the authors concerning Case 1, the game in which firms have identical budget constraints. For this case, R-A claim that a unique mixed strategy Nash equilibrium exists in the game (Rapoport and Amaldoss 2000, Proposition 1) and that their experimental results lend support to several of the properties of this solution at the aggregate, although not the individual, level. In particular, R-A claim that the experimental analysis seems to indicate that the players "mixed their strategies" and that the frequency of the choice of an expenditure equal to the budget constraint increased as the prize obtained from winning increased. On the other hand, several properties of the equilibrium identified by R-A were not supported by the experimental results. One prediction of the equilibrium identified in R-A's Proposition 1 is that the expected payoff of each player is independent of the (common) value of the prize to the players. R-A's experiments with prizes of different values (but identical budget constraints) showed that the mean realized payoffs to the players increased significantly in the value of the prize. R-A also found that the incidence of subjects choosing a zero investment in the game was higher than predicted

¹In Dechenaux, Kovenock, and Lugovskyy (2003) we refer to the game in which both firms lose the prize in the event of a tie in expenditure as the "full dissipation" case. The case examined by Che and Gale (1998), in which firms either split the prize or fairly randomize to determine the winner of the prize is referred to as the "partial dissipation" case.

by the equilibrium, a fact that they conjectured could be a consequence of either minimax behavior or simply "mixing strategies but investing zero more often than predicted by the theory" (R-A (2000), p. 497).

This note utilizes the results of Dechenaux, Kovenock, and Lugovskyy (2003) (henceforth D-K-L) to show that, contrary to the claim of R-A's Proposition 1, the Nash equilibrium for the Investment game with symmetric budget constraints examined by R-A is not generally unique. D-K-L (2003) show that for strategy spaces comprised of an equally spaced grid of an even number of strategies strictly greater than 2 (starting with a lower bound of zero and an upper bound less than the lower of the players' valuations of the prize) there exist three mixed strategy equilibria in the two-player Investment game analyzed by R-A. One of these equilibria is that identified by R-A, a symmetric equilibrium with full support and players completely dissipating all expected gains from winning the prize. The other two equilibria in the context of the symmetric game examined by R-A (2000) involve mixed strategies with an alternating structure. One firm randomizes placing positive probability at investment levels $c_i = 0, 2, 4, \dots, e - 1$ and the other firm randomizes placing positive probability at investment levels $c_j = 1, 3, \dots, e$, where e is the common budget constraint of the two firms.

We believe that the failure of R-A to account for the existence of other equilibria in their Investment game is the result of the erroneous claim used in the proof of their Proposition 1 that both firms earn an expected profit of zero from the Investment game (and hence earn an expected utility equal to their initial endowment e).² Indeed, in the asymmetric equilibria that we identify the firm investing odd unit amounts $(1, 3, \dots, e)$ earns a positive expected payoff from the game that is increasing in the value of the prize while the firm investing even unit amounts earns zero. Hence, in these alternating equilibria dissipation of the value of the prize is partial and the expected payoff of one of the players is increasing in the value of the prize.

The existence of these alternating equilibria and the possibility that some subset of experimental subjects may sometimes play the corresponding equilibrium strategies leaves open the possibility that the experimental evidence examined in R-A (2000) is more consistent with Nash equilibrium behavior than is evident from the R-A analysis. Like the symmetric equilibrium, the alternating equilibria have the property that the frequency of the choice of an expenditure equal to the budget constraint increases as the prize obtained from winning increases. Moreover, since the alternating equilibria have the property that one player receives a positive payoff that is increasing in the common value of the prize, while the other player receives zero (and therefore the utility of his endowment), aggregate behavior under these alternating equilibria will exhibit a mean payoff increasing in the value of the

²The error in R-A's Proposition 1 arises because proving that the system provided in (A1) and (A2) of the proof of Proposition 1 (R-A (2000), p. 518) has a unique solution is not sufficient to prove uniqueness of a Nash equilibrium in mixed strategies. This system assumes that the equilibrium payoff to the two players is the value of their budget, which may not hold in equilibrium. The system (A1) also is written with equality in every row. However, if the probability of choosing a particular pure strategy c is zero, the equation corresponding to that strategy in (A1) should in fact be a weak inequality. (See, for instance, Baye, Kovenock, and de Vries (1994) for a rigorous treatment of the programming framework for solving for the Nash equilibria of all-pay auctions with discrete strategy spaces.) By imposing (A1) as it is stated, R-A implicitly eliminate all mixed strategies that do not have full support (that is, have $p(c) > 0$ for $c = 0, 1, 2, \dots, e$).

