Parameter Estimation for the EOQ Lot-Size Model: Minimax and Expected Value Choices

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This article provides formulas for estimating the parameters to be used in the basic EOQ lot-size model. The analysis assumes that the true values of these parameters are unknown over known ranges and perhaps nonstationary over time. Two measures of estimator "goodness" are derived from EOQ sensitivity analysis. Formulas are given for computing the minimax choice and the minimum expected value choice for the parameter estimates using both measures of estimator "goodness." A numerical example is included.

1. INTRODUCTION

This article provides formulas for estimating the parameters to be used in the basic EOQ lot-size model: the demand rate, the fixed order (setup) cost, and the marginal holding cost rate. In theory these parameters are assumed to be stationary, known quantities. However, in realistic circumstances management is likely to be uncertain about the numerical values of these parameters. Furthermore, in practice these parameters are likely to be nonstationary over time due to seasonality of demand, changes in the firm’s cost of capital, changes in order (setup) activity and/or stuffing, etc. Consequently, the values of these parameters are usually known only within some given ranges. For example, demand rate may be known to be between 10,000 and 40,000 units/time, the marginal holding cost rate may be known to be between $1 and $4/unit-time, and fixed order cost may be known to be between $9 and $100 per order. In some cases probability distributions for each parameter over its given range may also be known.

In order to implement the EOQ model, it is necessary to select estimates of these parameters which reflect their nonstationarity and/or management’s uncertainty about them. Furthermore, these estimates should have "good" performance characteristics. The goal of this article is to consider two measures of estimator "goodness" and to provide formulas for computing the (a) minimax choice and (b) minimum expected value choice for the parameter estimates given these measures. Our purpose is not to recommend either measure of "goodness" or either criterion used to optimize the measures. The specific measure and criterion selected by a manager will depend, among other things, on the manager’s aversion to risk.

The basis of this work is an analysis of the sensitivity of inventory cost rate to errors in parameter estimation. Such analysis is not new (see [1] or [4], for example), but, unfortunately, not well known. We briefly review this analysis in Section 2. Section 3 applies this analysis to the problem at hand, considers two measures of estimator "goodness," interprets the minimax and expected value criteria as applied to these parameters, and estimates the minimum expected value choice and the minimax choice for parameter estimates using these measures.
measures, and presents formulas for the appropriate estimates. An example problem is presented in Section 4. The derivations are given in the Appendix.

2. EOQ SENSITIVITY ANALYSIS

The EOQ model assumes that the demand rate for a given product is a stationary, known quantity \( D \). It is further assumed that there is a stationary fixed order (setup) cost \( S \), incurred each time a lot is ordered, and a stationary marginal holding cost rate \( C \), applied against average inventory. Given these assumptions, the average setup and holding cost rate over an infinite time horizon given lot size \( Q \), denoted \( ACR(Q) \), is

\[
ACR(Q) = \frac{(SD)}{Q} + \frac{(CQ^2)}{2}.
\]

(1)

The lot size \( Q \) which minimizes Equation (1), denoted \( Q^* \), is

\[
Q^* = \sqrt{\frac{2SD}{C}}.
\]

(2)

and the corresponding optimal average cost rate, \( ACR(Q^*) \), is

\[
ACR(Q^*) = \frac{(SD)}{Q^*} + \frac{(CQ^*^2)}{2} = \frac{(SDC)}{2(Q^*)^2} + \frac{(SDC)}{2} = \frac{(SDC)}{2}.
\]

(3)

Sensitivity analysis proceeds as follows (see [1] or [4]). Assume that all of the EOQ assumptions hold but that one or more of the stationary parameters, \( D \), \( S \), or \( C \), is estimated incorrectly. Denote these estimates \( \bar{D} \), \( \bar{S} \), and \( \bar{C} \), respectively. The corresponding lot size, denoted \( \bar{Q} \), will be

\[
\bar{Q} = \sqrt{\frac{2\bar{S}\bar{C}}{\bar{C}}}, \quad \varepsilon = \sqrt{\frac{2\bar{C}}{\bar{C}}}.
\]

(4)

where

\[
\varepsilon = \sqrt{\frac{2\bar{S}}{\bar{C}}}. \quad (5)
\]

The average cost rate faced by the firm as a consequence of these parameter choices, \( ACR(\bar{Q}) \), is, of course, based on the true parameter values. Thus,

\[
ACR(\bar{Q}) = \frac{(SD)}{\bar{Q}} + \frac{(C\bar{Q}^2)}{2}.
\]

