

KRANNERT SCHOOL OF MANAGEMENT

Purdue University
West Lafayette, Indiana

Incomplete Financial Markets With Real Assets And
Endogenous Credit Limits

By

Matthew Hoelle
Marina Pireddu
Antonio Villanacci

Paper No. 1271
Date: June, 2012

Institute for Research in the
Behavioral, Economic, and
Management Sciences

Incomplete financial markets with real assets and endogenous credit limits

Matthew Hoelle

Department of Economics,
Purdue University
e-mail: mhoelle@purdue.edu

Marina Pireddu

Dipartimento di Matematica e Applicazioni,
Università degli Studi di Milano-Bicocca
e-mail: marina.pireddu@unimib.it

Antonio Villanacci

Dipartimento di Matematica per le Decisioni,
Università degli Studi di Firenze
e-mail: antonio.villanacci@unifi.it

June 28, 2012

Abstract

In this paper we analyze the effects of restricted participation in a two-period general equilibrium model with uncertainty in the second period and real assets. Similar to certain arrangements in the market for bank loans, household borrowing is restricted by a household-specific wealth dependent upper bound on credit lines in all states of uncertainty in the second period.

We first establish that, generically in the set of the economies, equilibria exist and are finite and regular. We then show that equilibria are generically suboptimal. Finally, we provide a robust example demonstrating that the equilibrium allocations can be Pareto improved through a tightening of the participation constraints.

Keywords: general equilibrium; restricted participation; financial markets; generic regularity; real assets; Pareto suboptimality

JEL classification: D50, D53, D61

1 Introduction

In this paper we analyze a two-period general equilibrium model with incomplete financial markets and two key elements: (i) the competitive trading of real assets, where real assets have payouts in terms of vectors of commodities, and (ii) household-specific inequality constraints that restrict participation in the financial markets.

While there exists a thorough literature on each of these key elements in isolation (see a summary of the theoretical properties in Villanacci et al. [2002]), the literature has very little to say when both elements of reality are considered together. More precisely, some contributions which deal only with the existence problems are indeed available. Apart from the seminal paper by Radner [1972], Seghir and Torres-Martinez [2011] and Gori et al. [2011] do provide existence proofs in the case of some financial constraints depending on some endogenous variables. The main limitation of those results is that their analytical approaches seem not applicable to show a basic, crucial result for any further analysis on properties of equilibria in a general equilibrium context: generic regularity.

In fact, to the best of our knowledge, the only model of restricted participation with real assets in which generic regularity is proven is from Polemarchakis and Siconolfi [1997]. Yet, in that paper, the restriction sets for the asset choices are difficult to interpret. Specifically, each household is exogenously associated with a linear subspace of the possible wealth transfers. Its restriction set is then described by the orthogonal projection of that subspace on the (price dependent) image of the return matrix.

What we offer is an analysis of economically meaningful constraints on households' participation in financial markets and a general approach to determine which constraints can be analyzed in a real asset setting.

The participation constraints described in our model, consistent with what occurs in the market for bank loans, impose an upper bound on a household's future debt in all states of uncertainty in the final period. The upper bound on debt is household-specific and is determined so that a household's wealth after the repayment of the loan is never lower than a base level of consumption expenditure. In fact, banks can get information on a household's future income and can then determine the size of a loan that the household can be reasonably expected to repay. We show generic existence of equilibria and generic regularity, the latter being an indispensable tool to both describe equilibria and to prove several important normative properties of equilibria.

The first property we show is that the equilibria are generically suboptimal. Additionally, we conjecture that a form of generic constrained suboptimality holds. We are verifying this conjecture in a companion paper, in which we restrict attention to the significant set of economies in which a sufficiently high number of participation constraints are binding. Generically in that set of economies, these equilibria are Pareto improvable through a local change of the participation constraints. The general strategy that is used in that framework is described in Carosi et al. [2009].

In the current paper, we describe a more direct approach to constrained suboptimality for a specific economy. In particular, we present a robust example in the economy space whose associated equilibria are such that using a more restrictive credit policy results in a Pareto improvement. In other words, reducing the freedom of households to trade financial assets can make all households better off.

We believe our analysis contains both technically and economically significant features. We first discuss our technical contributions to the analysis of the problem of generic existence and generic regularity of equilibria.

The seminal contribution for generic existence in a model with real assets is the paper by Duffie and Shafer [1985]. To prove generic existence, they first set the dimension of the return space equal to the number of available assets, define the resulting equilibrium a "pseudo equi-

librium”, and show the existence of such an equilibrium. Finally, they prove that these pseudo equilibria are true equilibria for a generic subset of household endowments and asset payouts. With the inequality constraints that we use to model households’ restricted participation, the household demand functions are in general not C^1 , and therefore the equilibrium manifold is not C^1 either. This fact prevents us from using the smooth analysis arguments from Duffie and Shafer [1985]. Rather, we employ a fixed point argument based on the approach of Dierker [1974] for the Walrasian model and later generalized and formalized by Husseini et al. [1990] for the incomplete markets model with real assets. To the best of our knowledge, we provide the first application of the methodology from Husseini et al. [1990].

Duffie and Shafer [1985] use a “fixed dimension return space” approach¹ with results in terms of the *kernel* of a well chosen linear function. In our model, characterizing equilibria using this fixed dimension return space approach would allow us to verify the existence of appropriately defined pseudo equilibria, but we are then unable to show that the pseudo equilibria are generically true equilibria.² To circumvent this problem, we take a different approach by presenting a natural characterization of fixed dimension return space equilibria in terms of the *image* of an appropriately chosen linear function.

For those interested in the technical aspects of our proof, we preview the strategy used to obtain the generic existence result. As previously discussed, once the definition of equilibrium is introduced, we define a fixed dimension return space type of equilibrium (see the image - symmetric equilibrium introduced in Definition 2 and its price normalized version in Definition 4). Then, as done in the approach followed by Duffie and Shafer [1985], we use a Mr. 1 trick, i.e., we get rid of the explicit presence of the financial side of the economy using the introduction of a specific household, Mr. 1, who behaves as a Walrasian consumer (see the image - Mr. 1 equilibrium described in Definition 5). After showing that all three concepts are equivalent, we prove that the one in Definition 4 is a “true” equilibrium if a (standard) full rank condition of the return matrix holds true - see Proposition 7.

The technical reason to introduce two different types of auxiliary equilibria is described by the following logic. Using the Dierker [1974] approach in the form of the theorem proved by Husseini et al. [1990], we are able to show the existence of an image - Mr. 1 equilibrium. Yet, as mentioned above, we are then unable to complete the next step as we are unable to verify that the projection function from the equilibrium set to the economy space is proper. This is a required step for the genericity argument. We can verify properness by using the equivalent concept of (normalized price) image-symmetric equilibrium in Definition 4.

We now discuss the economically significant features of both our model and our proof methodology. In terms of the type of participation constraints we employ, we believe they are realistic and economically meaningful. They are meant to represent the market for bank loans. Consider that legal requirements (or uncontractable social norms) are present that guarantee households a base level of consumption expenditure. Thus, for all states of uncertainty, a household is only able to repay previous debts if this leaves the household with at least this base level of consumption expenditure. Knowing this, the financial markets only permit borrowing up the point where the household is able to repay the loan and not be reduced to consumption expenditure

¹This terminology is borrowed from Bich and Cornet [1998].

²Specifically, it is not clear how to show properness of the projection from the equilibrium set to the economy space, since it is not possible to uncouple two multipliers and prove that they converge separately.

below the base level.

To show the economic relevance of the constraints, we consider a simple example of our model. For the particular economy chosen, the equilibrium allocation can be Pareto improved by tightening the participation constraints in some states, without loosening the participation constraints in any other states, for all households. This result may seem counter-intuitive, but demonstrates the importance of general equilibrium price effects in financial markets. Thus, restricting credit access may in fact make all households better off.

More generally, the analysis presented in the paper provides what we believe are crucial conditions on the *type* of constraints for which generic regularity can be verified, at least following what currently seems to be the only successful approach: the fixed dimension return space approach. As will be discussed further in Section 2,³ both the kernel approach (that is used by Duffie and Shafer [1985]) and the image approach (that we use here) require that the constraints on the financial side of the economy are rewritten in terms of constraints on the real side, specifically in terms of the values of the excess demands in each state. In the former approach, the financial side simply disappears from the household maximization problems. In the latter one, we must introduce fictitious asset demands and we recognize that constraints imposed on the fictitious asset demands may not be equivalent to constraints imposed on the true asset demands.

Future research is required to confirm this conjecture about the type of constraints that can be employed in models with real assets. Given this negative result, any attempts to obtain regularity for interesting models of collateral and default may be in vain. The reason is that any known approaches to modeling collateral and default involve restrictions that differ from the types of restrictions that we described above as being successful. Again, future research is required in this direction.

The rest of the paper is organized as follows. In Section 2, we present the set up of the model. In Section 3, we introduce some equivalent definitions of fixed dimension return space equilibria. In Section 4, we state the results of existence of these equilibria, together with the generic existence, generic regularity and generic suboptimality of true equilibria. Section 5 contains the numerical example and the Appendix collects all of the proofs.

2 Set up of the model

Our model builds on the standard two-period, general equilibrium model of pure exchange with uncertainty. In the commodity markets, $C \geq 2$ different physical commodities are traded, denoted by $c \in \mathcal{C} = \{1, 2, \dots, C\}$. In the final period, only one among $S \geq 1$ possible states of the world, denoted by $s \in \{1, 2, \dots, S\}$, will occur. The initial period is denoted $s = 0$ and we define the set of all states $\mathcal{S} = \{0, 1, \dots, S\}$ and the set of uncertain states $\mathcal{S}' = \{1, \dots, S\}$. In the initial period, asset markets open and $A \geq 1$ assets are traded, denoted by $a \in \mathcal{A} = \{1, 2, \dots, A\}$.

We assume $A \leq S$. Finally, there are $H \geq 2$ households, denoted by $h \in \mathcal{H} = \{1, 2, \dots, H\}$. The time structure of the model is as follows: in the initial period, households exchange commodities and assets, and consumption takes place. In the final period, uncertainty is resolved, households honor their financial obligations, exchange commodities, and then consume com-

³See especially the part immediately after condition (12).

modities.

We denote $x_h^c(s) \in \mathbb{R}_{++}$ as the consumption of commodity c in state s by household h and $e_h^c(s) \in \mathbb{R}_{++}$ as the endowment of commodity c in state s owned by household h .⁴ We define

$$\begin{aligned} x_h(s) &= (x_h^c(s))_{c \in C} \in \mathbb{R}_{++}^C, & x_h &= (x_h(s))_{s \in S} \in \mathbb{R}_{++}^G, & x &= (x_h)_{h \in \mathcal{H}} \in \mathbb{R}_{++}^{GH}, \\ e_h(s) &= (e_h^c(s))_{c \in C} \in \mathbb{R}_{++}^C, & e_h &= (e_h(s))_{s \in S} \in \mathbb{R}_{++}^G, & e &= (e_h)_{h \in \mathcal{H}} \in \mathbb{R}_{++}^{GH}, \end{aligned}$$

where $G = C(S+1)$. Household h 's preferences are represented by a utility function $u_h : \mathbb{R}_{++}^G \rightarrow \mathbb{R}$. As in most of the literature on smooth economies we assume that, for every $h \in \mathcal{H}$,

$$u_h \in C^2(\mathbb{R}_{++}^G); \tag{1}$$

$$\text{for every } x_h \in \mathbb{R}_{++}^G, Du_h(x_h) \gg 0; \tag{2}$$

$$\text{for every } v \in \mathbb{R}^G \setminus \{0\} \text{ and } x_h \in \mathbb{R}_{++}^G, v D^2 u_h(x_h) v < 0; \tag{3}$$

$$\text{for every } \underline{x}_h \in \mathbb{R}_{++}^G, \{x_h \in \mathbb{R}_{++}^G : u_h(x_h) \geq u_h(\underline{x}_h)\} \text{ is closed in the Euclidean topology of } \mathbb{R}^G. \tag{4}$$

Let us denote by \mathcal{U} the set of vectors $u = (u_h)_{h \in \mathcal{H}}$ of utility functions satisfying assumptions (1), (2), (3), and (4). We denote by $p^c(s) \in \mathbb{R}_{++}$ the price of commodity c in state s , by $q^a \in \mathbb{R}$ the price of asset a and by $b_h^a \in \mathbb{R}$ the quantity of asset a held by household h . Moreover we define

$$\begin{aligned} p(s) &= (p^c(s))_{c \in C} \in \mathbb{R}_{++}^C, & p &= (p(s))_{s \in S} \in \mathbb{R}_{++}^G, & q &= (q^a)_{a \in A} \in \mathbb{R}^A, \\ b_h &= (b_h^a)_{a \in A} \in \mathbb{R}^A, & b &= (b_h)_{h \in \mathcal{H}} \in \mathbb{R}^{AH}. \end{aligned}$$

We denote by $y^{a,c}(s) \in \mathbb{R}$ the units of commodity c delivered by one unit of asset a in state s and we define

$$y^a(s) = (y^{a,c}(s))_{c \in C} \in \mathbb{R}^C, \quad y(s) = (y^a(s))_{a \in A} \in \mathbb{R}^{CA}, \quad y = (y(s))_{s \in S'} \in \mathbb{R}^{CAS}.^5$$

Note in particular that, in state s , asset a promises to deliver a vector $y^a(s)$ of commodities.

For any $m, n \in \mathbb{N} \setminus \{0\}$, let $\mathbb{M}(m, n)$ be the set of real $m \times n$ matrices and $\mathbb{M}^f(m, n)$ be the set of real $m \times n$ matrices with full rank (equal to $\min\{m, n\}$). Define the return matrix function as follows

$$\begin{aligned} \mathcal{R} : \mathbb{R}_{++}^G \times \mathbb{R}^{CAS} &\rightarrow \mathbb{M}(S, A), \\ (p, y) &\mapsto \begin{bmatrix} p(1)y^1(1) & \dots & p(1)y^a(1) & \dots & p(1)y^A(1) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ p(s)y^1(s) & \dots & p(s)y^a(s) & \dots & p(s)y^A(s) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ p(S)y^1(S) & \dots & p(S)y^a(S) & \dots & p(S)y^A(S) \end{bmatrix} = \begin{bmatrix} r^1(p, y) \\ \vdots \\ r^s(p, y) \\ \vdots \\ r^S(p, y) \end{bmatrix}. \end{aligned}$$

⁴Given $v = (v_1, \dots, v_N), w = (w_1, \dots, w_N) \in \mathbb{R}^N$, we write $v \gg w$ if $v_i > w_i, \forall i \in \{1, \dots, N\}$, $v \geq w$ if $v_i \geq w_i, \forall i \in \{1, \dots, N\}$. and $v > w$ if $v \geq w$ and $v \neq w$. We further define the sets $\mathbb{R}_+^N = \{v \in \mathbb{R}^N : v \geq 0\}$ and $\mathbb{R}_{++}^N = \{v \in \mathbb{R}^N : v \gg 0\}$.

⁵We consider possibly negative yields. Notice however that all the results we obtain are still valid in the case of positive yields.

For future use we also define, for every $p \in \mathbb{R}_{++}^G$,

$$\Phi(p) = \begin{bmatrix} p(0) & & & \\ & p(1) & & \\ & & \ddots & \\ & & & p(S) \end{bmatrix} \in \mathbb{M}(S+1, G).$$

A credit limit is the maximum amount that a household is allowed to borrow. A number of aspects are taken into account when evaluating the credit worthiness of a borrower, but there are two basic components. One is the borrower's current and projected ability to repay a loan. This can be determined by looking at things like income, other debts that the borrower is carrying, expenses, and future employment opportunities. Second is the borrower's inclination to repay debts, which is inferred from the borrower's credit history, i.e., past repayment patterns.

The above considerations are formalized into our restricted participation framework and the constraints that we impose limit the amount of future debt. In fact, we say that the amount household h can borrow, i.e., $q(-b_h)$, must be such that the household can repay what is due in all states in the final period. That amount due is

$$r^s(p, y)(-b_h), \quad \forall s \in \mathcal{S}'. \quad (5)$$

Additionally, we assume that each household will not be able (due to legal restrictions) or will not be willing to consume less than a given proportion $\gamma_h(s)$ of its wealth in state $s \in \mathcal{S}'$. In other words, there is a base level of consumption expenditure required for all households:

$$\gamma_h(s)p(s)e_h(s).$$

Then we require that the amount due in (5) has to be smaller than the difference between a household's endowment level and the base level of consumption expenditure:

$$(1 - \gamma_h(s))p(s)e_h(s).$$

Therefore, defining $\alpha_h(s) = 1 - \gamma_h(s)$, the borrowing constraints we impose are:

$$\forall h \in \mathcal{H}, \quad \forall s \in \mathcal{S}', \quad -r^s(p, y)b_h \leq \alpha_h(s)p(s)e_h(s), \quad (6)$$

where $\alpha_h(s) \in (0, 1)$, $\forall h \in \mathcal{H}$ and $\forall s \in \mathcal{S}'$.