prize. This property is consistent with the pattern appearing in R-A's experimental data, but is not a property of the symmetric equilibrium that they identify. Furthermore, as long as the value of the prize r satisfies $r > e + 1$, which holds in both of the symmetric experimental treatments analyzed by R-A, the frequency of zero investments in the alternating equilibria is larger than that in the symmetric equilibrium. The frequency of zero investments in the alternating equilibria is also larger than the frequency of any specific investment level strictly between zero and the budget constraint e , a property that appears to be supported by the experimental data. Hence, one may not need to appeal to minimax or non-equilibrium behavior to explain the pattern of zero expenditure choices in the data.³

In section 2 we review the characterization of the set of Nash equilibria of the two-player Investment game appearing in Dechenaux, Kovenock, and Lugovskyy (2003). Section 3 concludes by examining the implications of this characterization for the experimental results reported in Rapoport and Amaldoss (2000).

2 The model and theoretical results

Suppose there are two firms indexed by i and j . Each firm receives a similar integer endowment (or budget) $e > 1$ and each firm's value for the prize is $r > e$. Let $C_k \equiv C = \{0, 1, 2, \dots, e\}$ denote firm k 's (pure) strategy space, $k \in \{i, j\}$. If a firm invests a higher amount than its rival, it receives the prize and pays the amount it invested. Otherwise, it receives nothing but still pays the amount it invested. Thus firm k 's payoff function is:

$$U_k(c_k, c_{-k}) = \begin{cases} r + e - c_k & \text{if } c_k > c_{-k}, \\ e - c_k & \text{if } c_k \leq c_{-k}. \end{cases} \quad \text{for } k \in \{i, j\}.$$

In the mixed extension of the game, let $p_k(c)$ denote a probability distribution over the elements of C . $U_k(p_k, p_{-k})$ is then firm k 's expected payoff, $k \in \{i, j\}$.

According to R-A's Proposition 1, the game described above has a unique mixed strategy equilibrium (see R-A, p. 488). This equilibrium is symmetric and is such that each firm earns an expected profit equal to e . We show below that R-A's Proposition 1 is erroneous. Under fairly general conditions, and more importantly, under the conditions satisfied in R-A's experiments for this game, there exist three equilibria. In the two equilibria R-A fail to

³R-A also observe strong sequential dependencies in the choices of subjects in their experiments. They claim (R-A (2000), p. 500) that one implication of the Nash equilibrium that they identify is that "the mixed-strategy solution calls for perfect randomization of choices between successive trials." They go on to note that "in order to eliminate sequential dependencies, the subjects' pairing...was deliberately changed from trial to trial." However, in an idealized environment in which it is common knowledge that subjects never meet more than once in a session and outcomes from each play of the game are known only to the pair playing, equilibrium does not require independent randomization across plays of the game. Randomizing once according to the particular Nash equilibrium strategy at the beginning of the session and playing the same realization in every play of the game would constitute equilibrium behavior. To the extent that R-A's experiments do not replicate this idealized environment, in particular, since players meet more than once in each session, this one-time randomization does not constitute equilibrium behavior. However, it is not clear to us that subjects in the experiment do not take the experimental environment to be a close enough approximation to the idealized anonymous environment of perfect strangers to justify one-time randomization. At least a subset of the subjects in the R-A experiments exhibit behavior consistent with this explanation.

identify, one firm's expected payoff is equal to r (and its rival's is equal to e), and is therefore increasing in the reward r . These results are summarized in Claim 1 below.

Claim 1 *Assume $e > 1$. If e is even, there exists a unique equilibrium (p_i^*, p_j^*) characterized by:*

$$p_i^*(c) = p_j^*(c) = \begin{cases} \frac{1}{r} & \text{if } c = 0, 1, \dots, e-1 \\ 1 - \frac{e}{r} & \text{if } c = e \end{cases} \quad (1)$$

Moreover, (p_i^*, p_j^*) satisfies $U_i^* = U_j^* = e$.

