A NEWSVENDOR’S PROCUREMENT PROBLEM WHEN SUPPLIERS ARE UNRELIABLE

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Abstract

We consider the problem of a newsvendor that is served by multiple suppliers, where any given supplier is defined to be either perfectly reliable or unreliable. By perfectly reliable we mean a supplier that delivers an amount identically equal to the amount desired, as is the case in the most basic variant of the newsvendor problem. By unreliable, we mean a supplier that with some probability delivers an amount strictly less than the amount desired. Our results indicate the following effects of unreliability: From the perspective of the newsvendor, the aggregate quantity ordered is higher than otherwise would be ordered if the newsvendor’s suppliers were completely reliable. From the perspective of end customers, however, the service level provided is lower than otherwise would be provided if the newsvendor’s suppliers were completely reliable. And, from the perspective of the suppliers, although reliability affects how much is ordered from a selected supplier, cost generally takes precedence over reliability when it comes to selecting suppliers in the first place. Even perfect reliability is no guarantee for qualification since, in an optimal solution, a given supplier will be selected only if all less expensive suppliers are selected, regardless of the given supplier’s reliability level. Nevertheless, the relative size of a selected supplier’s order depends on its reliability.
1 INTRODUCTION

By definition, a newsvendor must decide how much of its product to order from its supplier to stock for a single selling season, prior to observing the random demand for the product. Correspondingly, the newsvendor’s sales for the season are constrained both by the demand that materializes and by the quantity that the newsvendor actually supplies. The economic implications are dichotomous: On the one hand, if realized demand exceeds the supply, then the newsvendor will sell its entire stock, but at the expense of having excess demand go unsatisfied; on the other hand, if the supply exceeds the realized demand, then the newsvendor will satisfy demand completely, but at the expense of having leftovers. The familiar critical fractile solution to this trade-off indicates that the newsvendor, to maximize expected profit for the selling season, should endeavor to provide just enough supply so that the probability of meeting demand is equal to the ratio of the marginal understocking cost to the sum of the marginal overstocking plus understocking costs. Since the newsvendor’s supplier cannot reasonably be expected to supply more than its own available capacity, this answer to the newsvendor’s original how much decision translates into the following operational policy: If the desired quantity, as dictated by the critical fractile, is no more than the supplier’s capacity, then the amount desired should be ordered; otherwise, an amount equal to the supplier’s capacity should be ordered.

In this basic definition of the model, the newsvendor implicitly is served by a single supplier having a capacity that is deterministic from the newsvendor’s perspective. Nonetheless, it is this explicit characterization of the newsvendor’s supply source that provides the framing for two important and insightful extensions to the basic model. In the first extension, there exist multiple suppliers, but each supplier’s capacity remains deterministic. A key insight derived from this case is the following natural and intuitive ranking among available suppliers: In the absence of administrative costs for managing the portfolio of suppliers, the newsvendor should order as much as possible from the least expensive available supplier up to an aggregate inventory target that is dictated by the critical fractile associated with that supplier. If the capacity of the least expensive available supplier prohibits the newsvendor from reaching its target, then the newsvendor should iteratively repeat the process using the next least expensive
available supplier, with a corresponding lower aggregate inventory target, until an order is placed that does not exhaust the supplier’s capacity.

In the second extension, there exists only a single supplier, but that supplier’s capacity is uncertain. In this case, an intuitive insight is as follows: Given that the newsvendor is not guaranteed to receive the amount ordered from its supplier, if the newsvendor is liable for paying only for the quantity actually delivered by the supplier up to the amount ordered, then, as a general rule, the newsvendor should order no less than the amount that would have been ordered from the supplier if the supplier’s capacity were deterministic. Perhaps more interesting, closer examination indicates that this general rule actually depends on the nature of the supplier’s uncertain capacity. In particular, if the supplier’s random capacity cannot be influenced in any way by the size of the newsvendor’s order, as in Ciarallo et al. (1994), then the newsvendor should not alter its decision from the base case in which the supplier’s capacity is deterministic. However, if the supplier’s random capacity can be influenced by the size of the newsvendor’s order, as in Henig and Gerchak (1990), then the newsvendor should alter its base decision by ordering extra. In short, a supplier’s random capacity, taken in isolation, does not affect the newsvendor’s base decision; it only affects the newsvendor’s expected base profits. Hence, the newsvendor should alter its base decision and inflate its order only if, by doing so, it will affect the supplier’s random capacity.

Given these two natural extensions to the basic newsvendor model, the central question of this paper is what happens if the two extended models were combined? To what extent (or how) do the key insights from each of the individual extensions continue to apply? For example, the multiple supplier variant essentially prescribes that the newsvendor should not order a unit of its product from a given supplier if the newsvendor knows for certain that the unit can be obtained from a different source for less expense. But, what if the newsvendor does not know for certain if a unit can be obtained from a different source (for less expense)? If this is the case, then from which suppliers should the newsvendor place orders, and from which should the newsvendor not place orders? Likewise, the uncertain capacity variant essentially establishes how much should be ordered if no alternative supply source exists. But, what if alternative sources, albeit possibly unreliable sources, do exist? If this is the case, then how much should be ordered
from each supplier? In summary, given a newsvendor that has available multiple suppliers, where a given supplier’s capacity is not necessarily deterministic, we ask (1) whether or not an order should be placed with a given supplier, and (2) if so, then for how much?

To address these questions, we develop a multiple-supplier modeling framework built around two notions: procurement and reliability. Central to our construction is the definition of the newsvendor as the decision maker, and the suppliers as outside sources. Thus, in our model, the newsvendor is representative of a purchaser, and the suppliers can be thought of as producers subject to individual production capabilities. In this context, procurement refers to the situation in which the quantity delivered by a given supplier, and thus paid for by the newsvendor, is no greater than the quantity ordered by the newsvendor. In contrast, if the newsvendor were instead defined to operate in a production setting, then the newsvendor conceivably could be faced with having to pay for a quantity that may be greater than the usable quantity delivered (i.e., produced).

Given that in our model suppliers are defined to be outside the system, we develop the idea of reliability by explicitly defining, for each supplier, a production function from the perspective of the newsvendor, and then by comparing the output of a given supplier’s production function with the amount ordered. From this definition, we thus can designate any given supplier as being either perfectly reliable or unreliable, where a particular supplier’s reliability level reflects the newsvendor’s perceived likelihood that the supplier will deliver a marginal unit desired from it. Specifically, if the newsvendor perceives that a supplier can deliver any marginal unit ordered from it with probability equal to one, then that supplier is considered to be perfectly reliable. Otherwise, the supplier is considered to be unreliable. If a supplier is unreliable, then that supplier is further classified as having either an exogenous production function, in which case the newsvendor perceives the supplier’s random production capability to be independent of the quantity ordered, or an endogenous production function, in which case the newsvendor perceives the supplier’s random production capability to be affected by the quantity ordered.

One contribution of our model is the specific insights that it yields. In particular, we conclude, in the words of Hill (2000), that cost is an order qualifier but reliability can be interpreted as an order winner. Consequently, suppliers with costs that are high relative to other suppliers could be left without a share of
the newsvendor’s aggregate order, regardless of their relative reliability levels. For these suppliers, reliability improvements would be to no avail; only through improving costs could they break through the threshold to gain a share of the newsvendor’s aggregate order. In contrast, suppliers with costs that are low relative to other suppliers cannot be left without a share of the newsvendor’s aggregate order in lieu of more reliable but more expensive suppliers. If a supplier does become a participating supplier, however, that supplier can use reliability improvements to increase market share, even to the point of barring more expensive suppliers from participation. To complement these insights, we also demonstrate that, when compared to a newsvendor that does not have to deal with an unreliable supply source, a newsvendor that does have unreliable suppliers will order more (in aggregate), but provide a lower customer service level.

Another contribution of our model is that it embraces a relatively general reliability construct that includes as special cases some of the more traditional forms interspersed throughout both the newsvendor literature and the random yield literature. (Authoritative reviews of the newsvendor and random yield literatures are provided by Porteus (1990) and by Yano and Lee (1995), respectively. More recent updates include those by Khouja (1999) and Minner (2002), respectively.) Examples of reliability constructs appearing in these literatures include, among others, the case of random capacity (Ciarallo et al. (1994)), the case of binomial yield (Chen et al. (2001)), the case of stochastic proportional yield (Henig and Gerchak (1990); and “Model II” of Anupindi and Akella (1993)), the special case of all-or-nothing delivery (Gerchak (1996); and the single-period application of “Model I” of Anupindi and Akella (1993)), and combinations thereof (Wang and Gerchak (1996)). Typically in these papers, the primary focus is on characterizing the structure of the optimal policy, given that each of the supply sources are utilized, so that it may serve as a building block for dynamic inventory models. In contrast, our focus (like, to some degree, the focus of Chen et al. (2001) and Gerchak (1996)) is, first, to characterize optimal supplier selection and its implications for the newsvendor, for its outside suppliers, and for its customer market; and, second, to perform comparative statics on the optimal solution, with a particular emphasis on investigating how changes in supplier cost or reliability affect the newsvendor’s ordering decisions and
customer service level. Ultimately, we find that the newsvendor model with unreliable suppliers has similar structural properties as the two extensions to the basic newsvendor model described above.

In Section 2, we formally define the procurement and reliability constructs that serve as the fundamental building blocks for our decision model, using a table along the way to succinctly and explicitly connect those constructs to the literature. Then, we characterize useful functional properties that follow as logical consequences of our definitions. In Section 3, we develop the newsvendor decision model and establish properties of its optimal solution. Then, in Section 4, we explore implications of the optimal solution, developing, in particular, insights on supplier selection. In Section 5 we study a two-supplier solution to investigate how the newsvendor should react to changes in fundamental problem parameters. And in Section 6 we conclude the paper with a discussion of the scope and applicability of our model and assumptions.

2 PROCUREMENT, RELIABILITY, AND SERVICE LEVEL DEFINED

In this section we define the terms, notation, assumptions, and functional relationships that lay the groundwork for the decision model developed and analyzed in the next section. Specifically, we formalize our notions of procurement and reliability, which essentially provide the inputs into our decision model, and within that context, we specify service level, which is useful for shaping and interpreting the output derived from the decision model. Table 2 at the end of this section summarizes the technical content introduced, explained, and developed throughout the section.

To set the stage, we consider suppliers to be either perfectly reliable or unreliable, where the distinction is viewed from the newsvendor’s perspective. If the newsvendor knows with certainty that a supplier will deliver fully its desired order quantity, regardless of its size, then that supplier is considered to be perfectly reliable. Otherwise, that supplier is considered to be unreliable and to have a random production capability. We will refer to what the newsvendor understands to be true about a given supplier’s production capability as the supplier’s production function.
2.1 Defining Procurement

Qualitatively, we mean for procurement to represent a setting characterized by two fundamental traits: (1) the newsvendor pays a given supplier only for the quantity that the supplier actually delivers; and (2) the quantity delivered by a given supplier is no greater than the quantity ordered by the newsvendor. In Section 6, we briefly discuss implications if variations of either or both of these defining characteristics are considered.

To formalize the procurement setting studied in this paper, we develop notation to distinguish between the newsvendor’s order quantity, the supplier’s production capability, and the quantity delivered. Specification of more detailed production functions to further characterize a given supplier’s production capability will follow in Sections 2.2 and 2.3, where supplier reliability is defined. Here, the key idea is that the quantity delivered is equal to the minimum of the order quantity and the production capability.

Accordingly, let:

\[ Q_i = \text{quantity ordered from supplier } i. \]
\[ K_i = \text{production capability of supplier } i. \]
\[ S_i \equiv \min\{Q_i, K_i\} = \text{quantity supplied (i.e., delivered) by supplier } i. \]
\[ c_i S_i = \text{total purchase cost of quantity delivered by supplier } i \text{ (i.e., } c_i \text{ is a constant per-unit purchase price charged by supplier } i). \]

In addition, let:

\[ n = \text{total number of suppliers available}. \]
\[ Q \equiv \{Q_1, \ldots, Q_n\} = \text{vector of quantities ordered from all available suppliers}. \]
\[ Q_T = \sum_{i=1}^{n} Q_i = \text{total quantity ordered among all available suppliers}. \]
\[ S_T = \sum_{i=1}^{n} S_i = \text{total quantity delivered among all available suppliers}. \]

2.2 Defining Perfect Reliability

In our model, supplier i is considered to be perfectly reliable if, from the newsvendor’s perspective, there is no chance that the amount delivered will be less than the amount desired. Accordingly, it is sufficient to define a perfectly reliable supplier through the following production function:

If supplier i is perfectly reliable, then \( K_i \equiv \infty. \)
Basically, to the newsvendor, perfect reliability means that \( S_i = Q_i \), regardless of the size of \( Q_i \). Consequently, given the definition of \( S_i \) as \( S_i \equiv \min\{Q_i, K_i\} \), the notion of perfect reliability is captured simply by defining \( K_i = \infty \). In other words, a perfectly reliable supplier is defined here to precisely characterize the supplier in the most basic variant of the newsvendor model. As such, a perfectly reliable supplier can be used naturally as a benchmark against which to compare all unreliable suppliers.

2.3 Defining Unreliability

In our model, supplier \( i \) is considered to be unreliable if, from the newsvendor’s perspective, there exists the possibility that the amount delivered will be less than the amount desired. Accordingly, we define an unreliable supplier in broad terms by the stipulation of a random production function that may or may not depend on the amount ordered; then in precise terms (later) by the technical specifications of that random production function.

If supplier \( i \) is unreliable, then \( K_i \) may or may not depend on \( Q_i \); but, either way, \( K_i \) is subject to \( R_i \), a nonnegative random variable characterized by cumulative distribution function (cdf), \( G_i(r) \), and corresponding probability density function (pdf), \( g_i(r) \). Assume that the \( R_i \) are independent, that \( G_i(r) \) is continuous and differentiable, and that \( G_i(r) > 0 \) for \( r > 0 \).

The assumptions stipulated here to begin the specification of unreliability are technical ones, useful for simplifying the presentation of our analysis. However, they are not necessarily required for the analysis itself or the qualitative results derived from it. At its core, our analysis relies on two structural properties. One, we require that the newsvendor’s expected profit function be differentiable. And, two, we require that non-decreasing functions of \( R_i \) have non-negative covariance. When coupled with analogous properties of the demand distribution, continuity and differentiability of \( G_i(r) \) is sufficient to guarantee that the expected profit function is differentiable. In section 6, we discuss alternative combinations that also yield differentiability of the profit function as well as highlight and discuss combinations that do not. Similarly, independence of the \( R_i \) is one regularity condition sufficient for establishing the requisite covariance property, but it is not necessary. More generally, the covariance property is secured as long as the \( R_i \) are associated (Lehmann, 1966; Esary et al., 1967; Barlow and Proschan, 1975), which is another idea we discuss more fully in Section 6.
The key here is that the production function of unreliable supplier i is modulated by an $R_i$ that influences the supplier’s output. How specifically this influence is defined, however, depends on how supplier i is further classified, which is the idea articulated next, with Sections 2.3.1 and 2.3.2.

