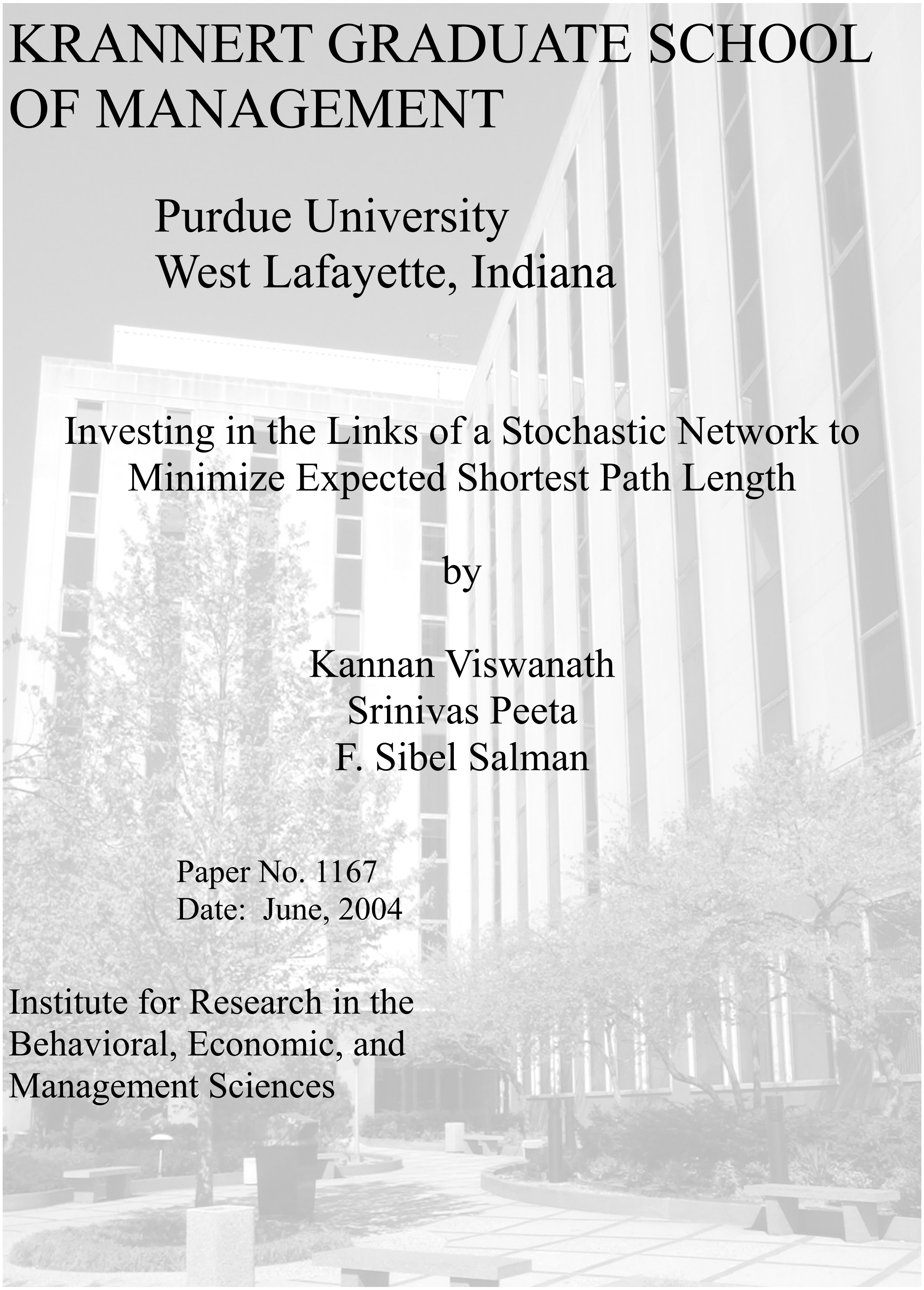

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Investing in the Links of a Stochastic Network to Minimize Expected Shortest Path Length

by

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Investing in the Links of a Stochastic Network to Minimize Expected Shortest Path Length

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ABSTRACT

We consider a network whose links are subject to independent, random failures due to a disruptive event. The survival probability of a link is increased, if it is strengthened by investment. A given budget is to be allocated among the links with the objective of optimizing the post-event performance of the network. Specifically, we seek to minimize the expected shortest path length between a specified origin node and destination node in the network. This criterion is defined through the use of a fixed penalty cost for those network realizations in the expectation, that do not have a path connecting the origin node to the destination node. This problem type arises in the pre-disaster planning phase, where a decision-maker seeks to reduce the vulnerability of a transportation network to disasters, by upgrading its weakest elements. We model the problem as a two-stage stochastic program in which the underlying probability distribution of the random variables is dependent on the first stage decision variables. Using a path-based approach we construct its equivalent deterministic program and derive structural results for the objective function. We then propose an approximate solution procedure based on a first order approximation to the objective function. The procedure is tested by numerical experiments on a small-size network. The test results show that it yields very good performance on the instances solved.

Keywords: network vulnerability, decision dependent probability distribution, two-stage stochastic program, multilinear function

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1 INTRODUCTION

This paper investigates the problem of allocating a given budget among the links of a network that are subject to independent, random failures caused by a disruptive event. The purpose of investing in a link is to increase its survival probability. The performance criterion that we seek to optimize is the expected shortest distance (or least travel cost) from a specified origin node to a specified destination node in the network. This criterion is defined through the use of a fixed penalty cost for those network realizations in the expectation, that do not have a path connecting the origin node to the destination node.

The motivation for this problem stems from a planning context for managing natural disasters, but the modeling framework to be presented, is also applicable for devising protection plans against terrorist attacks. Transportation networks are vulnerable to natural disasters such as earthquakes, hurricanes, floods etc. The functionality of the network following the disaster event is a critical determinant of the effectiveness of emergency response. To illustrate this point we cite Nicholson and Du [1997] – "*Clearly, restoration of virtually all other lifeline systems is very dependent upon people and equipment being able to move to the sites where damage has occurred, and damage to the transportation system inhibits repairs to the other lifeline systems*." The functionality of the network has implications for the operations of business, as well. Companies that adopt the just in time paradigm to reduce the inventory carrying costs rely on the availability of an efficient and reliable transportation system. Any disruption in the scheduling of goods or services due to a disaster event causes losses to the business parties involved. Until the restoration of the transportation network to its completely functional state, the losses or the additional costs incurred during the disrupted state would depend on the additional travel time required for transportation.

Herein arises the importance of pre-disaster planning, which seeks to mitigate the effect of a disaster on the network as a whole. The most common approach adopted to reduce the vulnerability of a transportation network is to strengthen its weakest elements, by increasing its survival probability. In the current context, it refers to retrofitting of bridges. However, the retrofitting process is costly and time-consuming, and hence it is impractical to retrofit all the bridges in the network. Therefore the budget available serves as a constraint in the determination of the set of links that are to be invested for retrofitting. Also the investment decisions need to be made under *uncertainty* with the view of optimizing a suitable, post-event network performance criterion.

The specific problem setting we address in this paper concerns a network with a single origin-destination (*O-D*) pair, where the performance criterion relates to the path length/travel cost from O to D. Without loss of generality we shall use the term path length to refer to the cost incurred by travel. The sequence of events for this problem can be described as follows. Investment decisions for the links must be made a priori without the knowledge of the actual network that would survive a disruptive event, from hereon referred as network realization. When the event occurs, links fail independent of each other with specified probabilities depending on whether they were invested in or not. Once a link fails, it becomes impassable and is in a non-operational state. Now, complete information on the state of the links is available to the decision-maker, subsequently the path with the shortest path length in the network realization is chosen for travel from the origin node (O) to the destination node (D), if such a path exists. If the network realization is infeasible i.e. does not have connectivity from O to D, then the shortest path length is equal to a fixed penalty cost pre-determined by the decision-maker. The performance criterion of interest is the expected value of the shortest path length. Our objective is to determine the investment decision for the links of the network that minimizes this value.

This problem is formulated as a two-stage stochastic program, in which the first stage corresponds to making investment decisions for links under a budget constraint, and the second stage corresponds to selecting the path with the shortest length for travel from O to D, in the network realization.

The remainder of the paper is structured as follows. In Subsection 1.1, we motivate the performance criterion using an example. Subsection 1.2, discusses related previous research. In Section 2, the mathematical model is developed. Subsection 2.1 introduces the preliminary notation and necessary assumptions, used in the paper. In Subsection 2.2, we present the two-stage stochastic program. In Section 3, we derive its equivalent deterministic program. In Section 4, we present structural results for the objective function. In Subsection 4.1, we derive the multilinear functional form of $F(y)$. In

Subsection 4.2, we characterize the coefficients of this function. In Subsection 4.3, we prove the monotone decreasing property of F(y). In Section 5, we develop a solution approach to the problem. In Subsection 5.1, we propose and justify the first order approximation to the objective function. In Subsection 5.2, the first order approximation idea is extended to an iterative scheme, stochastic subgradient method. In Section 6, we present results from a computational study on a small-size network and discuss some of insights, based on sensitivity analysis of problem parameters. Finally, in Section 7 we have our concluding comments.

