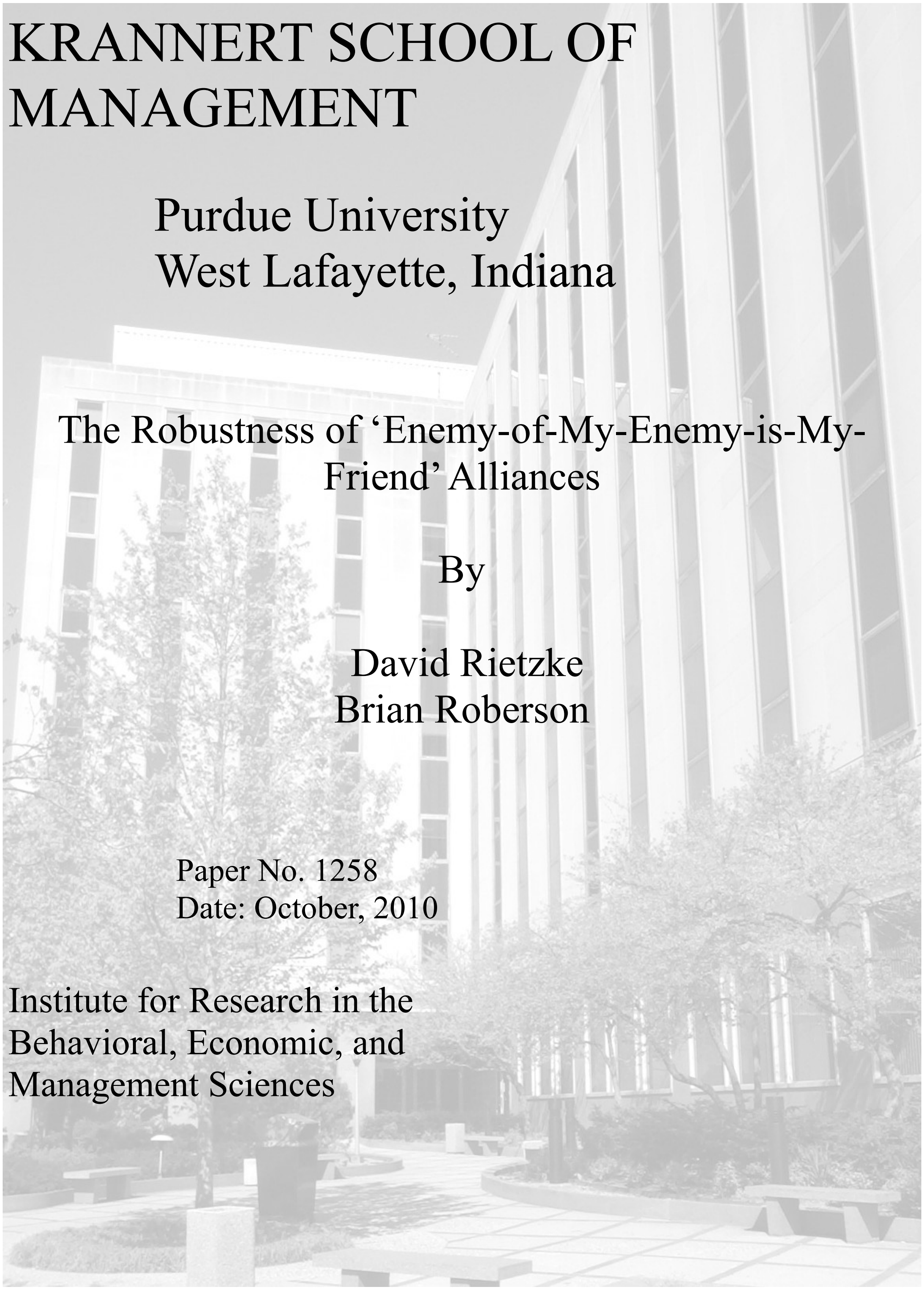

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The Robustness of 'Enemy-of-My-Enemy-is-My-Friend' Alliances

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Abstract

This paper examines the robustness of alliance formation in a three-player, two-stage game in which each of two players compete against a third player in disjoint sets of contests. Although the players with the common opponent share no common interests, we find that under the lottery contest success function (CSF) there exists a range of parameter configurations in which the players with the common opponent have incentive to form an alliance involving a pre-conflict transfer of resources. Models that utilize the lottery CSF typically yield qualitatively different results from those arising in models with the auction CSF (Fang 2002). However, under the lottery and the auction CSFs, the parameter configurations within which players with a common opponent form an alliance are closely related. Our results, thus, provide a partial robustness result for 'enemy-of-my-enemy-is-my-friend' alliances.

JEL Classification: C72, D74

Keywords: Alliance Formation, Contests, Economics of Alliances, Conflict

1 Introduction

Utilizing a two-stage game of alliance formation and multi-battle conflict, this paper examines the robustness of 'enemy-of-my-enemy-is-my-friend' alliances across two benchmark contest success functions (henceforth CSFs), the lottery and auction CSFs. Our focus is on alliance formation in the case that alliance members lack common interests other than a common opponent. In this framework, battle lines are exogenous as in the economics of alliances literature originating with Olson (1965) and Olson and Zeckhauser (1966).[1] The combination of exogenous battle lines with alliance benefits that are both rival and excludable provides a point of departure from the alliance as a public good approach and a formalization of the logic that the 'enemy of my enemy is my friend.' In equilibrium, pre-conflict transfers of resources among alliance members affect the strategic choices of the common enemy, thereby creating strategic externalities. Thus, even if ex post transfers between the two alliance members are not feasible, there exists a range of parameters in which strategic externalities are sufficient to generate individually rational ex ante transfers of resources.