(6)

Substituting (4) in (6) and using the fact that \( SD\bar{Q}^* = CQ^*^2 \), we have

\[
ACR(\bar{Q}) = \frac{(S\varepsilon\bar{C})}{\bar{Q}} + \frac{(C\varepsilon\bar{C})}{2} = ACR(Q^*) + \frac{(C\varepsilon\bar{C})}{2}.
\]

(7)

The effect of errors in parameter estimation may now be measured as the difference between the average cost rate the firm faced, \( ACR(\bar{Q}) \), and the average cost rate they would have faced had there been no errors in estimation, \( ACR(Q^*) \). Denote this difference as \( \Delta(\bar{Q}) \), where

\[
\Delta(\bar{Q}) = ACR(\bar{Q}) - ACR(Q^*)
\]

\[
= \left[ \frac{1}{2} \left( \varepsilon^2 + 1 \right) - 1 \right] ACR(Q^*)
\]

(9)

The second measure of estimator goodness that we consider involves the ratio of \( ACR(\bar{Q}) \) to \( ACR(Q^*) \). This measure is used by several introductory texts (see [1]
[2], [3], or [4], for example) in their treatment of EOQ sensitivity analysis. Furthermore, many managers choose to think in terms of "relative" cost increases (e.g., "a 10% increase in cost rate") instead of "absolute" changes (e.g., "a $10 increase in cost rate"). Thus, a ratio measure \( R(\hat{Q}) \) may be defined as

\[
R(\hat{Q}) = \frac{ACR(\hat{Q})}{ACR(Q^*)} \quad (10)
\]

or

\[
R(\hat{Q}) = \frac{1}{2} \left( e^{\gamma/2} + e^{-\gamma/2} \right), \quad (11)
\]

Note that \( \Delta(\hat{Q}) \) is measured in dollars/time; \( R(\hat{Q}) \) is dimensionless. It will be evident from our analysis that the measures \( \Delta(\hat{Q}) \) and \( R(\hat{Q}) \) can give rise to very different inventory policies and corresponding costs.

Several observations may be made about \( \Delta(\hat{Q}) \) and \( R(\hat{Q}) \). First, the average cost rate consequences of "moderate" errors in parameter estimation are "small." For example, if \( \epsilon = 2 \) (e.g., if \( D = D^* \) and \( C = C^* \), then \( R(\hat{Q}) \approx 1.061 \), corresponding to \( \Delta(\hat{Q}) \approx 0.061 \ AC(\hat{Q}) \), which implies an average cost rate of only about 6% above optimum. Second, both \( R(\hat{Q}) \) and \( \Delta(\hat{Q}) \) are minimized (at \( \epsilon = 1 \) and 0, respectively) at \( \epsilon = 1 \), for which \( Q = Q^* \). Note that equality of parameters and their estimates is a sufficient condition for \( \epsilon = 1 \), but not a necessary one. Third, note that both functions \( R(\hat{Q}) \) and \( \Delta(\hat{Q}) \) are "symmetric" about \( \epsilon = 1 \) such that any given value of \( \epsilon \) and its reciprocal yield the same cost consequences. For example, \( \epsilon = 2 \) and \( \epsilon = 0.5 \) yield identical results.

3. ERROR RATES: MINIMAX AND EXPECTED VALUE PARAMETER CHOICES

The error rates \( R(\hat{Q}) \) and \( \Delta(\hat{Q}) \) provide measures for evaluating the goodness of parameter estimates in the EOQ lot-size model. Two cases will be described: (1) the stationary case in which the lot-size parameters are stationary but unknown over given ranges; and (2) the nonstationary case in which these parameters are nonstationary (and perhaps unknown) over given ranges.

A. The Stationary Case

In the stationary case, \( S, D, \) and \( C \) are stationary but unknown quantities over the following ranges:

\[
\bar{S} \leq S \leq \hat{S}, \quad (12)
\]

\[
\bar{C} \leq C \leq \hat{C}, \quad (13)
\]

\[
\bar{D} \leq D \leq \hat{D}, \quad (14)
\]

where we assume that at least one of the pairs of upper and lower bounds are unequal. The most common instances of this case arise because of uncertainty in estimating the firm's true stationary cost of ordering, \( S \), or true stationary holding cost rate \( C \). The values of these parameters are not typically found in a firm's accounting documents. Instead they must be determined, usually on an ad hoc basis, for inventory control purposes only. This determination is difficult because \( S \) and \( C \) must represent incremental (i.e., marginal) costs of ordering and holding inventory of the given item.
These incremental rates are likely to be quite different from the firm’s corresponding average rates of ordering and carrying inventory, which are estimated easily by dividing total cost of the ordering function (inventory holding function) by the total number of orders placed (average inventory in units) (see [2] for a further discussion).