2.3.1 Specification of Exogenous Production Function

In our model, unreliable supplier i’s production function is considered to be exogenous if it is independent of the amount ordered. That is,

If supplier i is unreliable with an exogenous production function, then $K_i \equiv R_i$.

Essentially, we are defining the reliability level of a supplier with an exogenous production function to be in terms of the supplier’s capacity so that “unreliable” refers to the phenomenon in which the supplier delivers the amount ordered from it if and only if the amount ordered is no greater than the supplier’s uncertain capacity; and “more reliable” refers to a capacity level that is stochastically larger1.

2.3.2 Specification of Endogenous Production Function

In our model, unreliable supplier i’s production function is considered to be endogenous if it depends on the amount ordered. That is,

If supplier i is unreliable with an endogenous production function, then $K_i = K_i(Q_i,R_i)$, where the following technical assumptions provide the details for $K_i(Q_i,R_i)$:

TA1. Assume that $K_i(Q_i,R_i)$ is continuous and differentiable; and that $K_i(Q_i,0) = K_i(0,R_i) = 0$.

TA2. Assume that, for a given $Q_i > 0$, $\frac{\partial K_i(Q_i,R_i)}{\partial R_i} > 0$.
Note that a direct implication of this assumption is that, for any given $Q_i > 0$, there exists a unique $R_i$ that satisfies the implicit function $K_i(Q_i,R_i) = Q_i$. Correspondingly, let $z_i = z_i(Q_i)$ denote that value of $R_i$. Then the following properties are true for $Q_i > 0$:
(a) $z_i > 0$;
(b) $K_i(Q_i,z_i) = Q_i > 0$;
(c) $K_i(Q_i,R_i) < Q_i \iff R_i < z_i$;
(d) $\Pr\{K_i(Q_i,R_i) > Q_i\} = 1 - G(z_i)$.

TA3. Let $k_i(Q_i,R_i) = \frac{\partial K_i(Q_i,R_i)}{\partial Q_i}$ represent supplier i’s yield rate function; i.e., the (random) change in supplier i’s (random) production output resulting from a change in $Q_i$.

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1 Notice that this definition of an exogenous production function also applies to a supplier having a deterministic but limited capacity. In such a case, however, the supplier’s “random” capacity would be characterized by a degenerate probability mass function, not by a continuous probability density function. Accordingly, such a supplier would violate the technical assumption at the beginning of Section 2.3 stipulating that all the $G_i(r)$ are differentiable; hence, it is beyond the current scope of our analysis. Nonetheless, as alluded to above, we will revisit this issue in Section 6.
TA3.1. Assume that, for a given \( R_i \), \( k(Q_i,R_i) \geq 0 \) and \( \frac{\partial k(Q_i,R_i)}{\partial Q} \leq 0 \). That is, \( K(Q_i,R_i) \) is non-decreasing and concave with respect to \( Q_i \).

TA3.2. Assume that \( \frac{\partial k(Q_i,R_i)}{\partial R_i} \geq 0 \) and that \( R_i < z_i \Rightarrow k(Q_i,R_i) < 1 \).

Essentially, TA3.2 means that we are defining the reliability level of a supplier with an endogenous production function to be in terms of the supplier’s yield rate function so that “unreliable” refers to the phenomenon in which an increase in order quantity yields a less than equivalent increase in production output, and “more reliable” refers to a yield rate function that is stochastically larger. Note that this description of reliability for a supplier with an endogenous production function is consistent with the definition of reliability for a supplier with an exogenous production function (in Section 2.3.1). Table 1 identifies examples of production functions that meet the specifications stipulated in this section. It also provides interpretations as well as references for those that are common to the random yield literature.

### Table 1. Examples of Applicable Production Functions

<table>
<thead>
<tr>
<th>Exogenous Production Function: ( K \equiv R )</th>
<th>Interpretation:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case of random capacity (e.g., Ciarallo et al., (1994)).</td>
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</table>

<table>
<thead>
<tr>
<th>Endogenous Production Functions: ( K \equiv K(Q,R) )</th>
<th>Comments:</th>
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<tbody>
<tr>
<td>( K(Q,R) = \frac{QR}{Q + \alpha} )</td>
<td>TA1: ( K(Q,R) ) is continuous, differentiable; ( K(Q,0) = K(0,R) = 0. )</td>
</tr>
<tr>
<td>( \frac{\partial K(Q,R)}{\partial R} = \frac{Q}{Q + \alpha} )</td>
<td>TA2: ( Q &gt; 0 \Rightarrow \frac{\partial K(Q,R)}{\partial R} &gt; 0. )</td>
</tr>
<tr>
<td>( z = 1 )</td>
<td></td>
</tr>
<tr>
<td>( \frac{QR + 2\alpha}{(QR + \alpha)^2} )</td>
<td></td>
</tr>
<tr>
<td>TA3.1: ( k(Q,R) \geq 0; \frac{\partial k(Q,R)}{\partial Q} \leq 0. )</td>
<td></td>
</tr>
<tr>
<td>( R &lt; z \Rightarrow k(Q,R) &lt; 1. )</td>
<td></td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Interpretation:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case of stochastic proportional yield (e.g., Henig and Gerchak, (1990)).</td>
</tr>
<tr>
<td>Case of rationed random capacity</td>
</tr>
<tr>
<td>A case of random rationing of random capacity</td>
</tr>
<tr>
<td>(Another) case of random rationing of random capacity</td>
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</tbody>
</table>
2.3.3 Properties of Unreliability

Next, Lemma 1 and its proof provide useful properties derived from the definition of unreliability:

**Lemma 1.**

(a) \( z_i = z_i(Q_i) \) is non-decreasing and continuous.

(b) Where the derivative exists, let \( s_i(Q_i,R_i) \equiv \partial S_i/\partial Q_i \) represent supplier \( i \)'s marginal supply function. Then, for a given \( Q_i > 0 \), \( s_i(Q_i,R_i) \) is non-decreasing, but not necessarily continuous, as a function of \( R_i \).

(c) If \( Q_i > 0 \), then \( \Pr\{K_i > Q_i\} \leq E[s_i(Q_i,R_i)] < 1 \) and \( E[s_i(Q_i,R_i)] \) is non-increasing and continuous as a function of \( Q_i \).

**Proof.** Part (a). By definition, \( K_i(Q_i,z_i) = Q_i \). Since \( K_i(Q_i,R_i) \) is continuous by TA1, \( z_i(Q) \) is continuous. Therefore, taking the total derivative,

\[
\frac{1}{dQ_i} \frac{dz_i}{dz_i} = 1.
\]

First, \( k_i(Q_i,R_i) \) is continuous by TA1 and \( k_i(Q_i,R_i) < 1 \) for \( R_i < z_i \) by TA3.2. This implies that \( k_i(Q_i,z_i) \leq 1 \).

Second, \( \partial K_i(Q_i,z_i)/\partial z_i > 0 \) by TA2. Therefore, \( dz_i/dQ_i \geq 0 \).

Part (b). There are two cases to consider. If unreliable supplier \( i \) has an exogenous production function, then, by definition, \( S_i = \min\{Q_i,K_i\} = \min\{Q_i,R_i\} \). That is,

\[
S_i = \begin{cases} R_i & \text{if } R_i < Q_i \\ Q_i & \text{if } R_i \geq Q_i \end{cases}.
\]

This implies

\[
s_i(Q_i,R_i) = \begin{cases} 0 & \text{if } R_i < Q_i \\ 1 & \text{if } R_i \geq Q_i \end{cases},
\]

which is non-decreasing in \( R_i \), but is not continuous.

If unreliable supplier \( i \) has an endogenous production function, then, by definition, \( S_i = \min\{Q_i,K_i(Q_i,R_i)\} \). Thus, from TA2,

\[
S_i = \begin{cases} K_i(Q_i,R_i) & \text{if } R_i < z_i \\ Q_i & \text{if } R_i \geq z_i \end{cases},
\]

which implies

\[
s_i(Q_i,R_i) = \begin{cases} k_i(Q_i,R_i) & \text{if } R_i < z_i \\ 1 & \text{if } R_i > z_i \end{cases}.
\]

For \( R_i < z_i \), \( s_i(Q_i,R_i) \) is non-decreasing in \( R_i \) because \( \partial k_i(Q_i,R_i)/\partial R_i \geq 0 \) by TA3.2. At \( R_i = z_i \), \( s_i(Q_i,R_i) \) is non-decreasing because \( 1 \geq k_i(Q_i,z_i) \) by TA3.2 and TA1. And, for \( R_i > z_i \), \( s_i(Q_i,R_i) \) remains non-decreasing because it is independent of \( R_i \). Therefore, \( s_i(Q_i,R_i) \) is non-decreasing everywhere. However, \( s_i(Q_i,R_i) \) is not necessarily continuous because it is not necessarily true that \( k_i(Q_i,z_i) = 1 \).

Part (c). If unreliable supplier \( i \) has an exogenous production function, then, from (2),

\[
E[s_i(Q_i,R_i)] = 1 - G_i(Q_i) = \Pr\{K_i > Q_i\} < 1.
\]
Therefore, \( E[s_i(Q_i,R_i)] = \Pr\{K_i > Q_i\} < 1 \), and \( E[s_i(Q_i,R_i)] \) is continuous, and \( dE[s_i(Q_i,R_i)]/dQ_i = -g_i(Q_i) \leq 0 \).

If unreliable supplier \( i \) has an endogenous production function, then, from (4),

\[
E[s_i(Q_i,R_i)] = \int_0^{z_i} k_i(Q_i,r)g_i(r)dr + [1 - G_i(z_i)] = 1 - \int_0^{z_i} [1 - k_i(Q_i,r)]g_i(r)dr.
\]

First, \( E[s_i(Q_i,R_i)] \) is continuous because \( z_i \) is continuous by Lemma 1(a).  Second, \( E[s_i(Q_i,R_i)] \geq 1 - G_i(z_i) = \Pr\{K_i > Q_i\} \) because \( k_i(Q_i,r) \geq 0 \) by TA3.1.  Third, \( E[s_i(Q_i,R_i)] < 1 \) because \( k_i(Q_i,r) < 1 \) by TA3.2.  And fourth, \( dE[s_i(Q_i,R_i)]/dQ_i \leq 0 \) because

\[
\frac{dE[s_i(Q_i,R_i)]}{dQ_i} = \int_0^{z_i} \frac{\partial k_i(Q_i,r)}{\partial Q_i}g_i(r)dr - \int_0^{z_i}[1 - k_i(Q_i,z_i)]g_i(z_i)\frac{dz_i}{dQ_i},
\]

where \( \frac{\partial k_i(Q_i,r)}{\partial Q_i} \leq 0 \) by TA3.1, \( [1 - k_i(Q_i,z_i)] \geq 0 \) by TA3.2 and TA1, and \( dz_i/dQ_i \geq 0 \) by Lemma 1(a).

2.4 Defining Demand

By definition, demand for the newsvendor’s product is random:

Let \( D \) be a nonnegative demand random variable independent of \( R_i \), \( i = 1,\ldots,n \), and characterized by cdf, \( F(x) \), and corresponding pdf, \( f(x) \).  Assume that \( F(x) \) is continuous and differentiable, and that \( F(x) > 0 \) for \( x > 0 \).

Again, when coupled with analogous conditions on the \( G_i(r) \), continuity and differentiability of \( F(x) \) is sufficient to guarantee that the newsvendor’s expected profit function is differentiable, which, recall, is a property central to our analysis.  In section 6, we discuss the compatibility of our analysis with alternative couplings of demand and unreliability distributions.

2.5 Defining Service Level

We adopt Type I service level as one benchmark for comparisons because it is convenient both for developing and for interpreting results from our analysis.  In general, Type I service level is defined as the probability that demand is no greater than supply (see, for example, Silver and Peterson, 1985).

Let \( SL(Q_i) \) denote the newsvendor’s service level as a function of \( Q_i \), and let \( S_{T,i} = S_T - S_i = \sum_{j \neq i} S_j \).  Then,

\[
SL(Q) = \Pr\{D \leq S_T\} = \Pr\{D \leq S_i + S_{T,i}\}.
\]

Lemma 2 summarizes properties of \( SL(Q) \) important within the context of our analysis.  The derivations of these properties are provided in the appendix.
Lemma 2.  
(a) If supplier i is perfectly reliable, then \( SL(Q) = E[F(Q_i + S_{Ti})] \leq F(Q_T) \).
(b) If supplier i is unreliable, then
   (i) \( Q_i = 0 \Rightarrow SL(Q) = E[F(S_{Ti})] \leq F(Q_T) \);
   (ii) \( Q_i > 0 \Rightarrow Pr[K_i > Q_i]E[F(Q_i + S_{Ti})] < SL(Q) < E[F(Q_i + S_{Ti})] \leq F(Q_T) \).

2.6 Summary

The purpose of Section 2 was to develop the analytical machinery underlying the decision model to be introduced and analyzed next in Section 3. Given the technical nature of this material, we conclude this section with Table 2, which provides a convenient reference for notation and assumptions.