Under the lottery CSF at each component contest, each player's probability of winning a given component contest is proportional to his share of the total resources allocated to that contest. This corresponds to the commonly used case of the general-ratio form Tullock CSF with $r = 1$.[2] Alternatively, if the outcome in each of the component contests, or battlefields, is determined by the auction CSF, then within each contest the winner is the player that allocates the higher level of resources to that contest. This corresponds to the limiting case of the general-ratio form Tullock CSF with $r = \infty$. These two benchmark CSFs capture extreme points in the spectrum of exogenous noise in the CSFs in the component contests. The auction CSF has no noise, and the lottery CSF has a large amount of noise.[3]

In this paper, we characterize — for both the case that ex post transfers among alliance members are feasible and the case that they are not — the parameter configurations in which, under the lottery CSF, the players with the common opponent have incentive to form, prior to the conflict with the common opponent, an alliance involving an individually rational ex ante transfer of resources. The corresponding characterization for the auction CSF is

[1] See also Sandler and Cauley (1975), Sandler (1977, 1999), Murdoch and Sandler (1982, 1984), Arce M. and Sandler (2001), and Ihori and McGuire (2007).

[2] Under the general ratio-form CSF, if player A allocates a_i to the contest and player B allocates b_i to the contest, then $\frac{a_i^r}{a_i^r+b_i^r}$ is the probability that player A wins the contest. The case where $r = 1$ corresponds to the lottery CSF and $r = \infty$ to the auction CSF. The parameter, r, specifies the level of noise or randomness in the contest. As r increases the amount of exogenous noise in the CSF decreases.

[3] For more information on the level of noise implied in the lottery CSF, see Konrad and Kovenock (2009).

given by Kovenock and Roberson (2010a).[4] In comparing the regions of alliance formation under the lottery and auction CSFs, we find that the ranges of parameter configurations in which the players with the common opponent have incentive to form an alliance involving a pre-conflict transfer of resources are nearly identical. This applies to both alliances with and without commitment. In those parameter configurations in which the alliance formation boundaries differ under the two CSFs, the alliance formation boundary under the all-pay auction CSF model is less stringent than the one for the lottery CSF (i.e. alliance formation is more likely when the CSF has less exogenous noise). This paper, thus, provides a partial robustness result for 'enemy-of-my-enemy-is-my-friend' alliances.

In general, large fluctuations in the level of exogenous noise in the general ratio-form contest success function result in qualitatively different equilibrium behavior. In particular, Baye, Kovenock, and de Vries (1994) and Alcalde and Dahm (2009) both show that in the case of a single contest with two players,[5] there is a structural break in the nature of the equilibrium strategies at $r = 2$. If $r \geq 2$, then there exists an equilibrium which exhibits properties that are similar to the unique equilibrium in the two-player all-pay auction, with the auction CSF (see Hillman and Riley 1989 and Baye, Kovenock and de Vries 1996). The condition that $r \geq 2$ implies that the amount of noise in the CSF must be sufficiently small. When $r < 2$ (as with the lottery CSF), this relationship breaks down. In this context, Fang (2002) shows that several results that arise under the auction CSF, such as Baye, Kovenock, and de Vries's (1993) exclusion principle[6] and Che and Gale's (1998) result on caps on lobbying,[7] fail to hold under the lottery CSF.

Furthermore, in two-player games involving competition over a set of component contests that are structurally linked there are a number of important differences between the outcomes arising under the lottery CSF and those arising under the auction CSF.[8] This includes games where the objective is to win a majority of contests (for the lottery CSF see Snyder 1989 and Klumpp and Polborn 2006 and for the auction CSF see Szentes and Rosenthal 2003 and Laslier 2003), games of attack and defense of a network of targets (for the lottery CSF see Clark and Konrad 2007 for the auction CSF see Kovenock and Roberson 2010c), and the case

[4]Under the auction CSF, this game becomes a multi-player, multi-front variation of the Colonel Blotto game.

[5]Alcalde and Dahm (2009) allow for two or more players.

[6]Baye, Kovenock, and de Vries's (1993) exclusion principle states that a revenue maximizing auctioneer may have incentive to exclude a bidder with a high valuation for the prize in order to foster competition among a set of bidders with lower valuations for the prize.

[7]Che and Gale (1998) show that an exogenous cap on bids may provide a competitive advantage to a player with a lower valuation and thereby increase the expected revenue from the contest.

[8]See Kovenock and Roberson (2010b) for a survey of this literature.

of a budget constraint on expenditures across the set of component contests (for the lottery CSF see Friedman 1958 and Robson 2005 for the auction CSF see Roberson 2006 and Hart 2008). In general, the difference in the outcomes for these multiple contest games centers on the fact that under the auction CSF a mixed strategy is a multidimensional joint distribution function and the linkages between the contests lead to endogenous correlation structures in the equilibrium mixed strategies. Conversely, under the lottery CSF the assumption of exogenous noise generates a situation in which the first-order conditions may be used to characterize the pure-strategy equilibrium. This marginal analysis frequently leads to tradeoffs that are qualitatively different from those arising under the auction CSF.

Although the exogenous battle lines feature of the model examined here is most closely aligned with the economics of alliances literature, there are a number of related approaches to the alliance formation problem.[9] Closely related to our focus is the literature on truels. For example Dimico and Seidmann (2010) examine a three-player model involving three component contests in which the outcome at each component contest is determined by the aggregate resource levels of the groups engaged in that component contest.[10] Also related are the three-player one-cake and three-player three-cake problems examined by Skaperdas (1998). However, each of these formulations of the alliance problem differs from our focus in that battle lines are endogenous and any combination of players may form an alliance that competes against the remaining player. Thus, these models show how players optimally choose a common opponent and the nature of the resulting competition. In contrast, by emphasizing exogenous battle lines and the resulting alliance transfers, the focus of our model is on situations in which rational actors face an exogenously determined common opponent. Clearly, there are number of instances in which historical and/or geopolitical realities create such an environment. In such cases, our analysis shows that the strategic externalities arising from the presence of a common enemy are largely robust to fluctuations to the specification of the contest success function.