In the stationary case all of the assumptions of the EOQ lot-size model apply. Consequently the error rates directly measure the cost consequences of estimation errors.

One criterion for selecting the estimates ($\hat{S}$, $\hat{C}$, $\hat{D}$) is to select estimates that minimize the maximum value of the corresponding error rate. Specifically, we define the problems

\[
\min_{\hat{S}, \hat{C}, \hat{D}} \left\{ \max_{\omega \in \Omega} R(\hat{Q}) \right\},
\]

(P1)

and

\[
\min_{\hat{S}, \hat{C}, \hat{D}} \left\{ \max_{\omega \in \Omega} \Delta(\hat{Q}) \right\},
\]

(P2)

This is the well-known minmax criterion, likely to be used by risk-averse managers who desire to select policies that avoid the worst possible outcome, regardless of its likelihood and regardless of the average overall quality of the policy alternatives.

When the joint probability distribution on the parameter is known, an alternative criterion is to select estimates ($\hat{S}$, $\hat{C}$, $\hat{D}$) that minimize the expected value of the corresponding error rate. Thus, we have

\[
\min_{\hat{S}, \hat{C}, \hat{D}} \left\{ E[R(\hat{Q})] \right\},
\]

(P3)

and

\[
\min_{\hat{S}, \hat{C}, \hat{D}} \left\{ E[\Delta(\hat{Q})] \right\},
\]

(P4)

where $E[\cdot]$ is the expected value operator with respect to the joint probability distribution of $S$, $C$, and $D$.

Table 1 gives the values of $\max R(\cdot)$, $\max \Delta(\cdot)$, $E[R(\cdot)]$, and $E[\Delta(\cdot)]$ for any given $\hat{Q}$ (see Appendix for details). For simplicity of presentation we have defined $Z = SD$, $\bar{Z} = SD$, and $\bar{Z} = SD$.

Table 2 provides the formulas for $\hat{Q}_m$, $\hat{Q}_m^*$, $\hat{Q}_m^*$, and $\hat{Q}_m^*$, the optimal choices of $\hat{Q}$ for problems P1, P2, P3, and P4, respectively.

Several comments are in order. First, note that since $\hat{Q} = (2SD/\hat{C})^{1/2}$, the corresponding optimal choices for the ratio $SD/\hat{C}$ may be found by squaring each entry of Table 2 and dividing by 2. Second, given this, note that in all cases two of the three parameter estimates, $\hat{S}$, $\hat{D}$, or $\hat{C}$, may be chosen completely arbitrarily provided that the third is selected so that the specified ratio is obtained. Third, note that one minimizes the choice of $SD/\hat{C}$ for the measure $R(\hat{Q})$ is the geometric mean (i.e., square root of the product) of the corresponding range values [i.e., $\overline{S} = (\bar{S})^{1/2}$, $\overline{D} = (\bar{D})^{1/2}$, $\overline{C} = (\bar{C})^{1/2}$]. Fourth, note that if $S$ and $D$ are probabilistically independent then $E[Z] = E[S]E[D]$. Thus, for example, if $S$, $D$, and $C$ are uniformly distributed over their respective ranges and independent of each other then one choice of $(\hat{S}, \hat{D}, \hat{C})$ which minimizes $E[\Delta(\hat{Q})]$ is the arithmetic means of the corresponding range values [i.e., $\overline{S} = (\bar{S} + \bar{S})/2$, $\overline{C} = (\bar{C} + \bar{C})/2$, $\overline{D} = (\bar{D} + \bar{D})/2$].
Table 1. Formulas for \( \max \{ \hat{R}(\hat{Q}), \Delta(\hat{Q}) \}, E[R(\hat{Q})], \) and \( E[\Delta(\hat{Q})] \).