Including the participation constraint parameters along with the parameters governing the asset structure and the household endowments and preferences, we define the set of economies as

$$\mathcal{E} = \mathbb{R}_{++}^{GH} \times \mathcal{U} \times \mathbb{R}^{CAS} \times (0, 1)^{SH},$$

with generic element (e, u, y, α) , where $\alpha = (\alpha_h)_{h \in \mathcal{H}} = (\alpha_h(s))_{h \in \mathcal{H}, s \in \mathcal{S}'}$.

Definition 1. A vector $(x^*, p^*, b^*, q^*) \in \mathbb{R}_{++}^{GH} \times \mathbb{R}_{++}^G \times \mathbb{R}^{AH} \times \mathbb{R}^A$ is an *equilibrium* for the economy $(e, u, y, \alpha) \in \mathcal{E}$ if

1. $\forall h \in \mathcal{H}$, (x_h^*, b_h^*) solves the following problem:

given $(p^*, q^*, e, u, y, \alpha)$

$$\max_{(x_h, b_h) \in \mathbb{R}_{++}^G \times \mathbb{R}^A} u_h(x_h)$$

$$s.t. \quad p^*(0)(x_h(0) - e_h(0)) + q^*b_h \leq 0 \quad (7.1)$$

(7)

$$p^*(s)(x_h(s) - e_h(s)) - (p^*(s)y^a(s))_{a=1}^A b_h \leq 0, \quad \forall s \in \mathcal{S}' \quad (7.2)$$

$$- (p^*(s)y^a(s))_{a=1}^A b_h \leq \alpha_h(s)p^*(s)e_h(s), \quad \forall s \in \mathcal{S}' \quad (7.3)$$

2. (x^*, b^*) satisfies the market clearing conditions

$$\sum_{h=1}^H (x_h^* - e_h) = 0 \quad (8)$$

and

$$\sum_{h=1}^H b_h^* = 0. \quad (9)$$

Let's further consider the restrictions we chose to analyze and the technical reasons why these constraints can be analyzed using our proof methodology. From Duffie and Shafer [1985], the standard way to tackle the problem of generic existence of equilibria in the real asset model is to fix the dimension of the return space so that the return matrix does not suffer a drop in rank. We briefly describe this process. Define

$$\Phi^1(p) = \begin{bmatrix} p(1) & & \\ & \ddots & \\ & & p(S) \end{bmatrix} \quad \text{and} \quad z_h^1 = x_h^1 - e_h^1,$$

with

$$x_h^1 = (x_h(s))_{s \in \mathcal{S}'} \in \mathbb{R}_{++}^{CS}, \quad e_h^1 = (e_h(s))_{s \in \mathcal{S}'} \in \mathbb{R}_{++}^{CS}.$$

In this fixed dimension return space approach, the budget constraints in the final period (7.2) can be equivalently expressed as:

$$\Phi^1(p) z_h^1 \in L, \quad (10)$$

where L is an A dimensional subspace of \mathbb{R}^S . Condition (10) is equivalent to any of the following conditions:

1.

$$\exists M(L) \in \mathbb{M}^f(S-A, S) \text{ such that } M(L) \cdot \Phi^1(p) z_h^1 = 0, \quad (11)$$

where $M(L)$ is such that $\ker M(L) = L$;

2.

$$\exists N(L) \in \mathbb{M}^f(S, A) \text{ and } \exists b_h \in \mathbb{R}^A \text{ such that } \Phi^1(p) z_h^1 = N(L) \cdot b_h, \quad (12)$$

where $N(L)$ is such that $\text{Im } N(L) = L$.

Duffie and Shafer [1985] use condition (11). Here, we use condition (12) as well. With either condition, it is not clear how to impose constraints directly on b_h . In the first condition, b_h does not appear. In regard to the second condition, we show that for a “fictitious” (regular) equilibrium, up to permutations of states, there exists $E \in \mathbb{M}(S - A, A)$ such that

$$\begin{bmatrix} I \\ E \end{bmatrix} b_h = \mathcal{R}(p, y) b'_h = \begin{bmatrix} R^*(p, y) \\ \widehat{R}(p, y) \end{bmatrix} b'_h,$$

where $R^*(p, y)$ has full rank, b_h is the asset demand in a fictitious equilibrium, and b'_h is the asset demand in the true equilibrium. Therefore, again up to permutations, we get that

$$b'_h = [R^*(p, y)]^{-1} b_h.^6$$

The above condition indicates that imposing restrictions on the fictitious equilibrium asset demand b_h does not imply that the same restrictions will hold for the true asset demand b'_h . For example, the restriction $b_h \geq 0$ does not imply that $b'_h = [R^*(p, y)]^{-1} b_h \geq 0$. That explains why the fixed dimension return space approach is likely not applicable for restrictions written directly in terms of b_h .

On the other hand, constraints on the physical quantity b_h^a have little meaning as the future yields depend upon future commodity prices. So constraints could be written for each asset on the value $q^a b_h^a$, but this presupposes that either lenders are not able to gain information about the other assets in a household’s portfolio or do not care about this information.

The latter assumption is absurd as future repayment likelihoods depend upon all asset positions of a household, while the former assumption imposes an unrealistic information gap in this market for bank loans. Thus, it appears more economically meaningful to consider constraints imposed upon the payouts of all assets of a household in the final period.⁷

Finally, our methods allow us to conjecture that (similar to Polemarchakis and Siconolfi [1997]) restricting excess demand in all states to a linear subspace of the column span of the returns matrix also suffices to guarantee generic existence and regularity. The restriction that we have in mind is:

$$\Phi^1(p) z_h^1 \in L_h(p), \tag{13}$$

where $L_h(p)$ is a household-specific endogenous return space, which is a linear subspace of L . The constraints of this form (13) fit with the fixed dimension return space approach, as we can simply replace L in the previous analysis with $L_h(p)$. In fact, similar restrictions have been considered in Balasko et al. [1990], in the case of nominal assets. We do not consider constraints (13) any further in the present paper, but we are working with them in a companion paper.

⁶The asset demand in the fictitious equilibrium is equal to the asset demand in the true equilibrium, up to a change of basis. The elements of the basis are the columns of $R^*(p, y)$, which is an invertible matrix with A states following the permutation of states.

⁷Observe that the methods of this paper would be equally effective in obtaining generic existence and regularity, if we were to consider participation constraints as inequalities on initial period portfolio value $q b_h = -p(0)(x_h(0) - e_h(0))$.

3 Fixed dimension return space equilibria

As explained in the previous sections, we present some definitions of equilibria in which the dimension of the feasible wealth transfer space L is fixed and equal to the number of available assets, A .

The main difference between the concept of pseudo equilibrium by Duffie and Shafer [1985] and the one proposed below in Definitions 2 and 4 is that in the former the space L appearing in the household maximization problem is written as the *kernel* of a linear function, while in our household maximization problem (see (15)) the space L is the *image* of a linear function.

Below, after introducing some preliminary definitions and facts, we present three equivalent definitions of equilibria that are useful for our analysis. Indeed, as explained in Section 1, the different steps in the proofs of Theorems 11 and 12 require a different definition of equilibrium.

We denote by $\mathcal{G}_{A,S}$ the set of A dimensional vector subspaces of \mathbb{R}^S . It can be shown that $\mathcal{G}_{A,S}$ is a Hausdorff, compact, and second countable (and therefore sequentially compact) metric space and also a C^∞ abstract manifold of dimension $A(S - A)$.⁸

Denoting by Σ the set of permutations of $\{1, \dots, S\}$, with generic element $\sigma \in \Sigma$, by P_σ the corresponding permutation matrix and by I_M the M -dimensional identity matrix, then for every $L \in \mathcal{G}_{A,S}$, there exists $\sigma^{-1} \in \Sigma$, a neighborhood $V_{\sigma^{-1}}$ of L , and a diffeomorphism⁹

$$\psi_{\sigma^{-1}} : V_{\sigma^{-1}} \rightarrow \mathbb{M}(S - A, A) \quad (14)$$

such that $L = \text{Im } P_\sigma \begin{bmatrix} -\psi_{\sigma^{-1}}(L) \\ I_A \end{bmatrix} = \ker [I_{S-A} \mid \psi_{\sigma^{-1}}(L)] \cdot P_{\sigma^{-1}}$. Some related basic results are listed in Appendix 6.1.

Define

$$A_h = \begin{bmatrix} \alpha_h(1) & & \\ & \ddots & \\ & & \alpha_h(S) \end{bmatrix}.$$

Definition 2. A vector $(x^*, p^*, b^*, q^*, L^*) \in \mathbb{R}_{++}^{GH} \times \mathbb{R}_{++}^G \times \mathbb{R}^{AH} \times \mathbb{R}^A \times \mathcal{G}_{A,S}$ is an **image-symmetric equilibrium** for the economy $(e, u, y, \alpha) \in \mathcal{E}$ if

1. $\forall h \in \mathcal{H}$, (x_h^*, b_h^*) solves the following problem:
given $(p^*, q^*, L^*, e, u, y, \alpha)$

$$\max_{(x_h, b_h) \in \mathbb{R}_{++}^G \times \mathbb{R}^A} u_h(x_h)$$

$$s.t. \quad -p^*(0)(x_h(0) - e_h(0)) - q^*b_h = 0 \quad (15.1)$$

$$-\Phi^1(p^*)(x_h^1 - e_h^1) + P_\sigma \begin{bmatrix} -\psi(L^*) \\ I_A \end{bmatrix} b_h = 0 \quad (15.2) \quad (15)$$

$$P_\sigma \begin{bmatrix} -\psi(L^*) \\ I_A \end{bmatrix} b_h + A_h \cdot \Phi^1(p^*) e_h^1 \geq 0 \quad (15.3)$$

⁸See Kato [1995], page 198.

⁹From now on, for ease of notation, we will simply write ψ in place of $\psi_{\sigma^{-1}}$. Notice that we chose to start with σ^{-1} instead of σ because in this way the definitions of equilibria below get simplified.

2. (x^*, b^*) satisfies market clearing conditions (8) and (9);
3. $\text{Im}\mathcal{R}(p^*, y) \subseteq L^*$, i.e.,

$$\text{vec}[I_{S-A} \mid \psi(L^*)] \cdot P_{\sigma^{-1}} \cdot \mathcal{R}(p^*, y) = 0. \quad (16)$$

Define

$$p^\backslash(s) = (p^c(s))_{c \neq 1} \in \mathbb{R}_{++}^{C-1}, \text{ for any } s \in \mathcal{S}, \text{ and } p^\backslash = (p^\backslash(s))_{s \in \mathcal{S}} \in \mathbb{R}_{++}^{G-(S+1)},$$

and similarly, for any $h \in \mathcal{H}$, $x_h^\backslash(s)$, x_h^\backslash , $e_h^\backslash(s)$ and e_h^\backslash . Additionally, define

$$p^{\backslash(01)} = (p^c(s))_{(s,c) \neq (0,1)} \in \mathbb{R}_{++}^{G-1}, \quad \Delta_{++}^{G-1} = \{p \in \mathbb{R}_{++}^G : \sum_{s=0}^S \sum_{c=1}^C p^c(s) = 1\},$$

and, for any $h \in \mathcal{H}$,

$$e_h^\circ = (e_h^1(s))_{s \in \mathcal{S}} \in \mathbb{R}_{++}^{S+1}, \quad e_h^{\backslash(01)} = (e_h^c(s))_{(s,c) \neq (0,1)} \in \mathbb{R}_{++}^{G-1}, \quad x_h^{\backslash(01)} = (x_h^c(s))_{(s,c) \neq (0,1)} \in \mathbb{R}_{++}^{G-1}.$$

Moreover, $\mathbf{1}_N$ denotes an N dimensional vector whose components are all equal to 1; if no confusion arises, we will write $\mathbf{1}$ in the place of $\mathbf{1}_N$.

Remark 3. Observe that the number of admissible price normalizations for the equilibrium concept presented in Definition 2 is $S + 1$ (one for each spot) and there are $S + 1$ Walras' laws. Therefore, the number of significant equations (i.e., conditions (8) and (9) "without $S + 1$ Walras' laws") is equal to the number of significant variables (i.e., spot by spot normalized good prices p^\backslash and asset prices q).

The above observations are formalized in Definition 4 below.

Definition 4. A vector $(x^*, p^{\backslash}, b^*, q^*, L^*) \in \mathbb{R}_{++}^{GH} \times \mathbb{R}_{++}^{G-(S+1)} \times \mathbb{R}^{AH} \times \mathbb{R}^A \times \mathcal{G}_{A,S}$ is a **normalized price image-symmetric equilibrium** for the economy $(e, u, y, \alpha) \in \mathcal{E}$ if

1. $\forall h \in \mathcal{H}$, (x_h^*, b_h^*) solves Problem (15) given $(p^* = (1, p^{\backslash}(s))_{s=0}^S, q^*, L^*, (e, u, y, \alpha))$;
2. b^* satisfies market cleaning conditions (9) and

$$\sum_{h=1}^H (x_h^{\backslash} - e_h^{\backslash}) = 0;$$

3. Condition (16) holds true.

We now introduce the needed definition of image - Mr. 1 equilibrium.

Definition 5. A vector $(x^*, p^*, L^*) \in \mathbb{R}_{++}^{GH} \times \Delta_{++}^{G-1} \times \mathcal{G}_{A,S}$ is an **image - Mr. 1 equilibrium**¹⁰ for the economy $(e, u, y, \alpha) \in \mathcal{E}$ if

- 1a. $\forall h \in \mathcal{H} \setminus \{1\}$, x_h^* solves the following problem:

¹⁰Among the various kinds of fixed dimension return space equilibria we introduce, the one that bears the most resemblance to the original concept by Duffie and Shafer [1985] is Definition 5. However, we have elected to call it "image - Mr. 1 equilibrium" instead of "pseudo equilibrium" to highlight the main difference between this notation and the notion of image-symmetric equilibrium in Definition 2.

given $(p^*, L^*, e, u, y, \alpha)$,

$$\max_{x_h \in B_h^{\setminus b}(p^*, L^*)} u_h(x_h) \quad (17)$$

where $B_h^{\setminus b}(p^*, L^*) =$

$$\{x_h \in \mathbb{R}_{++}^G : \exists b_h \in \mathbb{R}^A \text{ such that } -p^*(0)(x_h(0) - e_h(0)) - \mathbf{1} \cdot P_\sigma \begin{bmatrix} -\psi(L^*) \\ I_A \end{bmatrix} b_h = 0 \quad (18.1)$$

$$-\Phi^{\mathbf{1}}(p^*)(x_h^{\mathbf{1}} - e_h^{\mathbf{1}}) + P_\sigma \begin{bmatrix} -\psi(L^*) \\ I_A \end{bmatrix} b_h = 0 \quad (18.2)$$

$$P_\sigma \begin{bmatrix} -\psi(L^*) \\ I_A \end{bmatrix} b_h + A_h \Phi^{\mathbf{1}}(p^*) e_h^{\mathbf{1}} \geq 0 \quad (18.3) \quad \} \quad (18)$$

1b. x_1^* solves the following problem:

given $(p^*, L^*, e, u, y, \alpha)$,

$$\max_{x_1 \in \mathbb{R}_{++}^G} u_1(x_1)$$

$$\text{s.t.} \quad -p^*(0)(x_1(0) - e_1(0)) - \mathbf{1} \cdot \Phi^{\mathbf{1}}(p^*)(x_1^{\mathbf{1}} - e_1^{\mathbf{1}}) = 0 \quad (19.1) \quad (19)$$

$$\Phi^{\mathbf{1}}(p^*)(x_1^{\mathbf{1}} - e_1^{\mathbf{1}}) + A_1 \Phi^{\mathbf{1}}(p^*) e_1^{\mathbf{1}} \geq 0 \quad (19.2)$$

2. x^* satisfies market clearing conditions

$$\sum_{h=1}^H (x_h^{*\setminus(01)} - e_h^{\setminus(01)}) = 0; \quad (20)$$

3. Condition (16) holds true.

In Appendix 6.2, we show that Definitions 2, 4, and 5 are in fact “allocation equivalent”, as formally stated below.