If e is odd, there exist three Nash equilibria. One equilibrium is characterized by (1) and satisfies $U_i^ = U_j^* = e$. The two other equilibria are given by (p_k^*, p_{-k}^*) , $k \in \{i, j\}$ where:*

$$p_k^*(c) = \begin{cases} \frac{2}{r} & \text{if } c = 1, 3, \dots, e-2 \\ 1 - \left(\frac{e-1}{r}\right) & \text{if } c = e \\ 0 & \text{if } c = 0, 2, \dots, e-1 \end{cases} \quad (2)$$

and

$$p_{-k}^*(c) = \begin{cases} 1 - \left(\frac{e-1}{r}\right) & \text{if } c = 0 \\ \frac{2}{r} & \text{if } c = 2, 4, \dots, e-1 \\ 0 & \text{if } c = 1, 3, \dots, e \end{cases} \quad (3)$$

In such equilibria, expected payoffs are given by $U_k^ = r > e$ and $U_{-k}^* = e$, $k \in \{i, j\}$.*

The "symmetric" equilibrium of equation (1) and the "alternating" equilibria characterized by equations (2) and (3) have several properties of empirical interest. First of all, aggregating across the two players, both types of equilibria place greater probability at an investment equal to the endowment, e , than at any one of the investment levels $1, 2, \dots, e-1$. However, the alternating equilibria place the same aggregate probability at 0 as at e , whereas the symmetric equilibrium places the same probability at 0 as at each of the levels $1, 2, \dots, e-1$. The expected payoffs aggregated across the two players also differ across the two types of equilibria. In the symmetric equilibrium, the two players dissipate all expected gains from the competition for the prize, and each earns an expected payoff equal to their common endowment level, e . This expected payoff is, of course, invariant with respect to the value of the prize, r . In the alternating equilibria one player earns an expected payoff equal to r while the other earns an expected payoff equal to e . Aggregating across the two players, the mean expected payoff of a player playing an alternating equilibrium is $\frac{r+e}{2}$, an expression which is increasing in the common value of the prize, r . The following section examines which of these properties is supported by R-A's reported experimental results.

3 Implications for the Analysis of Experimental Data

Rapoport and Amaldoss's (2000) Proposition 1 is used as a theoretical prediction against which the data from experiments are evaluated. In particular, the authors consider two games with the following parameters:

$$\begin{aligned} \text{Low Reward Condition (Game L)} & : e = 5, r = 8 \\ \text{High Reward Condition (Game H)} & : e = 5, r = 20 \end{aligned}$$

In R-A (2000), theoretical frequency distributions of investments for both games are calculated in accordance with Proposition 1 and presented in R-A's Table 3 along with aggregated experimental data. Note that according to our Claim 1, the symmetric equilibrium defined by R-A's Proposition 1 is not unique in both games. Since $e = 5$, the two "alternating" equilibria defined in Claim 1 also exist in both games. Using equations (2) and (3), we calculate the equilibrium solutions for the alternating equilibria in both games.

In Game L, we have:

$$\begin{aligned} p_k^{L*} &= (p_k^{L*}(0), p_k^{L*}(1), p_k^{L*}(2), p_k^{L*}(3), p_k^{L*}(4), p_k^{L*}(5)) \\ &= (0, 0.25, 0, 0.25, 0, 0.5) \\ p_{-k}^{L*} &= (p_{-k}^{L*}(0), p_{-k}^{L*}(1), p_{-k}^{L*}(2), p_{-k}^{L*}(3), p_{-k}^{L*}(4), p_{-k}^{L*}(5)) \\ &= (0.5, 0, 0.25, 0, 0.25, 0) \end{aligned} \quad \text{for } k \in \{i, j\}.$$

Similarly, in Game H:

$$\begin{aligned} p_k^{H*} &= (p_k^{H*}(0), p_k^{H*}(1), p_k^{H*}(2), p_k^{H*}(3), p_k^{H*}(4), p_k^{H*}(5)) \\ &= (0, 0.1, 0, 0.1, 0, 0.8) \\ p_{-k}^{H*} &= (p_{-k}^{H*}(0), p_{-k}^{H*}(1), p_{-k}^{H*}(2), p_{-k}^{H*}(3), p_{-k}^{H*}(4), p_{-k}^{H*}(5)) \\ &= (0.8, 0, 0.1, 0, 0.1, 0), \end{aligned} \quad \text{for } k \in \{i, j\}.$$

We can use these solutions to calculate the frequency distributions of choices for alternating equilibria. Since the frequency distribution contains information about how often each pure strategy is played in each game, in the case of two players, for every investment $c \in C$, this distribution can be calculated as:

$$f^a = \frac{p_k^{a*}(c) + p_{-k}^{a*}(c)}{2} \quad \text{for } a \in \{L, H\} \quad \text{and } k \in \{i, j\}. \quad (4)$$