<table>
<thead>
<tr>
<th>( \hat{R}(\hat{Q}) ) [ \text{Max} ]</th>
<th>( \hat{E}[\hat{R}(\hat{Q})] ) [ \text{E}[\hat{r}] ]</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \max \left{ \frac{z^2}{2\lambda C}, \frac{z^2}{2\lambda C} \hat{Q}^{-1} + \frac{\hat{Q}}{2\lambda C} \right} ) [ \text{P1} ]</td>
<td>( 2^{-1/2} \left( \frac{z}{\lambda} \right)^{1/2} \hat{Q}^{-1} + 2^{-1/2} \left( \frac{z}{\lambda} \right)^{1/2} \hat{Q} ) [ \text{P3} ]</td>
</tr>
<tr>
<td>( \hat{\Delta}(\hat{Q}) ) [ \text{Max} ]</td>
<td>( \hat{E}[\hat{\Delta}(\hat{Q})] ) [ \text{E}[\Delta] ]</td>
</tr>
<tr>
<td>( \max \left{ \frac{b^2}{2\lambda C}, \frac{b^2}{2\lambda C} \hat{Q}^{-1} + \frac{\hat{Q}}{2\lambda C} \right} ) [ \text{P2} ]</td>
<td>( \frac{\hat{E}[\hat{Q}]^{-1}}{2} \hat{Q}^{-1} + \frac{\hat{E}[\hat{C}]}{2} \hat{Q} - 2(\hat{E}[\hat{C}])^{1/2} ) [ \text{P4} ]</td>
</tr>
</tbody>
</table>

B. The Nonstationary Case

In the nonstationary case \( S, D, \) and \( C \) vary in some time-varying—and perhaps unknown—manner over the ranges \((12)-(14)\). In this case the EOQ lot-size assumptions are not satisfied. However, with the appropriate interpretations, \( \hat{S}(\hat{Q}) \) and \( \hat{R}(\hat{Q}) \) do provide a measure of the cost rate error in using stationary estimates for the nonstationary parameters. We provide these interpretations below.

For purposes of illustration assume that \( S \) and \( C \) are known, stationary quantities but that \( D \) varies (with respect to time) over the range \([\bar{D}, \overline{\bar{D}}]\). The value of \( D \) at any time \( t \) is given by a (known or unknown) function \( D(t) \). The following three lot-size policies are relevant to our interpretations. Define:

\( P^* \): A lot-size policy that minimizes average setup and holding cost rate over an infinite time horizon. Except in special cases (see [1], for example) \( P^* \) is unknown.

\( Q^*(t) \): An optimal nonstationary EOQ lot-size policy at time \( t \) where \( Q^*(t) \) is placed at time \( t \). \( Q^*(t) = \left( 2\hat{D}(t)/C \right)^{1/2} \), where \( \hat{D}(t) \) is an "optimal choice" of demand rate to be used in the EOQ formula at time \( t \). We remark that in

Table 2. Minimax and minimum expected value choices for \( \hat{Q} \).

<table>
<thead>
<tr>
<th>Minimax</th>
<th>( \hat{Q} )</th>
<th>( \hat{E}[\hat{Q}] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{Q}_\text{min} ) [ \text{Min} ]</td>
<td>[ \frac{\left( 2\hat{D}/C \right)^{1/2}}{\hat{C}} ] [ \text{P1} ]</td>
<td>[ \frac{2\hat{E}[\hat{Q}]}{\hat{E}[\hat{C}]} ] [ \text{P3} ]</td>
</tr>
<tr>
<td>( \hat{Q}_\text{max} ) [ \text{Max} ]</td>
<td>[ \frac{\left( 2\hat{D}/C \right)^{1/2} + \hat{Z}/C}{\hat{C}} ] [ \text{P2} ]</td>
<td>[ \frac{2\hat{E}[\hat{Z}]}{\hat{E}[\hat{C}]} ] [ \text{P4} ]</td>
</tr>
</tbody>
</table>
general, \( \bar{B}(t) \) cannot be found (since it depends upon a nonstationary and perhaps unknown future).

\( Q \): A stationary EOQ lot-size policy, where if an order is placed at time \( t \) the lot size is \( Q = (2SD/C)^{1/2} \). \( \bar{B} \) is a stationary estimate of the nonstationary demand rate.

Finally, let \( f(\cdot) \) be the average setup and holding cost rate over an infinite time horizon when using policy \( \cdot \). Given the above

\[ f(P^*) \leq f(Q^*(), 0) \leq f(\bar{Q}); \]

the first inequality follows from the definition of \( P^* \), the second inequality follows from the definition of \( Q^*() \) and the fact that \( Q \) is a feasible choice of \( Q^*() \).