Proposition 6. For a given economy $(e, u, y, \alpha) \in \mathcal{E}$, the following statements are equivalent:

1. x is an image-symmetric equilibrium allocation;
2. x is a normalized price image-symmetric equilibrium allocation;
3. x is an image - Mr. 1 equilibrium allocation.

Proposition 7 describes the relationship between fixed dimension return space equilibria and “true” equilibria.

Proposition 7. If $(x^*, p^*, b^*, q^*, L^*) \in \mathbb{R}_{++}^{GH} \times \mathbb{R}_{++}^G \times \mathbb{R}^{AH} \times \mathbb{R}^A \times \mathcal{G}_{A,S}$ is an image-symmetric equilibrium for the economy $(e, u, y, \alpha) \in \mathcal{E}$ and

$$\text{rank } \mathcal{R}(p^*, y) = A, \quad (21)$$

then there exist b^{**} and q^{**} such that $(x^*, p^*, b^{**}, q^{**})$ is an equilibrium for \mathcal{E} .

Proof. See Section 6.3. □

4 Generic existence, regularity, and suboptimality

In this section, we first show existence of an image - Mr. 1 equilibrium - see Theorem 11. Then, we can obtain the generic existence of a true equilibrium, after showing the generic regularity and generic full rank condition of the return matrix for normalized price image-symmetric equilibria - see Theorem 12.

As a preliminary step towards the application of a Brouwer like fixed point theorem to prove Theorem 11 - see Appendix 6.5 - we show some basic properties of the demand function associated with Definition 5.

Omitting for simplicity the dependence on utility functions, define

$$\begin{aligned} \beta_1 : \Delta_{+++}^{G-1} \times \mathbb{R}_{+++}^G \times (0, 1)^S &\rightrightarrows \mathbb{R}_{+++}^G, \\ \beta_1(p, e_1, \alpha_1) = \{x_1 \in \mathbb{R}_{+++}^G : & -p(x_1 - e_1) \geq 0 \\ & \Phi^1(p)(x_1^1 - e_1^1) + A_1 \Phi^1(p) e_1^1 \geq 0 \\ & u_1(x_1) - u_1((1 - \alpha_1(s))e_1(s))_{s=0}^S \geq 0\}, \end{aligned}$$

where we have set $\alpha_1(0) = \frac{3}{4}$, and for every $h \in \mathcal{H} \setminus \{1\}$

$$\begin{aligned} \beta_h : \Delta_{+++}^{G-1} \times \mathbb{R}_{+++}^G \times (0, 1)^S \times \mathcal{G}_{A,S} &\rightrightarrows \mathbb{R}_{+++}^G \times \mathbb{R}^A, \\ \beta_h(p, e_h, \alpha_h, L) = \{(x_h, b_h) \in \mathbb{R}_{+++}^G \times \mathbb{R}^A : & -p(0)(x_h(0) - e_h(0)) - \mathbf{1} \cdot P_\sigma \begin{bmatrix} -\psi(L) \\ I_A \end{bmatrix} b_h \geq 0 \\ & -\Phi^1(p)(x_h^1 - e_h^1) + P_\sigma \begin{bmatrix} -\psi(L) \\ I_A \end{bmatrix} b_h \geq 0 \\ & P_\sigma \begin{bmatrix} -\psi(L) \\ I_A \end{bmatrix} b_h + A_h \Phi^1(p) e_h^1 \geq 0 \\ & u_h(x_h) - u_h(\frac{1}{4}e_h) \geq 0\}. \end{aligned}$$

Remark 8. *It is obvious that if (x_h, b_h) is a solution to*

$$\max_{(x_h, b_h) \in \mathbb{R}_{+++}^G \times \mathbb{R}^A} u_h(x_h) \quad \text{s.t.} \quad (x_h, b_h) \in \beta_h(p, e_h, \alpha_h, L), \quad (22)$$

for some $h \in \mathcal{H} \setminus \{1\}$, then x_h is a solution to (17); and conversely if x_h is a solution to (17), then there exists b_h such that (x_h, b_h) is a solution to (22).

Lemma 9. *For any $h \in \mathcal{H}$, β_h is nonempty valued, convex valued, compact valued, closed and lower hemi continuous.*

Proof. The proof is presented in Appendix 6.4. □

Proposition 10. *The demand correspondences associated with Problems (17) and (19) are continuous functions.*

Proof. It follows from Remark 8, Lemma 9, the Maximum Theorem, and assumption (3). □

Theorem 11. *For every economy, an image - Mr. 1 equilibrium exists.*

Proof. The proof is presented in Appendix 6.5. □

Consider the Hausdorff topological vector space

$$\mathcal{V} = \mathbb{R}_{++}^{GH} \times [C^2(\mathbb{R}_{++}^G)]^H \times \mathbb{R}^{CAS} \times (0, 1)^{SH}, \quad (23)$$

endowed with the product topology of the natural topologies on each of the spaces in the Cartesian product. In particular, on the C^2 function space, we consider the C^2 compact-open topology. Assume that $\mathcal{E} \subseteq \mathcal{V}$ is endowed with the topology induced by \mathcal{V} .

Theorem 12. *There exists an open and dense set $\mathcal{D} \subseteq \mathcal{E}$ such that, for every $(e, u, y, \alpha) \in \mathcal{D}$, there is a (positive) finite number of associated equilibria which locally smoothly depend on the elements of \mathcal{D} .*

Proof. First of all, observe that from Proposition 6 and Theorem 11, a normalized price image-symmetric equilibrium exists. Moreover, from Proposition 7, it is enough to show that generically rank condition (21) does hold true. The strategy of the proof is then to consider normalized price image-symmetric equilibria and proceed through the following steps:

1. the associated extended equilibrium system is such that border line cases are rare;
2. the return matrix has generic full rank;
3. the associated projection from the equilibrium set to the economy space is proper;
4. apply Glöckner Theorem 24.

Each of the above steps is formalized and proven in Appendix 6.6. □

The theorem below states the typical inefficiency of equilibria.

Theorem 13. *If $A < S$, then there exists an open and dense set $\tilde{\mathcal{D}} \subseteq \mathcal{E}$ such that, for every $(e, u, y, \alpha) \in \tilde{\mathcal{D}}$, every corresponding equilibrium allocation is not Pareto Optimal.*

The proof of the above theorem follows a standard argument and therefore it is omitted.

Observe that in the statement of the above theorem, the qualification $A < S$ is indeed a necessary condition. If it were the case $A = S$, starting from a regular economy in the complete market model with an associated Pareto Optimal equilibrium, and then adding “insignificant” constraints, it would be immediate to construct an open set of economies in the restricted participation model with the property that at least one associated equilibrium is still Pareto Optimal.

5 A numerical example

Given the proof of generic regularity of equilibria in Theorem 12, we can now compute an equilibrium of our model using algorithms that utilize the theory of differential topology. Specifically, the two equilibria computed in this section are numerically determined using homotopy methods, i.e., the HOMPACT suite of subroutines for Fortran 90, and Kubler [2007]. These methods require generic regularity to work successfully.

With these two equilibria, a comparative statics analysis yields interesting conclusions. In particular, the example shows that by tightening credit constraints, an anonymous planner intervention can actually effect a Pareto improvement. The planner intervention works through

adjustments in the parameters governing the participation restriction (6): $(\alpha_h(s))_{h \in \mathcal{H}, s \in \mathcal{S}'} \in (0, 1)^{SH}$. For this example, these parameters are household independent, so they are simply $(\alpha(s))_{s \in \mathcal{S}'} \in (0, 1)^S$. The planner intervention is also household independent and its tools are given by $\tau(s) \in \left(-1, -1 + \frac{1}{\alpha(s)}\right)$, $s \in \mathcal{S}'$, so that the new parameters in (6) are defined as

$$\hat{\alpha}(s) = (1 + \tau(s)) \cdot \alpha(s) \in (0, 1), \quad \forall s \in \mathcal{S}'.$$

Obviously, an intervention with $\tau = (\tau(s))_{s \in \mathcal{S}'} = \mathbf{0}$, where $\mathbf{0} = (0, \dots, 0) \in \mathbb{R}^S$, implies no change in either the parameters or the resulting equilibrium. Define the equilibrium obtained following planner intervention as $(\hat{x}, \hat{p}, \hat{b}, \hat{q})$, in contrast to the original equilibrium (x, p, b, q) prior to planner intervention.

The example in this section demonstrates the following fact. For some values $\tau(s) \leq 0$, $s \in \mathcal{S}'$, the resulting equilibrium allocation \hat{x} Pareto dominates the original equilibrium allocation x .

That is, for this particular economy, more regulation on the credit markets is employed in order to make all households better off. Notice that, due to the generic regularity result and the way the algorithm works, the example is robust to perturbation.

The economy is defined by:

- $H = 3$ households;
- $C = 2$ commodities traded in each state;
- $S = 4$ possible states of uncertainty tomorrow;
- $A = 2$ real assets.

The household endowments are given by:

	$e_h^1(0)$	$e_h^2(0)$	$e_h^1(1)$	$e_h^2(1)$	$e_h^1(2)$	$e_h^2(2)$	$e_h^1(3)$	$e_h^2(3)$	$e_h^1(4)$	$e_h^2(4)$
$h = 1$	1	7	1	0.5	7	1	7.5	6	1	7
$h = 2$	2	4	7	6.5	1	7	2	0.5	8	2
$h = 3$	9	1	4	5	4	4	2.5	5.5	3	3
Sum	12	12	12	12	12	12	12	12	12	12

The household utility functions are given by:

$$u_h(x_h) = \gamma_h(0) \cdot \log(x_h^1(0)) + (1 - \gamma_h(0)) \cdot \log(x_h^2(0)) \\ + \frac{1}{4} \sum_{s \in \mathcal{S}'} [\gamma_h(s) \cdot \log(x_h^1(s)) + (1 - \gamma_h(s)) \cdot \log(x_h^2(s))],$$

where

	$\gamma_h(0)$	$\gamma_h(1)$	$\gamma_h(2)$	$\gamma_h(3)$	$\gamma_h(4)$
$h = 1$	2/3	1/2	1/4	3/4	1/2
$h = 2$	1/3	1/4	3/4	1/2	1/4
$h = 3$	2/3	3/4	1/2	1/4	3/4

The assets are real assets, so each asset has payouts in terms of a vector of commodities in each state $s \in \mathcal{S}'$. These vector of payouts are given by:

States \ Assets	$a = 1$	$a = 2$
$s = 1$	(4, 0.5)	(0.5, 3.6)
$s = 2$	(3.9, 0.5)	(3.7, 0.5)
$s = 3$	(0.5, 3.8)	(0.5, 3.8)
$s = 4$	(0.5, 3.7)	(3.9, 0.5)

Finally, the parameters (identical for all households) governing the participation restriction (6) are given by:

$$\begin{aligned}\alpha(1) &= 0.033 \\ \alpha(2) &= 0.020 \\ \alpha(3) &= 0.029 \\ \alpha(4) &= 0.033.\end{aligned}$$

For this economy, the equilibrium¹¹ is given by:

- Consumption

	$x_h^1(0)$	$x_h^2(0)$	$x_h^1(1)$	$x_h^2(1)$	$x_h^1(2)$	$x_h^2(2)$	$x_h^1(3)$	$x_h^2(3)$	$x_h^1(4)$	$x_h^2(4)$
$h = 1$	2.966	3.680	0.881	0.871	4.951	3.240	8.275	5.441	2.017	5.422
$h = 2$	1.615	4.778	6.266	7.305	3.133	4.811	1.338	0.988	6.107	4.084
$h = 3$	7.426	3.546	4.851	3.822	3.913	3.946	2.383	5.569	3.873	2.487

- Assets

	b_h^1	b_h^2
$h = 1$	0.334	-0.314
$h = 2$	0.225	-0.240
$h = 3$	-0.556	0.550

- Prices

$p^1(0) = 1$	$p^2(0) = 0.622$
$p^1(1) = 1$	$p^2(1) = 0.973$
$p^1(2) = 1$	$p^2(2) = 0.982$
$p^1(3) = 1$	$p^2(3) = 1.205$
$p^1(4) = 1$	$p^2(4) = 0.781$
$q^1 = 5.886$	$q^2 = 5.941$

- Utility values

$$\begin{aligned}u_1(x_1) &= 2.2473 \\ u_2(x_2) &= 2.4142 \\ u_3(x_3) &= 3.1676.\end{aligned}\tag{24}$$

¹¹A unique equilibrium is guaranteed by our use of the Cobb-Douglas utility functions.

Above is the original equilibrium. Following planner intervention, we will obtain a new equilibrium. The planner intervenes according to:

$$\begin{aligned}\tau(1) &= 0 \\ \tau(2) &= -0.22 \\ \tau(3) &= 0 \\ \tau(4) &= -0.17.\end{aligned}$$

This means that the parameters $\hat{\alpha}(1)$ and $\hat{\alpha}(3)$ remain unchanged compared to $\alpha(1)$ and $\alpha(3)$, but $\hat{\alpha}(2)$ is 22% lower compared to $\alpha(2)$ and $\hat{\alpha}(4)$ is 17% lower compared to $\alpha(4)$:

$$\begin{aligned}\hat{\alpha}(1) &= 0.033 \\ \hat{\alpha}(2) &= 0.016 \\ \hat{\alpha}(3) &= 0.029 \\ \hat{\alpha}(4) &= 0.028.\end{aligned}$$

The credit constraints have just been tightened.

The equilibrium following planner intervention (again, unique¹²) is:

- Consumption

	$\hat{x}_h^1(0)$	$\hat{x}_h^2(0)$	$\hat{x}_h^1(1)$	$\hat{x}_h^2(1)$	$\hat{x}_h^1(2)$	$\hat{x}_h^2(2)$	$\hat{x}_h^1(3)$	$\hat{x}_h^2(3)$	$\hat{x}_h^1(4)$	$\hat{x}_h^2(4)$
$h = 1$	3.029	3.624	0.866	0.859	4.841	3.328	8.302	5.395	2.096	5.392
$h = 2$	1.619	4.784	6.215	7.363	3.233	4.717	1.327	1.007	6.005	4.174
$h = 3$	7.358	3.595	4.917	3.776	3.923	3.952	2.366	5.596	3.897	2.428

- Assets

	\hat{b}_h^1	\hat{b}_h^2
$h = 1$	0.301	-0.284
$h = 2$	0.237	-0.251
$h = 3$	-0.537	0.533

- Prices

$\hat{p}^1(0) = 1$	$\hat{p}^2(0) = 0.627$
$\hat{p}^1(1) = 1$	$\hat{p}^2(1) = 0.973$
$\hat{p}^1(2) = 1$	$\hat{p}^2(2) = 0.985$
$\hat{p}^1(3) = 1$	$\hat{p}^2(3) = 1.180$
$\hat{p}^1(4) = 1$	$\hat{p}^2(4) = 0.795$
$\hat{q}^1 = 6.233$	$\hat{q}^2 = 6.305$

- Utility values

$$\begin{aligned}u_1(\hat{x}_1) &= 2.2603 \\ u_2(\hat{x}_2) &= 2.4259 \\ u_3(\hat{x}_3) &= 3.1686.\end{aligned}\tag{25}$$

¹²Again, a unique equilibrium is guaranteed by our use of the Cobb-Douglas utility functions.

Comparing the utility values in (24) and (25), a Pareto improvement has been achieved. The utility increases are 0.58% for household $h = 1$, 0.48% for household $h = 2$, and 0.03% for household $h = 3$.

We now explain the intuition behind this Pareto improvement. Taken in isolation, a binding constraint of the form (6) for a single household h and for a particular state $s \in \mathcal{S}'$ has well-established properties. A reduction in the parameter $\alpha_h(s)$ restricts the budget set for household h , because the constraint has become tighter. This results in lower utility for household h .

However, consider what happens, as in the above example, when a reduction in the parameter $\alpha_h(s)$ results in constraints binding in some states in which they previously did not bind. Specifically, the following table illustrates this endogenous effect for the above example:

Constraint (6) is binding in states		
	Before intervention	After intervention
$h = 1$	$s = 4$	$s = 4$
$h = 2$	$s = 3$	$s = 3$ and $s = 4$
$h = 3$	$s = 1$	$s = 2$

As can be seen from the table, for households $h = 2$ and $h = 3$, different constraints are binding after the intervention compared to before the intervention. When the states of binding constraints “switch” following an intervention, the property described above for isolated constraints is no longer valid. In particular, two effects now play a leading role in determining the equilibrium. First, portfolio effects are present as households adjust their portfolios across the states of uncertainty where now the constraints may bind for different states. Second, general equilibrium effects are present, whereby one household’s adjustments to the newly binding constraints must affect the other households, through the relative commodity prices and asset prices, in order for the market clearing conditions to be satisfied.