The rest of the paper is laid out as follows. Section 2 provides a formal description of the two-stage alliance formation game. In section 3 we provide sufficient conditions for the existence of a self-enforcing alliance without commitment when the component contests have the lottery CSF and compare those conditions to the ones arising under the auction CSF. In section 4 we relax the assumption that ex post transfers are not possible and examine the nature of alliances when the alliance members act cooperatively as a single entity. Section 5

[9]For surveys of the alliance literature see Bloch (2009) and Konrad (2009).

[10]See also Shubik (1954), Kilgour and Brams (1997) and Bossert, Brams, and Kilgour (2002).

concludes.

2 The Model

We examine a two-stage game that builds upon Friedman's (1958) multi-item contest model.[11] There are three players, $\{A, x, y\}$, and two disjoint sets of contests in which each component contest has the lottery CSF. The disjoint sets of contests correspond to separate fronts in a larger conflict. There are a number of classic examples of this type of conflict (e.g. World War II in which Germany faced the Allied forces and the Soviet Union on disjoint fronts).

Players A, x, and y are initially endowed with a fixed amount of a one-dimensional resource denoted R_A, R_x, and R_y, respectively. Resource endowments are strictly positive and cannot be consumed. Therefore any unused resources have no value. It is assumed that all players are risk neutral. Let $I_j = \{1, 2, \ldots, n_j\}$ denote the set of n_j contests in which player A is competing with player $j \in \{x, y\}$. Winning contest $i \in I_j$ has value $v_{i,j}$ to both players A and j. Denote the total value of the set I_j of contests as: $V_j \equiv \sum_{i \in I_j} v_{i,j}$.

In the first stage, players x and y may choose to form an alliance. Rather than examine the transfers that arise under a particular transfer game, our focus is on sufficient conditions for a reallocation of resources among the alliance members to be Pareto improving for the alliance members. That is, a self-enforcing alliance without commitment is said to exist if there exist resource transfers, among the players with the common opponent, that increase the payoffs of both players in their respective second-stage set of contests against the common opponent. Clearly, there exist a number of possible transfer games that would implement such Pareto improving transfers (see Kovenock and Roberson 2010a for further details). Following a positive net transfer of ϵ from j to $-j$, the resulting budget constraints faced by j and $-j$ are $R_j^{'} \equiv R_j - \epsilon$ and $R_{-j}^{'} \equiv R_{-j} + \epsilon$, respectively.

In the second stage, player A observes the updated budget constraints faced by players x and y and then x and y individually compete with A in their respective sets of contests subject to the updated budget constraints determined in the first stage. The timing of the two-stage game is summarized as follows:

1. The two players facing the common opponent (players x and y) have the opportunity to reallocate resources among themselves.

[11] See also Robson (2005) who examines several extensions of Friedman (1958) including allowing for the noise parameter, r, to take any value in (0,1].

2. Player A observes players x's and y's updated resource constraints, and then players x and y individually compete against the common opponent, A, in their respective sets of contests.

3 Alliances Without Commitment

Given the nature of the game, a subgame perfect equilibrium may be derived using backward induction. We begin by solving for the Nash equilibrium in stage two, given that players x's and y's updated resource constraints are given by $R_x^{'}$ and $R_y^{'}$ respectively. Given the optimal strategies in stage two, we then investigate the stage one resource transfers.

Stage Two

The second stage of this game consists of two disjoint sets of two-player contests between A and $j \in \{x, y\}$. Player A allocates R_A across the two disjoint sets of contests I_x and I_y. Each player $j \in \{x, y\}$ allocates his entire budget across his set I_j of contests. Resource allocations are made simultaneously. In the set of contests, I_j, players A and j each maximize their expected payoff subject to the constraint that the sum of their individual resource allocation is no greater than their respective resource endowment. Because unused resources have no value, in any equilibrium this constraint is binding.

Friedman (1958) characterizes the optimal resource allocations in the two-player multi-item contest with the lottery CSF. Let $R_{A,j}$ denote the total amount of resources that player A allocates across the set of contests, I_j. Let $r_{A,i,j}$ denote player A's resource allocation to contest $i \in I_j$, and let $r_{i,j}$ denote the resource allocation by player $j \in \{x, y\}$ to contest $i \in I_j$. As the first stage results in players x and y facing budget constraints of $R_x^{'}$ and $R_y^{'}$ respectively, the pure-strategy Nash equilibrium resource allocations to contest $i \in I_j$ are given by: $r_{i,j}^{*} = (v_{i,j}/V_j)R_j^{'}$ and $r_{A,i,j}^{*} = (v_{i,j}/V_j)R_{A,j}$ for players j and A respectively. Intuitively, each player's allocation to battlefield i is equal to the value of that battlefield relative to the total value of all the battlefields in multi-item contest I_j multiplied by that player's total resource endowment. Let $u_{A,i,j}$ and $U_{A,j}$ denote player A's payoff in contest $i \in I_j$ and the sum of his payoffs across the set of contests I_j, respectively. Similarly, for $j \in \{x, y\}$ let $u_{i,j}$ and U_j denote player j's payoff in contest $i \in I_j$ and the sum of his payoffs across the set of contests I_j, respectively. In equilibrium, the total expected payoffs for players $j \in \{x, y\}$ and A are:

$$E[U_j^*] = \sum_{i \in I_j} E[u_{i,j}^*] = \sum_{i \in I_j} \frac{r_{i,j}^*}{r_{i,j}^* + r_{A,i,j}^*} v_{i,j} = \frac{R_j^{'}}{R_j^{'} + R_{A,j}^*} V_j \tag{1}$$

and

$$E[U_{A,j}^*] = \sum_{i \in I_j} E[u_{A,i,j}^*] = \sum_{i \in I_j} \frac{r_{A,i,j}^*}{r_{i,j}^* + r_{A,i,j}^*} v_{i,j} = \frac{R_{A,j}^*}{R_j^{'} + R_{A,j}^*} V_j \tag{2}$$

Observe that the fraction of the total value of the set I_j that a player expects to win is equal to the ratio of the total amount of resources that he allocates to the set divided by the sum of all resources allocated to the set. Because player A's optimal allocation of resources to each component contest $i \in I_j$ depends only on the relative value of that battlefield and player A's total allocation of resources to the set I_j of contests, $R_{A,j}$, characterizing player A's optimal division of resources to the two sets of contests I_x and I_y implicitly provides the optimal allocations of resources to each of the component contests. We now solve for player A's optimal division of resources to the sets I_x and I_y of contests.

