Economic Order Quantities for Products with Finite Demand Horizons

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Abstract: Industrial engineers have for many years recommended the implementation of EOQ policies in applications in which the strict assumptions of the EOQ model are not satisfied. There are two reasons for this: 1) the EOQ model is a relatively robust model; that is, it is insensitive to at least some of its assumptions; and 2) the lack of a suitable alternative policy. This paper presents a simple generalization of the EOQ model which allows the model to be directly applicable to products with a finite product life. The model is: 1) computationally simple to use; and 2) yields policies which are appreciably less costly than currently employed alternatives.

Industrial engineers have long been familiar with the classical economic order quantity (EOQ) model (2) and the corresponding economic manufacturing quantity (EMQ) model (1). One shortcoming of both of these models is the assumption that demand will continue infinitely into the future (an infinite time horizon). Obviously, no product meets the infinite horizon assumption. Every product manufactured or sold has a finite product life (a finite demand horizon). This product life may be very short, as in some short term custom subcontract work; moderately long, as in spare parts for 1972 automobiles; or very long, as in the case of 60 watt incandescent light bulbs. The effect of this finite demand horizon on the optimal inventory manufacturing and/or stocking policy can be quite significant, particularly in the case of short demand horizons.

This paper presents a simple generalization of the EOQ and EMQ models which explicitly accounts for the finite demand horizon. This model is computationally simple (on the order of difficulty of the standard EOQ or EMQ models) and yields policies which are always less costly than any alternative policy.

Consider a firm which must supply a uniform demand, $D$ per unit time, for some item over the finite time interval $[0, T]$ without backorders or lost sales. The firm has costs:

\[ K = \text{set-up or order cost, } K > 0; \]
\[ h = \text{holding cost per unit per unit of time, } h > 0. \]

We will assume that a setup taking $S \geq 0$ time units must occur between production start (orders) and that $P > D (P = \infty)$ is the production rate. This notation permits us to treat the ordering problem as a special case of the manufacturing problem (by setting $P = \infty$ and $S = 0$). We will also assume that the desired ending inventory is zero and that beginning inventory, $I_0$, is $S D$ (zero for the merchandising application; a setup time's supply in the manufacturing application). If $I_0$ is greater than $S D$, we will solve the problem for the interval $[(I_0 - S D)/D, T]$ and implement the policy at time $(I_0 - S D)/D$, doing nothing until that time.

The objective is to select a manufacturing (order) policy which minimizes total incremental cost, $C(O, T)$, over the finite time interval $[0, T]$. We shall denote the cost of the optimal policy $C^*(O, T)$.

Given $k > 0$, we may restrict our attention to only those policies in which manufacture (delivery) of product occurs when inventory equals zero, since all other policies have higher costs. It remains to deter-
mine the number of deliveries (orders), \( n \), and their lot size.

Let \( t_i \) be the interval between the \((i-1)\)st and the 
\( i \)th production run (order). Let \( C(t_i) \) be the total 
incremental cost for the interval \( t_i \). Then for any given \( n \)
\[
C(O, T) = \sum_{i=1}^{n} C(t_i) 
\]

where
\[
C(t_i) = K + AD(1 - D/P)t_i^2/2. 
\]

(2)

\( C(t_i) \) is derived as the sum of the order cost, \( K \), and the 
inventory cost, \( AD(1 - D/P)t_i^2/2 \), in the time interval, \( t_i \). 
The inventory cost is \( \$ \)/unit/year times the area under one sawtooth of the typical EOQ (EMQ) 
time-inventory graph. The area of the sawtooth is, of 
course, one-half the base times the height. The base 
of the sawtooth is \( t_i \), the length of the interval. The 
height of the sawtooth is equal to the rate of inventory 
accumulation, \( D - P \), times the length of time inventory 
accumulates, \( D/P \) (derived below). Thus 
the area of the sawtooth is \( (1/2)D(D/P)(D - P) \), and 
the holding cost after rearrangement is \( AD(1 - D) 
P/P^2 \). The form of the optimal policy can be deter-
mined by viewing the problem as the quadratic pro-
gram:

\[
\min C(O, T) 
\]

(3)

\[
t_i = \sum_{i=1}^{n} t_i = T 
\]

(4)

\[
s(1 - D/P) t_i \leq S \quad \text{for} \quad i = 1, \ldots, n 
\]

(5)

where \( n \), the number of deliveries (orders), is unknown.