6 Appendix

6.1 Basic results for kernels and images

For any $m, n \in \mathbb{N}_+$, let $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ be the vector space of linear functions from \mathbb{R}^m to \mathbb{R}^n . For any $l \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$, let $[l]$ denote the matrix associated with l with respect to the canonical bases in \mathbb{R}^n and \mathbb{R}^m .

The following result is well known.

Proposition 14. *If $L \in \mathcal{G}_{A,S}$, then*

1. a. $\exists l_1 \in \mathcal{L}(\mathbb{R}^A, \mathbb{R}^S)$ such that $\text{Im } l_1 = L$;
- b. $\exists \sigma \in \Sigma$ and $\exists M \in \mathbb{M}(S - A, A)$ such that

$$[l_1] = P_\sigma \begin{bmatrix} -M \\ I_A \end{bmatrix};$$

- c. $M = \psi_{\sigma^{-1}}(L)$, where $\psi_{\sigma^{-1}}$ is a chart of an atlas of $\mathcal{G}_{A,S}$.

2. a. $\exists l_2 \in \mathcal{L}(\mathbb{R}^S, \mathbb{R}^{S-A})$ such that $\ker l_2 = L$;

b.

$$[l_2] = [I_{S-A}|M] \cdot P_\sigma^{-1}.$$

3. Let $Y, Y' \in \mathbb{M}^f(S, A)$ be given. Then

$$\text{Im}Y = \text{Im}Y' \Leftrightarrow \text{there exists a unique } C \in \mathbb{M}^f(A, A) \text{ such that } Y' = YC.$$

6.2 Proof of Proposition 6

First of all, observe that the need to “hide” b_h in Definition 5 is due to the fact that we want to apply a fixed point theorem written in terms of x and p only (see Theorem 22). Setting $\tilde{b} = (b_h)_{h=2}^H \in \mathbb{R}^{A(H-1)}$, Definition 5 is obviously equivalent to the following definition.

Definition 15. A vector $(x^*, p^*, \tilde{b}^*, L^*) \in \mathbb{R}_{++}^{GH} \times \Delta_{++}^{G-1} \times \mathbb{R}^{A(H-1)} \times \mathcal{G}_{A,S}$ is an **image - Mr. 1 with asset demand equilibrium** for the economy $(e, u, y, \alpha) \in \mathcal{E}$ if

1a. $\forall h \in \mathcal{H} \setminus \{1\}$, (x_h^*, b_h^*) solves the following problem:
given $(p^*, L^*, e, u, y, \alpha)$

$$\max_{(x_h, b_h) \in \mathbb{R}_{++}^G \times \mathbb{R}^A} u_h(x_h)$$

$$\begin{aligned} \text{s.t.} \quad & -p^*(0)(x_h(0) - e_h(0)) - \mathbf{1} \cdot P_\sigma \begin{bmatrix} -\psi(L^*) \\ I_A \end{bmatrix} b_h = 0 \\ & -\Phi^{\mathbf{1}}(p^*)(x_h^{\mathbf{1}} - e_h^{\mathbf{1}}) + P_\sigma \begin{bmatrix} -\psi(L^*) \\ I_A \end{bmatrix} b_h = 0 \\ & P_\sigma \begin{bmatrix} -\psi(L^*) \\ I_A \end{bmatrix} b_h + A_h \Phi^{\mathbf{1}}(p^*) e_h^{\mathbf{1}} \geq 0 \end{aligned}$$

- 1b. x_1^* solves Problem (19);
2. x^* satisfies market clearing conditions (20);
3. Condition (16) holds true.

Lemma 16. The maximization problems presented in Definitions 2, 4 and 15 are characterized by Kuhn-Tucker conditions.

Proof. Sufficiency of Kuhn-Tucker conditions follows from the fact that the utility functions are concave and the constraints affine.

Necessity of Kuhn-Tucker conditions follows from the so called “Weak reverse convex constraint qualification” (see Mangasarian [1969], point 4, page 172 and Theorem 6, page 173). \square

Notice that, since the third constraint in the maximization problems of Definitions 2, 4 and 15 is the same¹³, it is enough and notationally lighter to show “equivalence” among those definitions

¹³The only exception is given by the (second) constraint for Mr. 1 in Definition 15, which does not contain b_h . However, it is immediate to see that the third constraint in the maximization problems of Definitions 2, 4 and 15 for $h \in \mathcal{H} \setminus \{1\}$ can be rewritten, just using the second constraint therein, like the constraint for Mr. 1. Hence, the last constraints are equivalent in the above mentioned definitions.

without that constraint. Consider then the equilibrium concepts presented in Definitions 2, 4 and 15 all of them without the last inequality constraint.

We are going to show equivalence among those concepts of equilibria showing that the three associated extended systems are “allocation” equivalent, in the sense that the equilibrium allocations satisfying such systems are the same.

More precisely, we proceed as follows.

1. We write the extended system associated with symmetric fixed dimension return space equilibrium of Definition 2 - see System (27).

2. We show that it is possible to normalize prices of the numeraire commodity 1 in each state in Definition 2, i.e., that Definitions 2 and 4 are allocation equivalent - see Lemma 17.

3. We write the extended system associated with image - Mr. 1 with asset demand equilibrium of Definition 15 - see System (28).

4. We show that it is possible to put commodity prices in the simplex and normalize S multipliers in the extended system associated with Definition 2 - see Lemma 18.

5. Using Step 4, we show that Definitions 2 and 15 are allocation equivalent - see Lemma 19.

We now go through the five steps listed above.

1.

Define

$$\mathcal{R}(L) = P_\sigma \begin{bmatrix} -\psi(L) \\ I_A \end{bmatrix} = \begin{bmatrix} r^1(L) \\ \vdots \\ r^S(L) \end{bmatrix}. \quad (26)$$

For sake of simplicity, here we consider the case $\sigma = Id$. Moreover, in order to lighten notation, we will sometimes denote the excess demand of household h by z_h instead of $x_h - e_h$.

The extended system associated with Definition 2 is as follows:

$$\left\{ \begin{array}{l} \dots \\ Du_h(x_h) - \lambda_h \Phi(p) = 0 \\ \lambda_h \begin{bmatrix} -q \\ \mathcal{R}(L) \end{bmatrix} = 0 \\ -\Phi(p) z_h + \begin{bmatrix} -q \\ \mathcal{R}(L) \end{bmatrix} b_h = 0 \\ \dots \\ \sum_h z_h = 0 \\ \sum_h b_h = 0 \\ \text{vec}[I \mid \psi(L)] \cdot \mathcal{R}(p, y) = 0 \end{array} \right. \quad (27)$$

2.

Lemma 17. *If (x, λ, b, p, q, L) , where $\psi(L) = (m_{sa})_{s \in \{1, \dots, S-A\}, a \in A}$, is an extended image-symmetric equilibrium associated with (e, u, y) , then $(x, \lambda', b', p', q', L')$ is an extended image-*

symmetric equilibrium associated with (e, u, y) , where

$$\lambda'_h = (\lambda_h(s) p^1(s))_{s \in \mathcal{S}}, \quad b'_h = \left(\frac{b_h^a}{p^1(S-A+a)} \right)_{a \in \mathcal{A}}, \quad p' = \left(\frac{p(s)}{p^1(s)} \right)_{s \in \mathcal{S}},$$

$$q' = \left(q^a \frac{p^1(S-A+a)}{p^1(0)} \right)_{a \in \mathcal{A}}, \quad \psi(L') = \left(m_{sa} \frac{p^1(S-A+a)}{p^1(s)} \right)_{s \in \{1, \dots, S-A\}, a \in \mathcal{A}}.$$

Observe that

$$(p'^1(s))_{s \in \mathcal{S}} = \mathbf{1}_{S+1}$$

and thus $(x, \lambda', b', p', q', L')$ is a normalized price image-symmetric equilibrium.

We omit the proof of Lemma 17 as well as that of Lemma 18 below, because they are straightforward.

3.

The extended system associated with Definition 15 is as follows:

$$\left\{ \begin{array}{l} Du_1(x_1) - \mu_1(0)p = 0 \\ pz_1 = 0 \\ \dots \\ Du_h(x_h) - \mu_h \Phi(p) = 0 \\ \mu_h \begin{bmatrix} -\mathbf{1}_S \cdot \mathcal{R}(L) \\ \mathcal{R}(L) \end{bmatrix} = 0 \\ -\Phi(p)(x_h - e_h) + \begin{bmatrix} -\mathbf{1}_S \cdot \mathcal{R}(L) \\ \mathcal{R}(L) \end{bmatrix} b_h = 0 \\ \dots \\ \sum_h z_h^{\setminus(01)} = 0 \\ \text{vec}[I \mid \psi(L)] \cdot \mathcal{R}(p, y) = 0 \\ \sum_{c,s} p^c(s) - 1 = 0 \end{array} \right. \quad (28)$$

4.

Lemma 18. *If (x, λ, b, p, q, L) , where $\psi(L) = (m_{sa})_{s \in \{1, \dots, S-A\}, a \in \mathcal{A}}$, is an extended image-symmetric equilibrium associated with (e, u, y) , then $(x, \lambda', b', p', q', L')$ is an extended image-symmetric equilibrium associated with (e, u, y) , where*

$$\lambda'_h = \left(\frac{\lambda_h(s)}{\lambda_1(s)} \sum_{c,s} p^c(s) \lambda_1(s) \right)_{s \in \mathcal{S}}, \quad b'_h = \left(\frac{\lambda_1(S-A+a)}{\sum_{c,s} p^c(s) \lambda_1(s)} \cdot b_h^a \right)_{a \in \mathcal{A}}, \quad p' = \left(\frac{p(s) \cdot \lambda_1(s)}{\sum_{c,s} p^c(s) \lambda_1(s)} \right)_{s \in \mathcal{S}},$$

$$q' = \left(q^a \frac{\lambda_1(0)}{\lambda_1(S-A+a)} \right)_{a \in \mathcal{A}}, \quad \psi(L') = \left(m_{sa} \frac{\lambda_1(s)}{\lambda_1(S-A+a)} \right)_{s \in \{1, \dots, S-A\}, a \in \mathcal{A}}.$$

Thus

$$\lambda'_1 = \left(\sum_{c,s} p^c(s) \lambda_1(s) \right) \cdot \mathbf{1}_{S+1} = \lambda'_1(0) \cdot \mathbf{1}_{S+1}, \quad \sum_{c,s} p^c(s) = 1, \quad q' = \mathbf{1}_S \cdot \mathcal{R}(L'). \quad (29)$$

In this case we will call $(x, \lambda', b', p', q', L')$ an extended normalized (p, λ) image-symmetric equilibrium.

5.

The extended system associated with a normalized (p, λ) image - symmetric equilibrium is displayed below.

$$\left\{ \begin{array}{l} Du_1(x_1) - (\lambda_1(0) \cdot \mathbf{1}_{S+1}) \Phi(p) = 0 \\ (\lambda_1(0) \cdot \mathbf{1}_{S+1}) \begin{bmatrix} -q \\ \mathcal{R}(L) \end{bmatrix} = 0 \quad \text{or} \quad q = \mathbf{1}_S \cdot \mathcal{R}(L) \\ -\Phi(p) z_1 + \begin{bmatrix} -q \\ \mathcal{R}(L) \end{bmatrix} b_1 = 0 \\ \dots \\ Du_h(x_h) - \lambda_h \Phi(p) = 0 \\ \lambda_h \begin{bmatrix} -q \\ \mathcal{R}(L) \end{bmatrix} = 0 \\ -\Phi(p) z_h + \begin{bmatrix} -q \\ \mathcal{R}(L) \end{bmatrix} b_h = 0 \\ \dots \\ \sum_h z_h^\wedge = 0 \\ \sum_h b_h = 0 \\ \text{vec}[I \mid \psi(L)] \cdot \mathcal{R}(p, y) = 0 \\ \sum_{c,s} p^c(s) - 1 = 0 \\ (\lambda_1(s) = \lambda_1(0))_{s=1}^S \end{array} \right.$$

It is immediate to prove the following lemma.

Lemma 19. 1. *If (x, λ, b, p, q, L) is an extended normalized (p, λ) image - symmetric equilibrium, then $(x, \mu_1(0), (\mu_h)_{h=2}^H, (b_h)_{h=2}^H, p, L)$ is an extended image - Mr. 1 with asset demand equilibrium, with $\mu_1(0) = \lambda_1(0)$ and, for $h \geq 2$, $\mu_h = \lambda_h$.*

2. *If $(x, \mu_1(0), (\mu_h)_{h=2}^H, (b_h)_{h=2}^H, p, L)$ is an extended image - Mr. 1 with asset demand equilibrium, then (x, λ, b, p, q, L) is an extended normalized (p, λ) image - symmetric equilibrium, with $q = \mathbf{1} \cdot \begin{bmatrix} -E \\ I \end{bmatrix}$, $b_1 = -\sum_{h=2}^H b_h$, $\lambda_1 = \mu_1(0) \cdot \mathbf{1}_{S+1}$ and, for $h \geq 2$, $\lambda_h = \mu_h$.*

6.3 Proof of Proposition 7

Proof. Since $\text{rank } \mathcal{R}(p^*, y) = A$ and $L^* \in \mathcal{G}_{A,S}$, then

$$\dim \text{Im} \mathcal{R}(p^*, y) = A = \dim L^*. \quad (30)$$

Since, by definition of image - symmetric equilibrium, $\text{Im} \mathcal{R}(p^*, y) \subseteq L^*$, then $\text{Im} \mathcal{R}(p^*, y) = L^*$. Hence, from Proposition 14 in Appendix 6.1, $\exists \sigma \in \Sigma$, a neighborhood $V_{\sigma^{-1}}$ of L and a function $\psi : V_{\sigma^{-1}} \rightarrow \mathbb{M}(S - A, A)$ such that

$$\text{Im} \mathcal{R}(p^*, y) = L^* = \text{Im} P_\sigma \begin{bmatrix} -\psi(L^*) \\ I_A \end{bmatrix} \quad (31)$$

and there exists a full rank matrix $C \in \mathbb{M}(A, A)$ such that

$$\mathcal{R}(p^*, y) = P_\sigma \begin{bmatrix} -\psi(L^*) \\ I_A \end{bmatrix} \cdot C. \quad (32)$$

Moreover,

$$[0_{A \times (S-A)} | I_A] P_{\sigma^{-1}} \cdot \mathcal{R}(p^*, y) = [0_{A \times (S-A)} | I_A] P_{\sigma^{-1}} P_\sigma \begin{bmatrix} -\psi(L^*) \\ I_A \end{bmatrix} \cdot C = [0_{A \times (S-A)} | I_A] \begin{bmatrix} -\psi(L^*) \\ I_A \end{bmatrix} \cdot C,$$

and then

$$C = [0_{A \times (S-A)} | I_A] P_{\sigma^{-1}} \cdot \mathcal{R}(p^*, y).$$

Observe that:

1. (x_h, b_h) is a solution to Problem (7) in the definition of equilibrium if and only if it is a solution to

$$\max_{(x_h, b_h)} u_h(x_h)$$

$$s.t. \quad -\Phi(p^*) z_h + \begin{bmatrix} -q^* \\ \mathcal{R}(p^*, y) \end{bmatrix} b_h = 0 \quad (1) \quad (33)$$

$$\mathcal{R}(p^*, y) b_h + A_h \cdot \Phi^1(p^*) e_h^1 \geq 0 \quad (2)$$

which is the same as Problem (7) apart from the fact that conditions (1) and (2) there are replaced by equalities.