Theorem 1. *If the combination of the updated resource constraints $(R_x^{'}, R_y^{'}, R_A)$ and the total values for the two sets of contests I_x and I_y (i.e. V_x and V_y respectively) satisfy the condition $((R_j^{'} R_{-j}^{'})^{1/2}/(R_A + R_{-j}^{'})) < (V_j/V_{-j})^{1/2} < ((R_A + R_j^{'})/(R_j^{'} R_{-j}^{'})^{1/2})$ for some $j \in \{x, y\}$, then to the set I_j of contests player A allocates*

$$R_{A,j}^* = \frac{\sqrt{\frac{V_j R_j^{'}}{V_{-j} R_{-j}^{'}}} \left(R_A + R_{-j}^{'}\right) - R_j^{'}}{1 + \sqrt{\frac{V_j R_j^{'}}{V_{-j} R_{-j}^{'}}}} \tag{3}$$

and to the set I_{-j} of contests player A allocates

$$R_{A,-j}^* = R_A - \frac{\sqrt{\frac{V_j R_j^{'}}{V_{-j} R_{-j}^{'}}} \left(R_A + R_{-j}^{'}\right) - R_j^{'}}{1 + \sqrt{\frac{V_j R_j^{'}}{V_{-j} R_{-j}^{'}}}}. \tag{4}$$

Conversely, if $(V_j/V_{-j})^{1/2} \leq ((R_j^{'} R_{-j}^{'})^{1/2}/(R_A + R_{-j}^{'}))$ then player A allocates no resources to the set I_j of contests and all of his resources, R_A, to the set I_{-j} of contests. In this case, it is clear that no self-enforcing alliance without commitment forms.

Proof. Given that the updated budget constraints faced by j and $-j$ are $R_j^{'}$ and $R_{-j}^{'}$, respectively, player A's optimization problem is given by:

$$\max_{R_{A,j},R_{A,-j}} E(U_{A,j}) + E(U_{A,-j}) \quad s.t. \quad R_{A,j} + R_{A,-j} \leq R_A$$

or equivalently

$$\max_{R_{A,j},R_{A,-j}} \frac{R_{A,j}}{R_j^{'} + R_{A,j}} V_j + \frac{R_{A,-j}}{R_{-j}^{'} + R_{A,-j}} V_{-j} \quad s.t. \quad R_{A,j} + R_{A,-j} \leq R_A$$

Because the sum $E(U_{A,j}) + E(U_{A,-j})$ is increasing in both $R_{A,j}$ and $R_{A,-j}$ player A optimally sets $R^*_{A,j} + R^*_{A,-j} = R_A$. Let $E(U_A) \equiv E(U_{A,j}) + E(U_{A,-j})$. It is worth noting that $E(U_A)$ is strictly concave in $(R_{A,j}, R_{A,-j})$. Thus, the first-order conditions are both necessary and sufficient for an interior local maximum. At an interior equilibrium, the first-order conditions imply that player A chooses $R_{A,j}$ and $R_{A,-j}$ such that

$$\frac{V_j R_j^{'}}{(R_j^{'} + R^*_{A,j})^2} = \frac{V_{-j} R_{-j}^{'}}{(R_{-j}^{'} + R^*_{A,-j})^2}$$

Solving the first-order conditions for $R^*_{A,j}$ yields:

$$R^*_{A,j} = \frac{\sqrt{\frac{V_j R_j^{'}}{V_{-j} R_{-j}^{'}}}(R_A + R_{-j}^{'}) - R_j^{'}}{1 + \sqrt{\frac{V_j R_j^{'}}{V_{-j} R_{-j}^{'}}}}$$

Finally, see that player A need not allocate a strictly positive level of resources to both I_j and I_{-j}. It may be readily confirmed that if $(V_j/V_{-j})^{1/2} \leq ((R_j^{'} R_{-j}^{'})^{1/2}/(R_A + R_{-j}^{'}))$ then the marginal return to player A for allocating his last unit of resources to the set I_{-j} of contests exceeds the marginal payoff from the first unit of resources allocated to set I_j. In this case, player A optimally allocates zero resources to the set I_j of contests and all of his resources, R_A, to the set I_{-j}. In such a case, player j's payoff is equal to V_j. Obviously, if the initial endowments warrant player A allocating no resources to set I_j, then because player j cannot strictly increase his payoff, by transferring resources to player $-j$, a self-enforcing alliance without commitment does not arise. □

Because self-enforcing alliances without commitment arise only in interior equilibria, we focus on the case where player A allocates a strictly positive level of resources to both sets of contests I_j and I_{-j}. Substituting equations (3) and (4) into equation (1) it follows that the equilibrium total expected payoff for player j is

$$E(U_j^*) = \frac{V_j R_j' + \sqrt{V_x R_x' V_y R_y'}}{R_x + R_y + R_A} \tag{5}$$

and for player $-j$ is

$$E(U_{-j}^*) = \frac{V_{-j} R_{-j}' + \sqrt{V_x R_x' V_y R_y'}}{R_x + R_y + R_A}. \tag{6}$$

Player A's total expected payoff is the sum of his payoffs from both sets of contests I_x and I_y. Substituting equations (3) and (4) into equation (2) yields player A's equilibrium total expected payoff:

$$E(U_A^*) = E(U_{A,x}^*) + E(U_{A,y}^*) = \frac{R_A(V_x + V_y) + \left(\sqrt{R_y' V_x} - \sqrt{R_x' V_y}\right)^2}{R_x + R_y + R_A}. \tag{7}$$

Examining equation (7), it is apparent that player A's equilibrium total expected payoff is increasing in the difference between $(R_y' V_x)^{1/2}$ and $(R_x' V_y)^{1/2}$. Conversely, because this is a constant-sum game, the sum of player x's and y's equilibrium total expected payoffs decreases as this difference increases.