Ideally, management would select policy \( P^*() \) provided \( P^*() \) was known. However, as noted above \( P^*() \) is only known for some special cases. In the absence of \( P^*(), \) management might select policy \( Q^*() \) provided it was known. Indeed, many installed EOQ lot-size procedures endeavor to implement policy \( Q^*() \); e.g., each time \( t_q \) a lot is ordered, a “guess” at the optimal \( B(t) \) is generated using historical time series. Alternatively, a \( \bar{Q} \) lot-size policy could be used.

All other things being equal, the “closer” \( Q \) policy is to \( Q^*() \) [or, equivalently, the closer \( \bar{B} \) is to \( B(t) \)] the smaller the difference between \( f(Q^*()) \) and \( f(\bar{Q}) \). One possible measure of the closeness of \( Q \) to \( Q^*() \) at any time \( t \) is \( Q^*() - Q(t) = (Q^*() - \bar{Q}) + (\bar{Q} - Q(t)) \). Another possible measure is \( (Q^*() - \bar{Q}) + (\bar{Q} - Q(t)) \). However, both of these measures are monotonic in \( t \). Consequently, neither measure reflects the tradeoffs of reduced average order cost rate versus increased average holding cost rate (or vice versa) between the \( Q^*() \) and \( \bar{Q} \) policies. What is required is an inventory cost-based measure of the proximity of \( Q \) to \( Q^*() \); \( R(Q) \) and \( \Delta(Q) \) defined above are such measures. They may be interpreted in either of two ways: First, as the ratio (or difference) between the average cost rates of \( Q^*() \) and \( Q \) policies at time \( t \) provided that true demand is fixed from \( t \) through infinity. Second, as defined, \( f(\cdot) \) the average setup and holding cost rate over an infinite horizon. Thus, \( f(\cdot) \) can be viewed as

\[ f(\cdot) = \lim_{t \to \infty} \int_{0}^{t} C(t, \cdot)b(t) \, dt, \]  

where \( C(t, \cdot) \) is the instantaneous rate of setup and holding cost accumulation at time \( t \) using policy \( \cdot \). Now suppose that the demand rate changes only at the discrete points in time \( t_1, t_2, \ldots \) so that the demand rate in time interval \( [t_i, t_{i+1}] \) is \( \bar{B}(t_i) \), \( i = 0, 1, \ldots \). Given this, an approximation to the \( Q^*() \) policy is to use \( \bar{B}(t) = \bar{B}(t_i) \) for \( t \in [t_i, t_{i+1}], \) \( i = 0, 1, \ldots \). From this, an approximation to \( f(Q^*()) \) would be

\[ \lim_{i \to \infty} \sum_{i=1}^{n} \frac{ACR(\bar{B}(t_i))(t_{i+1} - t_i)}{(t_i - t_{i-1})} \]  

Furthermore, an approximation to \( f(\bar{Q}) \) would be

\[ \lim_{i \to \infty} \sum_{i=1}^{n} \frac{ACR(\bar{Q})(t_{i+1} - t_i)}{t_{i+1} - t_i} \]
The "goodness" of these approximations depends on the lengths of the intervals \( t_{i+1} - t_i \) (see Fig. 2 in [5]), but generally improves as the intervals increase. Given (20) and (21), the measures \( D \) and \( R \) can be viewed as the difference and ratio, respectively, of the rate of change of the approximations to the total accumulated cost curves associated with policies \( Q^*(t) \) and \( \bar{Q} \).

Under either interpretation \( D \) and \( R \) will themselves be nonstationary over time. Consequently, their maximum and expected values are values of interest. Given \( R(Q) \) and \( \Delta(Q) \) as measures of closeness between \( Q^*(t) \) and \( \bar{Q} \) in the nonstationary case, we wish to select stationary parameter estimates \((\bar{S}, \bar{D}, \bar{C})\) to minimize the maximum \( R(Q) \) or \( \Delta(Q) \) or minimize the corresponding expected values [see (17) and (18)]. The corresponding optimal \( \bar{Q} \) policies are given in Table 2.

### 4. EXAMPLE PROBLEM

The following example, although not intended to be generally representative, indicates the differences in policies and costs which can emerge from the application of the various measures and criteria considered above.