1.1 The Expected Shortest Path Length Objective: A Motivating Example

Consider a two link network as shown in Figure 1. The length of link e is denoted by t_e, for e = 1, 2. The probability of survival of link e due to a disruptive event, with and without strengthening the link is denoted by p_e and q_e respectively, for e = 1, 2. The investment required for strengthening link e is denoted by c_e, for e = 1, 2. Upon the occurrence of the event, the network that survives will be either connected between *O* and *D*, or not. Without any investment, the *O-D* reliability, that is the probability that *O* and *D* is connected, is 0.88 and the expected value of the shortest path length evaluated over feasible network realizations is 3.27273.

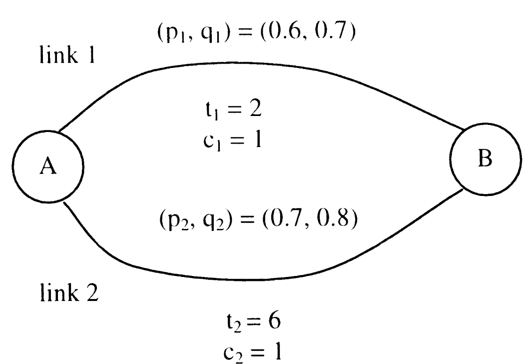

Figure 1 Two link network

Now, a decision-maker seeks to strengthen a transportation network under a budget restriction.

Suppose the given budget is one unit. Therefore, only one of the links can be strengthened. Moreover, let us assume the decision-maker is interested just in maximizing the *O-D* reliability. Then, the optimal choice is to invest in link 2 as seen from Table 1 and this increases the reliability to 0.92, an increase of 4.55%. However, the implementation of this decision would worsen the expected value of the shortest path length from *O* to *D* over feasible network realizations to a value of 3.392 (see Table 1). This is an increase of 3.62%. Intuitively this means, in the event that the surviving network remains connected, there is a greater likelihood that the path with the higher travel time would have to be used to travel from *O* to *D* in comparison with the case without investment.

Link invested in	*O-D* reliability	% increase in reliability	Expected value of the shortest path length over feasible realizations	% decrease in the expected value of the shortest path length over feasible realizations
1	0.91	3.4091	2.92308	10.6837
2	0.92	4.5455	3.3913	-3.623

Table 1

Also from Table 1, it can be seen that investing in link 1 not only increases the *O-D* reliability to 0.91, an increase of 3.41%, but also decreases the expected value of the shortest distance over feasible realizations to 2.923, a decrease of 10.68%. Observe that the increase in reliability due to either investment choice is comparable and, also investing in link 1 alone improves the expected shortest path length over feasible realizations. In other words, investing in link 1 guarantees a higher probability of lower travel time under the conditioning event, that the network realization is connected. Since path length is an important determinant of effectiveness of earthquake response, it is perhaps judicious choice is to invest in link 1.

This example clearly brings out the issue of considering a tradeoff between path length and connectivity in making the investment decision. More precisely, it suggests two objectives that are of interest to the decision-maker, namely, the expected value of the shortest path length over feasible network realizations and the *O-D* reliability. This is because, connectivity is necessary, but is not sufficient to ensure effective response, as the latter criterion does not consider the quality of service vis-à-vis travel time, in the event of a disaster. This paper seeks to develop a model towards this end. The single objective of interest is defined to be the expected shortest path length. We accomplish this by combining the two objectives through the use of a penalty cost as the weight factor for unreliability. In this example, for 7 ≤ penalty cost ≤ 46, it is optimal to invest in link 1, for penalty cost > 46, it is optimal to invest in link 2.

1.2 Overview of Past work

In this section we overview the past work, which consists of two parts. The first part reviews work that has focused on the problem of investing in the links of a stochastic network. The second part discusses selected models of network interdiction that serve as a comparison for our modeling approach.

The literature on the problem of investing in a stochastic network to optimize a performance criterion is sparse. Wollmer [1980] first studied the problem of transporting commodities from its source nodes to its demand nodes in which flow of the commodities in the arcs consumes a resource (viz arc capacity) whose initial supply is random. The objective was to identify an investment policy, which increases the resource of the arcs in the network so as to minimize the expected total cost. The total cost was defined as the sum of the investment used to increase the supply of resources and the minimum cost for the generalized multicommodity network flow problem after the realization of the random variables. A two-stage stochastic program under uncertainty was formulated in which the first-stage variables represented the amount to be invested in each resource and the second stage variables corresponded to arc flows of the different commodities. A procedure based on a cutting plane technique that exploits network structure was also proposed.

Wallace [1987] studied the problem of investing in the arcs of a network to maximize expected maximum flow from a source node to a sink node. The initial capacity of each arc of the given network is a random variable. Additional arc capacity can be achieved by investing in it; the increase in capacity being a linear function of the investment. The problem was formulated as a two-stage linear program under uncertainty. A solution procedure and bounds for the second stage program were proposed. Later, Wollmer [1991] studied a variation of the problem of investing in maximum flow networks under uncertainty. The objective was to maximize a linear combination of the expected maximum flow between two specified nodes and the negative of the investment cost. Here as well, the problem was formulated as a two-stage linear program under uncertainty and its solution procedure uses a cutting plane technique that exploits the formulation structure.

The work described here concern with investing in a stochastic network to improve network performance by increasing capacity. Like in many stochastic programming applications, the models developed, concern with problems in which randomness is not affected by decisions [Sen, 2001]. In this study we present a class of stochastic programs in which the probability distribution of the random variables is dependent on the decisions, thereby influencing the expected performance. The only other paper, to our knowledge, which has explicitly stated it as a feature in their model, is by Karaesmen and van Ryzin [2003]. Their problem is in the area of revenue management, where they seek to determine overbooking levels with the option of using substitutes to satisfy the demand of a certain reservation class.

From a modeling perspective, we now overview studies in a related area, namely, network interdiction, which has seen a surge in research. Cormican et al. [1998] investigate a stochastic version of the problem where an interdictor using limited budget attempts to destroy the arcs of a capacitated network, through which a network user subsequently maximizes flow from a source node to a sink node. The randomness arises due to the uncertainty in the interdiction success for the links, and are assumed to be independent, binary-valued random variables. The objective was to determine the set of arcs to interdict so as to minimize the expected maximum flow between the two nodes. Other model variants were formulated, including one with random arc capacities. Israeli and Wood [2002] address the problem of interdicting a network under a limited budget, so as to maximize the shortest path length between two specified nodes, for a network user. The problem setting is deterministic and a link interdiction either makes it impassable or increases its travel cost. Recently, Hemmecke et al. [2003], Held et al. [2003] address the problem of interdicting the flow of information or goods between two specified nodes in a network whose configuration is random but the realization probability of each configuration is known. The objective of interest is to maximize the probability that the minimum path length between the two nodes exceeds a certain value by using the available budget to interdict the arcs. A link interdiction increases its travel cost and there is no uncertainty in the interdiction success. Potential applications of the models discussed, include interdicting the supply network of enemy troops, drug trafficking and disrupting the functioning of an adversary's economy. Salmeron et al. [2003] sought to identify the critical components of an electric grid for hardening, against a terrorist attack in a deterministic setting. They developed a mathematical program to determine the optimal attack plan that terrorists would employ under resource constraints to maximize the power loss to the system being served. They propose that the method would help in understanding the strategy of the adversaries while developing a protection plan.

Our study contrasts to the earlier models in network interdiction in that these models address the budget allocation problem from the perspective of the interdictor whereas ours addresses it from the network user's perspective. The study that is close to the problem we address here is the one by Salmeron et al., although it does not consider explicitly the budget constraint for the network user. Furthermore, it does not consider uncertainty in the disruption that could be caused to the grid. The other key difference is that in our case, the associated probability distribution is a function of the investment decisions.

2 MATHEMATICAL MODEL

This section develops the mathematical model for the problem addressed in this paper. Subsection 2.1 introduces the notation, and assumptions used in the model development. Subsection 2.2 presents the two-stage stochastic program as our mathematical model.

2.1 Notation and Assumptions

We are given a directed network $G = (N, E)$ with node set N and arc set E, where the index i denotes a node and the index e denotes an arc in G. Let O represent the origin node and D the destination node in G. From here on, we will use the terms arc and link interchangeably without any distinction.

Assumption 1 *Each link in G appears in at least in one of the paths from O to D.*

If there is any link which does not appear in any path from O to D then it can be dropped from G as it is not useful for consideration for investment. Thus the network G consists only of links that can be potentially invested in.

Associated with each arc is a non-negative transportation cost t_e. Let p_e $(0 < p_e < 1)$ denote a non-negative survival probability of link e. This probability can be increased to q_e by expending a positive cost c_e. We are given a limited budget B for investment in the links to increase their survival probabilities. Since after the occurrence of a disaster each link would be either operational or non-operational, we use a binary-valued random variable ξ_e to denote the state of the link e. That is, $\xi_e = 1$, if link e is operational after the disaster event; and $\xi_e = 0$, otherwise. The vector of the random variables ξ_e for all links e in E is denoted by $\xi = (\xi_e)$, representing the state of the network, and takes values in its support, $\Xi \subseteq \{0, 1\}^{|E|}$ according to the decision-dependent probability distribution, $\mathrm{P}(\cdot \mid y)$. A unit flow must be sent from O to D in the network realization. If there does not exist a path connecting O to D in the network realization, a fixed penalty cost M is incurred. We let the random variable π represent the shortest path in the network realization that is used to travel from O to D, let the set $K = \{ \pi_1, \pi_2, \ldots, \pi_{|K|} \}$ be the set of paths from O to D in G with $\mathrm{T}(\pi_k)$ being the path length (travel cost) corresponding to path π_k such that $0 < T_{min} = \mathrm{T}(\pi_1) \leq \mathrm{T}(\pi_2) \leq \ldots \leq \mathrm{T}(\pi_{|K|}) = T_{max}$.