<table>
<thead>
<tr>
<th>Table 2. Summary of Notation and Assumptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>n = Number of suppliers available.</td>
</tr>
<tr>
<td>( Q_i ) = Decision variable denoting quantity ordered from supplier i.</td>
</tr>
<tr>
<td>( Q = {Q_1, \ldots, Q_n}; Q_T = \sum_{i=1}^{n} Q_i ) = Order-quantity vector and total quantity ordered, respectively.</td>
</tr>
<tr>
<td>( Q^<em>_i; Q^</em> = {Q^*_i} ) = Optimal value of ( Q_i ) and corresponding vector of optimal decisions, respectively (see Section 3).</td>
</tr>
<tr>
<td>D = Nonnegative and independent demand random variable.</td>
</tr>
<tr>
<td>( F(x); f(x) ) = Cdf and pdf, respectively, characterizing ( D ), where ( F(x) ) is continuous and differentiable, and ( F(x) &gt; 0 ) for ( x &gt; 0 ).</td>
</tr>
<tr>
<td>( R_i ) = Nonnegative and independent random variable representing reliability of supplier i, given that supplier i is unreliable.</td>
</tr>
<tr>
<td>( G_i(r); g_i(r) ) = Cdf and pdf, respectively, characterizing ( R_i ), where ( G_i(r) ) is continuous and differentiable, and ( G_i(r) &gt; 0 ) for ( r &gt; 0 ).</td>
</tr>
</tbody>
</table>
| \( K_i(Q_i,R_i) \) = Endogenous production function of supplier i, given that supplier i is unreliable, where:
   (i) \( K_i(Q_i,R_i) \) is continuous and differentiable;
   (ii) \( K_i(Q_i,0) = K_i(0,R_i) = 0 \);
   (iii) \( \partial K_i(Q_i,R_i)/\partial R_i > 0 \) for \( Q_i > 0 \). |
| \( z_i \) = Unique value of \( r \) that satisfies \( K_i(Q_i,r) = Q_i \), given that \( Q_i > 0 \). |
| \( k_i(Q_i,R_i) = \partial K_i(Q_i,R_i)/\partial Q_i \) = Yield rate function of supplier i, given that supplier i is unreliable with an endogenous production function, where:
   (i) \( k_i(Q_i,R_i) \geq 0 \) and \( k_i(Q_i,R_i)/\partial Q_i \leq 0 \);
   (ii) \( \partial k_i(Q_i,R_i)/\partial R_i \geq 0 \);
   (iii) \( k_i(Q_i,R_i) < 1 \) for \( R_i < z_i \). |
| \( K_i = \begin{cases} \infty & \text{if i is perfectly reliable} \\ R_i & \text{if i is unreliable with exogenous production function} \\ K_i(Q^*_i, R_i) & \text{if i is unreliable with endogenous production function} \end{cases} \) |

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Table 2. Summary of Notation and Assumptions (concluded)

<table>
<thead>
<tr>
<th>Notation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_i = \min{Q_i, K_i} )</td>
<td>Quantity delivered by supplier ( i ).</td>
</tr>
<tr>
<td>( S_T = \sum_{i=1}^{n} S_i ); ( S_{T-i} = S_T - S_i )</td>
<td>Total quantity delivered by all suppliers and total quantity delivered by all suppliers other than supplier ( i ), respectively.</td>
</tr>
<tr>
<td>( s_i(Q_i, R_i) )</td>
<td>Marginal quantity delivered by unreliable supplier ( i ) ( = \begin{cases} \partial K_i / \partial Q_i &amp; \text{if } K_i &lt; Q_i \ 1 &amp; \text{if } K_i &gt; Q_i \end{cases} )</td>
</tr>
<tr>
<td>( \text{SL}(Q) = \Pr{D \leq S_T} )</td>
<td>Type I service level provided by the newsvendor.</td>
</tr>
<tr>
<td>( c_i )</td>
<td>Per-unit purchase cost of units procured from supplier ( i ).</td>
</tr>
<tr>
<td>( p, v, \pi )</td>
<td>Newsvendor’s per-unit selling price, salvage value, and penalty cost of loss goodwill, respectively (see Section 3).</td>
</tr>
<tr>
<td>( \phi_i = (p + \pi - c_i)/(p + \pi - v) )</td>
<td>Newsvendor critical fractile associated with supplier ( i ) (see Section 3).</td>
</tr>
</tbody>
</table>

3 THE NEWSVENDORS DECISION MODEL AND SOLUTION

Within the context of the modeling framework introduced in Section 2, we use this section to formulate the newsvendor’s decision model and characterize its solution. Then, in Sections 4 and 5, we more fully explore the properties of that solution to develop deeper insight into the questions of whether or not the newsvendor should place an order with supplier \( i \), and if so, then for how much.

Subject to the notation, definitions, and functional relationships developed in Section 2, we characterize the newsvendor’s decision problem as follows: Prior to the beginning of a single selling season, the newsvendor places orders for its product from among any of \( n \) independent suppliers, where a given supplier is defined as in Section 2. We index these suppliers from least to most expensive so that \( c_1 \leq c_2 \leq \ldots \leq c_n \), where, recall, \( c_i \) denotes the per-unit purchase cost of inventory delivered by supplier \( i \) (\( S_i \)). The combined amount of inventory delivered by all \( n \) suppliers (\( S_T \)) thus provides the newsvendor with its total amount of stock available to fulfill its demand (\( D \)) for the season. The newsvendor then sells as much of this stock as demand will support (\( \min\{D, S_T\} \)) at the per-unit selling price \( p > c_n \). If the newsvendor’s available stock exceeds realized demand, then leftovers (\( [S_T - D]^+ \)) are salvaged for the per-unit value \( v < c_1 \) (where a negative \( v \) denotes a disposal cost); conversely, if realized demand exceeds the available stock, then shortages (\( [D - S_T]^+ \)) are assessed a per-unit penalty cost \( \pi \) (to signify the cost of lost goodwill). Finally, for convenience, let \( \phi_i = (p + \pi - c_i)/(p + \pi - v) \) denote the standard newsvendor
critical fractile (i.e., ratio of underage costs to the sum of underage and overage costs) associated with supplier i, and note that the indexing of suppliers above implies that

\[ \phi_1 \geq \phi_2 \geq \ldots \geq \phi_n. \]  

(6)

The newsvendor’s objective is to choose a vector of non-negative order quantities \( (Q = \{Q_1, \ldots, Q_n\}) \) to maximize \( \Pi(Q) \), its expected profit for the selling season, which is equal to the sum of its expected sales and salvage revenues less its expected shortage and purchase costs:

\[
\Pi(Q) = E[p \cdot \min \{D, S_T\} + v \cdot (S_T - D)^+ + \pi (D - S_T)^+ - \sum_{i=1}^{n} c_i S_i].
\]  

(7)

However, by first applying the identities, \( \min \{D, S_T\} = S_T - (S_T - D)^+ = D - (D - S_T)^+ \), and by then substituting \( S_{T-i} = S_T - S_i \) from Table 2, (7) can be written as

\[
\Pi(Q) = (p + \pi - v)E[\sum_{j \neq i}^{n} \phi_j S_j + \phi_i S_i - (S_i + S_{T-i} - D)^+] - \pi E[D].
\]  

(8)

In (8), the term \( \pi E[D] \) is an additive constant and the term \( (p + \pi - v) \) is a multiplicative constant. Thus, neither affects the newsvendor’s decision vector. Therefore, without loss of generality, we rescale the parameters \( p, v, \) and \( \pi \) such that \( p - v = 1 \) and \( \pi = 0 \) to convert (8) into the following convenient and compact statement of the newsvendor’s decision problem:

\[
\begin{align*}
\text{maximize} & \quad \Pi(Q) = \phi_i E[S_i] - E[(S_i + S_{T-i} - D)^+] + E[\sum_{j \neq i}^{n} \phi_j S_j] \\
\text{s.t.} & \quad Q_i \geq 0 \text{ for all } i
\end{align*}
\]  

(9)

One benefit of (9) is the interpretation that it yields: From the perspective of determining how much, if any, to order from supplier \( i \), the last term in (9) can be ignored because it is independent of \( Q_i \). Thus, only the combined effect of the first two terms, namely \( \Pi_i(Q) = \phi_i E[S_i] - E[(S_i - (D - S_{T-i}))^+] \), is material to the decision. Notice, however, that \( Q_i \) affects \( \Pi_i(Q) \) only through \( S_i \), and not through \( S_{T-i} \). Thus, for a given \( S_{T-i} \), \( \Pi_i(Q) \) can be interpreted as a standard newsvendor profit function in which \( S_i \), the amount received from supplier \( i \), denotes the resource supply decision while \( D - S_{T-i} \), the residual demand remaining after taking into consideration all other supply sources, denotes the effective random demand for that resource. The one complication here is that the supply decision \( (S_i) \) itself might be random.
Given (9), we use the remainder of this section to develop key properties of the newsvendor’s optimal solution. To that end, we let \( Q^* \) denote the optimal value of \( Q_i \), and we let \( Q^* = \{ Q_i^* \} \) denote the corresponding vector of optimal decisions. Then we proceed in three steps: First, we establish that \( \Pi(Q) \) is differentiable. Second, we exploit the differentiability of \( \Pi(Q) \) by applying Karush-Kuhn-Tucker (KKT) conditions to characterize \( Q^* \) in an interpretable form. And third, we convert that characterization of \( Q^* \) into the key building-block properties that then provide the basis for the resulting analyses of Sections 4 and 5.

We begin with Lemma 3:

**Lemma 3.** For a given \( n \), \( \Pi(Q) \) is differentiable.

*Proof.* See Appendix.

As alluded to in Section 2, it is Lemma 3, and not the assumption that all of the random variables in (9) are characterized by differentiable distribution functions, that is central to our analysis. As a result, everything else being equal, if Lemma 3 continues to hold even when some of the distribution functions incorporated within (9) are not differentiable, then the propositions and insights developed later in Sections 4 and 5 continue to apply. Again, we point to Section 6 for a deeper exploration into the scope of Lemma 3.

Since \( \Pi(Q) \) is differentiable for a given \( n \), let \( M_i(Q) = \partial \Pi(Q)/\partial Q_i \). Then, \( Q^* \) must satisfy the following KKT conditions for \( i = 1, \ldots, n \):

\[
\begin{align*}
M_i(Q^*) &\leq 0 \quad (10) \\
M_i(Q^*) Q_i^* & = 0 \quad (11)
\end{align*}
\]

where, applying Lemma 3 to (9),

\[
M_i(Q) = \begin{cases} 
\phi_i - E[F(Q_i + S_{T-i})] & \text{if supplier } i \text{ is perfectly reliable} \\
\phi_i [E[s_i(Q_i, R_i)] - E[s_i(Q_i, R_i)F(S_i + S_{T-i})]] & \text{if supplier } i \text{ is unreliable.}
\end{cases}
\]

In (12), the second term follows from (9) because the function \( E[(y - D)^+] \), which represents expected leftovers in a standard newsvendor model, yields the derivative \( \partial E[(y - D)^+] / \partial y = F(y) \).
To provide some interpretation for (12), note from (11) that, for $Q_i^*$ to be an interior point solution, it is necessary for $M_i(Q^*) = 0$. This, applied to (12), yields the following:

$Q_i^* > 0 \Rightarrow \begin{cases} 
E[F(Q_i + S_{T-1})]_{Q^*} = \phi_i & \text{if supplier i is perfectly reliable} \\
E\left[\frac{s_i(Q_i, R_i)}{E[s_i(Q_i, R_i)]}F(S_i + S_{T-1})\right]_{Q^*} = \phi_i & \text{if supplier i is unreliable}.
\end{cases}$

In (13), $\phi_i$ is the familiar critical fractile associated with supplier $i$, and the expectation term is representative of a weighted service level. With respect to the set of weights associated with a given supplier, note that only a single weight applies if the supplier is perfectly reliable; whereas no fewer than two weights apply if the supplier is unreliable. Nonetheless, as a general rule, (13) indicates that if an order is placed with supplier $i$, then the optimal amount to order is the amount that equates the newsvendor’s service level, adjusted by weights to account for supplier i’s reliability profile, to $\phi_i$, the service level that would be optimal for a classic newsvendor to provide if supplier $i$ were (1) perfectly reliable and (2) the only available supplier.

Although (13) is somewhat amenable for interpretation, it is less appealing for direct analysis. For this reason, we conclude this section with Lemmas 4 and 5, the proofs for which are in the appendix:

**Lemma 4.** For a given $n$, if supplier $i$ is perfectly reliable, then
(a) $Q_i^* = 0 \Rightarrow SL(Q^*) \geq \phi_i$;
(b) $Q_i^* > 0 \Rightarrow SL(Q^*) = \phi_i$.

**Lemma 5.** For a given $n$, if supplier $i$ is unreliable, then
(a) $Q_i^* = 0 \Rightarrow SL(Q^*) \geq \phi_i$;
(b) $Q_i^* > 0 \Rightarrow E[s_i(Q^*, R_i)]\phi_i < SL(Q^*) < \phi_i$.

As will be seen in the next section, these two lemmas hold the key to characterizing the structure of the newsvendor’s optimal policy and its implications for choosing and managing a supplier portfolio.

### 4 Supplier Selection: Cost versus Reliability

Since suppliers differ on cost and reliability, in this section we examine how these two attributes affect the outcome of the newsvendor's supplier selection decision, which refers to the process of choosing...
suppliers with which to place orders. To characterize these results, we define a supplier to be active if it is optimal for the newsvendor to place an order with that supplier. Conversely, we define a supplier to be inactive if it is optimal for the newsvendor not to place an order with that supplier. Given these definitions, we find that, as a general rule, cost takes precedence over reliability when it comes to indexing suppliers for selection. In particular, we use this section to establish, and to discuss implications related to, the following result: In an optimal solution, if a given supplier is inactive, then all more expensive suppliers will be inactive.

We begin by asking how the existence of unreliability affects the newsvendor’s optimal ordering policy. To help address this question, we compare the newsvendor’s optimal policy to the optimal policy associated with the special case in which all suppliers are defined to be perfectly reliable. In this benchmark case, which also is a special case of the newsvendor variant described in the introduction in which there exist multiple suppliers each characterized as having deterministic but limited capacity, an optimal solution is to order only from supplier 1 (the least expensive supplier), and to order the amount that equates service level to the corresponding critical fractile. See, for example, Porteus (1990). Using this benchmark, Proposition 1 below establishes that, when compared to an otherwise equivalent newsvendor that does not have to deal with unreliable suppliers, a newsvendor that does have to deal with unreliable suppliers will order more, but provide less (in terms of the service level provided to customers). Moreover, adding a supplier to the pool decreases (weakly) the amount ordered from each of the original suppliers while increasing (weakly) the amount ordered from the new supplier.

Proposition 1. For a given n, let $Q^*_b$ denote the optimal order quantity for the benchmark case in which supplier $i$ is perfectly reliable for all $i$. That is, let $Q^*_b = F^{-1}(\phi_i)$. Then

(a) $SL(Q^*_b) \leq \phi_i = F(Q^*_i) \leq E[F(Q^*_i)]$ ; and

(b) $Q^*_{i,n} \leq Q^*_{i,n+1}$, where the notation $Q^*_{i,n}$ is used here in place of $Q^*_i$ to explicitly reflect the case in which there are a total of $n$ suppliers available.

Proof. See Appendix.

Proposition 1 provides some insight into how the newsvendor’s aggregate how much decision is affected by unreliability. However, the more strategic question is, when should an order be placed (and
when should an order not be placed) with a given supplier? A concise answer to this question, interestingly, is that an order should be placed with a given supplier only if an order is placed with all less expensive suppliers:

**Proposition 2.** For a given n, consider any two suppliers i and j that are such that $c_j > c_i$. If $Q_i^* = 0$, then $Q_j^* = 0$.