2. Kuhn-Tucker conditions do characterize solutions to the above maximization problem. Therefore, the extended system associated with equilibria consistently modified with the above observation is as follows.

$$\left\{ \begin{array}{l} \dots \\ Du_h(x_h) - \lambda_h \Phi(p) = 0 \\ \lambda_h \begin{bmatrix} -q \\ \mathcal{R}(p, y) \end{bmatrix} + \eta_h \mathcal{R}(p, y) = 0 \\ -\Phi(p) z_h + \begin{bmatrix} -q \\ \mathcal{R}(p, y) \end{bmatrix} b_h = 0 \\ \min \{ \alpha_h(s) p(s) e_h(s) + r^s(p, y) b_h, \eta_h(s) \} = 0 \\ \dots \\ \sum_h z_h = 0 \\ \sum_h b_h = 0 \end{array} \right. \quad (34)$$

Recalling the definition of $\mathcal{R}(L)$ in (26), the extended system associated with image - symmetric equilibrium is as follows.

$$\left\{ \begin{array}{l} \dots \\ Du_h(x_h) - \lambda_h \Phi(p) = 0 \\ \lambda_h \begin{bmatrix} -q \\ \mathcal{R}(L) \end{bmatrix} + \eta_h \mathcal{R}(L) = 0 \\ -\Phi(p) z_h + \begin{bmatrix} -q \\ \mathcal{R}(L) \end{bmatrix} b_h = 0 \\ \min \{ \alpha_h(s) p(s) e_h(s) + r^s(L) b_h, \eta_h(s) \} = 0 \\ \dots \\ \sum_h z_h = 0 \\ \sum_h b_h = 0 \\ \mathcal{R}(p, y) - \mathcal{R}(L) \cdot C = 0 \end{array} \right.$$

The desired result follows from the comparison of the two systems, choosing

$$q^{**} = q^* \cdot C \text{ and } b_h^{**} = C^{-1} b_h^*.$$

□

6.4 Proof of Lemma 9

We present the desired proof for $h \neq 1$. For $h = 1$, the argument is an easier version of the presented proof.

1. β_h is nonempty valued.

$$(e_h, 0) \in \beta_h(p, e_h, \alpha_h, L).$$

2. β_h is convex valued.

The constraint functions are either affine or quasi-concave (with respect to (x_h, b_h)).

3. β_h is compact valued.

Namely, $\beta_h(p, e_h, \alpha_h, L)$ is closed in \mathbb{R}^{G+A} because, by 4. below, β_h is closed.

Suppose it is not bounded. Then $\exists \left(x_h^{[n]}, b_h^{[n]} \right)_{n \in \mathbb{N}} \in (\beta_h(p, e_h, \alpha_h, L))^\infty$ such that $\left\| \left(x_h^{[n]}, b_h^{[n]} \right) \right\| \rightarrow +\infty$. Consider

$$\left(\frac{\left(x_h^{[n]}, b_h^{[n]} \right)}{\left\| \left(x_h^{[n]}, b_h^{[n]} \right) \right\|} \right)_{n \in \mathbb{N}}$$

whose elements belong to the unit sphere in \mathbb{R}^{G+A} . Then, up to a subsequence,

$$\frac{\left(x_h^{[n]}, b_h^{[n]} \right)}{\left\| \left(x_h^{[n]}, b_h^{[n]} \right) \right\|} \rightarrow (\bar{x}_h, \bar{b}_h) \neq 0 \text{ with } \bar{x}_h > 0.$$

In fact, the last inequality follows from the fact that, for every $n \in \mathbb{N}$, $u_h \left(x_h^{[n]} \right) - u_h \left(\frac{1}{4} e_h \right) \geq 0$. Hence,

$$\begin{aligned} & -p(0) \left(\frac{x_h^{[n]}(0)}{\|(x_h^{[n]}, b_h^{[n]})\|} - \frac{e_h(0)}{\|(x_h^{[n]}, b_h^{[n]})\|} \right) - \mathbf{1} \cdot P_\sigma \begin{bmatrix} -\psi(L) \\ I_A \end{bmatrix} \frac{b_h^{[n]}}{\|(x_h^{[n]}, b_h^{[n]})\|} \geq 0 \\ & -\Phi^{\mathbf{1}}(p) \cdot \frac{x_h^{[n]\mathbf{1}}}{\|(x_h^{[n]}, b_h^{[n]})\|} + \Phi^{\mathbf{1}}(p) \cdot \frac{e_h^{\mathbf{1}}}{\|(x_h^{[n]}, b_h^{[n]})\|} + P_\sigma \begin{bmatrix} -\psi(L) \\ I_A \end{bmatrix} \frac{b_h^{[n]}}{\|(x_h^{[n]}, b_h^{[n]})\|} \geq 0 \end{aligned}$$

and taking limits

$$\begin{aligned} & -p(0) \bar{x}_h(0) - \mathbf{1} \cdot P_\sigma \begin{bmatrix} -\psi(L) \\ I_A \end{bmatrix} \bar{b}_h \geq 0 \quad (1) \\ & -\Phi^{\mathbf{1}}(p) \cdot \bar{x}_h + P_\sigma \begin{bmatrix} -\psi(L) \\ I_A \end{bmatrix} \bar{b}_h \geq 0 \quad (2) \end{aligned} \tag{35}$$

or

$$\begin{aligned} & -\mathbf{1} \cdot P_\sigma \begin{bmatrix} -\psi(L) \\ I_A \end{bmatrix} \bar{b}_h \geq p(0) \bar{x}_h(0) \geq 0 \quad (1) \\ & P_\sigma \begin{bmatrix} -\psi(L) \\ I_A \end{bmatrix} \bar{b}_h \geq \Phi^{\mathbf{1}}(p) \cdot \bar{x}_h > 0. \quad (2) \end{aligned} \tag{36}$$

Then from (36), we get

$$\begin{bmatrix} -\mathbf{1} \cdot P_\sigma \begin{bmatrix} -\psi(L) \\ I_A \end{bmatrix} \\ P_\sigma \begin{bmatrix} -\psi(L) \\ I_A \end{bmatrix} \end{bmatrix} \cdot \bar{b}_h > 0,$$

contradicting the fact that, as it is easy to prove, if $\lambda \in \mathbb{R}_{++}^S$ and $R \in \mathbb{M}(S, A)$, then there exists no $b \in \mathbb{R}^A$ such that $\begin{bmatrix} -\lambda R \\ R \end{bmatrix} b > 0$.

4. β_h is closed.

We want to show that

$$\begin{aligned} & a. \quad \left(p^{[n]}, e_h^{[n]}, \alpha_h^{[n]}, L^{[n]} \right) \rightarrow (\bar{p}, \bar{e}_h, \bar{\alpha}_h, \bar{L}) \\ & \left\langle b. \quad \left(x_h^{[n]}, b_h^{[n]} \right) \in \beta_h \left(p^{[n]}, e_h^{[n]}, \alpha_h^{[n]}, L^{[n]} \right) \right\rangle \Rightarrow (\bar{x}_h, \bar{b}_h) \in \beta_h (\bar{p}, \bar{e}_h, \bar{\alpha}_h, \bar{L}). \\ & c. \quad \left(x_h^{[n]}, b_h^{[n]} \right) \rightarrow (\bar{x}_h, \bar{b}_h) \end{aligned}$$

From assumption $b.$, we have that

$$\begin{aligned}
& -p^{[n]}(0) \left(x_h^{[n]}(0) - e_h^{[n]}(0) \right) - \mathbf{1} \cdot P_\sigma \left[\begin{array}{c} -\psi(L^{[n]}) \\ I_A \end{array} \right] b_h^{[n]} \geq 0 \\
& -\Phi^{\mathbf{1}}(p^{[n]}) \left(x_h^{[n]\mathbf{1}} - e_h^{[n]\mathbf{1}} \right) + P_\sigma \left[\begin{array}{c} -\psi(L^{[n]}) \\ I_A \end{array} \right] b_h^{[n]} \geq 0 \\
& P_\sigma \left[\begin{array}{c} -\psi(L^{[n]}) \\ I_A \end{array} \right] b_h^{[n]} + A_h^{[n]} \Phi^{\mathbf{1}}(p^{[n]}) e_h^{[n]\mathbf{1}} \geq 0 \\
& u_h \left(x_h^{[n]} \right) - u_h \left(\frac{1}{4} e_h^{[n]} \right) \geq 0.
\end{aligned}$$

Notice that, since $L^{[n]} \rightarrow \bar{L}$, there exists σ such that $\bar{L} \in V_{\sigma-1}$ and for sufficiently large n , $L^{[n]} \in V_{\sigma-1}$, too. Hence in the above inequalities we wrote P_σ instead of $P_{\sigma^{[n]}}$. Taking limits and using assumptions (4), $a.$ and $c.$, we get the desired result.

5. β_h is lower hemi continuous.

We want to show that for any $(p^{[n]}, e_h^{[n]}, \alpha_h^{[n]}, L^{[n]})_{n \in \mathbb{N}} \in (\Delta_{++}^{G-1} \times \mathbb{R}_{++}^G \times (0, 1)^{SH} \times \mathcal{G}_{A,S})^\infty$ such that $(p^{[n]}, e_h^{[n]}, \alpha_h^{[n]}, L^{[n]}) \rightarrow (\bar{p}, \bar{e}_h, \bar{\alpha}_h, \bar{L})$, and for any $(\bar{x}_h, \bar{b}_h) \in \beta_h(\bar{p}, \bar{e}_h, \bar{\alpha}_h, \bar{L})$, there exists $(x_h^{[n]}, b_h^{[n]})_{n \in \mathbb{N}} \in (\mathbb{R}_{++}^G \times \mathbb{R}^A)^\infty$ such that

- $a.$ $\forall n \in \mathbb{N}$, $(x_h^{[n]}, b_h^{[n]}) \in \beta_h(p^{[n]}, e_h^{[n]}, \alpha_h^{[n]}, L^{[n]})$ and
- $b.$ $(x_h^{[n]}, b_h^{[n]}) \rightarrow (\bar{x}_h, \bar{b}_h)$.

We proceed as follows.

Step 1.

$$\tilde{\beta}_h : \Delta_{++}^{G-1} \times \mathbb{R}_{++}^G \times (0, 1)^{SH} \times \mathcal{G}_{A,S} \rightrightarrows \mathbb{R}_{++}^G \times \mathbb{R}^A,$$

$$\begin{aligned}
\tilde{\beta}_h(p, e_h, \alpha_h, L) = \{ & (x_h, b_h) \in \mathbb{R}_{++}^G \times \mathbb{R}^A : -p(0) (x_h(0) - e_h(0)) - \mathbf{1} \cdot P_\sigma \left[\begin{array}{c} -\psi(L) \\ I_A \end{array} \right] b_h > 0 \\
& -\Phi^{\mathbf{1}}(p) (x_h^{\mathbf{1}} - e_h^{\mathbf{1}}) + P_\sigma \left[\begin{array}{c} -\psi(L) \\ I_A \end{array} \right] b_h > 0 \\
& P_\sigma \left[\begin{array}{c} -\psi(L) \\ I_A \end{array} \right] b_h + A_h \Phi^{\mathbf{1}}(p) e_h^{\mathbf{1}} > 0 \\
& u_h(x_h) - u_h(\frac{1}{4}e_h) > 0 \}.
\end{aligned}$$

is lower hemi continuous.

Step 2. $\text{Cl} \tilde{\beta}_h = \beta_h$.

As Step 2. directly follows from the fact that β_h is closed, we prove Step 1. only.

First of all observe that $\tilde{\beta}_h$ is nonempty valued as $(\frac{1}{2}e_h, 0) \in \tilde{\beta}_h(p, e_h, \alpha_h, L)$.

Moreover,

$$\begin{aligned}
& -p^{[n]}(0) \left(\bar{x}_h(0) - e_h^{[n]}(0) \right) - \mathbf{1} \cdot P_\sigma \left[\begin{array}{c} -\psi(L^{[n]}) \\ I_A \end{array} \right] \bar{b}_h \rightarrow -\bar{p}(0) \left(\bar{x}_h(0) - \bar{e}_h(0) \right) - \mathbf{1} \cdot P_\sigma \left[\begin{array}{c} -\psi(\bar{L}) \\ I_A \end{array} \right] \bar{b}_h > 0 \\
& -\Phi^{\mathbf{1}}(p^{[n]}) \cdot \bar{x}_h^{\mathbf{1}} + \Phi^{\mathbf{1}}(p^{[n]}) \cdot e_h^{[n]\mathbf{1}} + P_\sigma \left[\begin{array}{c} -\psi(L^{[n]}) \\ I_A \end{array} \right] \bar{b}_h \rightarrow -\Phi^{\mathbf{1}}(\bar{p}) \cdot \bar{x}_h^{\mathbf{1}} + \Phi^{\mathbf{1}}(\bar{p}) \cdot \bar{e}_h^{\mathbf{1}} + P_\sigma \left[\begin{array}{c} -\psi(\bar{L}) \\ I_A \end{array} \right] \bar{b}_h > 0 \\
& P_\sigma \left[\begin{array}{c} -\psi(L^{[n]}) \\ I_A \end{array} \right] \bar{b}_h + A_h^{[n]} \Phi^{\mathbf{1}}(p^{[n]}) e_h^{[n]\mathbf{1}} \rightarrow P_\sigma \left[\begin{array}{c} -\psi(\bar{L}) \\ I_A \end{array} \right] \bar{b}_h + \bar{A}_h \Phi^{\mathbf{1}}(\bar{p}) \bar{e}_h^{\mathbf{1}} > 0 \\
& u_h(\bar{x}_h) - u_h\left(\frac{1}{4}e_h^{[n]}\right) \rightarrow u_h(\bar{x}_h) - u_h\left(\frac{1}{4}\bar{e}_h\right) > 0.
\end{aligned}$$

Notice that inequalities are strict because, by assumption, $(\bar{x}_h, \bar{b}_h) \in \tilde{\beta}_h(\bar{p}, \bar{e}_h, \bar{\alpha}_h, \bar{L})$. Then, $\exists N \in \mathbb{N}$ such that $\forall n > N$

$$\begin{aligned}
& -p^{[n]}(0) \left(\bar{x}_h(0) - e_h^{[n]}(0) \right) - \mathbf{1} \cdot P_\sigma \left[\begin{array}{c} -\psi(L^{[n]}) \\ I_A \end{array} \right] \bar{b}_h > 0 \\
& -\Phi^{\mathbf{1}}(p^{[n]}) \cdot \bar{x}_h^{\mathbf{1}} + \Phi^{\mathbf{1}}(p^{[n]}) \cdot e_h^{[n]\mathbf{1}} + P_\sigma \left[\begin{array}{c} -\psi(L^{[n]}) \\ I_A \end{array} \right] \bar{b}_h > 0 \\
& P_\sigma \left[\begin{array}{c} -\psi(L^{[n]}) \\ I_A \end{array} \right] \bar{b}_h + A_h^{[n]} \Phi^{\mathbf{1}}(p^{[n]}) e_h^{[n]\mathbf{1}} > 0 \\
& u_h(\bar{x}_h) - u_h\left(\frac{1}{4}e_h^{[n]}\right) > 0.
\end{aligned} \tag{37}$$

For $n < N$, choose an arbitrary $(x_h^{[n]}, b_h^{[n]}) \in \tilde{\beta}_h(p^{[n]}, e_h^{[n]}, \alpha_h^{[n]}, L^{[n]}) \neq \emptyset$.

Since $\forall n > N$, inequalities in (37) do hold true, we also have that $\forall n > N$, $\exists \varepsilon^{[n]} \in \mathbb{R}_{++}$ such that $\forall (\xi, \zeta) \in B((\bar{x}_h, \bar{b}_h), \varepsilon^{[n]})$ we have that

$$\begin{aligned}
& -p^{[n]}(0) \left(\xi - e_h^{[n]}(0) \right) - \mathbf{1} \cdot P_\sigma \left[\begin{array}{c} -\psi(L^{[n]}) \\ I_A \end{array} \right] \zeta > 0 \\
& -\Phi^{\mathbf{1}}(p^{[n]}) \cdot \xi^{\mathbf{1}} + \Phi^{\mathbf{1}}(p^{[n]}) \cdot e_h^{[n]\mathbf{1}} + P_\sigma \left[\begin{array}{c} -\psi(L^{[n]}) \\ I_A \end{array} \right] \zeta > 0 \\
& P_\sigma \left[\begin{array}{c} -\psi(L^{[n]}) \\ I_A \end{array} \right] \zeta + A_h^{[n]} \Phi^{\mathbf{1}}(p^{[n]}) e_h^{[n]\mathbf{1}} > 0 \\
& u_h(\xi) - u_h\left(\frac{1}{4}e_h^{[n]}\right) > 0.
\end{aligned} \tag{38}$$

For any $n > N$, choose

$$\begin{aligned}
x_h^{[n]} &= \bar{x}_h + \frac{1}{\sqrt{G+A}} \min \left\{ \frac{\varepsilon^{[n]}}{2}, \frac{1}{n} \right\} \cdot \mathbf{1}, \\
b_h^{[n]} &= \bar{b}_h + \frac{1}{\sqrt{G+A}} \min \left\{ \frac{\varepsilon^{[n]}}{2}, \frac{1}{n} \right\} \cdot \mathbf{1}.
\end{aligned}$$

Then, $d\left((\bar{x}_h, \bar{b}_h), (x_h^{[n]}, b_h^{[n]})\right) = \min\left\{\frac{\varepsilon^{[n]}}{2}, \frac{1}{n}\right\} < \varepsilon^{[n]}$. Therefore, from (38), $(x_h^{[n]}, b_h^{[n]}) \in \tilde{\beta}_h(p^{[n]}, e_h^{[n]}, \alpha_h^{[n]}, L^{[n]})$. Moreover,

$$0 \leq \lim_{n \rightarrow +\infty} d\left((\bar{x}_h, \bar{b}_h), (x_h^{[n]}, b_h^{[n]})\right) \leq \lim_{n \rightarrow +\infty} \frac{1}{n} = 0,$$

i.e., $(x_h^{[n]}, b_h^{[n]}) \rightarrow (\bar{x}_h, \bar{b}_h)$, as desired. \square

6.5 Proof of Theorem 11

The proof of Theorem 11 requires some preliminary results before proceeding.