Assumption 2 $M > T_{max}$.

Formally M is the opportunity cost of an infeasible network realization. It can be viewed as using an external path (outside G) which has a higher travel cost than T_{max} to travel from O to D. This also means that any path in G is preferred over this external path or equivalently any path is preferred over the choice no path available. By adopting a finite value for penalty cost, complete recourse is achieved. A similar type of assumption is a commonly used in the context of inventory management to handle backorders and lost sales.

We now state the decision variables used. We denote the investment vector by $y = (y_e)$, where the binary variable $y_e = 1$, if there is an investment in link e; and $y_e = 0$, otherwise. We denote the flow vector by $x(\xi) = (x_e(\xi))$, where $x_e(\xi) = 1$, if there is a unit flow through link e in the network realization ξ; and $x_e(\xi) = 0$, otherwise.

The problem of investing in links to minimize the expected shortest path length from O to D is formulated as a two-stage stochastic program. The first-stage decisions are the investment decisions for the links which need to be decided *here and now* without the knowledge of the random vector ξ. The second-stage decisions are the flow variables that need to be determined to minimize the path length from O to D, under the outcome of the network realization ξ. We denote the outcome by $\tilde{\xi} = (\tilde{\xi}_e)$.

2.2 Two-stage Stochastic Program

The two-stage stochastic program **P** is given by the expressions from (1) to (6).

Problem **P**

$$Z = \min_y F(y) = \min_y E_{\xi|y}[f(\xi)] \tag{1}$$

subject to

$$\sum_{e \in E} c_e y_e \le B \tag{2}$$

$$y_e = 0 \text{ or } 1 \qquad \forall\, e \in E \tag{3}$$

$$\sum_{e=(i,j)\in E} x_e(\xi) - \sum_{e=(j,i)\in E} x_e(\xi) = \begin{cases} 1, \text{if } i=O \\ -1, \text{if } i=D \\ 0, \text{otherwise} \end{cases} \qquad \forall\, i \in N \tag{4}$$

$$x_e(\xi) \le \xi_e \qquad \forall\, e \in E \tag{5}$$

$$x_e(\xi) = 0 \text{ or } 1 \qquad \forall\, e \in E \tag{6}$$

where,

$$f(\xi) = \begin{cases} M, \text{if } X(\xi) = \phi \\ \min_{\forall x(\xi)\in X(\xi)} \sum_{e\in E} t_e x_e(\xi), \text{otherwise} \end{cases} \tag{7}$$

and $X(\xi) = \{\, x(\xi) \mid$ subject to constraints (4), (5) and (6) $\}$, is the set of paths from O to D in the network realization ξ. $E_{\xi|y}[\cdot]$ is the expectation of the argument with respect to the random variable ξ for a given investment vector y. For notational compactness we denote it as F(y). The function $f(\xi)$ is the second-stage objective function. Note, it does not depend on y but its probability distribution is dependent on y. Its value is equal to the least transportation cost in the surviving network, if such a path exists; or the penalty cost M, if O and D are disconnected.

Constraint (2) is the budget restriction on the total investment. Constraint set (3) is the integrality restrictions on the first-stage decision variables. Constraint set (4) is the usual flow conservation constraint with source node *O* and destination node *D*. Constraint set (5) acts as capacity constraints for flow. It precludes flow in those links that are non-operational in ξ. Constraint set (6) is the integrality restrictions on the second-stage decision variables.

In general, two-stage stochastic programs in the literature treat the second-stage objective function as dependent on the first-stage decisions and the outcome of the random variable. These programs employ a notation similar to $E_\xi[\cdot]$ and f(y, ξ) for the expectation function and the second stage objective function respectively. It is here that our study departs from the existing research direction. It is the distribution of the random variable ξ, which is affected by the first stage decisions and hence the expectation of $f(\cdot)$. **P** belongs to a class of stochastic programs in which the probability measure depends on the first-stage decisions.

The expectation can be expanded as below,

$$F(y) = E_{\xi|y}[f(\xi)] = \sum_{\tilde{\xi} \in \Xi} P(\xi = \tilde{\xi} \mid y)\, f(\tilde{\xi}) \tag{8}$$

Where $P(\xi = \tilde{\xi} \mid y)$ is the probability that ξ assumes the particular realization $\tilde{\xi}$, for a given investment vector y. Due to the independence of link states the probability of a network state can be expressed in a product form as shown below,

$$P(\xi = \tilde{\xi} \mid y) = \prod_{\forall e \in E} P(\xi_e = \tilde{\xi}_e \mid y_e)$$

$$= \prod_{\forall e \in E} \{ \tilde{\xi}_e [(1-y_e)p_e + y_e q_e] + (1-\tilde{\xi}_e)[(1-y_e)(1-p_e) + y_e(1-q_e)] \} \tag{9}$$

in which y_e takes the value 0 or 1. The above equation clearly illustrates the nature of dependence of the probability of a network realization on the first-stage decision variables. If we denote the set of feasible investments as

$$Y = \{ y \mid \sum_{e \in E} c_e y_e \leq B,\ y \in \{0,1\}^{|E|} \}$$

then **P** can be expressed as:

$$Z = \min_{Y} \{ F(y) \mid y \in Y \} \tag{10}$$

The level of difficulty of **P** can be gauged by the fact that even the computation of F(y) is #P Complete [Ball et al., 1995] i.e. the counting analogue of NP Complete. Its exact computation would be nearly impossible for large size network problems. The difficulty is compounded by the discrete nature of the feasible solutions. Employing the trivial brute force technique that evaluates F(y) for all feasible solutions in order to select the optimal solution would be computationally prohibitive even for moderate size problems. Any alternative solution method for **P** that attempts to solve both the stages simultaneously would encounter the difficulty of solving an optimization problem within the expectation. This is because F(y) in its current form does not possess a closed-form expression in terms of the components of y. In the next section we overcome this.

3 THE EQUIVALENT DETERMINISTIC PROGRAM

We use a path-based approach to derive a closed-form expression for F(y) in terms of the components of y which subsequently yields the equivalent deterministic program of **P**. Specifically we treat the shortest path in a network realization to be in the sample space of paths instead of being a solution to a network flow problem on the network realization.

Firstly, we introduce the additional notation. Let S be the set of network realizations that have O-D connectivity viz. $X(\xi) \neq \emptyset \;\forall\; \xi \in S$, and let $S^c = \Xi / S$ be the set of network realizations that do not have O-D connectivity. Let $f_1(\xi) = f(\xi)$ when $\xi \in S$ and let $F_1(y) = E_{\xi|y}[f_1(\xi)]$. Let $I\{k, \tilde{\xi}\}$ be the indicator function which assumes the value 1 if the shortest path in the network realization $\tilde{\xi}$ ($\in \Xi$) is the kth shortest path in G and 0 otherwise. Conditioning F(y) over feasible and infeasible network realizations from equation (8), we have

$$F(y) = \sum_{\xi \in S} P(\xi = \tilde{\xi} \mid y)\, f(\tilde{\xi}) + \sum_{\xi \in S^C} P(\xi = \tilde{\xi} \mid y)\, f(\tilde{\xi})$$

$$= \Big\{ \sum_{\xi \in S} \frac{P(\xi = \tilde{\xi} \mid y)}{P(S \mid y)} f(\tilde{\xi}) \Big\} P(S \mid y) + M\, P(S^c \mid y) \quad \text{(using } P(S \mid y) = P(\xi \in S \mid y)\text{)} \qquad (11)$$

Now, $F_1(y) = E_{\xi|y}[f(\xi)|\xi \in S] = \dfrac{1}{P(S|y)} \sum_{\tilde{\xi} \in S} P(\xi = \tilde{\xi} \mid y)\, f(\tilde{\xi})$

$$= \frac{1}{P(S|y)} \sum_{\tilde{\xi} \in S} P(\xi = \tilde{\xi} \mid y)\Big(\min_{\forall x(\xi) \in X(\xi)} \sum_{e \in E} t_e x_e(\xi) \Big)$$

$$= \frac{1}{P(S|y)} \sum_{\tilde{\xi} \in S} P(\xi = \tilde{\xi} \mid y)\Big(\sum_{k=1}^{|K|} I\{k, \tilde{\xi}\} T(\pi_k) \Big)$$

Interchanging the order of summation over $\tilde{\xi}$ and k and observing that $I\{k, \tilde{\xi}\} = 0\ \forall\ k \in \{1, 2, \ldots, |K|\}$ and $\tilde{\xi} \in S^c$, we get

$$F_1(y) = \frac{1}{P(S|y)} \sum_{k=1}^{|K|} \Big(\sum_{\tilde{\xi} \varepsilon S} I\{k, \tilde{\xi}\} P(\xi = \tilde{\xi} \mid y) + \sum_{\tilde{\xi} \in S^c} I\{k, \tilde{\xi}\} P(\xi = \tilde{\xi} \mid y) \Big) T(\pi_k)$$

$$= \sum_{k=1}^{|K|} \frac{P(\pi_k | y)}{P(S \mid y)} T(\pi_k)$$

Here $P(\pi_k \mid y) = P(\pi = \pi_k \mid y)$ = denotes the probability that the shortest path in a network realization is π_k given that the investment decision is y.