Proof. Recall that $c_j > c_i$ implies that $\phi_i > \phi_j$, and assume that $Q_i^* = 0$. Then, Lemmas 4(a) and 5(a) indicate that $\text{SL}(Q^*) \geq \phi_i > \phi_j$, regardless of whether supplier i is perfectly reliable or unreliable. But, if $\text{SL}(Q^*) > \phi_j$, then Lemma 4(b) implies that $Q_j^* = 0$ if supplier j is perfectly reliable; and Lemma 5(b) implies that $Q_j^* = 0$ if supplier j is unreliable. Thus, regardless of supplier j’s reliability, $Q_j^* = 0$.  

Proposition 2, in effect, validates cost as the appropriate dimension on which to index a given pool of n available suppliers for possible selection. Basically, in terms of selecting suppliers, it is optimal for the newsvendor to start by choosing the least expensive supplier, and then to add suppliers to its selection set one by one, according to how inexpensive the supplier is. Consequently, the optimal number of active suppliers, say $n^*$, will be such that supplier i is active if and only if $c_i \leq c_{n^*}$. This basic structure of the optimal policy, which boils down to selecting the $n^*$ cheapest suppliers to activate, thus verifies that the insight from the multiple-supplier newsvendor variant in which each supplier has deterministic but limited capacity indeed is preserved under the more general notion of unreliability developed and explored in this paper.

The supplier-preference relationship yielded by Proposition 2 derives from the comparison of any two available suppliers that have different unit costs. But, what if two suppliers have the same unit cost? As Propositions 3 and 4 indicate next, if two suppliers have the same cost, then reliability – more specifically, perfect reliability – appears to provide the next layer of discrimination when searching for suppliers to activate:

**Proposition 3.** For a given n, consider any two suppliers i and j that are such that $c_j = c_i$, but $K_j \neq K_i$.

(a) If both suppliers are unreliable, then $Q_i^* > 0 \Leftrightarrow Q_j^* > 0$.

(b) If one supplier is unreliable while the other is perfectly reliable, then it is optimal not to order from the unreliable one.

2 We graciously thank the anonymous referee who prompted this question.
Proof. Recall that \( c_j = c_i \) implies that \( \phi_j = \phi_i \). Part (a). From Lemma 5, \( Q^*_i > 0 \) if and only if \( SL(Q^*_i) \leq \phi_i \), and \( SL(Q^*_j) \leq \phi_j \) if and only if \( Q^*_j > 0 \). Therefore, \( Q^*_i > 0 \) if and only if \( Q^*_j > 0 \). Part (b). Without loss of generality, assume that supplier \( i \) is perfectly reliable and supplier \( j \) is unreliable. Then, from Lemma 4, \( SL(Q^*_i) \geq \phi_i = \phi_j \); and, from Lemma 5(b), \( SL(Q^*_j) \geq \phi_j \Rightarrow 0Q^*_j = 0 \). Hence, \( Q^*_j = 0 \).

Proposition 4. For a given \( n \), consider any two suppliers \( i \) and \( j \) that are such that \( c_j = c_i \) and \( K_j = K_i \).

(a) If the suppliers are unreliable with exogenous production functions, then \( Q^*_i = Q^*_j \).

(b) If the suppliers are unreliable with endogenous production functions, then \( Q^*_i = Q^*_j \) is an optimal solution.

(c) If the two suppliers are perfectly reliable, then \( Q^*_i + Q^*_j = \text{constant} \).

Proof. Recall that \( c_j = c_i \) implies that \( \phi_j = \phi_i \). Part (a). If \( 0Q^*_i = 0 \), then, from Lemma 5(a), \( SL(Q^*_i) \geq \phi_i = \phi_j \); and, from Lemma 5(b), \( SL(Q^*_j) \geq \phi_j \Rightarrow 0Q^*_j = 0 \). Hence, \( 0Q^*_j = 0 \).

Part (b). If \( 0Q^*_i > 0 \), then, analogous to the proof of Part (a), Lemma 5 implies that \( 0Q^*_j = 0 \). If \( Q^*_i > 0 \), then, from Lemma 5(b), \( SL(Q^*_j) < \phi_j = \phi_i \), which, from Lemma 5(a), implies \( Q^*_j > 0 \). Thus, (11) and (A2) imply that \( E[F(Q^*_i + S^*_i - S^*_j - S^*_i)] = E[F(Q^*_i + Q^*_j + (S^*_i - S^*_j - S^*_i))] \).

Now, assume that \( Q^*_i < (>) Q^*_j \) and notice,

\[
E[F(Q^*_i + S^*_i + (S^*_i - S^*_j - S^*_i))] = E[F(Q^*_i + Q^*_j - \max |Q^*_j, R_j| + (S^*_i - S^*_j - S^*_i))]
\]

\[
< (>) E[F(Q^*_i + Q^*_j - \max |Q^*_j, R_j| + (S^*_i - S^*_j - S^*_i))]
\]

\[
= E[F(Q^*_i + Q^*_j + (S^*_i - S^*_j - S^*_i))],
\]

which contradicts (14). Therefore, \( Q^*_i > 0 \Rightarrow Q^*_i = Q^*_j \), which completes the proof of Part (a).

Part (c). From (A1),

\[
M_i(Q) = \phi_i - E[F(Q_i + Q_j + (S_T - S_i - S_j))] = \phi_j - E[F(Q_i + Q_j + (S_T - S_i - S_j))] = M_j(Q).
\]

Since \( M_i(Q) = M_j(Q) \) depends on \( Q_i \) and \( Q_j \) only through the sum \( Q_i + Q_j \), \( Q^*_i + Q^*_j = \text{constant} \).
According to Propositions 3 and 4, if both of two suppliers having the same unit cost are unreliable, then it is optimal to place an order either with both of them or with neither of them, but not with only one of them. In contrast, if one of the suppliers is unreliable while the other is perfectly reliable, then it is optimal to place an order either with only one of them (the perfectly reliable one) or with neither of them, but not with both of them. Moreover, if the two suppliers not only have the same cost, but also have the same reliability (i.e., the same production function), then ordering the same amount from both suppliers is an optimal solution (though, it is not necessarily uniquely optimal). Thus, at one extreme, if two equal-cost suppliers have equal reliabilities, then those suppliers split an order evenly between them; at the other extreme, if one of the two suppliers is perfectly reliable, then that supplier completely monopolizes the order allocated to the two of them. By interpolation, these results suggest that relative reliability impacts a given supplier’s market share such that, everything else being equal, the higher is the supplier’s reliability, the higher is the supplier’s share (which is an insight that is formalized by Proposition 9 for the two-supplier case considered in Section 5). Interestingly, however, these results also indicate that, from the perspective of determining which suppliers to activate (without concern for how much actually to order from a given active supplier), the relative reliability of two equal-cost suppliers is not really a factor. More pertinent is the issue of perfectly reliable versus not perfectly reliable.

To hone this insight further, we next take a closer look at the implications of perfect reliability. Specifically, we ask, does perfect reliability guarantee a supplier to be active? The answer we find, in a word, is no. According to Proposition 5 below, although perfect reliability of supplier i is enough to disqualify all more expensive suppliers from being active, it is not enough to avoid being disqualified by less expensive unreliable suppliers. In other words, in an optimal solution, it is possible that only unreliable suppliers will be active, even if a perfectly reliable supplier exists in the pool of available suppliers:

**Proposition 5.** For a given $n$, consider any two suppliers $i$ and $j$.

(a) If supplier $i$ is perfectly reliable, then $\phi_i > \phi_j \Rightarrow Q_j^* = 0$;

(b) If supplier $i$ is unreliable and active, then $E[x_i(Q_i^*, R_i)]\phi_i \geq \phi_j \Rightarrow Q_j^* = 0$. 
Proof. Part (a). If supplier \( i \) is inactive, then \( Q^*_i = 0 \) by Proposition 2. If supplier \( i \) is active, then \( SL(Q^*) = \phi_i > \phi_j \) by Lemma 4(b). But, if \( SL(Q^*) > \phi_i \), then, from Lemmas 4(b) and 5(b), \( Q^*_j = 0 \) regardless of supplier \( j \)'s reliability. Part (b). If unreliable supplier \( i \) is active, then, from Lemma 5(b), \( SL(Q^*) > E[s_t(Q^*_t, R_t)] \phi_i \geq \phi_j \). But, if \( SL(Q^*) > \phi_i \), then \( Q^*_j = 0 \) regardless of supplier \( j \)'s reliability (by Lemmas 4(b) and 5(b)).

Notice that Proposition 5(a) precisely echoes intuition from the introduction by indicating that there is no reason for a newsvendor to order a unit from a given supplier if the newsvendor knows for certain that it can obtain a unit elsewhere for less expense. In that vein, one interpretation of Proposition 5(b) is that it refines this intuition to indicate that the newsvendor should not order a unit from supplier \( j \) if it can obtain the unit from (less expensive) supplier \( i \), in expectation.

Proposition 5(a), together with Propositions 3(b) and 4(c), also indicate that no more than one perfectly reliable supplier need be active in an optimal solution. Thus, given a list of suppliers indexed so that \( \phi_i \geq \phi_{i+1} \) for all \( i \), a maximal set of potential suppliers can be established by scanning the list (beginning with \( i = 1 \)) until the first perfectly reliable supplier is found, if one exists. In other words, given an indexed list, \( n \) can be defined as the largest supplier index that is such that suppliers 1, ..., \( n-1 \) are all unreliable. Accordingly, in specifying an \( n \)-supplier problem, supplier \( n \) may be either perfectly reliable or unreliable, but suppliers 1 – \( n-1 \) should all be unreliable.

Moreover, if supplier \( n \) is perfectly reliable, then Proposition 5(b) indicates that supplier \( n \) may or may not be active in an optimal solution. It is interesting to note, however, that if perfectly reliable supplier \( n \), indeed, is active, then Lemma 4 establishes that \( SL(Q^*) = \phi_n \). Thus, one interpretation is that it is the perfectly reliable supplier’s role to provide a base service level: the purpose of the perfectly reliable supplier is to ensure that the newsvendor’s service level is brought up to the base service level \( \phi_n \). In this case, if the reliability of any one or more of the other \( n-1 \) suppliers were to change, for whatever reason, then a reallocation of ordering from among the suppliers undoubtedly would occur, but the quantity ordered from the perfectly reliable supplier would change accordingly to bring the newsvendor’s service level back up to \( \phi_n \). If ordering from the perfectly reliable supplier would force the service level to exceed
Proposition 5 also has technical implications. Basically, in contrast to (10) and (11), which establish necessary, but not sufficient, conditions for optimality, Proposition 5 helps establish a sufficient (but not necessary) condition for optimality. In particular, Proposition 5(b) can be combined with Proposition 1(b) and Lemma 1(c) to yield the following algorithm for determining the optimal solution:

Given \( n \), rank suppliers by cost so that \( \phi_i \geq \phi_{i+1} \) for \( i = 1, \ldots, n-1 \).

For \( m = 1, \ldots, n \),

**Step 1.** Solve \( m \)-supplier sub-problem. For that \( m \)-supplier sub-problem, let \( Q_{i,m}^* \) denote the optimal quantity to order from supplier \( i \), and let \( Q^*_m = \{Q_{i,m}^*\} \) denote the corresponding vector of optimal decisions.

**Step 2.** If \( Q_{i,m}^* = 0 \), then stop (optimal solution found). Else, go to **Step 3**.

**Step 3.** Compute \( B_m = \max_i \{E[s_i(Q_{i,m}^*, R_i)]\phi_i\} \).

**Step 4.** If \( B_m > \phi_{m+1} \), then stop (optimal solution found). Else, set \( m = m+1 \) and go to **Step 1**.

Optimal solution: \( Q^*_i = Q_{i,m}^* \) for \( i \leq m \); \( Q^*_i = 0 \) for \( i > m \).

The fundamental reason why this algorithm ensures optimality is Proposition 1(b), which establishes that

\[ Q_{i,m+1}^* \leq Q_{i,m}^* . \]

Thus, if \( Q_{i,m}^* = 0 \) for \( m < n \), then \( Q_{m,n}^* = 0 \), which validates **Step 2** as a sufficient stopping rule. Similarly, if \( B_m > \phi_{m+1} \), then it means that there exists an \( i \leq m \) that is such that

\[ E[s_i(Q_{i,m}^*, R_i)]\phi_i > \phi_j \]

for all \( j \geq m+1 \). Since \( E[s_i(Q_n, R_i)] \) is non-increasing in \( Q_i \) (by Lemma 1(c)), this means that

\[ E[s_i(Q_{i,m}^*, R_i)]\phi_i \geq E[s_i(Q_{i,m}^*, R_i)]\phi_j \]

Given Proposition 5(b), this then implies that

\[ Q_{j,n}^* = 0 \],

thus establishing **Step 4** as a sufficient stopping rule.

Essentially, \( B_m = \max_i \{E[s_i(Q_{i,m}^*, R_i)]\phi_i\} \) represents a barrier to entry derived from Proposition 5(b). Given Proposition 1(b) and Lemma 1(c), this barrier is such that each additional supplier becoming active makes it easier for a given active supplier to disqualify a more expensive supplier. However, since Proposition 5(b) corresponds to a sufficient but not necessary condition for optimality, the barrier \( B_m \) likewise is sufficient but not required to disqualify a given supplier. In contrast, for special cases in

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3 In light of the above algorithm, we adopt the notation of the \( m \)-supplier sub-problem for the remainder of this paper in order to explicitly reflect the dependency of the optimal solution on the number of suppliers under consideration. Accordingly, for \( m = 1, \ldots, n \), let \( Q_{i,m}^* = \{Q_{i,m}^*\} \) denote the optimal solution for the sub-problem in which only the subset of suppliers \( 1, \ldots, m \) are considered. Note that, given this adapted notation, \( Q_{i,n}^* \) and \( Q_{n}^* \) respectively correspond to \( Q_i^* \) and \( Q^* \) from Table 2.
which it can be shown that the service level for the m-supplier sub-problem, \( \text{SL}(\mathbf{Q}_m^*) \), is non-decreasing in \( m \), the definition of the barrier provided in Step 3 of the algorithm can be replaced by the sufficient and necessary barrier \( B_m = \text{SL}(\mathbf{Q}_m^*) \). The reason for this is summarized by the following proposition:

**Proposition 6.** If, for \( m = 1, \ldots, n \), \( \text{SL} (\mathbf{Q}_m^*) \) is non-decreasing in \( m \), then, for \( m < n \), \( Q_{m+1,n} > 0 \iff \text{SL}(\mathbf{Q}_m^*) < \phi_{m+1} \).

*Proof.* See Appendix.