Stage One

In stage one, players x and y have the opportunity to transfer resources among themselves. We focus on self-enforcing alliances without commitment in which only mutually beneficial exchanges of resources take place. If player j transfers a strictly positive amount, $\epsilon > 0$, of resources to player $-j$, then player A observes this alliance transfer and allocates his resources across the sets of contests I_j and I_{-j} taking into account player j's and $-j$'s updated budget constraints.

As outlined previously, a self-enforcing alliance without commitment forms only if, in the absence of an alliance transfer, player A would choose to allocate a strictly positive level of resources to each set of contests. Thus, we restrict our attention to interior equilibria. Without loss of generality, consider the case where player j transfers $\epsilon > 0$ to player $-j$. Note that equations (5) and (6) may now be written as:

$$E(U_j^\epsilon) = \frac{V_j(R_j - \epsilon) + \sqrt{V_j(R_j - \epsilon)V_{-j}(R_{-j} + \epsilon)}}{R_x + R_y + R_A} \tag{8}$$

$$E(U_{-j}^{\epsilon}) = \frac{V_{-j}(R_{-j}+\epsilon) + \sqrt{V_j(R_j-\epsilon)V_{-j}(R_{-j}+\epsilon)}}{R_x+R_y+R_A} \tag{9}$$

Theorem 2. *If the combination of the initial resource constraints* (R_x, R_y, R_A) *and the total values for the two sets of contests* I_x *and* I_y *(i.e.* V_x *and* V_y *respectively) satisfy the condition* $\left(\sqrt{R_j R_{-j}}/(R_A+R_{-j})\right) < \sqrt{(V_j/V_{-j})}$ *for some* $j \in \{x,y\}$*, then a self-enforcing alliance without commitment exists (in which player* j *transfers a positive level of resources to player* $-j$*) if and only if the condition* $\left((R_j - R_{-j})/\left(2\sqrt{R_j R_{-j}}\right)\right) > \sqrt{(V_j/V_{-j})}$ *holds.*

Proof. It may be readily verified that the conditions of Theorem 2 imply that, in the absence of any alliance transfers, player A allocates a strictly positive level of resources to each set of contests. Thus, any transfer from player j to player $-j$ yields the expected utilities, $E(U_j^{\epsilon})$ and $E(U_{-j}^{\epsilon})$, given equations (8) and (9) above. A Pareto improving transfer from player j to player $-j$ exists if and only if $\frac{\partial E(U_j^{\epsilon})}{\partial \epsilon}\Big|_{\epsilon=0} > 0$ and $\frac{\partial E(U_{-j}^{\epsilon})}{\partial \epsilon}\Big|_{\epsilon=0} > 0$. If both inequalities hold, then there exists a strictly positive transfer from player j to player $-j$ that strictly increases both players' expected payoffs. It is straightforward to show that $\frac{\partial E(U_j^{\epsilon})}{\partial \epsilon}\Big|_{\epsilon=0} > 0$ if and only if the initial conditions satisfy:

$$\frac{R_j - R_{-j}}{2\sqrt{R_j R_{-j}}} > \sqrt{\frac{V_j}{V_{-j}}} \tag{a}$$

Likewise, it can be easily verified that $\frac{\partial E(U_{-j}^{\epsilon})}{\partial \epsilon}\Big|_{\epsilon=0} > 0$ if and only if

$$\frac{R_j - R_{-j}}{2\sqrt{R_j R_{-j}}} > -\sqrt{\frac{V_{-j}}{V_j}} \tag{b}$$

Note that condition (a) implies condition (b). Thus, if player j is willing to transfer a portion of his endowment, player $-j$ is always willing to accept the gift. That is, for any transfer from player j to player $-j$, the direct benefit to player $-j$ from the additional resources alway overcomes the indirect effect from the strategic shift in player A's resources towards the set I_{-j} of contests. Moreover, note that because the right hand side of condition (a) is strictly positive, if a transfer of resources occurs, it must be the case that $R_j > R_{-j}$. Thus, transfers always flow from the resource wealthy player to the resource poor player. Therefore, if condition (a) holds, then no transfers from $-j$ to j will occur. □

As previously mentioned, the specific choice of the transfer depends on the specification of the transfer game. Condition (a) given above provides a necessary condition for a strictly positive transfer to arise in equilibrium. Surprisingly, the boundary for alliance formation established in condition (a) is equivalent to one of the boundaries established in Kovenock and Roberson (2010a) (heretofore referred to as KR).

KR examines the formation of self-enforcing alliances without commitment using two disjoint Colonel Blotto games (with auction CSFs in each component contest) in the second stage. Figure 1 shows, under the two CSFs, player A's expected payoffs in the set of contests against player j as a function of $R_{A,j}$. As illustrated in panel (b) of Figure 1, under the auction CSF player A's marginal payoff in each game is constant for $R_{A,j} \in [0, R_j^{'}]$. However, for $R_{A,j} > R_j^{'}$, player A experiences diminishing marginal returns. In contrast, as illustrated in panel (a), under the lottery CSF player A faces diminishing marginal returns for all values of $R_{A,j}$.