The above equation merely states that the expected value of the shortest path distance from O to D over feasible realizations is simply equal to the summation over the set of paths from O to D, of the product of, the likelihood of a path having the minimum length given that the network realization is feasible, and the corresponding path length. Substituting the above expression in (11) we get

$$F(y) = \sum_{k=1}^{|K|} P(\pi_k | y) T(\pi_k) + M\, P(S^c \mid y) \qquad (12)$$

This alternate expression is dependent only on the first stage decision variables y and it obviates the need to solve the second stage network flow problem. Both $P(\pi_k| y)$ and $P(S^c \mid y)$ can be expressed in closed form in terms of the components of y. Substituting (12) in (10), problem **P** becomes,

$$Z = \min_{y} \Big\{ \sum_{k=1}^{|K|} P(\pi_k | y) T(\pi_k) + M\, P(S^c \mid y) \mid y \in Y \Big\} \qquad (13)$$

which is the equivalent deterministic program [see Wets, 1974] of the original two-stage program, as F(y) has been expressed as an explicit analytical function of y. It is an integer nonlinear program as its objective function F(y) has product form expressions in terms of the components of y. As it will be seen in the next section, the advantage of expression (12) for F(y) is that, it enables direct evaluation of the marginal improvement in the objective function due to investment in link(s).

4 STRUCTURAL RESULTS

This section investigates the structure of the objective function, F(y). In Subsection 4.1 we derive the multilinear functional form of F(y). In Subsection 4.2, we characterize the coefficients of this function by providing expressions for them. In Subsection 4.3 we prove the monotone decreasing property of F(y). In Subsection 4.4 we present sufficient conditions for establishing the sign of the second order coefficients.

4.1 Multilinear Functional Form of F(*y*)

The structure of F(y) in equation (12) is complicated by the summation, of product form expressions, $P(\xi = \tilde{\xi} \mid y)$, as given in equation (9). Evaluating the function involves collecting the coefficients of the product of every possible combination of variables. We develop an alternate way to expand expression (12) that has intuition behind it and is more useful from the viewpoint of developing a solution procedure for **P**. Recalling expression (12) and substituting for $P(\pi_k \mid y)$ by $\sum_{\tilde{\xi} \in S} I\{k, \tilde{\xi}\}\ P(\xi = \tilde{\xi} \mid y)$, and for $P(S^C \mid y)$ by $\sum_{\tilde{\xi} \in S^C} P(\xi = \tilde{\xi} \mid y)$. We note that F(y) is a discrete function defined only at the vertices of the unit hypercube, $H = \{\ y \in \mathbb{R}^{|E|} \mid 0 \le y_e \le 1,\ e = 1..|E|\ \}$ in the space, $\mathbb{R}^{|E|}$ and therefore it is not differentiable. Given this discontinuous nature of the function, we temporarily relax the integrality restrictions on the y components or equivalently we permit partial investment in the links. Later, we show why this does not affect our derivation. This allows F(y) to be continuously differentiable in the domain H and enables us to consider its Taylor expansion in the neighborhood of some $y_0 \in Y \subset H$.

$$F(y) = F(y_0) + \sum_{e \in E} \frac{\partial F(y)}{\partial y_e}|_{y=y_0} (y_e - y_{0e}) + \frac{1}{2!} \sum_{e_1 \in E} \sum_{e_2 \in E} \frac{\partial^2 F(y)}{\partial y_{e_1} \partial y_{e_2}}|_{y=y_0} (y_{e_1} - y_{0e_1})(y_{e_2} - y_{0e_2})$$

$$+ \cdots + \frac{1}{|E|!} \sum_{e_1 \in E} \sum_{e_2 \in E} \cdots \sum_{e_{|E|} \in E} \frac{\partial^{|E|} F(y)}{\partial y_{e_1} \partial y_{e_2} \cdots \partial y_{e_{|E|}}}|_{y=y_0} (y_{e_1} - y_{0e_1})(y_{e_2} - y_{0e_2}) \cdots (y_{e_{|E|}} - y_{0e_{|E|}}) \tag{14}$$

To provide an understanding of the above expression let $y_0 = \mathbf{0}$ (the vector of zeroes), thus no investment in any of the links. The partial derivative of F(y) with respect to a set of link investment decisions represents the marginal improvement in the objective function just due to the effect of joint investment in those links alone. The first order derivative with respect to a link decision variable represents the improvement in the objective function due to a unit investment in that link alone, and a second order partial derivative with respect to investment decisions for two distinct links, represents the improvement in the objective function beyond their first-order effects due to simultaneously investment in both these links and so on for the rest of the partial derivatives.

It is sufficient to limit the number of terms in the Taylor series expansion, we use terms till the order of the derivative equals the maximum number of links that can be invested with the given budget. Denote the maximum number of links as R, it can be figured by investing in the ascending order of the link investment costs until the budget restriction is satisfied. Now, using $y_0 = \mathbf{0}$ as a feasible solution, we can express F(y) as equal to

$$F(\mathbf{0}) + \sum_{e \in E} \frac{\partial F(y)}{\partial y_e}|_{y=0}\, y_e + \frac{1}{2!} \sum_{e_1 \in E} \sum_{e_2 \in E} \frac{\partial^2 F(y)}{\partial y_{e_1} \partial y_{e_2}}|_{y=0}\, y_{e_1} y_{e_2} + \cdots$$

$$+ \frac{1}{R!} \sum_{e_1 \in E} \sum_{e_2 \in E} \cdots \sum_{e_R \in E} \frac{\partial^R F(y)}{\partial y_{e_1} \partial y_{e_2} \cdots \partial y_{e_R}}|_{y=0}\, y_{e_1} y_{e_2} \cdots y_{e_R} \tag{15}$$

If all the derivatives obtained are unbiased, then the above expression for F(y) is precise, despite the assumption that y be continuous. Again, to be shown later, it turns out that the expansion is in fact precise when $y_0 = \mathbf{0}$. For notational convenience we denote

$\frac{\partial F(y)}{\partial y_e}|_{y=y_o}$ as $g_e(y_0)$, $\frac{\partial^2 F(y)}{\partial y_{e_1}\partial y_{e_2}}|_{y=y_0}$ as $g_{e_1e_2}(y_0)$ and so on. Problem **P** can be reexpressed as,

$$Z = \min_{y} \{ F(\mathbf{0}) + \sum_{m=1}^{R} (\frac{1}{m!} \sum_{e_1 \in E} \cdots \sum_{e_R \in E} g_{e_1 \cdots e_m}(\mathbf{0})\ y_{e_1} \cdots y_{e_m}) \mid y \in Y \} \tag{16}$$

The above expression illustrates the multilinear functional form of F(y). It is linear in each of the components of y indivdiually but is not linear in more than one component simultaneously. Its coefficients measure the effect on the objective function due to interactions between the investment decisions.

4.2 Charecterization of the Coefficients

Firstly we make a couple of observations. There are no self-interactions i.e. $g_{ee}(y) = 0\ \forall\ e \in E,\ \forall\ y \in Y$ and the interactions are symmetric i.e. $g_{e_1e_2}(y) = g_{e_2e_1}(y)\ \forall e_1, e_2 \in E$ with $e_1 \neq e_2$, $\forall\ y \in Y$. For convenience let Δp_e as $q_e - p_e$.

Proposition 1 $g_e(y) = \frac{\partial\{\sum_{k=1}^{|K|} P(\pi_k \mid y)T(\pi_k)\}}{\partial y_e} - M\frac{\partial P(S \mid y)}{\partial y_e}$ $\forall e \in E,\ \forall y \in H$ *where,*

a) $\frac{\partial\{\sum_{k=1}^{|K|} P(\pi_k \mid y)T(\pi_k)\}}{\partial y_e} = E_{\xi|y}[\sum_{k=1}^{|K|} T(\pi_k)\ I(k,\xi)\frac{\xi_e[\Delta p_e] + (1-\xi_e)[-\Delta p_e]}{P(\xi_e \mid y_e)} \mid \xi \in S] P(S \mid y)$

b) $\frac{\partial P(S \mid y)}{\partial y_e} = E_{\xi|y}[\frac{\xi_e[\Delta p_e] + (1-\xi_e)[-\Delta p_e]}{P(\xi_e \mid y_e)} \mid \xi \in S]\ P(S \mid y)$.