Generally speaking, the results of this section establish cost as taking precedence over reliability when it comes to indexing suppliers for selection. Basically, we find that the most expensive suppliers do not control their own destinies in the sense that they cannot gain activation by making themselves more reliable (more expensive suppliers gain activation only if less expensive suppliers are not reliable enough); yet the least reliable suppliers do control their own destinies in the sense that they can gain activation by making themselves less expensive. Given this characterization, one interpretation is through the strategic language of Hill (2000): Whereas a given supplier’s reliability might be thought of as an order-winning criterion because a higher relative level of reliability might help shape how much is ordered from the supplier if an order is placed with that supplier, it is low relative cost that qualifies the supplier for a piece of the newsvendor’s business in the first place. Accordingly, even a perfectly reliable supplier can be disqualified from participation if that supplier’s cost is not competitive. We more completely explore the issue of qualifying versus winning next in Section 5 by studying in detail a case of two suppliers.

5 QUALIFYING VERSUS WINNING: A TWO-SUPPLIER CASE

In Section 4, we developed properties characterizing the newsvendor’s optimal solution, generally establishing that cost takes precedence over reliability when it comes to indexing suppliers for selection; and, as a result, only the \( n^* \) least expensive suppliers will be active, where \( n^* \) is such that (i) \( \phi_{n+1} \leq \text{SL}(\mathbf{Q}_{n^*}^*) \leq \phi_n \), and (ii) no supplier other than possibly supplier \( n^* \) is perfectly reliable. Moreover, we interpreted these results to mean that, in the language of Hill (2000), cost can be thought of as an order qualifier while reliability can be thought of as an order winner. In this section, we pursue this notion
further by analyzing in detail the case in which \( n = 2 \), assuming that \( \phi_1 > \phi_2 \). Specifically, we develop some intuition with regard to the following questions:

1. What does it take for supplier 2 to qualify for activation? That is, to what extent can supplier 2’s barrier to entry, \( B_1 \), be concisely characterized; and how does that barrier depend on supplier 1’s cost and reliability?

2. Once active, how do suppliers win orders; and what is the effect on the end market? Specifically, how do the optimal order quantities, the optimal total order quantity, and the optimal service level depend on the cost and reliability measures of supplier 1 and supplier 2?

To approach these questions, we require supplier 1 to be unreliable (since it would be impossible for supplier 2 to qualify for activation if supplier 1 were perfectly reliable). However, to ensure tractability and to sharpen the results of this section, we further stipulate that supplier 1 has an exogenous production function (so that \( K_1 = R_1 \)). Finally, to measure supplier 1’s reliability, we define \( \rho_1 \) as a parameter of \( \Gamma_1 \) and we assume that \( \partial \Gamma_1(\tau; \rho_1)/\partial \rho_1 < 0 \) so that a higher \( \rho_1 \) represents a higher level of reliability.

We begin addressing the questions of this section by establishing Proposition 7, which formally establishes the relationship \( SL(Q_2^*) \geq SL(Q_1^*) \) for this two-supplier case. As noted with Proposition 6 at the end of Section 4, one key implication of this relationship is that \( SL(Q_1^*) \) thus serves as a precise barrier to entry for supplier 2. That is, because of Proposition 6, Proposition 7 implies that supplier 2 will be active (in an optimal solution) if and only if \( \phi_2 > SL(Q_1^*) \). A second implication of Proposition 7 is that the newsvendor’s customers prefer supplier 2 to be active in the sense that the newsvendor’s service level is higher when both suppliers are active. This interpretation of Proposition 7 is representative of the conventional wisdom that consumers are the ultimate beneficiaries of a competitive environment.

**Proposition 7.** \[ SL(Q_2^*) \geq SL(Q_1^*) \]

**Proof.** See Appendix.

Given that Proposition 7, in effect, establishes that supplier 2 is activated if and only if \( \phi_2 > SL(Q_1^*) \), we delineate the role of \( SL(Q_1^*) \) as a necessary and sufficient barrier to entry by setting \( B_1(\phi_1, \rho_1) = SL(Q_1^*) \), where the notation here explicitly reflects that supplier 2’s barrier depends on, and only on, supplier 1’s cost and reliability measures. This delineation thus reflects that supplier 2 has only one lever (\( \phi_2 \)) to overcome its barrier to entry, whereas supplier 1 has two levers (\( \phi_1, \rho_1 \)) to strengthen the barrier.
Accordingly, the question of what it takes to qualify for activation can be investigated by studying $B_1(\phi_1, \rho_1)$ as follows:

**Proposition 8.** $B_1(\phi_1, \rho_1)$ increases as $\phi_1$ increases; $B_1(\phi_1, \rho_1)$ increases as $\rho_1$ increases.

**Proof.** See Appendix.

Since supplier 2 is active if and only if $\phi_2 > B_1(\phi_1, \rho_1)$, Proposition 8 implies that, everything else being equal, there exists a critical value of $\phi_1$, say $\hat{\phi}_1$, that is such that supplier 2 is active if and only if $\phi_1 < \hat{\phi}_1$. Similarly, there exists a critical value of $\rho_1$, say $\hat{\rho}_1$, that is such that supplier 2 is active if and only if $\rho_1 < \hat{\rho}_1$. We therefore conclude that, everything else being equal,

- the more cost efficient is supplier 1, the more likely it is that supplier 2 will be inactive;
- the more reliable is supplier 1, the more likely it is that supplier 2 will be inactive;
- the more cost efficient is supplier 2, the more likely it is that supplier 2 will be active;
- supplier 2’s reliability does not affect its likelihood of being active.

As the above discussion suggests, an active supplier can render more expensive suppliers inactive by strategically increasing reliability or decreasing cost. Our final question, then, is how changes to an active supplier’s cost or reliability affect the optimal solution if more expensive suppliers are not rendered inactive. Accordingly, we conclude our analysis with Proposition 9, which characterizes the results of a sensitivity analysis on $Q_{1,2}^*, Q_{2,2}^*, Q_{T,2}^* = Q_{1,2}^* + Q_{2,2}^*$, and $SL(Q_2^*)$ under the condition that both of the two suppliers considered in this section are active. Here, for convenience, we further stipulate that supplier 2, if unreliable, also has an exogenous production function (so that $K_2 \equiv R_2$) and we define $\rho_2$ analogous to $\rho_1$ to parameterize supplier 2’s reliability.

**Proposition 9.** Given that $\phi_1$, $\rho_1$, and $\phi_2$ remain such that $\phi_2 > B_1(\phi_1, \rho_1)$ so that both suppliers are active in an optimal solution:

(a) As a function of $\phi_1$: $Q_{1,2}^*$ is increasing while $Q_{2,2}^*$ is decreasing; and both $Q_{T,2}^*$ and $SL(Q_2^*)$ are non-decreasing.

(b) As a function of $\rho_1$: $Q_{1,2}^*$ is increasing while $Q_{2,2}^*$ is decreasing; $SL(Q_2^*)$ is non-decreasing, but $Q_{T,2}^*$ is non-increasing.

(c) As a function of $\phi_2$: $Q_{1,2}^*$ is decreasing, while $Q_{2,2}^*$ is increasing; and both $Q_{T,2}^*$ and $SL(Q_2^*)$ are non-decreasing.

(d) As a function of $\rho_2$ (if applicable): $Q_{1,2}^*$ is decreasing while $Q_{2,2}^*$ is increasing; $SL(Q_2^*)$ is non-decreasing, but $Q_{T,2}^*$ is non-increasing.
Proof. See Appendix.

A consequence of Proposition 9 is that supplier 1, either by lowering its cost or by improving its reliability, can increase the order it receives relative to the size of the order received by its rival. Consistent with the related inference drawn from Propositions 3 and 4, we interpret this coupling of effects to mean that supplier 1 increases its market share. Thus, when combined with the strategic implications of Proposition 8, the tactical implications of Proposition 9 yield useful insights into the trade-offs between cost and reliability.

6 CONCLUSIONS
We have considered the problem of a newsvendor that is served by multiple suppliers, where any given supplier is defined to be either perfectly reliable or unreliable. By perfectly reliable we mean a supplier that delivers an amount identically equal to the amount desired, as is the case in the most basic variant of the newsvendor problem. By unreliable, we mean a supplier that with some probability delivers an amount strictly less than the amount desired. If a given supplier is unreliable, then we interpret that supplier as having an exogenous production function if its reliability construct does not depend on the amount ordered, as is the case of a supplier with random capacity; and we interpret that supplier has having an endogenous production function if its reliability construct does depend on the amount ordered, as in the case of stochastic proportional yield. In short, we feature both demand and supply uncertainty in a single product, multiple supplier model, thus capturing key factors important for the development of vendor-selection criteria (see, for example, Elmaghraby, 2000).

Our results indicate the following effects of unreliability: From the perspective of the newsvendor, the aggregate quantity ordered is higher than otherwise would be ordered if the newsvendor’s activated suppliers were completely reliable. From the perspective of end customers, however, the service level provided is lower than otherwise would be provided if the newsvendor’s activated suppliers were completely reliable. And, from the perspective of the suppliers, although reliability affects how much is ordered from an active supplier, cost generally takes precedence over reliability when it comes to indexing suppliers for selection. Even perfect reliability is no guarantee for qualification since, in an
optimal solution, a given supplier will be active only if all less expensive suppliers are active, regardless of the given supplier’s reliability level. Nevertheless, the relative size of an active supplier’s order depends on its reliability. As a consequence, reliability improvements can be used strategically to disqualify more expensive suppliers from receiving an order.

Although we have assumed that the reliability random variables \( \{R_i\} \) are independent, this assumption is not essential to the analysis. Rather, it is the inequalities established by Lemma 5 that drive the structural properties of supplier selection that constitute our core results. These inequalities, however, follow from Lemmas A1 and A2 in the appendix, which themselves derive directly from the definition of associated random variables (see, for example, Karlin and Rinott, 1981). Since independent random variables are associated but associated random variables need not be independent (Esary, et al., 1967), the scope of our results extend more generally. For example, consider a scenario in which multiple suppliers ration capacity in a like way when their aggregate demand exceeds their aggregate capacity. Such a scenario could be represented within our modeling construct by interpreting the “\( \alpha \)” from Table 1 as a random shock that is common to multiple suppliers. In this case, each of the suppliers affected would have an endogenous production function that depended on a shared random variable \( \alpha \). As a result, these suppliers would have positively correlated (random) production functions; thus, Lemma 5 and the propositions that follow from it would continue to apply. In contrast, if a case existed in which suppliers had negatively correlated production functions (as is possible, for example, in Babich et al, 2005), then Lemma 5 would not directly apply.

As a second example of how the notion of association might be applied to extend the scope of our results, consider a situation in which a given supplier’s production function is characterized by a multivariate random vector \( \mathbf{R} = \{R^1, \ldots, R^t\} \). In such a scenario, we suggest thinking of the production function as the “bottleneck” of process functions so that the supply function could be defined directly in terms of the supplier’s processes as follows: \( S(Q, \mathbf{R}) = \min \{Q, K^1, \ldots, K^t\} \), where \( K^i = R^i \text{ or } K^i = K^i(Q, R^i) \), and where the \( R^i \)'s need not be independent. One application of this construct is illustrated by a supplier for which random capacity is coupled with stochastic proportional yield so that \( K^1 = QR^1 \) and \( K^2 = R^2 \), as in Wang and Gerchak (1996). If the multivariate nature of a given production function is incorporated by
generalizing the supply function $S$ in this way, then we conjecture that, everything else being equal, qualitative results analogous to those developed in this paper will continue to hold if it is assumed that each process function $K^j$ satisfies the technical specifications stipulated in Section 2.

We have also assumed that $f(.)$ as well as all of the $g_i(.)$’s are continuous. But, like the independent-suppliers assumption, this technical assumption is more critical to exposition than to analysis. Since our analysis essentially begins with the set of KKT conditions expressed by (10) and (11), and since these conditions apply only if the newsvendor’s profit function (9) is differentiable, it is Lemma 3 and not the probability density functions that is fundamental to our analysis. Accordingly, other sets of regularity conditions exist to yield the same effect. For example, if $R_i$ is a random variable characterized by a discrete probability mass function, then, everything else being equal, Lemma 3 continues to hold if $K_i$ is linear in $Q_i$. This particular set of regularity conditions includes as a special case an “all-or-nothing supplier” (since in that case the supplier can be characterized by the production function $K_i = R_i Q_i$, where $R_i = 1$ with probability $\beta$ and $R_i = 0$ with probability $1 - \beta$). Therefore, to complete this illustration, the results of this paper readily extend if, everything else being equal, one or more suppliers are modeled as all or nothing suppliers.

Similarly, if supplier $i$ has deterministic but limited capacity so that it can be characterized by an exogenous production function in which $R_i$ is defined by a degenerate mass function that is equal to 1 at a particular point (say, at $R_i = \kappa_i$) and is equal to 0 elsewhere, then Lemma 3 continues to hold, and hence, the KKT conditions continue to apply, as long as long as $Q_i \leq \kappa_i$. Therefore, one way to expand the scope of the analysis presented in this paper to incorporate such a supplier would be simply to append the set of KKT conditions expressed by (10) and (11) to include the condition $Q_i \leq \kappa_i$ as well. Although the addition of such a condition surely would affect the applicability of some of our results (e.g., Proposition 3), it would leave others fully applicable as currently stated (most notably, Proposition 2).

It is important to note, however, that Lemma 3 is not guaranteed in general. For example, if all reliabilities as well as demand are characterized by discrete mass functions, then (9) is not differentiable. In such a case, the KKT conditions do not accurately characterize the newsvendor’s optimal solution, thereby invalidating the structural results derived from them. To emphasize this point, consider the
following two-supplier illustration: Let \( D = 100 \) (with probability 1); and, for \( i = 1, 2 \), let supplier \( i \) be such that \( K_i = R_i Q_i \), where \( R_i = 1 \) with probability \( \beta_i \) and \( R_i = 0 \) with probability \( 1 - \beta_i \). Moreover, let \( \phi_1 = 3/2 > \phi_2 = 1/2 \); and let \( \beta_1 = 4/3 < \beta_2 = 4/4 \). In this case, (9) is not differentiable; thus the set of KKT conditions expressed by (10) and (11) do not apply. Accordingly, the propositions derived in Sections 4 and 5 need not hold true. Indeed, for this case, the optimal order quantities are \( Q_1^* = 0 \) and \( Q_2^* = 100 \), which are inconsistent with Proposition 2. For further insight into this illustration, we refer to Swaminathan and Shanthikumar (1999).