Define $\Delta_+^{G-1} = \{p \in \mathbb{R}_+^G : \sum_{s=0}^S \sum_{c=1}^C p^c(s) = 1\}$ and $\Pi^{G-1} = \{p \in \mathbb{R}^G : \sum_{s=0}^S \sum_{c=1}^C p^c(s) = 1\}$. In what follows, we take for given an economy (e, u, y, α) .

From Proposition 10, we can define the following continuous functions.

$$x_h : \Delta_{++}^{G-1} \times \mathcal{G}_{A,S} \rightarrow \mathbb{R}^G, \quad \text{for } h \in \mathcal{H},$$

$$x_1(p, L) = \arg \max (19),$$

$$x_h(p, L) = \arg \max (17), \quad \text{for } h \in \mathcal{H} \setminus \{1\},$$

and

$$z : \Delta_{++}^{G-1} \times \mathcal{G}_{A,S} \rightarrow \mathbb{R}^G, \quad (p, L) \mapsto \sum_{h \in \mathcal{H}} (x_h(p, L) - e_h). \quad (39)$$

Define also

$$\psi : \Delta_+^{G-1} \times \mathcal{G}_{A,S} \rightarrow \mathbb{R}^{SA}, \quad (p, L) \mapsto \mathcal{R}(p, y). \quad (40)$$

We say that a vector (p^*, L^*) is a reduced image - Mr. 1 equilibrium for the economy $(e, u, y, \alpha) \in \mathcal{E}$, if there exists x^* such that (x^*, p^*, L^*) is an image - Mr. 1 equilibrium for that economy.

Proposition 20. *A vector (p^*, L^*) is a reduced image - Mr. 1 equilibrium for the economy $(e, u, y, \alpha) \in \mathcal{E}$ if*

1. $z(p^*, L^*) = 0$ and
2. $\langle \psi(p^*, L^*) \rangle \subseteq L^*$.

In the next result we list some properties of the function z in (39) we will need in the proof of Lemma 23.

Lemma 21. *1. z is continuous;*

2. z satisfies Walras' law;

3. z is bounded from below;

4. z satisfies the boundary condition, i.e., if $(p^{[n]}, L^{[n]}) \rightarrow (\bar{p}, \bar{L})$ with $\bar{p} \in \partial \Delta_+^{G-1}$, then $\|z(p^{[n]}, L^{[n]})\| \rightarrow \infty$.

Proof. 1. It follows from Proposition 10.

2. It follows from household budget constraints.

3. From market clearing, for every s, c , $z^c(s)$ is bounded below by $-\sum_{h \in \mathcal{H}} e_h^c(s)$.

4. It follows from the budget constraint and the strict monotonicity of u_h . \square

In the proof of Theorem 11 we are going to use the following result contained in Husseini et al. [1990].

Theorem 22 (A Grassmannian Brouwer-like fixed point theorem). *Let H^N be an N -dimensional affine subspace, $C \subset H^N$ a compact convex subset with nonempty relative interior and let*

$$\Phi : C \times \mathcal{G}_{A,S} \rightarrow H^N, \quad \Psi : C \times \mathcal{G}_{A,S} \rightarrow \mathbb{R}^{AS}$$

be continuous functions such that $\Phi(\partial C, L) \subseteq C, \forall L \in \mathcal{G}_{A,S}$. Then there exists (\bar{p}, \bar{L}) such that

$$\Phi(\bar{p}, \bar{L}) = \bar{p}, \quad \langle \Psi(\bar{p}, \bar{L}) \rangle \subseteq \bar{L}.$$

A crucial role in the application of the above theorem is played by the following lemma. We present the proof of the lemma in the case, analyzed in the present paper, in which the return space is described as a Grassmannian manifold. In fact, Husseini et al. [1990] presented instead the proof in the case of Stiefel manifolds.

Lemma 23. *There exists a continuous function $\alpha : \Delta_+^{G-1} \times \mathcal{G}_{A,S} \rightarrow [0, 1]$ such that the function $\phi : \Delta_+^{G-1} \times \mathcal{G}_{A,S} \rightarrow \Pi^{G-1}$ defined by*

$$\phi(p, L) = \alpha(p, L) ((p^c(s) + p^c(s) z^c(s)(p, L))_{s,c} + (1 - \alpha(p, L))u), \quad (41)$$

where $u = (\frac{1}{G}, \dots, \frac{1}{G}) \in \mathbb{R}^G$, satisfies

1. $\phi(\partial \Delta_+^{G-1}, L) \subseteq \Delta_+^{G-1}, \forall L \in \mathcal{G}_{A,S}$;
2. $\phi(p, L) = p \Leftrightarrow z(p, L) = 0$.¹⁴

Proof. Define

$$V_j = \left\{ (p, L) \in \Delta_{++}^{G-1} \times \mathcal{G}_{A,S} : z_j(p, L) > 0, p_j < \frac{1}{G} \right\}, \quad j \in \{1, \dots, G\}$$

and $K = (\Delta_{++}^{G-1} \times \mathcal{G}_{A,S}) \setminus \left(\bigcup_{j=1}^G V_j \right)$. We are going to prove that K is closed in $\Delta_+^{G-1} \times \mathcal{G}_{A,S}$. Since $\mathcal{G}_{A,S}$ is a metric space, also $\Delta_+^{G-1} \times \mathcal{G}_{A,S}$ is a metric space. Thus it is enough to prove that K is sequentially closed, i.e., that the limit point of any convergent sequence of elements of K belongs to K .

Rewriting K as follows

$$K = \left\{ (p, L) \in \Delta_{++}^{G-1} \times \mathcal{G}_{A,S} : \forall j \in \{1, \dots, G\}, z_j(p, L) \leq 0 \text{ or } p_j \geq \frac{1}{G} \right\}$$

¹⁴Notice that, although z in (39) is defined only on $\Delta_{++}^{G-1} \times \mathcal{G}_{A,S}$, the function ϕ is defined on $\Delta_+^{G-1} \times \mathcal{G}_{A,S}$ because, by construction, $\alpha(\partial \Delta_+^{G-1} \times \mathcal{G}_{A,S}) = 0$ and thus $\phi(\partial \Delta_+^{G-1} \times \mathcal{G}_{A,S}) = u$.

and recalling that $\mathcal{G}_{A,S}$ is compact, and thus closed, it is clear that the only way in which the limit point (\bar{p}, \bar{L}) of a sequence $(p^{[n]}, L^{[n]})$ of elements of K does not belong to K is that $\bar{p} \in \partial\Delta_+^{G-1}$. However this is prevented by the boundary condition and the continuity of z on K . Indeed, if $\bar{p} \in \partial\Delta_+^{G-1}$, then there exists $j \in \{1, \dots, G\}$ such that $\bar{p}_j = 0$. Hence there exists $\bar{n} \in \mathbb{N}$ such that, for every $n \geq \bar{n}$, $p_j^{[n]} < \frac{1}{G}$. By definition of K and recalling that z is bounded from below, we then have $-\sum_{h \in \mathcal{H}} e_h(j) \leq z_j(p^{[n]}, L^{[n]}) \leq 0$ and thus, by the continuity of z , it holds that $-\sum_{h \in \mathcal{H}} e_h(j) \leq z_j(\bar{p}, \bar{L}) \leq 0$. On the other hand, by the boundary condition, $z_j(p^{[n]}, L^{[n]}) \rightarrow +\infty$. The contradiction is found.

Notice that $K \cap (\partial\Delta_+^{G-1} \times \mathcal{G}_{A,S}) = \emptyset$ and that $\partial\Delta_+^{G-1} \times \mathcal{G}_{A,S}$ is closed in $\Delta_+^{G-1} \times \mathcal{G}_{A,S}$. Recalling that any metric space is normal¹⁵ and that on normal spaces the Urysohn Lemma¹⁶ applies, there exists a continuous function $\alpha : \Delta_+^{G-1} \times \mathcal{G}_{A,S} \rightarrow [0, 1]$ such that $\alpha(K) = 1$ and $\alpha(\partial\Delta_+^{G-1} \times \mathcal{G}_{A,S}) = 0$. Let us then check that the function ϕ in (41) has Π^{G-1} as codomain and satisfies 1. and 2. As regards the codomain of ϕ , fix $(p, L) \in \Delta_+^{G-1} \times \mathcal{G}_{A,S}$. Then $\sum_{j=1}^G p_j = 1$ and recalling that z obeys Walras' law, it holds that

$$\begin{aligned} \sum_{j=1}^G \phi_j(p, L) &= \sum_{j=1}^G \left(\alpha(p, L)(p_j + p_j z_j(p, L)) + (1 - \alpha(p, L)) \frac{1}{G} \right) = \\ \alpha(p, L) \sum_{j=1}^G (p_j + p_j z_j(p, L)) + (1 - \alpha(p, L)) \sum_{j=1}^G \frac{1}{G} &= \alpha(p, L) \sum_{j=1}^G p_j z_j(p, L) + 1 = 1, \end{aligned}$$

i.e., $\phi(\Delta_+^{G-1} \times \mathcal{G}_{A,S}) \subseteq \Pi^{G-1}$, as desired.

In regard to 1., as we already know that $\phi(\partial\Delta_+^{G-1} \times \mathcal{G}_{A,S}) \subseteq \Pi^{G-1}$, in order to show that $\phi(\partial\Delta_+^{G-1} \times \mathcal{G}_{A,S}) \subseteq \Delta_+^{G-1}$, it is enough to check that $\phi_j(p, L) \geq 0$, for every $(p, L) \in \partial\Delta_+^{G-1} \times \mathcal{G}_{A,S}$ and $j \in \{1, \dots, G\}$. Since $\alpha(\partial\Delta_+^{G-1} \times \mathcal{G}_{A,S}) = 0$, it holds that, for $(p, L) \in \partial\Delta_+^{G-1} \times \mathcal{G}_{A,S}$, $\phi_j(p, L) = \frac{1}{G} \geq 0$.

Let us finally prove 2. Assume that $z(p, L) = 0$ and show that $\phi(p, L) = p$. If $z(p, L) = 0$ then $z_j(p, L) \leq 0$ for every j and so $(p, L) \in K$. Hence, $\alpha(p, L) = 1$ and thus $\phi_j(p, L) = p_j + p_j z_j(p, L) = p_j$, for every j , as desired.

Assume now that $\phi(p, L) = p$ and show that $z(p, L) = 0$. Notice that

$$\Delta_+^{G-1} \times \mathcal{G}_{A,S} = (\partial\Delta_+^{G-1} \times \mathcal{G}_{A,S}) \cup K \cup \left(\bigcup_{j=1}^G V_j \right)$$

and that $\partial\Delta_+^{G-1} \times \mathcal{G}_{A,S}$, K and $\bigcup_{j=1}^G V_j$ are pairwise disjoint. If $(p, L) \in \Delta_+^{G-1} \times \mathcal{G}_{A,S}$, then there are three cases to consider, i.e., $(p, L) \in \partial\Delta_+^{G-1} \times \mathcal{G}_{A,S}$, $(p, L) \in K$ and $(p, L) \in V_{j^*}$, for some $j^* \in \{1, \dots, G\}$. We claim that only in the second case it may happen that $\phi(p, L) = p$. Indeed, in the first case $\phi(p, L) = (\frac{1}{G}, \dots, \frac{1}{G}) \notin \Delta_+^{G-1}$. In the third case, by definition of V_{j^*} , we would have

$$p_{j^*} = \phi_{j^*}(p, L) = \alpha(p, L)(p_{j^*} + p_{j^*} z_{j^*}(p, L)) + (1 - \alpha(p, L)) \frac{1}{G} >$$

¹⁵A topological space X is called normal if for any pair of closed disjoint subsets C_1 and C_2 of X there exists a pair of open disjoint subsets O_1 and O_2 of X , with $O_1 \supset C_1$ and $O_2 \supset C_2$.

¹⁶We recall that the Urysohn lemma says that given two disjoint closed subsets C_1 and C_2 of a normal space X , there exists a continuous function $f : X \rightarrow [0, 1]$ such that $f(C_1) = 0$ and $f(C_2) = 1$.

$$\alpha(p, L)(p_{j^*} + p_{j^*} z(j^*)(p, L)) + (1 - \alpha(p, L))p_{j^*} = p_{j^*} + \alpha(p, L)p_{j^*} z(j^*)(p, L) \geq p_{j^*},$$

a contradiction. Thus $\phi(p, L) = p$ only if $(p, L) \in K$ and in this case, as $\alpha(p, L) = 1$, it follows that, for every $j \in \{1, \dots, G\}$, $p_j = \phi_j(p, L) = p_j + p_j z_j(p, L)$, from which, since $p_j > 0$, we have $z_j(p, L) = 0$, as desired. \square

Proof of Theorem 11. We want to apply Theorem 22, identifying Φ and Ψ with ϕ in (41) and ψ in (40), respectively. Π^{G-1} is an affine subspace of \mathbb{R}^G and $\Delta_+^{G-1} \subset \Pi^{G-1}$ is clearly a compact, convex subset with nonempty relative interior. ϕ is a continuous function from Lemma 21 and from the fact that α is a continuous function from Lemma 23. Again from the latter lemma, we have that $\phi(\partial\Delta_+^{G-1}, L) \subseteq \Delta_+^{G-1}$, $\forall L \in \mathcal{G}_{A,S}$. Finally, from Theorem 22 and Proposition 20, the desired result follows. \square

6.6 Proof of Theorem 12

Let \mathcal{V} be a topological Hausdorff vector space, $V \subseteq \mathcal{V}$ be an open set and $f : V \rightarrow \mathbb{R}^n$ be a function. We say that $f \in C^0(V, \mathbb{R}^n)$ if f is continuous, while $f \in C^1(V, \mathbb{R}^n)$ if it is continuous, there exists the limit

$$df(v, w) = \lim_{\varepsilon \rightarrow 0} \frac{f(v + \varepsilon w) - f(v)}{\varepsilon}, \quad \forall v \in V, w \in \mathcal{V},$$

and the function $df : V \times \mathcal{V} \rightarrow \mathbb{R}^n$ is continuous.

Given any (not necessarily open) set $X \subseteq \mathcal{V}$, and $f : X \rightarrow \mathbb{R}^n$, we say $f \in C^0(X, \mathbb{R}^n)$ if f is continuous with respect to the topology induced by \mathcal{V} on X , while, as in the finite dimensional setting, $f \in C^1(X, \mathbb{R}^n)$ if for every $v_0 \in X$ there exists an open neighborhood of v_0 in \mathcal{V} , say $V(v_0)$, and a function $\bar{f} : V(v_0) \rightarrow \mathbb{R}^n$ such that $\bar{f} \in C^1(V(v_0), \mathbb{R}^n)$ and, for every $v \in V(v_0) \cap X$, $f(x) = \bar{f}(x)$.

Those definitions allow to state the following implicit function theorem which is a simplified version of Theorem 2.3 in Glöckner [2006].