Constraint (5) deserves further explanation. If \( t_i \) is 
the interval between production runs, then during 
this interval \( D \) units must be produced (if less than 
\( D \) is produced, backorders or lost sales will be 
inventoried; if more than \( D \) is produced, positive inventory 
will be on hand when the next production run occurs). 
Production will take \( D/P \) time units. The time 
which remains before the next production run, \( t_i - 
D/P = t_i - (1 - D/P) \), must be long enough to permit 
a setup taking \( S \) time units; thus constraint (5). Of 
course, in the merchandising application, \( P = \infty \) 
and \( S = 0 \). In this instance (5) holds trivially.

From the quadratic programming formulation, it is 
clear (by Lagrange multipliers) that in the optimal 
solution all of the \( t_i \) are of the same length; that is, 
\( t_i = T/n \), where \( n \) is the number of setups (orders) is 
unknown. It remains to determine the optimal value 
of \( n, n^* \).

Let \( C^*(O, T) = \text{total incremental cost for the interval} \]
\( [0, T] \) \( \text{given} \ n \ \text{set-ups (orders)} \)

Since we know all of the production intervals will be 
the same length, \( T/n \), we may substitute \( t_i = T/n \) 
in (2) and then sum in (1) to obtain:

\[
C^*(O, T) = nK + AD(1 - D/P)T^2/2n 
\]

See Figure 1.

\[
\text{Fig. 1. Total incremental cost for the interval} \ [0, T] \ \text{given} \ \text{n set-ups (orders),} \ C^*(O, T) 
\]

If we ignore constraint [5], \( C^*(O, T) \), the cost of 
the optimal policy, as a function of \( T \), is the lower 
envelope of the set of cost functions \( C^*(O, T) \) for \( n = 1, 2, \ldots, \) 
etc. In order to further identify the envelope and to 
facilitate finding the optimal number of setups (orders) 
noted that \( C^*(O, T) \) and \( C^*(O, T) \) interest for \( T \) 
satisfying the following equivalent conditions:

\[
C^*(O, T) = C^*(O, T) \]

\[
nK + \frac{AD(1 - D/P)T^2}{2n} = (n + 1)K + \frac{AD(1 - D/P)T^2}{2(n + 1)} 
\]

\[
n(n + 1) = \frac{AD(1 - D/P)T^2}{2K} 
\]

\[
T = \left[ \frac{2n(n + 1)K}{AD(1 - D/P)} \right]^{1/2} 
\]

Note also that if \( T \) is less than \( T \), \( C^*(O, T) < C^* 
(0, T) \); and if \( T \) is greater than \( T \), \( C^*(O, T) > C^* 
(0, T) \). Thus

\[
C^*(O, T) = C^*(O, T) \]

for

\[
\left[ \frac{2n(n - 1)K}{AD(1 - D/P)} \right]^{1/2} \leq T \leq \left[ \frac{2n(n + 1)K}{AD(1 - D/P)} \right]^{1/2} 
\]

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Finally note (Figure 1) that for \(n\) and \(T\) satisfying [9],
\[
C^*(0, T) \leq C^*(0, T) \leq C^*(0, T) \leq C^*(0, T) \leq C^*(0, T),
\]
[10]

For example, on the interval over which \(C^*(0, T) = C^*(0, T) = C^*(0, T) = C^*(0, T) \leq C^*(0, T)\). This fact is useful in determining the optimal policy in the manufacturing application.

The Solution

In the ordering application, where constraint [5] can be ignored, we can see from [8] and [9] that the optimal number of orders, \(n^*\), is the smallest integer satisfying
\[
\frac{2n(n + 1)K}{D(1 - D/P)n} \leq \frac{D(1 - D/P)K}{2K}
\]
or,
\[
n(n + 1) \leq \frac{D(1 - D/P)K}{2K}
\]
[11]
The optimal order size, \(Q^*\), is given by
\[
Q^* = DT/n^*.
\]
[12]

In the manufacturing application, constraint [5] must be considered; that is, the interval between production runs, \(T/n\), must satisfy
\[
\frac{T}{n} \geq S
\]
or,
\[
n \leq \frac{T(1 - D/P)}{S}
\]
[13]

To determine \(n^*\) we determine the smallest integer, \(n\), satisfying [11], as in the EOQ application. If this \(n\) satisfies [13], then \(n^* = n\). If [13] is violated, we use relation [10] to identify \(n^*\) as the largest \(n\) satisfying [13]. In summary,
\[
n^* = \min \{n : (T(1 - D/P)/S)\}
\]
[14]
where \(n\) is the smallest integer satisfying [11] and \([k]\) is the largest integer smaller than \(k\). \(Q^*\) is given by [12].