Another example illustrating when the set of KKT conditions expressed by (10) and (11) is inaccurate (or incomplete), and therefore our analysis breaks down, consider the extension in which the newsvendor is constrained, either explicitly or implicitly, by the number of suppliers it may choose. If the number of suppliers that may be selected is constrained by an explicit limit (say, \( L \)), then the analysis here is affected because a third KKT condition is required, namely the condition indicating that, in an optimal solution, the total number of suppliers chosen must be no greater than \( L \). Equivalently, if the number of suppliers that may be selected is constrained by an implicit limit arising from, say, a fixed administrative cost associated with managing each active supplier, then the analysis here is affected because differentiability of the profit function (i.e., Lemma 1) is violated. Petruzzi, et al. (1994) illustrate this effect by providing examples in which the optimality of selecting the \( n^* \) least expensive suppliers breaks down when such a fixed cost is introduced. Note, however, that this particular effect once again parallels the motivating example of the introduction: In the multiple-supplier newsvendor variant in which each supplier’s capacity is deterministic but limited, the policy of choosing the \( n^* \) cheapest suppliers is not necessarily optimal if analogous fixed costs are incorporated in the model.

As a final example, consider the model of Chen et al. (2001). In their model, a newsvendor must manage suppliers whose unreliability results in the delivery of (and payment for) defective parts. In this modeling environment, the newsvendor implements an optimal inspection and repair regime to rectify defective units. As a consequence, the newsvendor’s procurement costs effectively become nonlinear and

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4 We gratefully acknowledge the anonymous referee who both constructed this illustration and pointed us to Swaminathan and Shanthikumar (1999).
concave, which is the same effect resulting from the fixed administrative cost described in the above example. From this perspective, it is natural to find that the optimal solution is one in which the set of activated suppliers is not necessarily limited to the least expensive ones.

We next turn our attention from a discussion of the technical assumptions forming the basis of our analysis and toward a final discussion of the procurement setting that provides the context for our model. In our model, the newsvendor pays each supplier only for the units actually delivered – it does not pay for the amount ordered if the delivery amount is less than the amount ordered, and it does not pay for the amount produced if the ordered amount is less than the amount produced. Accordingly, our model best represents the one end of a spectrum in which production capability for a given product is completely outsourced, as often is the case, for example, in retail. In contrast is the other end of that spectrum, which is characterized by the extreme case in which sourcing of a given product is allocated completely within a network of internal plants so that no production capability is outsourced. At this extreme, depending on the transfer pricing scheme of the integrated channel, it might be the case that the decision-making equivalent of the newsvendor must pay for the total number of units ordered (i.e., released into production), even if that amount exceeds the amount produced (i.e., delivered). If so, then our model does not apply without adaptation.

Perhaps more realistically, however, even an internal production network is likely to be complemented by some group of external suppliers. In such a setting, the model and insights of this paper apply directly as a building block for the combined procurement and production sourcing problem as follows: For any given vector of production quantities, the question of how much to order from each of the external suppliers is precisely the question analyzed in this paper. Hence, the conditionally optimal amounts to order from each supplier can be determined and interpreted by direct application of the algorithm presented in Section 4.

We conclude by exploring one other potentially viable area for applying our model as a building block for developing deeper insights. As motivation, consider the paper by Babich et al. (2005). In their paper, Babich et al. formulate a model in which a newsvendor pays for the amount it orders rather than the amount it receives from its suppliers, and they develop a game-theoretical analysis from the
perspective of the suppliers to study the impact that correlation among suppliers’ disruptions and default risks ultimately have on the newsvendor. As such, their model differs from ours both in construct and in context. Nonetheless, one interesting relation between their work and ours is their result indicating that the newsvendor would prefer for its suppliers’ disruptions to be positively rather than negatively correlated. This result thus suggests that our framework, which requires as a primitive that suppliers’ reliabilities be positively correlated, may be fruitfully applied to study issues of supply disruption.

Indeed, this is the case in the supply risk model of Federgruen and Yang (2005), who study supply disruptions by applying a newsvendor procurement model in which suppliers have stochastic proportional yields and are paid only for what they deliver. Hence, their model is a special case of ours in the sense that it directly satisfies the primitives of our framework. Nonetheless, their analysis provides a useful complement to ours because it focuses on applying asymptotic properties that allow for the efficient use of the normal distribution to speed up computation of the newsvendor’s optimal order quantities. In a similar vein, Tomlin (2006) has recently developed a framework for studying broader issues of supply disruptions by applying a notion of unreliable supply that is consistent with ours. By doing so, Tomlin underscores the promise for continued exploration into the exciting area of supply disruptions and its relation to newsvendor models with unreliable suppliers.

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REFERENCES


**APPENDIX**

**Proof of Lemma 2.** Part (a). If supplier i is perfectly reliable, then, by definition, \( K_i = \infty \), which implies that \( S_i = \min\{Q_i, K_i\} = Q_i \). Therefore, from (5), \( SL(Q) = \Pr\{D \leq Q_i + S_{-i}\} = E[F(Q_i + S_{-i})] \), where, from the definition of \( S_{-i} \),

\[
Q_i + S_{-i} = Q_i + \sum_{j \neq i}^n S_j \leq Q_i + \sum_{j \neq i}^n Q_j = Q_T.
\]

Part (b)(i). If \( Q_i = 0 \) and supplier i is unreliable, then, from (5), \( SL(Q) = \Pr\{D \leq S_{-i}\} = E[F(S_{-i})] \), where \( S_{-i} \leq Q_T \).

Part (b)(ii). If \( Q_i > 0 \) and supplier i is unreliable, then, from (5), \( SL(Q) = \Pr\{D \leq S_i + S_{-i}\}. \)

Therefore, from (1), if unreliable supplier i has an exogenous production function, \( SL(Q) \) can be written as follows:

\[
SL(Q) = \int_0^Q E[F(r + S_{-i})] g_i(r) dr + [1 - G_i(Q_i)]E[F(Q_i + S_{-i})].
\]

For \( 0 < r < Q_i \), \( 0 < F(r + S_{-i}) < F(Q_i + S_{-i}) \). Therefore, \( SL(Q) > [1 - G_i(Q_i)]E[F(Q_i + S_{-i})] = \Pr\{K_i > Q_i\}E[F(Q_i + S_{-i})], \) and \( SL(Q) < E[F(Q_i + S_{-i})] \leq F(Q_T) \). Similarly, from (3), if unreliable supplier i has an endogenous production function, \( SL(Q) \) can be written as follows:

\[
SL(Q) = \int_0^Q E[F(K_i(Q_i, r) + S_{-i})] g_i(r) dr + [1 - G_i(z_i)]E[F(Q_i + S_{-i})].
\]

From TA2(c), \( 0 < r < z_i \Rightarrow 0 < K_i(Q_i, r) < Q_i \). Thus, \( 0 < F(K_i(Q_i, r) + S_{-i}) < F(Q_i + S_{-i}) \) for \( 0 < r < z_i \), which implies that \( SL(Q) > [1 - G_i(z_i)]E[F(Q_i + S_{-i})] = \Pr\{K_i > Q_i\}E[F(Q_i + S_{-i})] \) and that \( SL(Q) < E[F(Q_i + S_{-i})] \leq F(Q_T) \).

**Proof of Lemma 3.** To establish that \( \Pi(Q) \) is differentiable for a given \( n \), it suffices to show that \( \partial \Pi(Q)/\partial Q_i \) is continuous for \( i = 1, \ldots, n \). To that end, let \( \Lambda(y) = E[y - D]^+ \) and note that \( d\Lambda(y)/dy = F(y) \), which is continuous. Thus, \( \Lambda(y) \) is a differentiable function.

Given the definition of \( \Lambda(y) \), there are three cases to consider, depending on whether supplier i is perfectly reliable, unreliable with an exogenous production function, or unreliable with an endogenous production function. If supplier i is perfectly reliable, then \( S_i = Q_i \). In this case, (9) can be written as

\[
\Pi(Q) = \phi_i Q_i - E[\Lambda(Q_i + S_{-i})] + E[\sum_{j \neq i}^n \phi_j S_j].
\]

Correspondingly,

\[
\partial \Pi(Q)/\partial Q_i = \phi_i - E[F(Q_i + S_{-i})], \tag{A1}
\]

which is continuous over \( Q_i \).

If supplier i is unreliable with an exogenous production function, then \( S_i = \min\{Q_i, R_i\} \). In this case, applying (1), (9) can be written as
\[\Pi(Q) = \phi_i \left( \int_0^Q r g_i(r) dr + [1 - G_i(Q_i)]Q_i \right) \]

\[- \left( \int_0^Q E[\Lambda(r + S_{T-i})]g_i(r) dr + [1 - G_i(Q_i)]E[\Lambda(Q_i + S_{T-i})] \right) + E[\sum_{j \neq i} \phi_j S_j] .\]

Correspondingly,

\[\frac{\partial \Pi(Q)}{\partial Q_i} = [1 - G_i(Q_i)](\phi_i - E[F(Q_i + S_{T-i})]) , \quad (A2)\]

which is continuous over \( Q_i \).

Finally, if supplier \( i \) is unreliable with an endogenous production function, then \( S_i = \min\{Q_i, K_i(Q_i, R_i)\} \). In this case, applying (3), (9) can be written as

\[\Pi(Q) = \phi_i \left( \int_0^{z_i} K_i(Q_i, r) g_i(r) dr + [1 - G_i(z_i)]Q_i \right) \]

\[- \left( \int_0^{z_i} E[\Lambda(K_i(Q_i, r) + S_{T-i})]g_i(r) dr + [1 - G_i(z_i)]E[\Lambda(Q_i + S_{T-i})] \right) + E[\sum_{j \neq i} \phi_j S_j] .\]

Correspondingly, from the definition of \( k_i(Q_i, R_i) \) and (4),

\[\frac{\partial \Pi(Q)}{\partial Q_i} = E[s_i(Q_i, R_i)]\phi_i \]

\[- \left( \int_0^{z_i} k_i(Q_i, r) E[F(K_i(Q_i, r) + S_{T-i})]g_i(r) dr + [1 - G_i(z_i)]E[F(Q_i + S_{T-i})] \right) , \quad (A3)\]

which again is continuous over \( Q_i \).

Since \( \frac{\partial \Pi(Q)}{\partial Q_i} \) is continuous in all three cases, \( \Pi(Q) \) is differentiable.

**Proof of Lemma 4.** Given \( n \), if supplier \( i \) is perfectly reliable, then, from (A1) and Lemma 2(a),

\[M_i(Q) \equiv \frac{\partial \Pi(Q)}{\partial Q_i} = \phi_i - SL(Q) . \quad (A4)\]

Therefore, from (10) and (11), \( Q_i^* = 0 \Rightarrow 0 \geq M_i(Q^*) = \phi_i - SL(Q^*) \Rightarrow SL(Q^*) \geq \phi_i \). And, from (11), \( Q_i^* > 0 \Rightarrow 0 = M_i(Q^*) = \phi_i - SL(Q^*) \Rightarrow SL(Q^*) = \phi_i . \)

**Proof of Lemma 5.** Given \( n \), if supplier \( i \) is unreliable, then there are two cases to consider, depending on whether supplier \( i \) has an exogenous production function or an endogenous production function.

If unreliable supplier \( i \) has an exogenous production function, then (A2) implies

\[M_i(Q) \equiv \frac{\partial \Pi(Q)}{\partial Q_i} = [1 - G_i(Q_i)](\phi_i - E[F(Q_i + S_{T-i})]) . \quad (A5)\]

If \( Q_i^* = 0 \), then (10) and (11) applied to (A5) imply that \( \phi_i \leq E[F(S^*_{T-i})] \), and Lemma 2(b)(i) implies that \( E[F(S^*_{T-i})] = SL(Q^*) \). Therefore, \( Q_i^* = 0 \Rightarrow \phi_i \leq SL(Q^*) \). If \( Q_i^* > 0 \), then (11) applied to (A5) implies that \( \phi_i = E[F(Q_i^* + S^*_{T-i})] \), and Lemma 2(b)(ii) implies that \( [1 - G_i(Q_i^*)]E[F(Q_i^* + S^*_{T-i})] < SL(Q^*) < E[F(Q_i^* + S^*_{T-i})] \). Therefore, \( Q_i^* > 0 \Rightarrow \phi_i = E[F(Q_i^* + S^*_{T-i})] > SL(Q^*) \); and \( Q_i^* > 0 \Rightarrow \phi_i = E[F(Q_i^* + S^*_{T-i})] < SL(Q^*)/[1 - G_i(Q_i^*)] \), where, from (2), \( 1 - G_i(Q_i^*) = E[s_i(Q_i^*, R_i)] \).
\[
M_i(Q) = \frac{\partial \Pi_i}{\partial Q_i} = E[s_i(Q_i, R_i)]\phi_i - \left( \int_0^{z_i} k_i(Q_i, r) E[F(K_i(Q_i, r) + S_{T-r})]g_i(r)dr + [1 - G_i(z_i)]E[F(Q_i + S_{T-r})] \right).
\]

On the one hand, if \( Q_i^* = 0 \), then (10) and (11) imply that \( 0 \geq M_i(Q_i^*) \) and Lemma 2(b)(i) implies that \( Q_i^* = 0 \Rightarrow 0 \geq E[s_i(0, R_i)][\phi_i - E[F(S_{T-r})]] = E[s_i(0, R_i)][\phi_i - SL(Q^*)] \). That is, \( Q_i^* = 0 \Rightarrow \phi_i \leq SL(Q_i^*) \).

On the other hand, if \( Q_i^* > 0 \), then (11) implies that \( M_i(Q_i^*) = 0 \), (5) implies that \( SL(Q_i^*) = \int_0^{z_i} E[F(K_i(Q_i^*, r) + S_{T-r})]g_i(r)dr + [1 - G_i(z_i^*)]E[F(Q_i^* + S_{T-r})] \), and TA2(a) implies that \( z_i^* > 0 \). These relationships, applied to (A6) thus establish two things: First, these relationships establish that \( 0 > i \phi_i - SL(Q_i^*) \), where the inequality follows because TA3.2 stipulates that \( k_i(Q_i^*, r) < 1 \) for \( r < z_i^* \). This inequality, in turn, implies \( SL(Q_i^*) > i \phi_i \), thereby completing the first part of the proof for Lemma 5(b). Second, the relationships establish that

\[
0 = E[s_i(Q_i^*, R_i)]\phi_i - \int_0^{z_i} k_i(Q_i^*, r) E[F(K_i(Q_i^*, r) + S_{T-r})]g_i(r)dr - \left( SL(Q_i^*) - \int_0^{z_i} E[F(K_i(Q_i^*, r) + S_{T-r})]g_i(r)dr \right)
= \phi_i - SL(Q_i^*) - \int_0^{z_i} \left( k_i(Q_i^*, r) - 1 \right) E[F(K_i(Q_i^*, r) + S_{T-r})] - \phi_i \bigg| g_i(r)dr .
\]

Therefore, to establish that \( SL(Q_i^*) < \phi_i \), and thereby complete the proof, it suffices to show that the integral in (A7) is strictly positive. To that end, we establish the following two intermediate results, both of which are derived directly from Karlin and Rinott (1981):

Lemma A1. (Karlin and Rinott, 1981). Let \( \psi(r) \) be a non-decreasing function, and let \( d\mu(r) \) be such that \( d\mu(r) < 0 \Leftrightarrow r < r^0 \), where \( r^0 \) is the unique value of \( r \) that satisfies \( d\mu(r) = 0 \). If there exists a \( z > 0 \) such that \( \int_0^z d\mu(r) = 0 \), then \( \int_0^z \psi(r)d\mu(r) \geq 0 \).