Theorem 24. *Let us consider $f : O \times V \rightarrow \mathbb{R}^n$, where O is an open subset of \mathbb{R}^n and V is an open subset of a topological Hausdorff vector space \mathcal{V} . Assume $f \in C^1(O \times V, \mathbb{R}^n)$ and let $(x_0, v_0) \in O \times V$ such that $f(x_0, v_0) = 0$ and $D_x f(x_0, v_0)$ is invertible.¹⁷ Then there exist $O(x_0) \subseteq O$ open neighborhood of x_0 , $V(v_0) \subseteq V$ open neighborhood of v_0 and $g : V(v_0) \rightarrow O(x_0)$ such that*

1. $g \in C^1(V(v_0), O(x_0))$,
2. $g(v_0) = x_0$,
3. $\{(x, v) \in O(x_0) \times V(v_0) : f(x, v) = 0\} = \{(x, v) \in O(x_0) \times V(v_0) : x = g(v)\}$.

Proof of Theorem 12.

Step 1.

¹⁷Note that if $f \in C^1(O \times V, \mathbb{R}^n)$ then, for every $v \in V$, $f(\cdot, v) : O \rightarrow \mathbb{R}^n$, $x \mapsto f(x, v)$, belongs to $C^1(O, \mathbb{R}^n)$ and thus, for every $(x, v) \in O \times V$, $D_x f(x, v)$ is well defined.

Define, for each $\sigma \in \Sigma$,

$$\Xi_\sigma = \mathbb{R}_{++}^{GH} \times \mathbb{R}^{(S+1)H} \times \mathbb{R}^{AH} \times \mathbb{R}^{SH} \times \mathbb{R}_{++}^G \times \mathbb{R}^A \times V_{\sigma^{-1}}, \quad (42)$$

with generic element

$$\xi = ((x_h, \lambda_h, b_h, \mu_h)_{h \in \mathcal{H}}, p, q, L) = (x, \lambda, b, \mu, p, q, L),$$

and the function

$$\mathcal{F}_\sigma : \Xi_\sigma \times \mathcal{E} \rightarrow \mathbb{R}^{\dim(\Xi_\sigma)},$$

$$\mathcal{F}_\sigma(\xi, e, u, y, \alpha) = \begin{bmatrix} (43.1) & D_{x_h(s)} u_h(x_h) - \lambda_h(s) p(s) \\ (43.2) & -\Phi(p)(x_h - e_h) + \begin{bmatrix} -q \\ P_\sigma \begin{bmatrix} -\psi(L) \\ I_A \end{bmatrix} \end{bmatrix} b_h \\ (43.3) & \lambda_h \begin{bmatrix} -q \\ P_\sigma \begin{bmatrix} -\psi(L) \\ I_A \end{bmatrix} \end{bmatrix} + \mu_h P_\sigma \begin{bmatrix} -\psi(L) \\ I_A \end{bmatrix} \\ (43.4) & \min \left\{ \mu_h, P_\sigma \begin{bmatrix} -\psi(L) \\ I_A \end{bmatrix} b_h + A_h \Phi^1(p) e_h^1 \right\} \\ (43.5) & \sum_{h=1}^H (x_h \setminus - e_h \setminus) \\ (43.6) & \sum_{h=1}^H b_h \\ (43.7) & p^1(s) - 1 \\ (43.8) & \text{vec}[I_{S-A} \mid \psi(L)] \cdot P_{\sigma^{-1}} \cdot \mathcal{R}(p, y) \end{bmatrix} \quad (43)$$

where $\psi : V_{\sigma^{-1}} \rightarrow \mathbb{M}(S-A, A)$ is the diffeomorphism in (14), with $V_{\sigma^{-1}} \subseteq \mathcal{G}_{A,S}$ open.

For simplicity and without loss of generality, from now on we consider the case $P_\sigma = Id$, so that \mathcal{F}_σ becomes $\mathcal{F} : \Xi \times \mathcal{E} \rightarrow \mathbb{R}^{\dim(\Xi)}$.

We now show that border line cases are rare. For every $h \in \mathcal{H}$, we define $\mathcal{S}_h^1, \mathcal{S}_h^2$ and $\widehat{\mathcal{S}}_h^1$ so that $\{1, \dots, S\} = \mathcal{S}_h^1 \cup \mathcal{S}_h^2$, with $(\mathcal{S}_h^1 \setminus \widehat{\mathcal{S}}_h^1) \cap \mathcal{S}_h^2 = \emptyset$ and $\widehat{\mathcal{S}}_h^1 \subseteq \mathcal{S}_h^1$, in order to have

$$s \in \mathcal{S}_h^1 \setminus \widehat{\mathcal{S}}_h^1 \Rightarrow m(s) \cdot b_h + \alpha_h(s) \cdot p(s) e_h(s) = 0$$

$$s \in \widehat{\mathcal{S}}_h^1 \Rightarrow m(s) \cdot b_h + \alpha_h(s) \cdot p(s) e_h(s) = 0 \quad \text{and} \quad \mu_h(s) = 0$$

$$s \in \mathcal{S}_h^2 \Rightarrow \mu_h(s) = 0,$$

where we have denoted by $m(s)$ the s -th row of $\begin{bmatrix} -\psi(L) \\ I_A \end{bmatrix}$.

Define $\hat{y} = (y^{a,1}(s))_{a \in \mathcal{A}, s \in \{1, \dots, S-A\}}$ and the full rank matrix

$$B = \begin{bmatrix} p^1(1) & & & & & & & & \\ & \ddots & & & & & & & \\ & & p^1(1) & & & & & & \\ & & & \ddots & & & & & \\ & & & & p^1(S-A) & & & & \\ & & & & & \ddots & & & \\ & & & & & & p^1(S-A) & & \\ & & & & & & & \ddots & \\ & & & & & & & & p^1(S-A) \end{bmatrix}.$$

The computation of the desired (partial) Jacobian matrix is presented in the table below, where the following conventions are adopted.

(a) The symbol \odot denotes a matrix which is insignificant for our argument, while no symbol means 0. For size convenience, just in the table below, we set

$$R(q) = \begin{bmatrix} -q \\ -\psi(L) \\ I \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} -\psi(L) \\ I \end{bmatrix}.$$

(b) The * next to a matrix indicates that it is a full row rank matrix.

(c) The desired full rank result is obtained as follows. In each super-row, use the starred matrix to clean up that super-row, being sure that in that super-column there are only zero matrices. An order in which the appropriate elementary (super) column operations have to be performed is the one indicated in the last column of the table.

$D^2 u_1^*$	$\begin{bmatrix} -p(0) \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ -\Phi^1(p) \end{bmatrix}$																\odot	5
$-\Phi(p)$			$R(q)$															\odot	3
	q^T	$R^T *$		R^T															7
			\odot															\odot	2
				$\begin{bmatrix} 0 & & I_{S_1^2} \end{bmatrix} *$															8
					\ddots														\vdots
						$D^2 u_H^*$	$\begin{bmatrix} -p(0) \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ -\Phi^1(p) \end{bmatrix}$										\odot	5
						$-\Phi(p)$			$R(q)$									\odot	3
							q^T	$R^T *$		R^T									7
									\odot									\odot	2
																			9
																			$\begin{bmatrix} 0 & & I_{S_H^2} \end{bmatrix} *$
$0I \setminus$					\cdots	$0I \setminus$												I^*	4
			I		\cdots														6
																		1^*	10
																			8
																			\vdots
																			9
																			$\begin{bmatrix} I_{S_H^1} & & 0 & & 0 \end{bmatrix} *$
					\ddots														B^*
																		\odot	1

where, for $h \in \mathcal{H}$, we have set Z_h^1 equal to the square diagonal matrix with elements $p(s)e_h(s)$, for $s \in \mathcal{S}_h^1$, on the diagonal. Moreover

$$\Phi(p^1) = \begin{bmatrix} p^1(0) & & & \\ & p^1(1) & & \\ & & \ddots & \\ & & & p^1(S) \end{bmatrix}.$$

Step 2.

After having shown that border line cases are rare, we are going to prove that in a full measure subset of $\mathbb{R}_{++}^{GH} \times \mathbb{R}^{CAS}$, the return matrix $\mathcal{R}(p, y)$ has full rank. Actually, we are going to show that its square A -dimensional submatrix $\widehat{\mathcal{R}}(p, y)$ has full rank, where

$$\widehat{\mathcal{R}}(p, y) = \begin{bmatrix} p(S-A+1)y^1(S-A+1) & \cdots & p(S-A+1)y^a(S-A+1) & \cdots & p(S-A+1)y^A(S-A+1) \\ \vdots & & \vdots & & \vdots \\ p(S-A+a)y^1(S-A+a) & \cdots & p(S-A+a)y^a(S-A+a) & \cdots & p(S-A+a)y^A(S-A+a) \\ \vdots & & \vdots & & \vdots \\ p(S)y^1(S) & \cdots & p(S)y^a(S) & \cdots & p(S)y^A(S) \end{bmatrix},$$

by showing that 0 is a regular value for the function $(F, G) : \Xi \times \mathcal{E} \times \mathbb{R}^A \rightarrow \mathbb{R}^{\dim(\Xi)+(A+1)}$, where

$$G : \Xi \times \mathcal{E} \times \mathbb{R}^A \rightarrow \mathbb{R}^{A+1}, \quad (\xi, e, u, y, \alpha, d) \mapsto (d \cdot \widehat{\mathcal{R}}(p, y), dd - 1).$$

Calling T the Jacobian matrix in the previous page, we then have to show that the following matrix

$$\begin{bmatrix} T & \star & 0 \\ 0 & N & \widehat{\mathcal{R}}(p, y) \\ 0 & 0 & 2d^T \end{bmatrix}$$

has full rank, where the last two columns are the derivatives with respect to \widehat{y} and d , respectively, with \widehat{y} defined hereinafter.

As $d = (d(S-A+a))_{a \in \mathcal{A}}$ is such that $dd = 1$, then there exists $\bar{a} \in \mathcal{A}$ such that $d(S-A+\bar{a}) \neq 0$. Then we set $\widehat{y} = (y^{a,1}(S-A+\bar{a}))_{a \in \mathcal{A}} \in \mathbb{R}^A$. Notice that

$$d \cdot \widehat{\mathcal{R}}(p, y) = \begin{bmatrix} \sum_{j=1}^A d(S-A+j)p(S-A+j)y^1(S-A+j) \\ \vdots \\ \sum_{j=1}^A d(S-A+j)p(S-A+j)y^a(S-A+j) \\ \vdots \\ \sum_{j=1}^A d(S-A+j)p(S-A+j)y^A(S-A+j) \end{bmatrix}$$

and thus

$$N = \begin{bmatrix} d(S-A+\bar{a})p^1(S-A+\bar{a}) & & & \\ & \ddots & & \\ & & d(S-A+\bar{a})p^1(S-A+\bar{a}) & \\ & & & \ddots \\ & & & & d(S-A+\bar{a})p^1(S-A+\bar{a}) \end{bmatrix}$$

which has clearly full rank. This concludes the proof of the step.

Step 3.

The proof is given in Proposition 25 below.

Step 4.

Apply Theorem 24. □

Recalling the definition of Ξ_σ in (42), we rewrite the function $\mathcal{F}_\sigma : \Xi_\sigma \times \mathcal{E} \rightarrow \mathbb{R}^{\dim(\Xi_\sigma)}$ in (43) as

$$\mathcal{F}_\sigma(\xi, e, u, y, \alpha) = \left[\begin{array}{l} (44.1) \quad D_{x_h(s)} u_h(x_h) - \lambda_h(s) p(s) \\ \quad - p(0)(x_h(0) - e_h(0)) - q b_h \\ (44.2) \quad - p(s)(x_h(s) - e_h(s)) - \sum_{a=1}^A m_{\sigma^{-1}(s)a} b_h^a, \quad \sigma^{-1}(s) \in \{1, \dots, S-A\} \\ \quad - p(s)(x_h(s) - e_h(s)) + b_h^{\sigma^{-1}(s)-(S-A)}, \quad \sigma^{-1}(s) \in \{S-A+1, \dots, S\} \\ (44.3) \quad - \lambda_h(0) q^{\sigma^{-1}(s)-(S-A)} - \sum_{\substack{\sigma^{-1}(s)=1 \\ \sigma^{-1}(s)=1}}^{S-A} (\lambda_h(s) + \mu_h(s)) m_{\sigma^{-1}(s)(\sigma^{-1}(s)-(S-A))} + \\ \quad + \lambda_h(s) + \mu_h(s), \quad \sigma^{-1}(s) \in \{S-A+1, \dots, S\} \\ (44.4) \quad \min \left\{ \mu_h, P_\sigma \begin{bmatrix} -\psi(L) \\ I_A \end{bmatrix} b_h + A_h \Phi^1(p) e_h^1 \right\} \\ (44.5) \quad \sum_{h=1}^H (x_h \setminus - e_h \setminus) \\ (44.6) \quad \sum_{h=1}^H b_h \\ (44.7) \quad p^1(s) - 1 \\ (44.8) \quad \text{vec}[I_{S-A} \mid \psi(L)] \cdot P_{\sigma^{-1}} \cdot \mathcal{R}(p, y) \end{array} \right] \quad (44)$$

where $L \in V_{\sigma^{-1}}$ and $\psi(L) = (m_{sa})_{s \in \{1, \dots, S-A\}, a \in \mathcal{A}} \in \mathbb{M}(S-A, A)$.

We also recall that a function $f : A \rightarrow B$, with A and B topological spaces, is proper if, for every $K \subseteq B$ compact set, $f^{-1}(K) \subseteq A$ is compact as well. We also recall that any proper and continuous function is closed, i.e., it maps closed sets onto closed sets.

Proposition 25. \mathcal{F}_σ is continuous on $\Xi_\sigma \times \mathcal{E}$ and

$$\pi : \bigcup_{\sigma \in \Sigma} \mathcal{F}_\sigma^{-1}(0) \rightarrow \mathcal{E}, \quad (\xi, e, u, y, \alpha) \mapsto \pi(\xi, e, u, y, \alpha) = (e, u, y, \alpha)$$

is proper.

Proof. The continuity of \mathcal{F}_σ is immediate. In order to show that π is proper, we have to prove that each sequence $(\xi^{[n]}, e^{[n]}, u^{[n]}, y^{[n]}, \alpha^{[n]})_{n \in \mathbb{N}}$ in $\bigcup_{\sigma \in \Sigma} \mathcal{F}_\sigma^{-1}(0)$, such that $(e^{[n]}, u^{[n]}, y^{[n]}, \alpha^{[n]})$ converges in \mathcal{E} , admits a converging subsequence in $\bigcup_{\sigma \in \Sigma} \mathcal{F}_\sigma^{-1}(0)$. Since $\mathcal{G}_{A,S}$ is sequentially compact, let us assume that

$$(e^{[n]}, u^{[n]}, y^{[n]}, \alpha^{[n]}, L^{[n]}) \rightarrow (\bar{e}, \bar{u}, \bar{y}, \bar{\alpha}, \bar{L}) \in \mathcal{E} \times \mathcal{G}_{A,S}.$$

Therefore there exists $\sigma \in \Sigma$ such that $\bar{L} \in V_{\sigma^{-1}}$ and for sufficiently large n , $L^{[n]} \in V_{\sigma^{-1}}$, too. Without loss of generality we can assume that $P_\sigma = Id$, so that \mathcal{F}_σ simply becomes

$$\mathcal{F}(\xi, e, u, y, \alpha) = \left[\begin{array}{l} (45.1) \quad D_{x_h(s)} u_h(x_h) - \lambda_h(s) p(s) \\ \quad \quad \quad - p(0)(x_h(0) - e_h(0)) - q b_h \\ (45.2) \quad -p(s)(x_h(s) - e_h(s)) - \sum_{a=1}^A m_{sa} b_h^a, \quad s \in \{1, \dots, S-A\} \\ \quad \quad \quad -p(s)(x_h(s) - e_h(s)) + b_h^{s-1(S-A)}, \quad s \in \{S-A+1, \dots, S\} \\ (45.3) \quad -\lambda_h(0) q^a - \sum_{s=1}^{S-A} (\lambda_h(s) + \mu_h(s)) m_{sa} + \lambda_h(S-A+a) + \mu_h(S-A+a) \\ (45.4) \quad \min \left\{ \mu_h, \left[\begin{array}{c} -\psi(L) \\ I_A \end{array} \right] b_h + A_h \Phi^1(p) e_h^1 \right\} \\ (45.5) \quad \sum_{h=1}^H (x_h \setminus - e_h \setminus) \\ (45.6) \quad \sum_{h=1}^H b_h \\ (45.7) \quad p^1(s) - 1 \\ (44.8) \quad \text{vec}[I_{S-A} \mid \psi(L)] \cdot \mathcal{R}(p, y) \end{array} \right] \quad (45)$$