Proof: Differentiating expression (12) w.r.t. y_e we have

$$g_e(y) = \frac{\partial F(y)}{\partial y_e} = \sum_{k=1}^{|K|} \frac{\partial P(\pi_k \mid y)}{\partial y_e} T(\pi_k) + M\frac{\partial P(S^c \mid y)}{\partial y_e} \qquad \forall\ e \in E,\ \forall\ y \in H \tag{17}$$

From the theorem of total probability we know that

$$P(S^c \mid y) + P(S \mid y) = 1 \qquad \forall\ y \in H$$

Differentiating both sides w.r.t. y_e we have

$$\frac{\partial P(S^c \mid y)}{\partial y_e} = -\frac{\partial P(S \mid y)}{\partial y_e} \qquad \forall\ e \in E,\ \forall\ y \in H$$

Substituting the above in equation (17) we have

$$g_e(y) = \sum_{k=1}^{|K|} \frac{\partial P(\pi_k | y)}{\partial y_e} T(\pi_k) - M\frac{\partial P(S | y)}{\partial y_e} \qquad \forall\, e \in E, \forall\, y \in H \tag{18}$$

This proves the main statement of the proposition. Now, we separately expand the two terms in the above expression, for the first term we have

$$\sum_{k=1}^{|K|} \frac{\partial P(\pi_k | y)}{\partial y_e} T(\pi_k) = \sum_{k=1}^{|K|} T(\pi_k) \sum_{\tilde{\xi}\, \varepsilon S} I(k, \tilde{\xi})\frac{\partial P(\xi = \tilde{\xi} | y)}{\partial y_e} \tag{19}$$

Now, $\frac{\partial P(\xi = \tilde{\xi} | y)}{\partial y_e} = \frac{\partial \{\prod_{\forall l \in E} P(\xi_l = \tilde{\xi}_l | y_l)\}}{\partial y_e}$

$$= \prod_{\forall l \in E | l \neq e} P(\xi_l = \tilde{\xi}_l | y_l) \frac{\partial P(\xi_e = \tilde{\xi}_e | y_e)}{\partial y_e}$$

$$= \prod_{\forall l \in E | l \neq e} P(\xi_l = \tilde{\xi}_l | y_l) \frac{\partial}{\partial y_e}\{ [\tilde{\xi}_e(1-y_e)p_e + y_e q_e] + (1-\tilde{\xi}_e)[(1-y_e)(1-p_e) + y_e(1-q_e)] \}$$

$$= \prod_{\forall l \in E | l \neq e} P(\xi_l = \tilde{\xi}_l | y_l) \{ \tilde{\xi}_e[\Delta p_e] + (1-\tilde{\xi}_e)[-\Delta p_e] \}$$

$$= P(\xi = \tilde{\xi} | y)\frac{\tilde{\xi}_e[\Delta p_e] + (1-\tilde{\xi}_e)[-\Delta p_e]}{P(\xi_e = \tilde{\xi}_e | y_e)} \tag{20}$$

Substituting the expression obtained from (20) in equation (19) the first term becomes

$= \sum_{k=1}^{|K|} T(\pi_k)\sum_{\tilde{\xi}\, \varepsilon S} I(k, \tilde{\xi})P(\xi = \tilde{\xi} | y) \frac{\tilde{\xi}_e[\Delta p_e] + (1-\tilde{\xi}_e)[-\Delta p_e]}{P(\xi_e = \tilde{\xi}_e | y_e)}$ which can be simplified as

$$= \{\sum_{\tilde{\xi}\, \varepsilon S} \frac{P(\xi = \tilde{\xi} | y)}{P(S | y)} \sum_{k=1}^{|K|} T(\pi_k) I(k, \tilde{\xi})\frac{\tilde{\xi}_e[\Delta p_e] + (1-\tilde{\xi}_e)[-\Delta p_e]}{P(\xi_e = \tilde{\xi}_e | y_e)}\}P(S | y)$$

$$= E_{\xi | y}[\sum_{k=1}^{|K|} T(\pi_k) I(k, \xi)\frac{\xi_e[\Delta p_e] + (1-\xi_e)[-\Delta p_e]}{P(\xi_e | y_e)} | \xi \in S]\, P(S | y)$$

This completes the proof of part (a). An identical derivation is valid for part (b).□

The following lemma proves that the continuous assumption does not affect the evaluation of the first order derivative under a certain condition. Define u_e as the unit vector having the 1 at component e and 0 at the remaining components.

Lemma 1 *$\forall e \in E$ and $\forall y \in Y$ with $y_e = 0$ $g_e(y) = F(y + u_e) - F(y)$.*

Proof: Following proposition 1, $g_e(y)$ =

$$\Delta p_e \{ \ E_{\xi|y} [\sum_{k=1}^{|K|} \ T(\pi_k) \ I(k,\xi) \frac{\xi_e - (1-\xi_e)}{P(\ \xi_e | \ y_e)} \ | \ \xi \in S] \ P(S \ | \ y)$$

$$- M \, E_{\xi|y} [\frac{\xi_e - (1-\xi_e)}{P(\ \xi_e | \ y_e)} \ | \ \xi \in S] \ P(S \ | \ y) \ \}$$

Now, $\{ \xi \in S \} = \sum_{k=1}^{|K|} I(k,\xi)$, which takes the value 1 or 0 depending whether or not O is connected to D. Unconditioning the second term in the previous expression, we have

$$g_e(y) = \Delta p_e \{ \ E_{\xi|y} [\sum_{k=1}^{|K|} \ T(\pi_k) \ I(k,\xi) \frac{\xi_e - (1-\xi_e)}{P(\ \xi_e | \ y_e)} \ | \ \xi \in S] \ P(S \ | \ y)$$

$$- M \, E_{\xi|y} [(\frac{\xi_e - (1-\xi_e)}{P(\ \xi_e | \ y_e)} \ \sum_{k=1}^{|K|} I(k,\xi)) \ \frac{1}{P(\xi \in S \ | \ y)}] \ P(S \ | \ y) \ \}$$

$$= \Delta p_e \{ \ E_{\xi|y} [\sum_{k=1}^{|K|} \ T(\pi_k) \ I(k,\xi) \frac{\xi_e - (1-\xi_e)}{P(\ \xi_e | \ y_e)} \ | \ \xi \in S] \ P(S \ | \ y)$$

$$- M \, E_{\xi|y} [(\frac{\xi_e - (1-\xi_e)}{P(\ \xi_e | \ y_e)} \ \sum_{k=1}^{|K|} I(k,\xi)) \ ] \ \}$$

Since the events $\xi_e = 0$ or 1 is mutually exclusive, we have

$$= \Delta p_e \{ \ E_{\xi|y} [\sum_{k=1}^{|K|} \ T(\pi_k) \ I(k,\xi) \frac{1}{P(\ \xi_e = 1 | \ y_e)} \ | \ \xi \in S, \ \xi_e = 1] \ P(S \ | \ y)$$

$$- M \, E_{\xi|y} [\sum_{k=1}^{|K|} \ I(k,\xi) \frac{1}{P(\ \xi_e = 1 | \ y_e)} | \ \xi_e = 1] \ \} \ -$$

$$\Delta p_e \{ \ E_{\xi|y} [\sum_{k=1}^{|K|} \ T(\pi_k) \ I(k,\xi) \frac{1}{P(\ \xi_e = 0 | \ y_e)} \ | \ \xi \in S, \ \xi_e = 0] \ P(S \ | \ y)$$

$$- M \, E_{\xi|y} [\sum_{k=1}^{|K|} \ I(k,\xi) \frac{1}{P(\ \xi_e = 0 | \ y_e)} | \ \xi_e = 0] \ \}$$

$$= \Delta p_e \{ \ E_{\xi|y} [\sum_{k=1}^{|K|} \ T(\pi_k) \ I(k,\xi) \ | \ \xi \in S, \ \xi_e = 1] \ P(S \ | \ y) \ - \ M \, E_{\xi|y} [\sum_{k=1}^{|K|} \ I(k,\xi) \ | \ \xi_e = 1] \ \} \ -$$

$$\Delta p_e \{ \ E_{\xi|y} [\sum_{k=1}^{|K|} \ T(\pi_k) \ I(k,\xi) \ | \ \xi \in S, \ \xi_e = 0] \ P(S \ | \ y) \ - M \, E_{\xi|y} [\sum_{k=1}^{|K|} \ I(k,\xi) \ | \ \xi_e = 0] \ \}$$

$$= \Delta p_e \{ \ E_{\xi|y} [\sum_{k=1}^{|K|} \ T(\pi_k) \ I(k,\xi) \ | \ \xi \in S, \ \xi_e = 1] \ P(S \ | \ y) \ - \ M \, \mathrm{rel}(G^*e) \ \} \ -$$

$$\Delta p_e \{ \ E_{\xi|y} [\sum_{k=1}^{|K|} \ T(\pi_k)\ I(k,\xi) \mid \xi \in S,\ \xi_e = 0]\ P(S \mid y)\ \ - M\ \text{rel}(G\text{-}e) \ \}$$

Here G^*e represents the network in which link e is made perfectly reliable in G. G-e represents the network in which link e is made perfectly unreliable in G.

$$= \Delta p_e \{ \ F_1(y \mid \xi_e = 1)\ P(S \mid y)\ \ - \ M\ \text{rel}(G^*e) \ \} \ - \ \Delta p_e \{ \ F_1(y \mid \xi_e = 0)\ P(S \mid y)\ \ - M\ \text{rel}(G\text{-}e) \ \}$$

$$= \Delta p_e \{ \ F(y \mid \xi_e = 1) - F(y \mid \xi_e = 0) \ \} \qquad (21)$$

$$= \{ (p_e + \Delta p_e)\ F(y \mid \xi_e = 1) + (1 - (p_e + \Delta p_e))\ F(y \mid \xi_e = 0) \}$$

$$- \{ p_e\ F(y \mid \xi_e = 1) + (1 - p_e)\ F(y \mid \xi_e = 0) \}$$

$$= F(y + u_e) - F(y) \quad \forall\ y \in Y \text{ with } y_e = 0 \ \square$$

The term $F(y \mid \xi_e = 1) - F(y \mid \xi_e = 0)$ in expression (21) represents the importance of link e to the objective function $F(y)$. Observe that this importance as well as $g_e(y)$ is independent of the link survival probability without and with investment, p_e or q_e respectively, and the investment decision variable y_e. It is a function of only the remaining components of y. This is because of the multilinear functional form of $F(y)$.