Proof. Given the definition of \( d\mu(r) \), if \( \int_0^z d\mu(r) = 0 \) for some \( z > 0 \), then it means that \( 0 < r^0 < z \).

Moreover, given the definitions of \( d\mu(r) \) and \( \psi(r) \), if \( r \leq r^0 \), then \( d\mu(r) \leq 0 \) and \( \psi(r) \leq \psi(r^0) \). Thus, \( r \leq r^0 \Rightarrow \psi(r)d\mu(r) \geq \psi(r^0)d\mu(r) \). Similarly, \( r \geq r^0 \Rightarrow \psi(r)d\mu(r) \geq \psi(r^0)d\mu(r) \). Accordingly, \( \psi(r)d\mu(r) \geq \psi(r^0)d\mu(r) \) for all \( r \). Therefore, \( \int_0^z \psi(r)d\mu(r) \geq \psi(r^0)\int_0^z d\mu(r) = 0 \).

Lemma A2. (Karlin and Rinott, 1981). Let \( \psi(r) \) and \( \eta(r) \) be two non-decreasing functions, and let \( g(r) \) be a probability density function. Then, \( \int_0^z \psi(r)\eta(r)g(r)dr \geq \int_0^z \psi(r)g(r)dr \int_0^z \eta(r)g(r)dr / G(z) \).

Proof. Let \( d\mu(r) = A(r)g(r)dr \), where

\[
A(r) = \frac{\eta(r)}{\int_0^z \eta(r)g(r)dr} - \frac{1}{G(z)} .
\]

Notice first that \( d\mu(r) = 0 \) if and only if \( A(r) = 0 \) and that \( A(r) \) is non-decreasing in \( r \) (because \( \eta(r) \) is non-decreasing). This means that there exists a unique \( r \), say \( r^0 \), that is such that \( d\mu(r^0) = 0 \) and \( d\mu(r) < 0 \Leftrightarrow r < r^0 \). Notice second that \( \int_0^r d\mu(r) = 0 \). Therefore, the conditions of Lemma A1 apply, implying that
Given Lemma A2, we now complete the proof of Lemma 5 by deriving Lemma A3 below to establish the final remaining desired result, namely that the integral in (A7) is strictly positive.

**Lemma A3.** \[ \int_0^{z_i} (k_i(Q^*_i,r) - 1)(E[F(K_i(Q^*_i,r) + S^*_r)] - \phi_i)g_i(r)dr > 0. \]

**Proof.** From TA3.2 and TA2, \( k_i(Q^*_i,r) - 1 \) and \( E[F(K_i(Q^*_i,r) + S^*_r)] - \phi_i \), respectively, are non-decreasing in \( r \). Therefore, by directly applying Lemma A2,

\[
\int_0^{z_i} (k_i(Q^*_i,r) - 1)(E[F(K_i(Q^*_i,r) + S^*_r)] - \phi_i)g_i(r)dr \geq \frac{\left( \int_0^{z_i} (1 - k_i(Q^*_i,r))g_i(r)dr \right)\left( \int_0^{z_i} (\phi_i - E[F(K_i(Q^*_i,r) + S^*_r)])g_i(r)dr \right)}{G_i(z^*_i)}. \quad (A8)
\]

Similarly, by directly applying Lemma A2 in a like manner,

\[
\int_0^{z_i} k_i(Q^*_i,r)(E[F(K_i(Q^*_i,r) + S^*_r)] - \phi_i)g_i(r)dr \geq \frac{\left( \int_0^{z_i} k_i(Q^*_i,r)g_i(r)dr \right)\left( \int_0^{z_i} (E[F(K_i(Q^*_i,r) + S^*_r)] - \phi_i)g_i(r)dr \right)}{G_i(z^*_i)}. \quad (A9)
\]

Now, returning to (A6) and recalling that \( Q^*_i > 0 \) implies both that \( M_i(Q^*_i) = 0 \) and that \( z^*_i > 0 \),

\[
0 = M_i(Q^*_i) > E[S_i(Q^*_i,R_i)][\phi_i - E[F(Q^*_i + S^*_r)]],
\]

where the inequality follows because TA2 establishes that \( K_i(Q^*_i,r) < Q^*_i \) for \( r < z^*_i \). Consequently, \( E[F(Q^*_i + S^*_r)] > \phi_i \). Substituting this inequality, in turn, for \( E[F(Q^*_i + S^*_r)] \) in (A6) establishes that

\[
0 = M_i(Q^*_i) < -\int_0^{z_i} k_i(Q^*_i,r)(E[F(K_i(Q^*_i,r) + S^*_r)] - \phi_i)g_i(r)dr,
\]

which, applying (A9), implies

\[
0 < \left( \int_0^{z_i} k_i(Q^*_i,r)g_i(r)dr \right)\left( \int_0^{z_i} (\phi_i - E[F(K_i(Q^*_i,r) + S^*_r)])g_i(r)dr \right)G_i(z^*_i).
\]

In other words,

\[
\int_0^{z_i} (\phi_i - E[F(K_i(Q^*_i,r) + S^*_r)])g_i(r)dr > 0.
\]

When applied to (A8), this inequality, together with the relationship stipulated by TA3.2, namely that \( r < z^*_i \Rightarrow k_i(Q^*_i,r) < 1 \), establishes that

\[
\int_0^{z_i} (k_i(Q^*_i,r) - 1)(E[F(K_i(Q^*_i,r) + S^*_r)] - \phi_i)g_i(r)dr > 0,
\]

which completes the proof. \( \square \)

**Proof of Proposition 1.** Part (a). Given \( n \), first note from (A4), (A5), and (A6) that \( M_i(0) > 0 \) for all \( i \).

Thus, ordering exactly 0 units from all \( n \) available suppliers does not satisfy necessary condition (10),
which implies that at least one supplier is active in an optimal solution. In other words, for any given \( n \), there exists an \( i \) that is such that \( Q_i^* > 0 \). Accordingly, from Lemmas 4(b) and 5(b), there exists an \( i \) that is such that \( SL(Q_i^*) \leq \phi_i \). But, from (6), \( \phi_i \leq \phi_i \). Therefore, \( SL(Q_i^*) \leq \phi_i \). Next, notice that (10) together with (A4) – (A6) imply that \( \phi_i \leq E[F(Q_i^* + S_{T-i}^*)] \). But, from Table 2, \( S_{T-i}^* = \sum_{i=2}^{n} S_i^* \) and \( S_i^* \leq Q_i^* \) for all \( i \). Therefore,

\[
Q_i^* + S_{T-i}^* = Q_i^* + \sum_{i=2}^{n} S_i^* \leq Q_i^* + \sum_{i=2}^{n} Q_i^* = Q_i^*,
\]

which implies that \( \phi_i \leq E[F(Q_i^*)] \).

Part (b). For the purpose of this proof, we substitute the following notation to make explicit the number of available suppliers: \( Q_n \) for \( Q \), \( ST_{i,n} \) for \( ST_{i} \), and \( M_i^n(Q) \) for \( M_i(Q) \). Accordingly, to establish Part (b), it suffices to show that \( M_i^n(Q_n) \geq M_i^{n+1}(Q_{n+1}) \) for \( i = 1, \ldots, n \). To that end, note that \( Q_{n+1} = \{ Q_n, Q_{n+1} \} \) and that \( ST_{i,n} \leq ST_{i,n+1} \), and consider that supplier \( i \) can be either perfectly reliable, unreliable with an exogenous production function, or unreliable with an endogenous production function. Thus:

If supplier \( i \) is perfectly reliable, then, from (A4) and Lemma 2(a),

\[
\begin{align*}
M_i^n(Q_n) &= \phi_i - E[F(Q_i + S_{T-i,n})] \geq \phi_i - E[F(Q_i + S_{T-i,n+1})] = M_i^{n+1}(Q_n).
\end{align*}
\]

If supplier \( i \) is unreliable with an exogenous production function, then, from (A5),

\[
\begin{align*}
M_i^n(Q_n) &= [1 - G_i(Q_i)](\phi_i - E[F(Q_i + S_{T-i,n})]) \geq [1 - G_i(Q_i)](\phi_i - E[F(Q_i + S_{T-i,n+1})]) = M_i^{n+1}(Q_{n+1}).
\end{align*}
\]

If supplier \( i \) is unreliable with an endogenous production function, then, from (A6),

\[
\begin{align*}
M_i^n(Q_n) &= E[s_i(Q_i, R_i)]\phi_i - \int_0^{z_i} [k_i(Q_i, r)E[F(K_i(Q_i, r) + S_{T-i,n})]g_i(r)dr + [1 - G_i(z_i)]E[F(Q_i + S_{T-i,n})]]
\geq E[s_i(Q_i, R_i)]\phi_i - \int_0^{z_i} [k_i(Q_i, r)E[F(K_i(Q_i, r) + S_{T-i,n+1})]g_i(r)dr + [1 - G_i(z_i)]E[F(Q_i + S_{T-i,n+1})]]
= M_i^{n+1}(Q_{n+1}).
\end{align*}
\]

Proof of Proposition 6. Given that, for \( m = 1, \ldots, n \), \( Q_m^* = \{ Q_{m,i}^* \} \) denotes the optimal solution to the subproblem in which only the subset of suppliers \( 1, \ldots, m \) are considered, we additionally adapt our notation for the purpose of this proof as follows: Given \( m \), let \( Q_m \) and \( ST_{i,m} \) replace, respectively, \( Q \) and \( ST_{i} \) from Table 2; and let \( M_i^m(Q) \) replace \( M_i(Q) \) from Section 3. Then, we proceed in two steps.

First, we establish that \( SL(Q_m^*) > \phi_{m+1} \Rightarrow Q_{m+1,n}^* = 0 \) as follows: From Lemmas 4(b) and 5(b), \( SL(Q_m^*) > \phi_{m+1} \Rightarrow Q_{m+1,n}^* = 0 \). Therefore, if \( SL(Q_m^*) \) is non-decreasing in \( m \) for \( m \leq n \), then \( SL(Q_m^*) > \phi_{m+1} \Rightarrow SL(Q_m^*) \geq SL(Q_m^*) > \phi_{m+1} \Rightarrow Q_{m+1,n}^* = 0 \).

Second, we establish that \( SL(Q_m^*) < \phi_{m+1} \Rightarrow Q_{m+1,n}^* > 0 \) as follows: We begin by making the observation that, for \( i = 1, \ldots, m \), \( M_i^m(Q_n) = M_i^m(Q_m) \) if \( Q_i = 0 \) for \( i = m+1, \ldots, n \). To demonstrate this, let \( Q_n = \{ Q_m, Q_{m+1}, \ldots, Q_n \} \); and then notice, from (12), that, for \( i = 1, \ldots, m \),

\[
M_i^m(Q_n) = M_i^m(Q_m).
M_i^n(Q^n) = M_i^n((Q_m^*, 0)) = \begin{cases} 
\phi_i - E[F(Q_i + S_{T-i,n})] & \text{if } i \text{ is perfectly reliable} \\
\phi_i E[s_i(Q_1, R_1)] - E[s_i(Q_1, R_1)F(S_i + S_{T-i,n})] & \text{if } i \text{ is unreliable} 
\end{cases}

\begin{align*}
M_i^n(Q^n) &= M_i^n((Q_m^*, 0)) = \begin{cases} 
\phi_i - E[F(Q_i + S_{T-i,m} + 0)] & \text{if } i \text{ is perfectly reliable} \\
\phi_i E[s_i(Q_1, R_1)] - E[s_i(Q_1, R_1)F(S_i + S_{T-i,m} + 0)] & \text{if } i \text{ is unreliable} 
\end{cases}
\end{align*}

Therefore, if Q_{1,n}^* = 0 for i = m+1, ..., n, then, for i = 1, ..., m, \{Q_i^*, m\} must satisfy the same set of KKT conditions that \{Q_i^*, m\} must satisfy for the m-supplier problem. In other words, if Q_{1,n}^* = 0 for i = m+1, ..., n, then Q_{i,n}^* = Q_{i,m}^* for i = 1, ..., m. Therefore, SL(Q_{1,n}^*) = SL((Q_m^*, 0)) = E[F(S_i + S_{T-i,m} + 0)] = SL(Q_m^*). Accordingly, if SL(Q_{1,n}^*) < \phi_{m+1}, then SL(Q_{1,n}^*) < \phi_{m+1}, which, from Lemmas 4(a) and 5(a), implies that Q_{m+1,n} > 0.

**Proof of Proposition 7.** Given that n = 2, if m = 1, then, from (11) and (A2), F(Q_{1,1}^*) = \phi_1. Similarly, if m = 2, then for any given Q_2 \geq 0, \phi_1 = E[F(Q_{1,2}^* + S_2)], where Q_{1,2}^*(Q_2) denotes the optimal Q_1 as a function of Q_2. Thus, Q_{1,2}^*(0) = Q_{1,1}^*. Consider, then, SL as a function of the single variable Q_2:

SL(Q_{1,2}^*(Q_2), Q_2) = E[F(S_{1,2} + S_2)] = \int_0^{Q_{1,2}^*(Q_2)} E[F(r + S_2)]g_1(r)dr + [1 - G_1(Q_{1,2}^*(Q_2))]\phi_1.  \hspace{1cm} (A10)

Thus,

$$
\frac{dSL(Q_{1,2}^*(Q_2), Q_2)}{dQ_2} = \frac{\partial SL(Q_{1,2}^*(Q_2), Q_2)}{\partial Q_2} + \frac{\partial SL(Q_{1,2}^*(Q_2), Q_2)}{\partial Q_{1,2}^*} \frac{dQ_{1,2}^*}{dQ_2} = \int_0^{Q_{1,2}^*(Q_2)} \frac{\partial E[F(r + S_2)]}{\partial Q_2}g_1(r)dr \geq 0,
$$

where the inequality follows because, from Table 2, S_2 is non-decreasing in Q_2 regardless of the specific form of supplier 2’s production function. In turn, this inequality implies that SL(Q_{1,n}^*) = SL(Q_{1,2}^*(Q_{2,2}^*), Q_{2,2}^*) \geq SL(Q_{1,2}^*(0, 0)) = SL(Q_{1,1}^*, 0) = SL(Q_{1,1}^*).