Under the auction CSF, KR demonstrate that if $(V_{-j}/V_j) \geq (R_{-j}^{'}/R_j^{'})$ and $1 - \sqrt{(V_{-j}R_j^{'}R_{-j}^{'}/V_j)} > R_j^{'}$ then player A optimally sets $R_{A,j}^{*} > R_j^{'}$ and $R_{A,-j}^{*} > R_{-j}^{'}$. In this case, player A equates his marginal returns across the two sets of contests I_x and I_y, and in each set of contests player A is in the region of diminishing marginal returns. For such parameter configurations, condition (a) outlined above provides a necessary and sufficient condition for mutually beneficial alliance transfers to arise. Note that for such parameter configurations, the alliance formation boundaries are identical under both the lottery and auction CSFs.

If $(V_{-j}/V_j) > (R_{-j}^{'}/R_j^{'})$ and $0 < 1 - \sqrt{(V_{-j}R_j^{'}R_{-j}^{'}/V_j)} \leq R_j^{'}$, then KR show that as player A equates his marginal returns across the two sets of contests I_x and I_y he enters the region of diminishing marginal returns in game G_{-j} (i.e. player A sets $R_{A,-j}^{*} > R_{-j}^{'}$) but will remain in the region of constant marginal returns in game G_j (i.e. player A sets $R_{A,j}^{*} < R_j^{'}$). In this case, mutually beneficial transfers from player j to player $-j$ exist if and only if

$$R_j + R_{-j} > 2\sqrt{\frac{V_j R_{-j}}{V_{-j} R_j}}. \tag{c}$$

For such parameter configurations, the alliance formation boundaries arising under the two CSFs diverge.

Figure 2 illustrates, for fixed values of R_A, V_j, and V_{-j}, the relationship between the alliance formation boundaries under the two CSFs. Equation (a) provides the boundary for alliance formation under the lottery CSF and is illustrated in Figure 2 by the bold (linear) line segment. Under the lottery CSF, alliances will form in the region above this line. This line is also the boundary for alliance formation under the auction CSF when $(V_{-j}/V_j) \geq (R_{-j}^{'}/R_j^{'})$

and $1-\sqrt{(V_{-j}R_j'R_{-j}'/V_j)} > R_j'$. In such cases, the boundary delineating the region in which self-enforcing alliances without commitment form is the same under both CSFs. The concave line segment dividing Region 2 from Region 3 provides the alliance formation boundary for the auction CSF model when $(V_{-j}/V_j) > (R_{-j}'/R_j')$ and $0 < 1-\sqrt{(V_{-j}R_j'R_{-j}'/V_j)} \leq R_j'$. That is, if the combination of the initial endowments (R_x, R_y) and the total values for the two sets of contests (V_x, V_y) lies in Region 3, then a self-enforcing alliance without commitment forms under the auction CSF but not under the lottery CSF.

Numerical Example

Suppose the initial endowments (R_x, R_y, R_A) are given by $(32, 2, 20)$ and the valuations, (V_x, V_y), are given by $(3, 12)$. In the absence of alliance transfers, player A's optimal strategy is to allocate 4 units of resources to the set of contests in which he competes against player x (I_x) and 16 units of resources to the set in which he competes against player y (I_y). Players x's and y's total expected payoffs are $(8/3)$ and $(4/3)$, respectively. Notice that if $j = x$ and $-j = y$, then in this case condition (a) is satisfied. Thus, self-enforcing alliances without commitment forms and the alliance transfers flow from player x to player y. Such alliance transfers are individually rational for both players x to y up until the alliance formation boundary where $(R_x' - R_y')/(2\sqrt{R_x'R_y'}) = (1/2)$. For example, consider an alliance transfer of one unit of resources from player x to player y. The updated resource constraints are $R_x' = 31$ and $R_y' = 3$. Given these updated resource constraints, player A optimally divides his resources among the two sets of contests according to $R_{Ax}^* \approx 1.934$ and $R_{Ay}^* \approx 18.067$. In this case, the new total expected payoffs are $E(U_x) \approx 2.834$ and $E(U_y) \approx 1.709$. Notice that such a transfer between the alliance members strictly increases both players' total expected payoffs.

4 Alliances With Commitment

We now contrast the above characterization of self-enforcing alliances without commitment, with that for alliances with full commitment (i.e. we now allow for ex post transfers among alliance members). That is, players x and y pool their resources in the first stage and then as a single entity, denoted xy, compete against player A in the second stage. Let $R_{xy} \equiv R_x + R_y$ and $V_{xy} \equiv V_x + V_y$. Finally, let $I_{xy} \equiv I_x \cup I_y$. Observe that this game is a two-player multi-item Tullock contest between xy and A on the set I_{xy}. In examining such an alliance, an issue that arises is how the winnings are distributed among the alliance members. Moreover, as frequently examined in the contest-theoretic literature on alliances, there is also an issue regarding credible commitment to the distribution scheme. We digress from these issues, and instead focus on the conditions under which the payoff to the alliance exceeds the sum of the payoffs to players x and y when they act as singletons. Let $E(U_x^a)$ and $E(U_y^a)$ be the equilibrium payoffs to players x and y, respectively, when they form an alliance with complete commitment and let $E(U_{xy}) \equiv E(U_x^a) + E(U_y^a)$. If $E(U_{xy}) > E(U_x^*) + E(U_y^*)$, then there exists a sharing rule in which $E(U_x^a) > E(U_x^*)$ and $E(U_y^a) > E(U_y^*)$. Thus, an alliance with complete commitment forms if and only if $E(U_{xy}) > E(U_x) + E(U_y)$.