Consider a situation in which the EOQ parameters are distributed independently and uniformly over the ranges:

\[
S = 9 \leq S \leq 100 = \bar{S},
\]

\[
C = 1 \leq C \leq 4 = \bar{C},
\]

\[
D = 10,000 \leq D \leq 40,000 = \bar{D}.
\]

The \( \bar{Q} \) values are computed from Table 2 to be

\[
\bar{Q}_{100} = 668.74, 
\]

\[
\bar{Q}_{104} = 1084.23, 
\]

\[
\bar{Q}_{74} = 740.99, 
\]

\[
\bar{Q}_{77} = 774.60. 
\]

Note that the measure/criterion choice can yield almost a factor of 2 change in inventory policy.

In order to illustrate the sensitivity of the measures in Table 1 to the choice of \( \bar{Q} \), we evaluated each of the entries in Table 1 at the four values of \( \bar{Q} \) given by (22)–(25). These evaluations appear in Table 3. The diagonal entries, denoted with an asterisk, represent the optimal choices. Note the considerable measured differences which are possible among these policies.

<table>
<thead>
<tr>
<th>( \bar{Q} )</th>
<th>( R(\bar{Q}) )</th>
<th>( \Delta(\bar{Q}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \bar{Q}_{100} ) = 668.74</td>
<td>1.963*</td>
<td>2722.85</td>
</tr>
<tr>
<td>( \bar{Q}_{104} ) = 1084.23</td>
<td>2.653</td>
<td>1402.94*</td>
</tr>
<tr>
<td>( \bar{Q}_{74} ) = 740.99</td>
<td>2.379</td>
<td>1808.15</td>
</tr>
<tr>
<td>( \bar{Q}_{77} ) = 774.60</td>
<td>2.562</td>
<td>1524.89</td>
</tr>
<tr>
<td></td>
<td>1.112</td>
<td>109.92</td>
</tr>
<tr>
<td></td>
<td>1.094</td>
<td>172.63</td>
</tr>
<tr>
<td></td>
<td>1.086*</td>
<td>179.37</td>
</tr>
<tr>
<td></td>
<td>1.090</td>
<td>170.77*</td>
</tr>
</tbody>
</table>
APPENDIX

In this appendix we establish that optimal solutions to Problems P1-34 are as given in Table 2. In each problem, we note that the true value of each of $S$, $C$, and $D$ lies in the ranges defined by (12)-(14). For simplicity, let $K$ be the subset of $R^1$ such that $K = [(S,C,D)]$ satisfies (12), $C$ satisfies (13), $D$ satisfies (14).

I. Analysis for the $R(\hat{Q})$ Measure: Problems P1 and P3. Using (2) and (4) in (11), we get

$$R(\hat{Q}) = \hat{Q} - 2^{-3/2}(SD)^{1/2}C^{-1/2} + 2^{-3/2}(SD)^{1/2}C^{-1/2}.$$  \hspace{1cm} (A1)

1. Analysis for Problem P1: Due to the form of $R(\hat{Q})$ in Equation (A1), we define a variable $y$, where for fixed $\hat{Q} > 0$,

$$\max_{0 < \hat{Q}, 0 < y < \hat{Q}} R(\hat{Q}) = \max_{0 < \hat{Q}, 0 < y < \hat{Q}} \hat{Q} - 2^{-3/2}(SD)^{1/2}C^{-1/2} + 2^{-3/2}(SD)^{1/2}C^{-1/2}.$$  \hspace{1cm} (A2)

subject to

$$y = \frac{SD}{C} \leq y < \frac{SD}{C} \leq \hat{y}.$$  \hspace{1cm} (A3)

Define $f_\hat{Q}(\hat{Q}, y) = \hat{Q} - 2^{-3/2}(SD)^{1/2}C^{-1/2} + 2^{-3/2}(SD)^{1/2}C^{-1/2}$. Thus, Problem P1 is equivalent to

$$\min_{0 < \hat{Q}, 0 < y < \hat{Q}} \max \{ f_\hat{Q}(\hat{Q}, y) | \text{subject to } y \leq y \leq \hat{y} \}.$$  \hspace{1cm} (A4)

For fixed $\hat{Q} > 0$, it can be shown that

$$f_\hat{Q}(\hat{Q}, y) = 1 \text{ at } y = \frac{\hat{Q}}{2},$$  \hspace{1cm} (A5)

and

$$f_\hat{Q}(\hat{Q}, y) \text{ (for } y > 0) \text{ is strictly decreasing (increasing) in } y \text{ at any }$$

$$y < \frac{\hat{Q}^2}{2} \text{ (} y > \frac{\hat{Q}^2}{2} \text{).}$$  \hspace{1cm} (A6)

Thus, it is clear that for fixed $\hat{Q} > 0$, $f_\hat{Q}(\hat{Q}, y)$ is maximized over $(y | y \leq y \leq \hat{y})$ at either $y$ or $\hat{y}$. As a consequence, Problem P1 is equivalent to

$$\min_{0 < \hat{Q}, 0 < y < \hat{Q}} \max \{ L_\hat{Q}(\hat{Q}, y) \}.$$  \hspace{1cm} (A7)

where $L_\hat{Q}(\hat{Q}, y)$ is $L_\hat{Q}(\hat{Q}, y)$ evaluated at $y = \hat{y}$ ($y = \hat{y}$).