It is straightforward to verify that for both symmetric games that R-A analyze, the predicted frequency of a zero investment is greater in alternating equilibria than it is in the symmetric equilibrium. From (1), the predicted frequency of a zero investment in the symmetric equilibrium is $\frac{1}{r}$. Using (2), (3), and (4) the predicted frequency of a zero investment in either alternating equilibrium is:

$$\frac{1}{2} \left(1 - \frac{e-1}{r} \right).$$

Comparing, we obtain:

$$\frac{1}{2} \left(1 - \frac{e-1}{r} \right) > \frac{1}{r} \iff r > e + 1,$$

a condition that is satisfied when e is equal to 5 and r is equal to 8 or 20.

We extend Table 3 of R-A to include the frequency distributions of investments for alternating equilibria. The results are presented in Table 1 below.

[TABLE 1 HERE]

Table 1 shows that taking alternating equilibria into consideration does not affect the theoretical predictions for the frequency of investments 1 to 4, but it widens the range for 0 and 5. In fact, if we compare the experimental data with the theoretical predictions for the symmetric equilibrium, the frequencies for investments 0 and 5 have the largest absolute deviations. It is clear from Table 1 that widening the range of possible frequencies for investments 0 and 5 allows for a significant reduction in the difference between observed and predicted investment frequencies.

In the experiments of R-A, subjects in Group 1 play 80 periods of Game L and then 80 periods of Game H, while subjects in Group 2 participate in the two games in the reverse order. In Table 1 the observations in bold indicate that the game was the second game for the subjects. If we assume that experience reduces confusion and focus only on the pool of experienced subjects, the experimental frequencies for 0 and 5 are then completely within the theoretically predicted range.

Taking alternating equilibria into consideration also extends the range of predicted mean expected payoffs. Table 2 in this paper shows that according to the symmetric equilibrium employed by R-A the predicted mean payoff is independent of the value of the prize. However, in alternating equilibria, the mean payoff is positively related to the value of the prize, which is also consistent with the description of the experimental data provided by R-A (2000, pp. 495-496).

[TABLE 2 HERE]

These results appear to indicate that R-A's failure to recognize the complete set of Nash equilibria in their Investment game may have caused the authors to too readily reject Nash equilibrium behavior as an explanation of behavior in the games they examine.

While this note addresses the results appearing in the two-player Investment game analyzed in Rapoport and Amaldoss (2000) it also demonstrates that the uniqueness claim in Proposition 1 of Rapoport and Amaldoss (2003) is false. R-A (2003) examines an n -player all-pay auction of which the two-player Investment game in R-A (2000) is a special case. Obviously, if uniqueness of equilibrium fails to hold for $n = 2$ the general statement of uniqueness of equilibrium for arbitrary n also fails. While an analysis of the n -player case is beyond the scope of the current manuscript,⁴ it is clear that arguments parallel to those made in this paper concerning the experimental implications of a larger-than-recognized set of Nash equilibria apply to the n -player case as well.

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⁴Baye, Kovenock, and de Vries (1996) show that for the symmetric all-pay auction with continuous strategy space and no budget constraints there is a unique equilibrium for $n = 2$ and a continuum of equilibria for $n = 3$. An increase in the set of equilibria may arise in moving from 2 to $n > 2$ players in the discrete game analyzed by R-A (2003).

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5 Appendix

Proof of Claim 1

We first show the equivalence between R-A's game and a similar game, but in which firms do not receive an endowment. For $k \in \{i, j\}$, let:

$$V_k(c_k, c_{-k}) = U_k(c_k, c_{-k}) - e, \quad (5)$$

then:

$$V_k(c_k, c_{-k}) = \begin{cases} r - c_k & \text{if } c_k > c_{-k}, \\ -c_k & \text{if } c_k \leq c_{-k}. \end{cases} \quad \text{for } k \in \{i, j\}.$$

By the Expected Utility Theorem, player k 's behavior is invariant to affine transformations of utility. Thus the game Γ' with strategy spaces C_i, C_j such that $C_j = C_i = C$, and payoff functions $V_k(c_k, c_{-k})$, $k \in \{i, j\}$, has exactly the same equilibria as the game Γ with strategy spaces C_i, C_j such that $C_j = C_i = C$, and payoff functions $U_k(c_k, c_{-k})$, $k \in \{i, j\}$. With a slight change in notation, the proof of Claim 1 for the game Γ' can be found in Dechenaux, Kovenock and Lugovskyy (2003), but we provide it here as well for completeness. Note that by (5), the equilibrium expected payoff in game Γ is equal to $U_k^* = V_k^* + e$ where V_k^* is firm k 's expected payoff in game Γ' , $k \in \{i, j\}$.