Theorem 3. *In an alliance with complete commitment the alliance sets $(R_x^{'}/R_y^{'}) = (V_x/V_y)$. Thus, transfers take place if and only if the initial endowments satisfy $(R_x^{'}/R_y^{'}) \neq (V_x/V_y)$. Specifically, player j transfers a positive level of resources to player $-j$ if and only if $(R_j^{'}/R_{-j}^{'}) > (V_j/V_{-j})$.*

Proof. In an alliance with complete commitment, the alliance maximizes their joint payoff subject to the alliance budget constraint: $R_x^{'} + R_y^{'} \leq R_{xy}$. Thus, the second stage amounts to a two player, multi-item Tullock contest between the alliance, xy , and player A on the set of contests, I_{xy}. Let $r_{xy,i}$ and $r_{A,i}$ denote the allocations to contest $i \in I_{xy}$ by the alliance xy and player A, respectively. Applying Friedman's (1958) characterization of equilibrium, the optimal resource allocation to each component contest $i \in I_{xy}$ is given by: $r_{xy,i}^* = (v_i/V_{xy})R_{xy}$ and $r_{A,i}^* = (v_i/V_{xy})R_A$. Thus, for each $j \in \{x, y\}$ the alliance, allocates $R_j^{'} = \sum_{i \in I_j}(v_i/V_{xy})R_{xy} = (V_j/V_{xy})R_{xy}$ to the set I_j of contests. It follows directly that the alliance sets $(R_x^{'}/R_y^{'}) = (V_x/V_y)$. As a result, alliance transfers between players x and y arise if and only if $(R_x^{'}/R_y^{'}) \neq (V_x/V_y)$. Specifically, player j transfers a strictly positive level of resources to player $-j$ when $(R_j^{'}/R_{-j}^{'}) > (V_j/V_{-j})$. □

Remarkably, KR show that under the auction CSF the equilibrium for the alliance with full commitment involves setting $(R_x^{'}/R_y^{'}) = (V_x/V_y)$. That is, the boundary for resource transfers in the alliance with full commitment is equivalent under the two CSFs. This boundary is illustrated in Figure 2 as the line $R_j^{'} = (V_j/V_{-j})R_{-j}^{'}$. If the the combination of the initial endowments (R_x, R_y) and the total values for the two sets of contests (V_x, V_y) lies on either side of this line, then an alliance with complete commitment forms under both CSFs. When this occurs the alliance transfers shift the resource constraints of players x and y toward this alliance boundary. It is also worth noting the relationship between the formation of self-enforcing alliances without commitment and alliances with full commitment. In both types of alliances, the alliance transfers shift the alliance members resource constraints towards the line $R_j' = (V_j/V_{-j})R_{-j}'$. But, in the case of the self-enforcing alliance without commitment, incentive compatibility prevents the alliance from reaching such allocations.

5 Conclusion

In examining the robustness of 'enemy-of-my-enemy-is-my-friend' alliances, we find that, for a non-empty subset of initial configurations, the boundary for alliance formation under the lottery CSF corresponds exactly with the alliance boundary arising under the auction CSF. Moreover, if the players can form an alliance with complete commitment, then the outcomes in the two models are equivalent. The boundary for self-enforcing alliances without commitment under the Lottery CSF diverges from the boundary under the auction CSF once the player's resource endowments cross a threshold. Over this range, the alliance region under the auction CSF is strictly larger, and as a consequence, there exist parameter configurations for which alliances form under the auction CSF but not under the lottery CSF. Given that in a host of single and multiple contest applications the lottery CSF and the auction CSF yield qualitatively different equilibrium predictions, we find it quite surprising that alliances formed by players who share no common interest other than a common opponent are largely robust to the specification of the contest success function.

References

[1] Alcalde, J. and M. Dahm (2009) Rent seeking and rent dissipation: a neutrality result, *Journal of Public Economics*, **94**, 1-7.

[2] Arce M., D.G., and T. Sandler (2001) A cooperative game theory of noncontiguous allies, *Journal of Public Economic Theory*, **3**, 391-411.

[3] Baye, M.R., D. Kovenock, and C.G. de Vries (1993) Rigging the lobbying process: an application of the all-pay auction, *American Economic Review*, **83**, 289-294.

[4] Baye, M.R., D. Kovenock, and C.G. de Vries (1994) The solution to the Tullock rent-seeking game when $R > 2$: mixed strategy equilibria and mean dissipation rates, *Public Choice*, **81**, 363-380.

[5] Baye, M.R., D. Kovenock, and C.G. de Vries (1996) The all-pay auction with complete information, *Economic Theory*, **8**, 291-305.

[6] Bloch, F. (2009) Endogenous formation of alliances in conflicts, in Garfinkel, M., & Skaperdas, S., eds. *Handbook of the Economics of Peace and Conflict*. forthcoming. Oxford: Oxford University Press.

[7] Bossert, W., S. Brams, and M. Kilgour (2002) Cooperative vs. non-cooperative truels: little agreement, but does that matter? *Games and Economic Behavior*, **40**, 185-202.

[8] Che, Y.K. and I. Gale (1998) Caps on political lobbying, *American Economic Review*, **88**, 643-651.

[9] Clark, D. and K. Konrad (2007) Asymmetric conflict: weakest link against best shot, *Journal of Conflict Resolution*, **51**, 457-469.

[10] Dimico, A. and D. Seidmann (2010) Patterns of conflict and preventive attacks, Working paper.

[11] Fang, H. (2002) Lottery versus all-pay auction models of lobbying, *Public Choice*, **112**, 351-371.

[12] Friedman, L. (1958) Game-theory models in the allocation of advertising expenditures, *Operations Research*, **6**, 699-709.

[13] Hart, S. (2008) Discrete Colonel Blotto and general lotto games, *International Journal of Game Theory*, **36**, 441-460.

[14] Hillman, A. and J. Riley (1989) Politically contestable rents and transfers, *Economics & Politics*, **1**, 17-39.

[15] Ihori, T., and M.C. McGuire (2007) Collective risk control and group security: the unexpected consequences of differential risk aversion, *Journal of Public Economic Theory*, **9**, 231-263.

[16] Kilgour, D. and S. Brams (1997) The truel, *Mathematics Magazine*, **70**, 315-326.

[17] Klumpp, T. and M. Polborn (2006) Primaries and the New Hampshire effect, *Journal of Public Economics*, **90**, 1073-1114.