Then it suffices to show that, up to a subsequence, $(\xi^{[n]})_{n \in \mathbb{N}}$ converges to a certain $\bar{\xi} \in \Xi$: indeed the condition $\mathcal{F}(\bar{\xi}, \bar{e}, \bar{u}, \bar{y}, \bar{\alpha}) = 0$ follows by the continuity of \mathcal{F} . As we are going to use a diagonal argument, every time we say that a sequence converges we mean it has a converging subsequence. Let us start with the convergence of $x^{[n]}$. For a fixed $h \in \mathcal{H}$, we know that, for every $n \in \mathbb{N}$, $(x_h^{[n]}, b_h^{[n]})$ is solution to the problem

$$\max_{(x_h, b_h)} u_h^{[n]}(x_h)$$

$$s.t. \quad -p^{[n]}(0) \left(x_h(0) - e_h^{[n]}(0) \right) - q^{[n]} b_h = 0 \quad (1)$$

$$-\Phi^1(p^{[n]}) \left(x_h^1 - e_h^{[n]1} \right) + \left[\begin{array}{c} -\psi(L^{[n]}) \\ I_A \end{array} \right] b_h = 0 \quad (2)$$

$$\left[\begin{array}{c} -\psi(L^{[n]}) \\ I_A \end{array} \right] b_h + A_h^{[n]} \Phi^1(p^{[n]}) e_h^{[n]1} \geq 0 \quad (3)$$

and then, since $(e_h^{[n]}, 0)$ belongs to the constraint set, it has to be $u_h^{[n]}(x_h^{[n]}) \geq u_h^{[n]}(e_h^{[n]})$. Since $(e_h^{[n]})_{n \in \mathbb{N}}$ converges to $\bar{e}_h \in \mathbb{R}_{++}^G$, it holds that the compact set $E_h = \{e_h^{[n]}\}_{n \in \mathbb{N}} \cup \{\bar{e}_h\}$ is a subset of \mathbb{R}_{++}^G and we have

$$u_h^{[n]}(x_h^{[n]}) \geq u_h^{[n]}(e_h^{[n]}) \geq \min_{x_h \in E_h} u_h^{[n]}(x_h) \geq \min_{x_h \in E_h} \bar{u}_h(x_h) - \varepsilon^{[n]},$$

for a suitable sequence $(\varepsilon^{[n]})_{n \in \mathbb{N}}$ in \mathbb{R}_{++} such that $\varepsilon^{[n]} \rightarrow 0$ if $n \rightarrow \infty$, by the definition of the topology on $C^2(\mathbb{R}_{++}^G)$. Indeed we can define, for every $n \in \mathbb{N}$,

$$\varepsilon^{[n]} = \max_{w \in E_h} |u_h^{[n]}(w) - \bar{u}_h(w)|.$$

Let $x_h^* \in E_h$ be such that $\min_{x_h \in E_h} \bar{u}_h(x_h) = \bar{u}_h(x_h^*)$, and let $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^G$ and $\delta > 0$ be small enough such that $x_h^* - 2\delta\mathbf{1} \in \mathbb{R}_{++}^G$. Obviously, since by (2), $\bar{u}_h(x_h^*) > \bar{u}_h(x_h^* - \delta\mathbf{1})$, there exists n_1 such that $n \geq n_1$ implies $\bar{u}_h(x_h^*) - \varepsilon^{[n]} \geq \bar{u}_h(x_h^* - \delta\mathbf{1})$ and thus, for every $n \geq n_1$,

$$u_h^{[n]}(x_h^{[n]}) \geq \bar{u}_h(x_h^* - \delta\mathbf{1}). \quad (46)$$

Of course, because of the validity of $S + 1$ Walras' laws in our model, we can also assume that, for every $n \geq n_1$,

$$0 \ll x_h^{[n]} \leq \sum_{h=1}^H e_h^{[n]} \leq \sum_{h=1}^H \bar{e}_h + \mathbf{1}.$$

Our purpose now is to prove that for infinite values of n it is $\bar{u}_h(x_h^{[n]}) \geq \bar{u}_h(x_h^* - 2\delta\mathbf{1})$. Let $\hat{x}_h \in [0, \sum_{h=1}^H \bar{e}_h + 1]$ be a cluster point of $(x_h^{[n]})_{n \geq n_1}$. Then we can assume $x_h^{[n]} \rightarrow \hat{x}_h$. Consider any $\tilde{x}_h \in \mathbb{R}_{++}^G$ such that $\bar{u}_h(\tilde{x}_h) = \bar{u}_h(x_h^* - 2\delta\mathbf{1})$. If we take n large enough, by (46), it is $u_h^{[n]}(x_h^{[n]}) - u_h^{[n]}(\tilde{x}_h) \geq 0$. Then, for n sufficiently large,

$$\begin{aligned} 0 &\leq u_h^{[n]}(x_h^{[n]}) - u_h^{[n]}(\tilde{x}_h) \leq D_{x_h} u_h^{[n]}(\tilde{x}_h)(x_h^{[n]} - \tilde{x}_h) \\ &= (D_{x_h} u_h^{[n]}(\tilde{x}_h) - D_{x_h} \bar{u}_h(\tilde{x}_h))(x_h^{[n]} - \tilde{x}_h) + D_{x_h} \bar{u}_h(\tilde{x}_h)(x_h^{[n]} - \tilde{x}_h). \end{aligned}$$

Taking the limit as $n \rightarrow \infty$ in the previous inequality, we get

$$D_{x_h} \bar{u}_h(\tilde{x}_h)(\hat{x}_h - \tilde{x}_h) \geq 0.$$

Then

$$\hat{x}_h \in \bigcap_{\tilde{x}_h \in \{y \in \mathbb{R}_{++}^G : \bar{u}_h(y) = \bar{u}_h(x_h^* - 2\delta\mathbf{1})\}} \{y \in \mathbb{R}^G : D_{x_h} \bar{u}_h(\tilde{x}_h)(y - \tilde{x}_h) \geq 0\}. \quad (47)$$

Since the right hand side of (47) is exactly $\{y \in \mathbb{R}^G : \bar{u}_h(y) \geq \bar{u}_h(x_h^* - 2\delta\mathbf{1})\}$, which is a subset of \mathbb{R}_{++}^G by (4), then $\hat{x}_h \in \mathbb{R}_{++}^G$ and the proof is complete. As regards the convergence of $\lambda^{[n]}$, from (45.1), (45.7) and (2) we find that, for every $h \in \mathcal{H}$ and $s \in \mathcal{S}$,

$$\lambda_h^{[n]}(s) = D_{x_h^1(s)} u_h^{[n]}(x_h^{[n]}) \rightarrow D_{x_h^1(s)} \bar{u}_h(\bar{x}_h) = \bar{\lambda}_h(s) \in \mathbb{R}_{++},$$

since $D_{x_h^1(s)} u_h^{[n]} \rightarrow D_{x_h^1(s)} \bar{u}_h$ uniformly on compact subsets of \mathbb{R}_{++}^G . Then, from (45.1) and (2), it follows that, for every $s \in \mathcal{S}$,

$$p^{[n]}(s) = \frac{D_{x_h(s)} u_h^{[n]}(x_h^{[n]})}{\lambda_h^{[n]}(s)} \rightarrow \frac{D_{x_h(s)} \bar{u}_h(\bar{x}_h)}{\bar{\lambda}_h(s)} = \bar{p}(s) \in \mathbb{R}_{++}^C$$

and thus $(p^{[n]})_{n \in \mathbb{N}}$ converges to an element $\bar{p} \in \mathbb{R}_{++}^C$.

By (45.2) we then immediately get the convergence of $b_h^{[n]}$ to an element $\bar{b}_h \in \mathbb{R}^A$.

Let us now check the convergence of $\mu_h^{[n]}$. Let us set $\mathcal{S}'_h = \{s \in \mathcal{S}' : \mu_h^{[n]}(s) \rightarrow 0\}$ and $\mathcal{S}''_h = \mathcal{S}' \setminus \mathcal{S}'_h$.

We have only to show that $\mu_h^{[n]}(s)$ is convergent for $s \in \mathcal{S}''_h$. From (45.3) it follows that

$$\lambda_h^{[n]}(0) q^{[n]} b_h^{[n]} = - \sum_{s=1}^{S-A} \sum_{a=1}^A (\lambda_h^{[n]}(s) + \mu_h^{[n]}(s)) m_{sa}^{[n]} b_h^{[n]a} + \sum_{a=1}^A \left(\lambda_h^{[n]}(S-A+a) + \mu_h^{[n]}(S-A+a) \right) b_h^{[n]a} =$$

$$\begin{aligned}
& - \sum_{s=1}^{S-A} \sum_{a=1}^A \lambda_h^{[n]}(s) m_{sa}^{[n]} b_h^{[n]a} + \sum_{a=1}^A \lambda_h^{[n]}(S-A+a) b_h^{[n]a} - \sum_{s \in \{1, \dots, S-A\} \cap \mathcal{S}'_h} \sum_{a=1}^A \mu_h^{[n]}(s) m_{sa}^{[n]} b_h^{[n]a} + \\
& \sum_{s \in \{S-A+1, \dots, S\} \cap \mathcal{S}'_h} \mu_h^{[n]}(s) b_h^{[n](s-(S-A))} - \sum_{s \in \{1, \dots, S-A\} \cap \mathcal{S}''_h} \sum_{a=1}^A \mu_h^{[n]}(s) m_{sa}^{[n]} b_h^{[n]a} + \sum_{s \in \{S-A+1, \dots, S\} \cap \mathcal{S}''_h} \mu_h^{[n]}(s) b_h^{[n](s-(S-A))}.
\end{aligned}$$

If for $s \in \mathcal{S}''_h$, $\mu_h^{[n]}$ is bounded above, then it admits a convergent subsequence and we are done. Suppose otherwise. Then, $\mu_h^{[n]}$ is not bounded above and there exists a subsequence converging to $+\infty$. Notice that if $s \in \mathcal{S}''_h$ there exists $n(s) \in \mathbb{N}$ such that $\mu_h^{[n]} > 0$, for all $n \geq n(s)$ and thus $\sum_a m_{sa}^{[n]} b_h^{[n]a} = \alpha_h^{[n]}(s) p^{[n]}(s) e_h^{[n]}(s)$, if $s \in \{1, \dots, S-A\}$ and $b_h^{[n](s-(S-A))} = -\alpha_h^{[n]}(s) p^{[n]}(s) e_h^{[n]}(s)$, if $s \in \{S-A+1, \dots, S\}$. Set then $n^* = \max\{n(s) : s \in \mathcal{S}''_h\}$ and for $n \geq n^*$ the above expression becomes

$$\begin{aligned}
& - \sum_{s=1}^{S-A} \sum_{a=1}^A \lambda_h^{[n]}(s) m_{sa}^{[n]} b_h^{[n]a} + \sum_{a=1}^A \lambda_h^{[n]}(S-A+a) b_h^{[n]a} - \\
& - \sum_{s \in \{1, \dots, S-A\} \cap \mathcal{S}'_h} \sum_{a=1}^A \mu_h^{[n]}(s) m_{sa}^{[n]} b_h^{[n]a} + \sum_{s \in \{S-A+1, \dots, S\} \cap \mathcal{S}'_h} \mu_h^{[n]}(s) b_h^{[n](s-(S-A))} - \\
& - \sum_{s \in \{1, \dots, S-A\} \cap \mathcal{S}''_h} \mu_h^{[n]}(s) \alpha_h^{[n]}(s) p^{[n]}(s) e_h^{[n]}(s) + \sum_{s \in \{S-A+1, \dots, S\} \cap \mathcal{S}''_h} \mu_h^{[n]}(s) (-\alpha_h^{[n]}(s) p^{[n]}(s) e_h^{[n]}(s)) = \\
& - \sum_{s=1}^{S-A} \sum_{a=1}^A \lambda_h^{[n]}(s) m_{sa}^{[n]} b_h^{[n]a} + \sum_{a=1}^A \lambda_h^{[n]}(S-A+a) b_h^{[n]a} - \sum_{s \in \{1, \dots, S-A\} \cap \mathcal{S}'_h} \sum_{a=1}^A \mu_h^{[n]}(s) m_{sa}^{[n]} b_h^{[n]a} + \\
& + \sum_{s \in \{S-A+1, \dots, S\} \cap \mathcal{S}'_h} \mu_h^{[n]}(s) b_h^{[n](s-(S-A))} - \sum_{s \in \mathcal{S}''_h} \mu_h^{[n]}(s) \alpha_h^{[n]}(s) p^{[n]}(s) e_h^{[n]}(s)
\end{aligned}$$

and thus from (45.2) we obtain

$$\begin{aligned}
& -\lambda_h^{[n]}(0) p^{[n]}(0) (x_h^{[n]}(0) - e_h^{[n]}(0)) = \lambda_h^{[n]}(0) q^{[n]} b_h^{[n]} = - \sum_{s=1}^{S-A} \sum_{a=1}^A \lambda_h^{[n]}(s) m_{sa}^{[n]} b_h^{[n]a} + \sum_{a=1}^A \lambda_h^{[n]}(S-A+a) b_h^{[n]a} - \\
& - \sum_{s \in \{1, \dots, S-A\} \cap \mathcal{S}'_h} \sum_{a=1}^A \mu_h^{[n]}(s) m_{sa}^{[n]} b_h^{[n]a} + \sum_{s \in \{S-A+1, \dots, S\} \cap \mathcal{S}'_h} \mu_h^{[n]}(s) b_h^{[n](s-(S-A))} - \sum_{s \in \mathcal{S}''_h} \mu_h^{[n]}(s) \alpha_h^{[n]}(s) p^{[n]}(s) e_h^{[n]}(s).
\end{aligned}$$

Letting $n \rightarrow \infty$ we find

$$\sum_{s=1}^{S-A} \sum_{a=1}^A \bar{\lambda}_h(s) \bar{m}_{sa} \bar{b}_h^a - \sum_{a=1}^A \bar{\lambda}_h(S-A+a) \bar{b}_h^a - \bar{\lambda}_h(0) \bar{p}(0) (\bar{x}_h(0) - \bar{e}_h(0)) = - \sum_{s \in \mathcal{S}''_h} \bar{\mu}_h(s) \bar{\alpha}_h(s) \bar{p}(s) \bar{e}_h(s)$$

and thus, if $\bar{\mu}_h(s) = +\infty$ for some s , we would find that the left hand side should be $-\infty$, which is impossible, as all its terms are finite. Thus $\bar{\mu}_h(s) \in \mathbb{R}$, for every s , as desired.

Finally, from (45.3) we easily get that also q^a is convergent, for every $a \in \mathcal{A}$. The proof is complete. \square

References

- Balasko Y., Cass D. and Siconolfi P., (1990), The structure of financial equilibrium with exogenous yields: the case of restricted participation, *Journal of Mathematical Economics*, 19, 195-216.
- Bich P. and Cornet B., (1998), Existence of financial equilibria: space of transfers of fixed dimension, mimeo.
- Dierker E., (1974), Topological methods in Walrasian economics, *Lecture Notes in Economics and Mathematical Systems*, 92, Springer-Verlag, Berlin.
- Duffie D. and Shafer W., (1985), Equilibrium with incomplete markets: I. A basic model of generic existence, *Journal of Mathematical Economics*, 14, 285-300.
- Gori M., Pireddu M. and Villanacci A., (2010), Existence of financial equilibria with endogenous borrowing restrictions and real assets, mimeo, University of Florence.
- Husseini S.Y., Lasry J.-M. and Magill M.J.P., (1990), Existence of equilibrium with incomplete markets, *Journal of Mathematical Economics*, 19, 39-67.
- Kato T., (1995), Perturbation theory for linear operators, *Classics in Mathematics*, Springer-Verlag, Berlin.
- Kubler F., (2007), Notes on numerical methods to solve non-linear equations, mimeo, University of Pennsylvania.
- Mangasarian O.L., (1969), *Nonlinear programming*, McGraw-Hill Book Company, New York, NY.
- Polemarchakis H. and Siconolfi P., (1997), Generic existence of competitive equilibria with restricted participation, *Journal of Mathematical Economics*, 28, 289-311.
- Radner R., (1972), Existence of equilibrium of plans, prices, and price expectations in a sequence of markets, *Econometrica*, 40, 289-303.
- Seghir A. and Torres-Martinez J.P., (2011), On equilibrium existence with endogenous restricted financial participation, *Journal of Mathematical Economics*, 47, 37-42.