We prove Claim 1 through a series of Lemmata. It follows from Nash (1950) that an equilibrium exists. Throughout the proof, let (p_i^*, p_j^*) be an equilibrium of the game. For a given equilibrium (p_i^*, p_j^*) , let $p_k^*(c)$ be the associated probability that firm k plays c , S_k the associated support of firm k 's distribution, and V_k^* firm k 's expected profit in that equilibrium, $k \in \{i, j\}$.

Lemma 1 *In any equilibrium (p_i^*, p_j^*) , $S_i \cup S_j = C$. Consequently, $V_k^* = 0$ for at least one k , $k \in \{i, j\}$.*

Proof. First, we show that (i) if a point $v \in C$, $v > 0$, is in the support of at least one firm, then all points $n \in C$ where $0 \leq n < v$ must be in the support of at least one firm. This implies that (ii) 0 is in the support of at least one firm. Then, we use (ii) to show that (iii), e is in the support of at least one firm. Combining (i) and (iii) completes the proof of the lemma.

We first show (i). Let v be a strictly positive integer with $v \in S_k$ for some $k \in \{i, j\}$. Suppose contrary to our claim that there exists an $n < v$ such that $n \notin S_i \cup S_j$. This implies that there exists some $u \leq v$ such that $u \in S_l$ for some $l \in \{i, j\}$ and $(u - 1) \notin S_i \cup S_j$.

Then, firm l 's expected payoff from playing u is given by:

$$V_l(u) = \sum_{z < u} p_{-l}^*(z)r - u,$$

and firm l 's expected profit from playing $(u - 1)$ is given by:

$$V_l(u - 1) = \sum_{z < (u-1)} p_{-l}^*(z)r - (u - 1).$$

But since $\sum_{z < (u-1)} p_{-l}^*(z) = \sum_{z < u} p_{-l}^*(z)$, we have $V_l(u) < V_l(u-1)$. Therefore, u cannot be in the support of firm l 's equilibrium distribution, a contradiction. Thus, we have established (i).

We now turn to (ii). First note that firm k 's expected profit from playing 0 is, regardless of its opponent's strategy:

$$V_k(0) = 0.$$

Suppose to the contrary that $0 \notin S_j \cup S_i$. Let $\underline{n} \equiv \min_n \{n | n \in S_i \cup S_j\}$. Suppose without loss of generality that $\underline{n} \in S_{l'}$ for some $l' \in \{i, j\}$. It follows from $\underline{n} > 0$, that:

$$V_{l'}(\underline{n}) = \sum_{z < \underline{n}} p_{-l'}^*(z)r - \underline{n} = 0 - \underline{n} < 0,$$

since $p_{-l'}^*(z) = 0, \forall z < \underline{n}$, a contradiction to the fact that $\underline{n} \in S_{l'}$. Thus $0 \in S_j \cup S_i$.

To complete the proof of the lemma, we show (iii). Suppose to the contrary that $e \notin S_i \cup S_j$. Then, any firm k 's expected profit from playing e is equal to:

$$V_k(e) = \sum_{z < e} p_{-k}^*(z)r - e = r - e > 0, \text{ for } k \in \{i, j\},$$

whereas its expected profit from playing 0 is:

$$V_k(0) = 0.$$

But, by (ii) above, 0 must be in the support of at least one firm k , a contradiction. Hence, we have established (iii).

Combining (i) and (iii), we have shown that $S_i \cup S_j = C$. Consequently, $V_k^* = V_k(0) = 0$ for at least one k , which proves the lemma. Q.E.D.