We note that both $L_\hat{Q}(\hat{Q}, y)$ and $f_\hat{Q}(\hat{Q}, y)$ are strictly convex functions of $\hat{Q}$ and attain their minima at $\hat{Q}^* = 2^{-\frac{1}{2}} SD$ and $\hat{Q}$ = $2^{-\frac{1}{2}}$ SD, respectively. Furthermore, $L_\hat{Q}(\hat{Q}) = L_\hat{Q}(\hat{Q}, 0) = \hat{Q}^* = 2^{-\frac{1}{2}} SD$. We now show that $\hat{Q}^*$ solves Problem P1.

Since $2^{-\frac{1}{2}} SD < \hat{Q}^* < 2^{-1/2} SD$, and due to the strict convexity of $L_\hat{Q}(\hat{Q})$ and $L_\hat{Q}(\hat{Q}, y)$, it follows that $\hat{Q}^*$ is a local minimum of $\max \{ L_\hat{Q}(\hat{Q}, y) \}$, $L_\hat{Q}(\hat{Q})$. But then $\max \{ L_\hat{Q}(\hat{Q}) \}$, $L_\hat{Q}(\hat{Q}, y)$ is itself a convex function which implies that $\hat{Q}^*$ is a global minimum of $\max \{ L_\hat{Q}(\hat{Q}, y) \}$, $L_\hat{Q}(\hat{Q})$.

Substituting the definitions of $\hat{y}$ and $\hat{y}$ from (A3) yields the value of $\hat{Q}^*$ given in Table 2.
ii. Analysis of Problem P3s. From (A1), \( E[R(\hat{Q})] = \hat{Q}^{-1} \cdot 2^{-12} E[(SD)^{12} (C^{-15})] + \hat{Q}^{-2} \cdot E[(SD)^{12} (C^{-15})] \). Clearly, \( R(\hat{Q}) \) is a convex function of \( \hat{Q} \). Setting the derivative to zero yields
\[
\hat{Q}^*_s = \left( 2 \cdot E[(SD)^{12} (C^{-15})] \right)^{15} \cdot E[(SD)^{12} (C^{-15})]^{-15},
\]
which is the value given in Table 2.

II. Analysis for the \( \Delta(\hat{Q}) \) Measure: Problems P2 and P4. As in Section 2,
\[
\Delta(\hat{Q}) = ACR(\hat{Q}) - ACR(\epsilon^*) .
\]  
Using (3) and (6) in (A8), we obtain
\[
\Delta(\hat{Q}) = \hat{Q}^{-1} (SD) + \hat{Q} (C^2) - (2SDC)^{12}. \tag{A9}
\]

i. Analysis for Problem P2: Substituting \( Z = SD \) in (A9) and denoting the right-hand side of (A9), for fixed \( \hat{Q} > 0 \), as \( f_d(\hat{Q};Z,C) \) we note that Problem P2 is equivalent to:
\[
\max_{\hat{Q} \in [0,\infty]} \text{subject to } \frac{f_d(\hat{Q};Z,C)}{C} \leq Z \leq \frac{C}{2}, \tag{A10}
\]
Let \( R(Z,C) \) be the subset of \( R(\hat{Q}) \) defined by all convex combinations of the extreme points \( (\hat{Q},C), (\hat{Q},C), (Z,C), (Z,C) \). For fixed \( \hat{Q} > 0 \), it can be shown that \( f_d(\hat{Q};Z,C) \neq 0 \) iff \( (Z,C) \) satisfies \( (2Z/C)^{12} = \hat{Q} \), otherwise
\[
f_d(\hat{Q};Z,C) > 0. \tag{A11}
\]
\[
\nabla f_d(\hat{Q};Z,C) = \begin{bmatrix}
\frac{\partial f_d(\hat{Q};Z,C)}{\partial Z} \\
\frac{\partial f_d(\hat{Q};Z,C)}{\partial C}
\end{bmatrix}
= \begin{bmatrix}
\hat{Q}^{-1} - (C/2Z)^{12} \\
2^{-12} \hat{Q} - (2Z/C)^{12}
\end{bmatrix}. \tag{A12}
\]
From (A12) it can be shown that
\[
\nabla f_d(\hat{Q};Z,C) = 0 \iff (2Z/C)^{12} = \hat{Q}, \tag{A13}
\]
and \( \frac{\partial f_d(\hat{Q};Z,C)}{\partial C} < 0 \) and \( \frac{\partial f_d(\hat{Q};Z,C)}{\partial C} > 0 \) iff \( (2Z/C)^{12} < (>) \hat{Q}. \tag{A14} \]