Proposition 2 $\forall\ y \in H\ \ g_{e_1e_2}(y)$

$$= \Delta p_{e_1} \Delta p_{e_2} \{ E_{\xi|y} [\sum_{k=1}^{|K|} \ T(\pi_k)\ I(k,\xi) \frac{\xi_{e_1} - (1-\xi_{e_1})}{P(\xi_{e_1} \mid y_{e_1})} \ \frac{\xi_{e_2} - (1-\xi_{e_2})}{P(\xi_{e_2} \mid y_{e_2})} \mid \xi \in S]\ P(S \mid y)$$

$$- M\ E_{\xi|y} [\frac{\xi_{e_1} - (1-\xi_{e_1})}{P(\xi_{e_1} \mid y_{e_1})} \ \frac{\xi_{e_2} - (1-\xi_{e_2})}{P(\xi_{e_2} \mid y_{e_2})} \mid \xi \in S]\ P(S \mid y) \} \ \text{if } e_1 \neq e_2$$

$$= 0 \ \ \text{if } e_1 = e_2$$

Proof: Now $g_{e_1e_2}(y) = \dfrac{\partial g_{e_1}(y)}{\partial y_{e_2}}$

$$= \ \Delta p_{e_1} \{ \sum_{k=1}^{|K|} \ T(\pi_k) \sum_{\tilde{\xi} \in S} I(k,\ \tilde{\xi}\ ) \ \frac{\xi_{e_1} - (1-\xi_{e_1})}{P(\xi_{e_1} \mid y_{e_1})} \ \frac{\partial P(\ \xi = \tilde{\xi} \mid y)}{\partial y_{e_2}}$$

$$- M \sum_{k=1}^{|K|} \ \sum_{\tilde{\xi} \in S} I(k,\ \tilde{\xi}\ ) \ \frac{\xi_{e_1} - (1-\xi_{e_1})}{P(\xi_{e_1} \mid y_{e_1})} \ \frac{\partial P(\ \xi = \tilde{\xi} \mid y)}{\partial y_{e_2}} \ \}$$

$$\forall\ e_1, e_2 \in E \text{ with } e_1 \neq e_2 \text{ and } \forall\ y \in H$$

$$= \Delta p_{e_1} \{ \sum_{k=1}^{|K|} \mathrm{T}(\pi_k) \sum_{\tilde{\xi} \varepsilon S} \mathrm{I}(k, \tilde{\xi}) \frac{\xi_{e_1} - (1-\xi_{e_1})}{\mathrm{P}(\xi_{e_1} | y_{e_1})} \frac{\xi_{e_2}[\Delta p_{e_2}] + (1-\xi_{e_2})[-\Delta p_{e_2}]}{\mathrm{P}(\xi_{e_2} | y_{e_2})} \mathrm{P}(\xi = \tilde{\xi} | y)$$

$$- M \sum_{k=1}^{|K|} \sum_{\tilde{\xi} \varepsilon S} \mathrm{I}(k, \tilde{\xi}) \frac{\xi_{e_1} - (1-\xi_{e_1})}{\mathrm{P}(\xi_{e_1} | y_{e_1})} \frac{\xi_{e_2}[\Delta p_{e_2}] + (1-\xi_{e_2})[-\Delta p_{e_2}]}{\mathrm{P}(\xi_{e_2} | y_{e_2})} \mathrm{P}(\xi = \tilde{\xi} | y) \}$$

$$= \Delta p_{e_1} \Delta p_{e_2} \{ \mathrm{E}_{\xi|y} [\sum_{k=1}^{|K|} \mathrm{T}(\pi_k) \mathrm{I}(k, \xi) \frac{\xi_{e_1} - (1-\xi_{e_1})}{\mathrm{P}(\xi_{e_1} | y_{e_1})} \frac{\xi_{e_2} - (1-\xi_{e_2})}{\mathrm{P}(\xi_{e_2} | y_{e_2})} | \xi \in S] \mathrm{P}(S | y)$$

$$- M \mathrm{E}_{\xi|y} [\frac{\xi_{e_1} - (1-\xi_{e_1})}{\mathrm{P}(\xi_{e_1} | y_{e_1})} \frac{\xi_{e_2} - (1-\xi_{e_2})}{\mathrm{P}(\xi_{e_2} | y_{e_2})} | \xi \in S] \mathrm{P}(S | y) \}$$

When $e_1 = e_2$ $g_{e_1 e_1}(y) = 0$ as $g_{e_1}(y)$ is independent of y_{e_1}. □

Lemma 2 $\forall e_1, e_2 \in E$ *with* $e_1 \neq e_2$ and $\forall y \in Y$ *with* $y_{e_1} = y_{e_2} = 0$ $g_{e_1 e_2}(y) = F(y + u_{e_1} + u_{e_2}) - F(y + u_{e_2}) - F(y + u_{e_1}) + F(y)$.

Proof: From Proposition 2 $g_{e_1 e_2}(y)$ can be further simplified along similar lines as in the previous lemma to yield $g_{e_1 e_2}(y)$ as

$$= \Delta p_{e_1} \Delta p_{e_2} \{ \mathrm{F}_1(y | \xi_{e_1} = 1, \xi_{e_2} = 1) \mathrm{P}(S | y) - M \,\mathrm{rel}(\mathrm{G}{*}e_1{*}e_2) \}$$

$$- \Delta p_{e_1} \Delta p_{e_2} \{ \mathrm{F}_1(y | \xi_{e_1} = 0, \xi_{e_2} = 1) \mathrm{P}(S | y) - M \,\mathrm{rel}(\mathrm{G}{-}e_1{*}e_2) \}$$

$$- \Delta p_{e_1} \Delta p_{e_2} \{ \mathrm{F}_1(y | \xi_{e_1} = 1, \xi_{e_2} = 0) \mathrm{P}(S | y) - M \,\mathrm{rel}(\mathrm{G}{*}e_1{-}e_2) \}$$

$$+ \Delta p_{e_1} \Delta p_{e_2} \{ \mathrm{F}_1(y | \xi_{e_1} = 0, \xi_{e_2} = 0) \mathrm{P}(S | y) - M \,\mathrm{rel}(\mathrm{G}{-}e_1{-}e_2) \}$$

$$= \Delta p_{e_1} \Delta p_{e_2} \{ \mathrm{F}(y | \xi_{e_1} = 1, \xi_{e_2} = 1) - \mathrm{F}(y | \xi_{e_1} = 0, \xi_{e_2} = 1) - \mathrm{F}(y | \xi_{e_1} = 1, \xi_{e_2} = 0) + \mathrm{F}(y | \xi_{e_1} = 0, \xi_{e_2} = 0) \} \quad (22)$$

Adding and subtracting appropriate terms the above expression becomes

$= \mathrm{F}(y + u_{e_1} + u_{e_2}) - \mathrm{F}(y + u_{e_2}) - \mathrm{F}(y + u_{e_1}) + \mathrm{F}(y)$ $\forall$ $e_1, e_2 \in E$ with $e_1 \neq e_2$ and $\forall$ $y \in Y$ with $y_{e_1} = y_{e_2} = 0$ □

The term $\mathrm{F}(y | \xi_{e_1} = 1, \xi_{e_2} = 1) - \mathrm{F}(y | \xi_{e_1} = 0, \xi_{e_2} = 1) - \mathrm{F}(y | \xi_{e_1} = 1, \xi_{e_2} = 0) + \mathrm{F}(y | \xi_{e_1} = 0, \xi_{e_2} = 0)$ in expression (22) measures the joint importance of links e_1 and e_2 in F(y). Also observe that $g_{e_1 e_2}(y)$ is independent of y_{e_1} and y_{e_2}, it is a function of only the remaining components of y. Again, this is because of the multilinear functional form of F(y). For proving the similar lemma for higher order derivatives we require an intermediate result.

Lemma 3 $\forall y \in H$ $g_{e_1 \cdots e_m}(y) = \Delta p_{e_1} \cdots \Delta p_{e_m} \sum_{\forall(\tilde{\xi}_1, \cdots, \tilde{\xi}_m) \in \{0,1\}^m} (-1)^{m-(\tilde{\xi}_1 + \cdots + \tilde{\xi}_m)} F(y \mid \xi_{e_1} = \tilde{\xi}_{e_1}, \cdots, \xi_{e_m} = \tilde{\xi}_{e_m})$

if $e_{m1} \neq e_{m2}$ $\forall e_{m1}, e_{m2} \in \{ e_1, \cdots, e_m\}$

$= 0$

if $\exists e_{m1}, e_{m2} \in \{ e_1, \cdots, e_m\}$ *such that* $e_{m1} = e_{m2}$

where $2 \leq m \leq R$ *and* $e_1, \cdots, e_m \in E$.