[18] Konrad, K.A. (2009) *Strategy and Dynamics in Contests*. New York: Oxford University Press.

[19] Konrad, K.A. and D. Kovenock (2009) Multi-Battle Contests, *Games and Economic Behavior*, **66**, 256-274.

[20] Kovenock, D. and B. Roberson (2010a) Coalitional Colonel Blotto games with application to the economics of alliances, *Journal of Public Economic Theory*, forthcoming.

[21] Kovenock, D. and B. Roberson (2010b) Conflicts with multiple battlefields, in Garfinkel, M., & Skaperdas, S., eds. *Handbook of the Economics of Peace and Conflict.* forthcoming. Oxford: Oxford University Press.

[22] Kovenock, D. and B. Roberson (2010c) The optimal defense of networks of targets, Purdue University working paper no. 1251.

[23] Laslier, J.F.(2003) Party objectives in the 'divide-a-dollar' electoral competition. in Austen-Smith, D. & Duggan, J., eds. *Social Choice and Strategic Decisions: Essays in Honor of Jeffrey S. Banks.* Berlin: Springer.

[24] Murdoch, J.C., and T. Sandler (1982) A theoretical and empirical analysis of NATO, *Journal of Conflict Resolution*, **26**, 237-263.

[25] Murdoch, J.C., and T. Sandler (1984) Complementarity, free riding, and the military expenditures of NATO allies, *Journal of Public Economics*, **25**, 83-101.

[26] Olson, M. (1965) *The Logic of Collective Action.* Cambridge, MA: Harvard University Press.

[27] Olson, M. and R. Zeckhauser (1966) An economic theory of alliances, *Review of Economic Statistics*, **48**, 266-279.

[28] Roberson, B. (2006) The Colonel Blotto game, *Economic Theory*, **29**, 1-24.

[29] Robson, A.R.W. (2005) Multi-item contests, Australian National University, working papers in economics and econometrics no. 446.

[30] Sandler, T. (1977) Impurity of defense: an application to the economics of alliances, *Kyklos*, **30**, 443-460.

[31] Sandler, T. (1999) Alliance formation, alliance expansion, and the core, *Journal of Conflict Resolution*, **43**, 727-747.

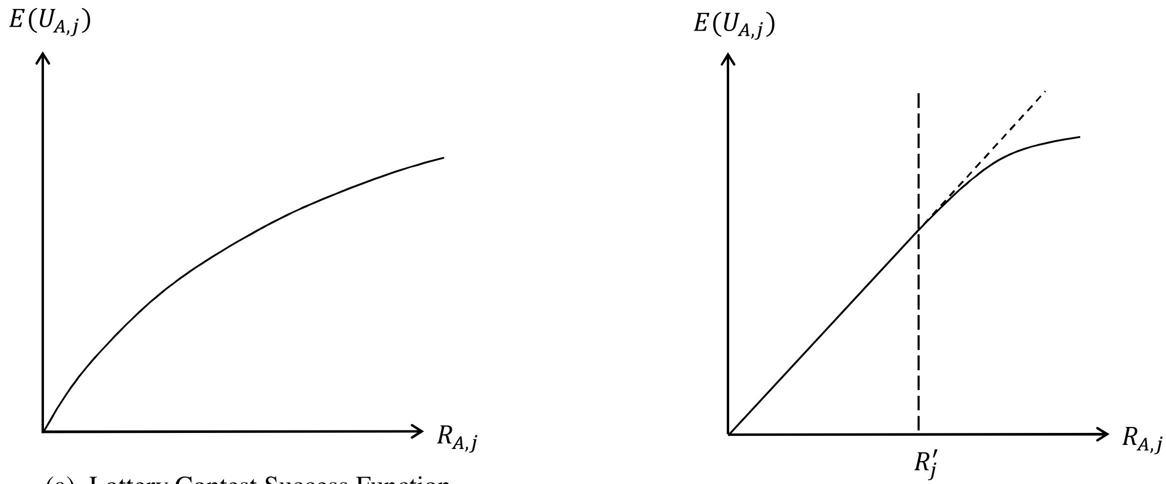
[32] Sandler, T., and J. Cauley (1975) On the economic theory of alliances, *Journal of Conflict Resolution*, **19**, 330-348.

[33] Shubik, M. (1954) Does the fittest necessarily survive?, from Shubik, M., ed. *Readings in Game Theory and Political Behavior*. Garden City, NY: Doubleday.

[34] Skaperdas, S. (1998) On the formation of alliances in conflict and contests, *Public Choice*, **96**, 25-42.

[35] Snyder, J. (1989) Election goals and the allocation of campaign resources, *Econometrica*, **57**, 637-660.

[36] Szentes, B. and R.W. Rosenthal (2003) Beyond chopsticks: symmetric equilibria in majority auction games, *Games and Economic Behavior*, **45**, 278-295.



(a) Lottery Contest Success Function

(b) All-Pay Auction Contest Success Function

Figure 1: Player A's expected payoff in the set I_j of contests under the lottery and auction CSFs

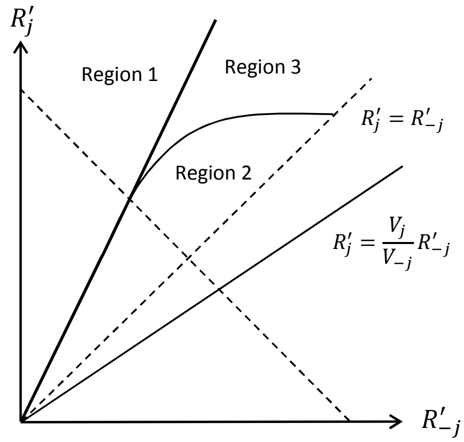


Figure 2: Boundaries of alliance formation for fixed values of V_j, V_{-j}, and R_A. (For the case that $\frac{V_j}{V_{-j}} < 1$).