For fixed \( \hat{Q} > 0 \), we consider the problem
\[
\max_{Z \in [0,\infty]} f_d(\hat{Q};Z,C). \tag{A11}
\]
From (A11) it follows that any \((Z,C)\) which satisfies \( (2Z/C)^{12} = \hat{Q} \) is not a maximum.

Let \( L \) be the subset of \( R(\hat{Q}) \) where \((Z,C) \in L \) satisfies \( (2Z/C)^{12} < \hat{Q} \). From (A14), the direction of the gradient of \( f_d(\hat{Q};Z,C) \) at any point in \( L \) implies that \( f_d(\hat{Q};Z,C) \) is maximized over \( L \) at the point \((Z,C) = (\hat{Q},C)\). Letting \( U \) be the subset of \( R(\hat{Q}) \) where
(Z, C) ∈ U satisfies (2\(Z/C\))^2 > 0, it similarly follows from (A14) that \(s(\hat{Q}, Z, C)\) is maximized over \(U\) at the point \((Z, C) = (\hat{Z}, \hat{C})\). Thus, it follows that for fixed \(\hat{Q} > 0\), the maximum of \(s(\hat{Q}, Z, C)\) over \(U\) is attained at either \((Z, C) = (\hat{Z}, \hat{C})\) or \((Z, C) = (Z, C_0)\).

Based on the above, Problem P2 is equivalent to

\[
\min_{\hat{Q} > 0} \{ g_1(\hat{Q}), h_0(\hat{Q}) \},
\]

where \(g_1(\hat{Q}) = \hat{Q}^{-1}Z + \hat{Q}(\hat{C}/2) - (2\hat{Z}/C)^{1/2} \) and \(h_0(\hat{Q}) = \hat{Q}^{-1}Z + \hat{Q}(\hat{C}/2) - (2\hat{Z}/C)^{1/2} \).

As in the analysis of Problem P1, both \(g_1(\hat{Q})\) and \(h_0(\hat{Q})\) are strictly convex functions of \(\hat{Q}\). Furthermore, \(g_1(\hat{Q})\) and \(h_0(\hat{Q})\) are minimized at \(\hat{Q} = (2\hat{Z}/C)^{1/2}\) and \(\hat{Q} = (2\hat{Z}/C)^{1/2}\), respectively.

With some additional algebra, it can be shown that with \(\hat{Q}_m > 0\) defined by

\[
\hat{Q}_m = 2\hat{Z}(\hat{Z}/C + \hat{C}/(C^{1/2} + C^{1/2})),
\]

that

\[
h_0(\hat{Q}_m) = g_1(\hat{Q}_m),
\]

and

\[
(2\hat{Z}/C)^{1/2} < \hat{Q}_m < (3\hat{Z}/C)^{1/2}. \quad (A16)
\]

It now follows, using arguments similar to those for Problem P1, that \(\hat{Q}_m\) solves Problem P2. (We remark that to establish (A15), one must consider separately the case \(C < \hat{C}\), where \(\hat{Q}_m\) is the positive root of a quadratic, and the case \(C = \hat{C}\).) \(\hat{Q}_m\) is given in Table 2.

ii. Analysis for Problem P4: For fixed \(\hat{Q} > 0\), from (A9) with \(Z = SD\), \(E(\hat{Q}) = \hat{C}^{-1}E(\hat{Z}) + 2\hat{C}^{-1}E(\hat{C}) - 2\hat{C}^{-1}E(\hat{C}^{1/2})\). Clearly, \(E(\hat{Q})\) is a convex function of \(\hat{Q}\). Setting the derivative to zero yields \(\hat{Q}_m = (2\hat{Z}/E(\hat{C}))^{1/2}\), which is the value given in Table 2.

REFERENCES