Proof: The proof is by induction. From equation (22) the expression is valid for m = 2. Now hypothesize that the expression is valid for m (> 2). Now,

$$g_{e_1 \cdots e_m e_{m+1}}(y) = \frac{\partial g_{e_1 \cdots e_m}(y)}{\partial y_{e_{m+1}}}$$

$$= \Delta p_{e_1} \cdots \Delta p_{e_m} \sum_{\forall(\tilde{\xi}_1, \cdots, \tilde{\xi}_m) \in \{0,1\}^m} (-1)^{m-(\tilde{\xi}_1 + \cdots + \tilde{\xi}_m)} \frac{\partial F(y \mid \xi_{e_1} = \tilde{\xi}_{e_1}, \cdots, \xi_{e_m} = \tilde{\xi}_{e_m})}{\partial y_{e_{m+1}}}$$

Now we apply the steps of Lemma 1 from beginning until step (21) to the network G modified according to the conditioning of the states,

$$\frac{\partial F(y \mid \xi_{e_1} = \tilde{\xi}_{e_1}, \xi_{e_2} = \tilde{\xi}_{e_2}, \cdots, \xi_{e_m} = \tilde{\xi}_{e_m})}{\partial y_{e_{m+1}}}$$

$$= \Delta p_{e_{m+1}} \{ F(y \mid \xi_{e_1} = \tilde{\xi}_{e_1}, \xi_{e_2} = \tilde{\xi}_{e_2}, \cdots, \xi_{e_m} = \tilde{\xi}_{e_m}, \xi_{e_{m+1}} = 1)$$

$$- F(y \mid \xi_{e_1} = \tilde{\xi}_{e_1}, \xi_{e_2} = \tilde{\xi}_{e_2}, \cdots, \xi_{e_m} = \tilde{\xi}_{e_m}, \xi_{e_{m+1}} = 0) \}$$

Substituting the above in the expression derived for $g_{e_1 \cdots e_m e_{m+1}}(y)$ we have

$$g_{e_1 \cdots e_m e_{m+1}}(y) =$$

$$\Delta p_{e_1} \Delta p_{e_2} \cdots \Delta p_{e_m} \Delta p_{e_{m+1}} \sum_{\forall(\tilde{\xi}_1, \tilde{\xi}_2, \cdots, \tilde{\xi}_m) \in \{0,1\}^m} (-1)^{m-(\tilde{\xi}_1 + \tilde{\xi}_2 + \cdots + \tilde{\xi}_m)} \{ F(y \mid \xi_{e_1} = \tilde{\xi}_{e_1}, \xi_{e_2} = \tilde{\xi}_{e_2}, \cdots, \xi_{e_m} = \tilde{\xi}_{e_m}, \xi_{e_{m+1}} = 1) - F(y \mid \xi_{e_1} = \tilde{\xi}_{e_1}, \xi_{e_2} = \tilde{\xi}_{e_2}, \cdots, \xi_{e_m} = \tilde{\xi}_{e_m}, \xi_{e_{m+1}} = 0) \}$$

$$= \Delta p_{e_1} \Delta p_{e_2} \cdots \Delta p_{e_m} \Delta p_{e_{m+1}} \sum_{\forall(\tilde{\xi}_1, \tilde{\xi}_2, \cdots, \tilde{\xi}_{m+1}) \in \{0,1\}^{m+1}} (-1)^{(m+1)-(\tilde{\xi}_1 + \tilde{\xi}_2 + \cdots + \tilde{\xi}_{m+1})} F(y \mid \xi_{e_1} = \tilde{\xi}_{e_1}, \xi_{e_2} = \tilde{\xi}_{e_2}, \cdots, \xi_{e_m} = \tilde{\xi}_{e_m}, \xi_{e_{m+1}} = \tilde{\xi}_{e_{m+1}})$$

The proof when the other condition is satisfied is trivial and is therefore omitted. □

We note from the previous result that only the cross-partial derivative terms can be non-zero. So from hereon when we use $e_1, e_2, \cdots, e_m$ it suffices that they are distinct links. The term $\sum_{\forall(\bar{\xi}_1,\cdots,\bar{\xi}_m)\in\{0,1\}^m}(-1)^{m-(\bar{\xi}_1+\cdots+\bar{\xi}_m)}F(y|\xi_{e_1}=\bar{\xi}_1,\cdots,\xi_{e_m}=\bar{\xi}_{e_m})$ measures the joint importance of the links $e_1, e_2, \cdots, e_m$ in F(y). Like before $g_{e_1\cdots e_m}(y)$ is independent of $y_{e_1}, \cdots, y_{e_m}$ and is dependent only on $\{y_1, \cdots, y_{|E|}\}\backslash\{y_{e_1}, \cdots, y_{e_m}\}$ $\forall y \in H$. The following corollary is obvious.

Corollary 1 *$g_{e_1\cdots e_m}(y)$ is linear in $\Delta p_{e_1}\cdots\Delta p_{e_m}$ where $1 \le m \le R$ $\forall e_1, e_2, \cdots, e_m \in E$ with $e_{m1} \neq e_{m2}$ $\forall e_{m1}, e_{m2} \in \{e_1, e_2, \cdots, e_m\}$ and $\forall y \in H$.*

Lemma 4 *$\forall e_1, e_2, \cdots, e_m \in E$ with $e_{m1} \neq e_{m2}$ $\forall e_{m1}, e_{m2} \in \{e_1, e_2, \cdots, e_m\}$ and $\forall y \in Y$ with $y_{e_1} = y_{e_2} = \cdots y_{e_m} = 0$ $g_{e_1\cdots e_m}(y) = \sum_{v\in V_m}(-1)^{m-(v_{e_1}+v_{e_2}+\cdots+v_{e_m})}F(y+v)$ where $1 \le m \le R$.*

Proof: The proof is by induction and is omitted. The approach is to add appropriate terms to the result of Lemma 3. In fact for the cases $m = 1$ and $m = 2$ the expression is valid from Lemma 1 and Lemma 2 respectively. □

We now have the following theorem.

Lemma 5 *$\forall y \in H$ $\forall e_1, \cdots, e_m \in E$ $g_{e_1\cdots e_m}(y) = 0$ where $m \ge R$.*

Proof: The proof is by induction and is omitted. In fact for the cases $m = 1$ and $m = 2$ the expression is valid from Lemma 1 and Lemma 2 respectively. □

We now have the following theorem.

Theorem 1 *The multilinear function, $F(\mathbf{0})+\sum_{m=1}^{R}(\frac{1}{m!}\sum_{e_1\in E}\cdots\sum_{e_R\in E} g_{e_1\cdots e_m}(\mathbf{0})\ y_{e_1}\cdots y_{e_m})$ interpolates through $(y, F(y))$ $\forall y \in Y$ in the space $\mathbb{R}^{|E|}\times\mathbb{R}$.*

Proof: The proof is by letting $y_0 = \mathbf{0}$ and applying Lemma 4 and Lemma 5. $g_e(\mathbf{0})$ quantifies the effect on the objective function due to investment in link e alone, $g_{e_1e_2}(\mathbf{0})$ quantifies the interaction effect on the objective function due to simultaneous investment in links e_1 and e_2 and so on. Thus the derivatives are precise. The theorem now follows. □

Thus Theorem 1 establishes the equivalence between Problem **P** and the optimization problem given in (16) and this forms the basis of our proposed solution procedure to be presented in Section 5.

4.3 Monotone Decreasing Property of F(y)

Before we describe the variation of F(y) with y, we have the following proposition on the sign of the first order derivative coefficients. But first we introduce the necessary notation. Let Ξ_{-e} denote the set of network realizations of the network (N, $E\backslash\{e\}$) = $G\backslash\{e\}$, i.e. the network G in which link e is made non-operational. Let ω_{-e} and $\tilde{\omega}_{-e}$ respectively denote the random variable for the network realization, and a particular network realization, of the network $G\backslash\{e\}$.