**Proof of Proposition 8.** Recall from the proof of Proposition 7 that F(Q_{1,1}^*) = \phi_1. This implies that Q_{1,1}^* is independent of \rho_1 and that \frac{dQ_{1,1}^*}{d\phi_1} > 0. Moreover,

$$
B_1(\phi_1, \rho_1) = SL(Q_{1,1}^*) = \int_0^{Q_{1,1}^*} F(r)g_1(r; \rho_1)dr + [1 - G_1(Q_{1,1}^*; \rho_1)]\phi_1 = \phi_1 - \int_0^{Q_{1,1}^*} f(r)G_1(r; \rho_1)dr,
$$

where the last equality follows from integration by parts. Accordingly,

$$
\frac{\partial B_1(\phi_1, \rho_1)}{\partial \phi_1} = [1 - G_1(Q_{1,1}^*; \rho_1)] > 0,
$$

and

$$
\frac{\partial B_1(\phi_1, \rho_1)}{\partial \rho_1} = -\int_0^{Q_{1,1}^*} f(r) \frac{\partial G_1(r; \rho_1)}{\partial \rho_1}dr > 0.
$$

**Proof of Proposition 9.** By assumption, Q_{1,2}^* > 0 and Q_{2,2}^* > 0. Therefore, from (11), Q_{1,2}^* and Q_{2,2}^* must satisfy $M_i^1(Q_2^*) = 0 = M_i^2(Q_2^*)$, where, here, the notation $M_i^1(Q)$ is used in place of $M_i(Q)$ from...
Section 3 to explicitly reflect the number of active suppliers. Given this, we consider separately the two possible cases for supplier 2.

Case 1. If supplier 2 is perfectly reliable, then, from (A2) and (A1), $M_1^2(Q_2^*) = 0 = M_2^*(Q_2^*)$ implies

$$\phi_1 = F(Q_{1,2}^* + Q_{2,2}^*) = F(Q_{T,2}^*) \quad \text{(A11)}$$

$$\phi_2 = SL(Q_2^*) = \int_0^{Q_{2,2}^*} F(r + Q_{2,2}^*) g_1(r; \rho_1) dr + [1 - G_1(Q_{1,2}^*; \rho_1)] \phi_1$$

$$= \phi_1 - \int_0^{Q_{2,2}^*} G_1(r; \rho_1) df(r + Q_{2,2}^*). \quad \text{(A12)}$$

From (A11), $Q_{T,2}^*$ is increasing in $\phi_1$ and is independent of both $\phi_2$ and $\rho_1$; and from (A12), $SL(Q_2^*)$ is increasing in $\phi_2$ and is independent of both $\phi_1$ and $\rho_1$. Moreover, by taking the total derivative of (A12) with respect to $\phi_1$,

$$0 = \frac{\partial Q_{2,2}^*}{\partial \phi_1} \left( \int_0^{Q_{2,2}^*} f(r + Q_{2,2}^*) g_1(r; \rho_1) dr \right) + [1 - G_1(Q_{1,2}^*; \rho_1)].$$

This implies that $\partial Q_{2,2}^*/\partial \phi_1 < 0$, which in turn implies that $\partial Q_{1,2}^*/\partial \phi_1 = (\partial Q_{T,2}^*/\partial \phi_1) - (\partial Q_{2,2}^*/\partial \phi_1) > 0$. Similarly, by taking the total derivative of (A12) with respect to $\phi_2$,

$$1 = \frac{\partial Q_{2,2}^*}{\partial \phi_2} \left( \int_0^{Q_{2,2}^*} f(r + Q_{2,2}^*) g_1(r; \rho_1) dr \right).$$

This implies that $\partial Q_{2,2}^*/\partial \phi_2 > 0$, which in turn implies that $\partial Q_{1,2}^*/\partial \phi_2 = (\partial Q_{T,2}^*/\partial \phi_2) - (\partial Q_{2,2}^*/\partial \phi_2) < 0$. Finally, by taking the total derivative of (A13) with respect to $\rho_1$,

$$0 = -\int_0^{Q_{2,2}^*} \frac{\partial G_1(r; \rho_1)}{\partial \rho_1} df(r + Q_{2,2}^*) - \frac{\partial Q_{2,2}^*}{\partial \rho_1} \left( \int_0^{Q_{2,2}^*} G_1(r; \rho_1) df(r + Q_{2,2}^*) \right) - \frac{\partial Q_{1,2}^*}{\partial \rho_1} f(Q_{T,2}^*) G_1(Q_{1,2}^*; \rho_1)$$

$$= -\int_0^{Q_{2,2}^*} \frac{\partial G_1(r; \rho_1)}{\partial \rho_1} df(r + Q_{2,2}^*) + \frac{\partial Q_{2,2}^*}{\partial \rho_1} \int_0^{Q_{2,2}^*} f(r + Q_{2,2}^*) g_1(r; \rho_1) dr,$$

where the second inequality follows first from integrating by parts and then from the above result indicating that $\partial Q_{T,2}^*/\partial \rho_1 = 0$. From this inequality, we conclude that $\partial Q_{2,2}^*/\partial \rho_1 < 0$. Therefore,

$$\partial Q_{2,2}^*/\partial \rho_1 = (\partial Q_{T,2}^*/\partial \rho_1) - (\partial Q_{2,2}^*/\partial \rho_1) > 0,$$

which establishes the last of the desired inequalities for the case in which supplier 2 is perfectly reliable.

Case 2. If supplier 2 is unreliable, then, from (A2), $M_1^2(Q_2^*) = 0 = M_2^*(Q_2^*)$ implies

$$\phi_1 = N_i(Q_{1,2}^*, Q_{j,2}^*, \rho_j), \quad \text{(A14)}$$

where, for $i = 1, 2; j = 1, 2; \text{ and } i \neq j$,

$$N_i(Q_{1,2}^*, Q_{j,2}^*, \rho_j) = \int_0^{Q_{1,2}^*} F(Q_{1,2}^* + r) dG_j(r; \rho_j) + [1 - G_j(Q_{j,2}^*; \rho_j)] F(Q_{1,2}^*)$$

$$= F(Q_{T,2}^*) - \int_0^{Q_{1,2}^*} G_j(r; \rho_j) df(Q_{1,2}^* + r).$$

Note that the following properties follow directly from the definition of $N_i(Q_{1,2}^*, Q_{j,2}^*, \rho_j)$.
\[
\frac{\partial N_i}{\partial Q_{i,2}} = [1 - G_j(Q_{j,2}^*; \rho_j)]f(Q_{i,2}^*) > 0;  \tag{A15}
\]

\[
\frac{\partial N_i}{\partial Q_{i,2}^*} = \int_0^{Q_{i,2}^*} f(Q_{i,2} + r)dG_j(r; \rho_j) + [1 - G_j(Q_{j,2}^*; \rho_j)]f(Q_{i,2}^*) > \frac{\partial N_i}{\partial Q_{i,2}^*} > 0;  \tag{A16}
\]

\[
\frac{\partial N_i}{\partial \rho_j} = -\int_0^{Q_{i,2}^*} \frac{\partial G_j(r; \rho_j)}{\partial \rho_j} dF(Q_{i,2}^*) > 0.  \tag{A17}
\]

Consider, then, the total derivative of (A14), taken with respect to \(\phi_j\) and \(\phi_i\), respectively:

\[
0 = \frac{\partial N_i}{\partial Q_{i,2}^*} \frac{\partial Q_{i,2}^*}{\partial \phi_j} + \frac{\partial N_i}{\partial Q_{i,2}^*} \frac{\partial Q_{i,2}^*}{\partial \phi_i} = \frac{\partial N_i}{\partial Q_{i,2}^*} \frac{\partial Q_{i,2}^*}{\partial \phi_j} - \left( \frac{\partial N_i}{\partial Q_{i,2}^*} - \frac{\partial N_i}{\partial \phi_j} \right) \frac{\partial Q_{i,2}^*}{\partial \phi_j},  \tag{A18}
\]

\[
1 = \frac{\partial N_i}{\partial Q_{i,2}^*} \frac{\partial Q_{i,2}^*}{\partial \phi_j} + \frac{\partial N_i}{\partial Q_{i,2}^*} \frac{\partial Q_{i,2}^*}{\partial \phi_i} = \frac{\partial N_i}{\partial Q_{i,2}^*} \frac{\partial Q_{i,2}^*}{\partial \phi_j} - \left( \frac{\partial N_i}{\partial Q_{i,2}^*} - \frac{\partial N_i}{\partial \phi_i} \right) \frac{\partial Q_{i,2}^*}{\partial \phi_i}.  \tag{A19}
\]

Given (A15) and (A16), (A18) implies that

\[-\text{sign}(\partial Q_{i,2}^*/\partial \phi_j) = \text{sign}(\partial Q_{i,2}^*/\partial \phi_i) = \text{sign}(\partial Q_{i,2}^*/\partial \phi_j),  \tag{A20}\]

and (A19) implies that

\[\partial Q_{i,2}^*/\partial \phi_i \geq 0 \implies \partial Q_{i,2}^*/\partial \phi_i > 0. \tag{A21}\]

However, if \(\partial Q_{i,2}^*/\partial \phi_i \geq 0\), then (A21) contradicts (A20). Therefore, we conclude that \(\partial Q_{i,2}^*/\partial \phi_j < 0\), which, from (A20), then implies that \(\partial Q_{i,2}^*/\partial \phi_i > 0\) and \(\partial Q_{i,2}^*/\partial \phi_i > 0\).

Similarly, taking the total derivative of \(\phi_i = N_j(Q_{j,2}^*, Q_{i,2}^*, \rho_j)\) with respect to \(\rho_i\) and applying (A15) – (A17) establishes that

\[-\text{sign}(\partial Q_{i,2}^*/\partial \rho_j) = \text{sign}(\partial Q_{i,2}^*/\partial \rho_i) = \text{sign}(\partial Q_{i,2}^*/\partial \rho_j), \tag{A22}\]

while taking the total derivative of \(\phi_j = N_j(Q_{j,2}^*, Q_{i,2}^*, \rho_j)\) with respect to \(\rho_i\) and applying (A14) – (A16) establishes that

\[\partial Q_{i,2}^*/\partial \rho_i \geq 0 \implies \partial Q_{i,2}^*/\partial \rho_i < 0. \tag{A23}\]

Since (A23) contradicts (A22) if \(\partial Q_{i,2}^*/\partial \rho_i \geq 0\), we conclude that \(\partial Q_{i,2}^*/\partial \rho_i < 0\), which, from (A22), implies that \(\partial Q_{i,2}^*/\partial \rho_i > 0\) and \(\partial Q_{i,2}^*/\partial \rho_i < 0\).

To complete the proof, we next analyze \(S\ell(Q_i^*)\) as a function of \(\phi_j\) and \(\rho_j\) for \(j = 1, 2\). To that end, recall that, by definition, \(S\ell(Q_i^*) = E[F(S_{i,2}^* + S_{j,2}^*)]\). Thus, using (A14),

\[
S\ell(Q_i^*) = \int_0^{Q_{i,2}^*} E[F(r_i + S_{2,2}^*)]dG_i(r_i; \rho_i) + [1 - G_i(Q_{i,2}^*; \rho_i)]f_i
\]

\[
= \int_0^{Q_{i,2}^*} \left( F(r_i + Q_{j,2}^*) - \int_0^{Q_{i,2}^*} G_j(r_j; \rho_j)f(r_i + r_j)dr_j \right) dG_i(r_i; \rho_i) + [1 - G_i(Q_{i,2}^*; \rho_i)]f_i  \tag{A24}\]

\[= S\ell(Q_{i,2}^*, Q_{j,2}^*, \rho_i, \rho_j, \phi_i).\]
Given (A24), note that
\[
\frac{\partial \text{SL}(Q_{1,2}^*, Q_{1,2}^*, \rho_j, \phi_j \phi_j)}{\partial Q_{1,2}^*} = \left( F(Q_{1,2}^*) - \int_0^{Q_{1,2}^*} G_j(r_j; \rho_j) f(Q_{1,2}^* + r_j) dr_j - \phi_j \right) g_1(Q_{1,2}^*; \rho_j) = 0; \tag{A25}
\]
\[
\frac{\partial \text{SL}(Q_{1,2}^*, Q_{1,2}^*, \rho_j, \phi_j \phi_j)}{\partial Q_{1,2}^*} = [1 - G_j(Q_{1,2}^*; \rho_j)] \int_0^{Q_{1,2}^*} f(r_j + Q_{1,2}^*) dG_i(r_j; \rho_j) > 0; \tag{A26}
\]
\[
\frac{\partial \text{SL}(Q_{1,2}^*, Q_{1,2}^*, \rho_j, \phi_j \phi_j)}{\partial \rho_j} = - \int_0^{Q_{1,2}^*} \int_0^{Q_{1,2}^*} \frac{\partial G_j(r_j; \rho_j)}{\partial \rho_j} f(r_j + r_j) dr_j dG_i(r_j; \rho_j) > 0. \tag{A27}
\]

Applying (A25) – (A27) to (A24) therefore implies, for \( j = 1 \) or \( 2 \):
\[
\frac{\partial \text{SL}(Q_j^* \phi_j)}{\partial \phi_j} = 0 - \frac{\partial Q_{1,2}^*}{\partial \phi_j} + \frac{\partial \text{SL}(Q_{1,2}^*, Q_{1,2}^*, \rho_j, \phi_j \phi_j)}{\partial Q_{1,2}^*} \frac{\partial Q_{1,2}^*}{\partial \phi_j} > 0,
\]
where this inequality follows because, recall, \( \partial Q_{1,2}^*/\partial \phi_j > 0 \); and
\[
\frac{\partial \text{SL}(Q_j^* \phi_j)}{\partial \rho_j} = 0 - \frac{\partial Q_{1,2}^*}{\partial \rho_j} + \frac{\partial \text{SL}(Q_{1,2}^*, Q_{1,2}^*, \rho_j, \phi_j \phi_j)}{\partial Q_{1,2}^*} \frac{\partial Q_{1,2}^*}{\partial \rho_j} + \frac{\partial \text{SL}(Q_{1,2}^*, Q_{1,2}^*, \rho_j, \phi_j \phi_j)}{\partial \rho_j} > 0,
\]
where this inequality follows because \( \partial Q_{1,2}^*/\partial \rho_j > 0 \). \qed