Convergence to the Infinite Horizon Policy

For finite horizon policies, which minimize total cost, \(C(0, T)/T\), also minimize average cost per unit time, \(C(0, T)/T\). Hence the policies developed above are optimal with respect to both the average and total cost criterion. For convergence our solution was developed using total costs. However, in order to see the convergence of optimal finite horizon policies to the optimal infinite horizon policy, consideration of the average cost of the optimal policy
\[
C^*(0, T)/T = \frac{n^*K}{T} + \frac{kD(1 - D/P)T}{2n^*}
\]
is useful. Figure 2 is a graph of \(C^*(0, T)/T\) assuming that the set-up constraint [5] is not binding. (If the set-up constraint is binding, interpretations of convergence to the optimal infinite horizon solution are confused because convergence in this case is convergence to the set-up constrained infinite horizon policy.)

Fig. 2. Average cost per unit time of optimal policy, \(C^*(0, T)/T\)

Two observations on Figure 2 are of interest. First, \(C^*(0, T)/T\) equals the ENQ (EOQ) cost per unit time, \(2KD(1 - D/P)K^2\), for \(T\) an integer multiple of \((2D/(1 - D/P)K)^2\). Second, the cost per unit time of the finite horizon constraint,
\[
C^*(0, T)/T = (2KD(1 - D/P)K^2)
\]
is not monotonic in \(T\). This difference does, however, have a monotonically decreasing upper bound for increasing \(T\). This can be seen by evaluating \(C^*(0, T)/T\) at each of its local maxima. The values of \(T\) for these local maxima are given by [7]. By substitution, \(C^*(0, T)/T\) at the \(k\)th local maximum equals
\[
\frac{kD(1 - D/P)K^2}{2} + \frac{k + 1}{k + 1} + \frac{k + 1}{k + 1}
\]
which decreases monotonically for increasing \(k\).

Finally, the decreasing upper bound on [15] can be
used to prove that
\[
\lim_{T \to \infty} \frac{C^*(Q, T)}{T} = \frac{(2KD_1 - D/D) \alpha^n}{[2/(2n+1)]}
\]
establishing the convergence of the optimal finite horizon policy to the optimal infinite horizon policy as \( T \) approaches infinity.

The practical impact of these observations is: 1) the optimal finite horizon EOQ policy will yield different policies and lower costs (see below) than the standard EOQ model in all instances except where the length of the finite horizon is some integer multiple of a "natural" EOQ cycle, \((2KD/(1-D/P))^{1/2} = (2KD/(1-D/P))^{1/2} = (2KD/(1-D/P))^{1/2}\), in which case the policies are identical, and 2) the discrepancy between the optimal

\[
\begin{array}{ccc|c|c|c|c}
T & \text{Optimal policy} & \text{Modified EOQ policy} & \text{Difference (30-2)} & \% \text{Difference} \\
0.25 & 3.125 & 3.526.55 & 1,400.55 & 125.25 \% \\
0.5 & 7.250 & 12.510.50 & 1,360.20 & 10.92 \% \\
1 & 24.500 & 25.400.40 & 1,040.80 & 4.25 \% \\
2 & 90.000 & 32.563.36 & 653.30 & 15.38 \% \\
5 & 125.475 & 122.531.00 & 45.00 & 8.82 \% \\
10 & 244.651 & 245.123.00 & 18.00 & 0.60 \% \\
\end{array}
\]

The table above shows the results of the calculations performed to determine the optimal policies for different values of \( T \).

### Finite Horizon Policy

The finite horizon policy concept is one where the demand is met by the EOQ policy until the beginning of the next period. This results in lower costs for lower demand levels, as shown in the table above.

### Comparison with Alternative Policies

The optimal policy derived above, by definition, yields lower costs than any alternative policy. In order to indicate the size of the cost difference, a sample problem was solved using the optimal policy and two alternative policies. The first alternative is a simple modification of the standard EOQ policy. In this policy, \( Q = (2KD/D)^{1/2} \) is ordered as many times as possible, and any remaining demand is met by a "remainder" order. For example, if \( D = 2000 \) per year, \( T = 1 \) year, and \( Q = (2KD/D)^{1/2} = 600 \), a single order of 600 units would be placed. If, in the same situation, \( Q = (2KD/D)^{1/2} > 2000 \), a single order for 2000 units would be placed. The second alternative policy is a monthly ordering policy.

### Sample Calculations

Sample calculations were performed on the problem:
\[
K = 3.125; \quad D = 20.000 \text{ per year}; \quad h = 40 \text{ per unit} \text{ per year}; \quad \text{and} \quad T = 0.25, 0.5, 1, 2, 5, \text{ and } 10 \text{ years.}
\]

The results are summarized in Table 1.

### References

3. Dr. Schwartz is an assistant professor of business at The Amos Tuck School of Dartmouth University. His primary research interest is in the areas of inventory theory with special concentration in warehouse-retailer models. He received a PhD from the Graduate School of Business, University of Chicago, and is a member of TIMS and ORSA.

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