Lemma 2 *In any equilibrium (p_i^*, p_j^*) , if $p_i^*(c) > 0$ and $p_j^*(c) > 0$, then $p_i^*(n) > 0$ and $p_j^*(n) > 0$, for every n such that $0 \leq n < c$.*

Proof. Suppose to the contrary that there exists an investment v such that $p_k^*(v) > 0$, $k = i, j$ and $n < v$ such that $p_l^*(n) = 0$, for some $l \in \{i, j\}$. This implies that there exists u , $u \leq v$ such that $u \in S_i \cap S_j$ and such that $(u-1) \notin S_l$ for some $l \in \{i, j\}$. From Lemma 1 it follows that since $p_l^*(u-1) = 0$, then $p_{-l}^*(u-1) > 0$. Furthermore, firm $-l$'s expected profit at u is equal to:

$$V_{-l}(u) = \sum_{z < u} p_l^*(z)r - u,$$

while its expected profit at $(u-1)$ is:

$$V_{-l}(u-1) = \sum_{z < u-1} p_l^*(z)r - (u-1).$$

Since $p_l^*(u-1) = 0$, $\sum_{z < u-1} p_l^*(z) = \sum_{z < u} p_l^*(z)$. It then follows that $V_{-l}(u-1) > V_{-l}(u)$, contradicting $u \in S_{-l}$. Q.E.D.

Lemma 3 *In any equilibrium (p_i^*, p_j^*) , if $V_l^* > 0$ for some $l \in \{i, j\}$, then:*

(i) $S_i \cap S_j = \emptyset$,

(ii) *Firm l randomizes over all pure strategies $c \in C$ for which c is odd and firm $-l$ randomizes over all pure strategies $c \in C$ for which c is even,*

(iii) $V_{-l}^* = 0$.

Proof. First $V_l^* > 0$ implies $0 \notin S_l$. Thus, from Lemma 1, $0 \in S_{-l}$. It follows that $1 \notin S_{-l}$ since firm $-l$ can increase its expected profit by moving mass from 1 to 0. Then, from Lemma 1, $1 \in S_l$. Suppose $2 \in S_l$. Then whether 2 is in S_{-l} or not, firm l can increase its expected payoff by moving all mass from 2 to 1. Thus $2 \notin S_l$, from which it follows that $2 \in S_{-l}$. Suppose now that $3 \in S_{-l}$. Then, whether 3 is in S_l or not, firm $-l$ can increase its expected payoff by moving all mass from 3 to 2 (recall that firm l puts no mass at 2). Since e is finite, it is straightforward to see that the same argument applies recursively to every $c \leq e$. That is, suppose $c \in S_m$ and $c \notin S_{-m}$, then $(c+1) \notin S_m$ and $(c+1) \in S_{-m}$, $\forall c \in C$ and $m \in \{i, j\}$.

(iii) follows immediately from the fact that $0 \in S_{-l}$. Q.E.D.

Lemma 4 *In any equilibrium (p_i^*, p_j^*) , if $V_i^* = V_j^* = 0$, then $S_i = S_j = C$.*

Proof. Let $\bar{n}_j \equiv \max_n \{n | n \in S_j\}$ and $\bar{n} \equiv \max\{\bar{n}_i, \bar{n}_j\}$. We claim that $\bar{n}_i = \bar{n}_j = \bar{n} = e$. Suppose $\bar{n}_i > \bar{n}_j$. Then it is clear that $V_l(\bar{n}_i) > 0$, a contradiction to $V_l^* = 0$. Thus $\bar{n}_i = \bar{n}_j = \bar{n}$. Applying Lemma 1, $\bar{n} = e$. The claim then follows from Lemma 2. Q.E.D.

Lemma 5 *There exists an equilibrium (p_i^*, p_j^*) with payoffs $V_i^* = V_j^* = 0$. In this equilibrium $S_j = S_i = C$ and:*

$$p_i^*(c) = p_j^*(c) = \begin{cases} \frac{1}{r} & \text{if } c = 0, 1, \dots, e-1, \\ 1 - \frac{e}{r} & \text{if } c = e. \end{cases}$$

Proof. If (p_i^*, p_j^*) is an equilibrium with payoffs $V_i^* = V_j^* = 0$, then from Lemma 4, it follows immediately that $S_j = S_i = C$. We establish existence by construction. Since $S_j = S_i = C$ and both firms obtain an expected profit of 0, if the following pair of matrix equations is satisfied (with equality) for $k \in \{i, j\}$, we have constructed such an equilibrium.

$$\begin{bmatrix} 1 & 1 & 1 & \dots & \dots & 1 \\ r & 0 & 0 & \dots & \dots & 0 \\ r & r & 0 & 0 & \dots & 0 \\ r & r & r & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & 0 \\ r & r & r & \dots & r & 0 \end{bmatrix} \begin{bmatrix} p_{-k}^*(0) \\ p_{-k}^*(1) \\ p_{-k}^*(2) \\ p_{-k}^*(3) \\ \dots \\ p_{-k}^*(e) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \\ \dots \\ e \end{bmatrix} \quad (6)$$

For a given $k \in \{i, j\}$, (6) is a system of $e + 1$ equations and $e + 1$ unknowns. If the leftmost matrix is non-singular, then (6) has a unique solution. The determinant of the leftmost matrix is equal to $(-1)^{e+2} (r)^e \neq 0$. Therefore (6) has a unique solution.

It is now straightforward to compute the solution to (6). We have

$$p_{-k}^*(0)r - 1 = V_k^* = 0,$$

which yields $p_{-k}^*(0) = \frac{1}{r} > 0$. The remaining probabilities can be solved by repeated substitution of $\frac{1}{r}$ in place of $p_{-k}^*(0)$ in (6), which yields $p_{-k}^*(c) = \frac{1}{r}$ for $c \in \{0, 1, \dots, e-1\}$ and $p_{-k}^*(e) = 1 - \frac{e}{r}$. Since $r > e$ by assumption, the solutions to (6) are probabilities. Uniqueness of an equilibrium satisfying the conditions in the lemma follows immediately from Lemma 4 and the fact that such an equilibrium must satisfy the system (6), for $k \in \{i, j\}$. Q.E.D.

Lemma 6 *An equilibrium with $V_l^* > 0$ for some $l, l \in \{i, j\}$, exists if and only if e is odd. If e is odd, then there exist exactly two such equilibria (p_l^*, p_{-l}^*) , $l \in \{i, j\}$, given by:*

$$p_l^*(c) = \begin{cases} \frac{2}{r} & \text{if } c = 1, 3, \dots, e-2 \\ 1 - \left(\frac{e-1}{r}\right) & \text{if } c = e \\ 0 & \text{if } c = 0, 2, \dots, e-1 \end{cases}$$

and

$$p_{-l}^*(c) = \begin{cases} 1 - \left(\frac{e-1}{r}\right) & \text{if } c = 0 \\ \frac{2}{r} & \text{if } c = 2, 4, \dots, e-1 \\ 0 & \text{if } c = 1, 3, \dots, e \end{cases}$$

Proof. We prove existence by construction. Let l be the firm obtaining $V_l^* > 0$ and let the other firm be $-l$. From Lemma 3, the equilibrium must be of the alternating form, and firm l must play pure strategies $c \in C$ for which c is an odd number.

Suppose e is even. From Lemma 3, it follows that e is in S_{-l} but not in S_l . Therefore $V_{-l}^* = r - e > 0$. But this and $V_l^* > 0$ contradict Lemma 3. Therefore no equilibria with $V_l^* > 0$ exist when e is even, which proves the "only if" part of the statement.

Suppose e is odd. Then by Lemma 3, $S_l = \{1, 3, \dots, e\}$ and $S_{-l} = \{0, 2, \dots, (e-1)\}$. It follows that $V_l^* = r - e$. From Lemma 3, for firm $-l$, the strictly positive $p_{-l}^*(c)$'s must be the solution to the following system of $\frac{e+1}{2}$ equations:

$$\begin{cases} V_l^* = p_{-l}^*(0)r - 1 \\ V_l^* = [p_{-l}^*(0) + p_{-l}^*(2)]r - 3 \\ \dots \\ V_l^* = [p_{-l}^*(0) + p_{-l}^*(2) + \dots + p_{-l}^*(e-1)]r - e \end{cases}$$

This system of equations can be written in matrix form:

$$\begin{bmatrix} r & 0 & 0 & \dots & 0 \\ r & r & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ r & r & r & \dots & r \end{bmatrix} \begin{bmatrix} p_{-l}^*(0) \\ p_{-l}^*(2) \\ \dots \\ p_{-l}^*(e-1) \end{bmatrix} = \begin{bmatrix} V_l^* + 1 \\ V_l^* + 3 \\ \dots \\ V_l^* + e \end{bmatrix}. \quad (7)$$

Note that the number of equations coincides with the number of unknowns. The determinant of the first matrix equals $(r)^{(e+1)/2} > 0$. This is sufficient to prove uniqueness of the solution. Moreover, in equilibrium $V_l^* = r - e = p_{-l}^*(0)r - 1$. Solving the system by repeated substitution yields $p_{-l}^*(0) = 1 - \left(\frac{e-1}{r}\right) > 0$ and $p_{-l}^*(c) = \frac{2}{r} > 0$ for $c \in \{2, \dots, (e-1)\}$. We check that the probabilities sum to one:

$$p_{-l}^*(0) + \sum_{c=2}^{e-1} p_{-l}^*(c) = 1 - \left(\frac{e-1}{r}\right) + \left(\frac{e-1}{2}\right) \frac{2}{r} = 1,$$

so they are indeed probabilities.

We now turn to firm l 's strategy. Note that firm $-l$ obtains an expected payoff of 0 in equilibrium, $V_{-l}^* = 0$. From Lemma 3, the strictly positive $p_l^*(c)$'s must solve the following system of $\frac{e-1}{2}$ equations:

$$\begin{cases} V_{-l}^* = p_l^*(1)r - 2 \\ V_{-l}^* = [p_l^*(1) + p_l^*(3)]r - 4 \\ \dots \\ V_{-l}^* = [p_l^*(1) + \dots + p_l^*(e-2)]r - (e-1) \end{cases}$$

This system of equations can be written in matrix form:

$$\begin{bmatrix} r & 0 & 0 & \dots & 0 \\ r & r & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ r & r & r & \dots & r \end{bmatrix} \begin{bmatrix} p_l^*(1) \\ p_l^*(3) \\ \dots \\ p_l^*(e-2) \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ \dots \\ (e-1) \end{bmatrix}. \quad (8)$$

Note that the number of equations coincides with the number of unknowns. The determinant of the first matrix equals $(r)^{(e-1)/2} > 0$. This is sufficient to prove uniqueness of the solution. Solving the system by repeated substitution yields $p_l^*(c) = \frac{2}{r} > 0$, where $\frac{2}{r} < 1$, for all $c \in \{1, 3, \dots, (e-2)\}$. Using the fact that probabilities must sum to 1 to solve for $p_l^*(e)$, we obtain $p_l^*(e) = 1 - \left(\frac{e-1}{r}\right) > 0$.

Uniqueness of an equilibrium (p_k^*, p_{-k}^*) satisfying $V_k^* > 0$ and $V_{-k}^* = 0$ follows from Lemma 3 and the fact that such an equilibrium must satisfy the systems (7) and (8), $k \in \{i, j\}$. Thus using Lemma 3, if e is odd, there exists exactly one equilibrium in which $V_i^* > 0$ and exactly one equilibrium in which $V_j^* > 0$. Q.E.D.

To complete the proof of the claim, it suffices to note that Lemma 1 implies that there are no equilibria in which both firms earn a strictly positive expected profit. Therefore, all equilibria of the game are characterized by Lemma 5 and Lemma 6. Q.E.D.

Game L	Empirical distribution			Equilibrium predictions		
Investment	Group 1	Group 2	Both groups	Symmetric equilibrium	Alternating equilibria	Range for both types of equilibria
$c = 0$	0.116	0.222	0.169	0.125	0.250	[0.125, 0.250]
$c = 1$	0.116	0.117	0.116	0.125	0.125	0.125
$c = 2$	0.096	0.079	0.088	0.125	0.125	0.125
$c = 3$	0.084	0.153	0.118	0.125	0.125	0.125
$c = 4$	0.094	0.086	0.090	0.125	0.125	0.125
$c = 5$	0.494	0.343	0.418	0.375	0.250	[0.250, 0.375]

Game H	Empirical distribution			Equilibrium predictions		
Investment	Group 1	Group 2	Both groups	Symmetric equilibrium	Alternating equilibria	Range for both types of equilibria
$c = 0$	0.135	0.147	0.141	0.050	0.400	[0.050, 0.400]
$c = 1$	0.065	0.044	0.055	0.050	0.050	0.050
$c = 2$	0.053	0.054	0.053	0.050	0.050	0.050
$c = 3$	0.040	0.066	0.053	0.050	0.050	0.050
$c = 4$	0.044	0.094	0.069	0.050	0.050	0.050
$c = 5$	0.662	0.594	0.628	0.750	0.400	[0.400, 0.750]

Table 1: Aggregate distribution of investment (observed and predicted for all types of equilibria). Predicted frequencies of investment for alternating equilibria were computed using f^a in equation (4).

Game	Observed mean payoff	Equilibrium prediction for expected payoff		
		Symmetric equilibrium	Alternating equilibria	Range for both types of equilibria
L ($r = 8$)	5.536	5	6.5	[5, 6.5]
H ($r = 20$)	6.845	5	12.5	[5, 12.5]

Table 2: Mean payoffs (observed and theoretically predicted for both types of equilibria).