Proposition 3 *$\forall y \in H$ with $y_e = 0$ $g_e(y) < 0$ $\forall e \in E$.*

Proof : From equation (21) we know that $g_e(y)$

$$= \Delta p_e \{ F(y \mid \xi_e = 1) - F(y \mid \xi_e = 0) \} \qquad e \in E \ \forall\, y \in H$$

Simplifying the above becomes

$$= \Delta p_e \{ \sum_{\tilde{\xi} \in \Xi \mid \tilde{\xi}_e = 1} P(\xi = \tilde{\xi} \mid y, \xi_e = 1)\, f(\tilde{\xi}) \; - \sum_{\tilde{\xi} \in \Xi \mid \tilde{\xi}_e = 0} P(\xi = \tilde{\xi} \mid y, \xi_e = 0)\, f(\tilde{\xi}) \}$$

With slight abuse of notation we let $\tilde{\omega}_{-e} \cup \{e\}$ be the network realization $\tilde{\omega}_{-e}$ to which link e is added and made operational, we have

$$= \Delta p_e \{ \sum_{\tilde{\omega}_e \in \Xi_{-e}} P(\omega_{-e} = \tilde{\omega}_{-e}, \xi_e = 1 \mid y, \xi_e = 1)\, f(\tilde{\omega}_{-e} \cup \{e\})$$

$$- \sum_{\tilde{\omega}_e \in \Xi_{-e}} P(\omega_{-e} = \tilde{\omega}_{-e}, \xi_e = 0 \mid y, \xi_e = 0) \mid y)\, f(\tilde{\omega}_{-e}) \}$$

Applying the assumption on independence of link states we have

$$= \Delta p_e \sum_{\tilde{\omega}_e \in \Xi_{-e}} P(\omega_{-e} = \tilde{\omega}_{-e} \mid y) \{ f(\tilde{\omega}_{-e} \cup \{e\}) - f(\tilde{\omega}_{-e}) \} \qquad (23)$$

Note that $f(\omega_{-e}) : \Xi_{-e} \to \mathbb{R}^+$ is a monotone decreasing set function, that is $f(\tilde{\omega}_{-e} \cup \{e\}) \leq f(\tilde{\omega}_{-e})$ $\forall\, \tilde{\omega}_{-e} \in \Xi_{-e}$ and $\forall\, e \in E$ or equivalently an addition of a link to $\tilde{\omega}_{-e}$ will never worsen the value of shortest distance from O to D. But this fact would prove only the non-positivity of $g_e(y)$, to prove that the inequality is strict we partition of Ξ_{-e} into two sets. The first set consists of network realizations in which, ω_{-e} is infeasible but $\omega_{-e} \cup \{e\}$ is feasible and the second set is the complement of the first. Due to Assumption 1 there exists at least one $\tilde{\omega}_{-e}$ belonging to the first set and due to

Assumption 2 f($\tilde{\omega}_{-e} \cup \{e\}$) - f($\tilde{\omega}_{-e}$) < 0 for this realization. Since P($\omega_{-e} = \tilde{\omega}_{-e} \mid y$) > 0 $\forall$ $\tilde{\omega}_{-e} \in \Xi_{-e}$, the right hand side of (23) is < 0. □

The following theorem describes the variation of F(y) with y.

Theorem 2 *The function F(y) is strictly decreasing with y.*

Proof : Let y_1 and y_2 be two investment decision vectors such that $y_1 < y_2$. The basic idea of the proof is to apply Proposition 3 sequentially by treating one investment component at a time instead of considering the vector $y_2 - y_1$, in order to evaluate the effect of the additional investment. Applying Lemma 1 to the first component position that has unity, calling it as e_1 in $y_2 - y_1$ and using Proposition 3 we have F($y_1 + u_{e_1}$) < F(y_1). Now apply this procedure component wise until the additional investment $y_2 - y_1$ has been made and the statement follows. □

Due to the monotone decreasing property of F(y), additional investment in the network always improves the performance criterion, which is consistent with intuition. If there is no budget restriction, then the optimal solution is to invest in all the links. We point out that this property may not be valid if $M \leq T_{\max}$ and hence there is a possibility where it is not optimal to utilize the entire budget.

4.4 Additional Results for the Second Order Coefficients

Here we seek to understand the interaction between investment decisions in two links, through the sign of the second order derivatives. Specifically, we establish conditions, which when satisfied would give us the sign of the derivative. Lemma 6 is useful in this regard, it provides us an alternate expression for $g_{e_1e_2}(y)$. But first we introduce the necessary notation. Let $\Xi_{-e_1e_2}$ denote the set of network realizations of the network (N, $E\backslash\{e_1, e_2\}$) = $G\backslash\{e_1, e_2\}$, i.e. the network G in which the links e_1 and e_2 are made non-operational. Let $\omega_{-e_1e_2}$ and $\tilde{\omega}_{-e_1e_2}$, respectively denote the random variable for the network realization, and a particular network realization, of the network $G\backslash\{e_1, e_2\}$.

Lemma 6 *$\forall$ e_1, e_2 $\in$ E $\forall$ y $\in$ H with $y_{e_1} = 0$ and $y_{e_2} = 0$ $g_{e_1e_2}(y)$ is* $\Delta p_{e_1} \Delta p_{e_2} \sum_{\tilde{\omega}_{e_1e_2} \in \Xi_{-e_1e_2}} \text{P}(\omega_{e_1e_2} = \tilde{\omega}_{e_1e_2} \mid y)\, \Delta f_{e_1e_2}(\tilde{\omega}_{e_1e_2})$, *where* $\Delta f_{e_1e_2}(\tilde{\omega}_{-e_1e_2}) = f(\tilde{\omega}_{-e_1e_2} \cup \{e_1, e_2\}) - f(\tilde{\omega}_{-e_1e_2} \cup \{e_1\}) - f(\tilde{\omega}_{-e_1e_2} \cup \{e_2\}) + f(\tilde{\omega}_{-e_1e_2})$.

Proof : The proof is to apply to equation (22), steps similar to that in the proof of Proposition 3 once w.r.t. e_1 and then w.r.t. e_2. □

Like before, in order to evaluate the sign of the second order derivatives we partition $\Xi_{-e_1e_2}$ in a way that is useful in concluding the sign of $\Delta f_{e_1e_2}(\tilde{\omega}_{-e_1e_2})$. This corresponds to different possible ways of classifying the values that can be taken by $f(\tilde{\omega}_{-e_1e_2})$, $f(\tilde{\omega}_{-e_1e_2}\cup\{e_1\})$, $f(\tilde{\omega}_{-e_1e_2}\cup\{e_2\})$ and $f(\tilde{\omega}_{-e_1e_2}\cup\{e_1,e_2\})$. The functions $f(\tilde{\omega}_{-e_1e_2}\cup\{e_1\})$, $f(\tilde{\omega}_{-e_1e_2}\cup\{e_2\})$ and $f(\tilde{\omega}_{-e_1e_2}\cup\{e_1,e_2\})$ respectively, help to measure the effect of adding to the network $\tilde{\omega}_{-e_1e_2}$, link e_1, link e_2, and links e_1 and e_2 simultaneously. So it is possible that each of them independent of each other may or may not improve $f(\tilde{\omega}_{-e_1e_2})$. Hence we need to consider 8 (2^3) subsets. We further classify $\tilde{\omega}_{-e_1e_2}$ whether or not it is a feasible network realization we have 16 subsets. So we let $\Xi_{-e_1e_2}=$ $\Xi_{-e_1e_2}(1)\cup\Xi_{-e_1e_2}(2)\cdots\cup\Xi_{-e_1e_2}(16)$, where the sixteen subsets are mutually exclusive. The following proposition specifies a sufficient condition for $g_{e_1e_2}(y)$ to be greater than zero.

Proposition 4 *If there does not exist any path in G that contains both e_1 and e_2 then* $\forall y \in H$ *with* $y_{e_1}=0$ *and* $y_{e_2}=0$ $g_{e_1e_2}(y)>0$.

Proof: We use Table 2 in proving the proposition. The first column of the table indicates the set to which $\tilde{\omega}_{-e_1e_2}$ belongs. The rest of the column headings are self-explanatory. Sets 1 and 9 correspond to the case when either adding e_1 or e_2 or both e_1 and e_2 does not improve $f(\tilde{\omega}_{-e_1e_2})$. Sets 2 and 10 correspond to the case when adding e_1 alone improves $f(\tilde{\omega}_{-e_1e_2})$. Sets 3 and 11 correspond to the case when adding e_2 alone improves $f(\tilde{\omega}_{-e_1e_2})$. Sets 4 and 12 correspond to the case when adding either e_1 or e_2 improves $f(\tilde{\omega}_{-e_1e_2})$ but the simultaneous addition of e_1 and e_2 does not improve it any further. Sets 5 and 13 correspond to the case when only the simultaneous addition of e_1 and e_2 improves $f(\tilde{\omega}_{-e_1e_2})$. Sets 6 and 14 correspond to the case when the addition of e_1 and the simultaneous addition of e_1 and e_2 improves $f(\tilde{\omega}_{-e_1e_2})$. Sets 7 and 15 correspond to the case when the addition of e_1 and the simultaneous addition of e_1 and e_2 improves $f(\tilde{\omega}_{-e_1e_2})$. Sets 8 and 16 correspond to the case when the addition of e_1 or e_2 improves

$f(\tilde{\omega}_{-e_1e_2})$ and so also does the simultaneous addition of e_1 and e_2 but its effect being different from the earlier 2 cases.

Set No.	Probability of the set	$f(\tilde{\omega}_{-e_1e_2})$	$f(\tilde{\omega}_{-e_1e_2}\cup\{e_1\})$	$f(\tilde{\omega}_{-e_1e_2}\cup\{e_2\})$	$f(\tilde{\omega}_{-e_1e_2}\cup\{e_1,e_2\})$	$\Delta f_{e_1e_2}(\tilde{\omega}_{-e_1e_2})$
1	≥ 0	T_{11}	$T_{12} (= T_{11})$	$T_{13} (= T_{11})$	$T_{14} (= T_{11})$	0
2	≥ 0	T_{21}	$T_{22} (< T_{21})$	$T_{23} (= T_{21})$	$T_{24}(= T_{22})$	0
3	≥ 0	T_{31}	$T_{32} (= T_{31})$	$T_{33} (< T_{31})$	$T_{34}(= T_{33})$	0
4	≥ 0	T_{41}	$T_{42} (< T_{41})$	$T_{43} (< T_{41})$	$T_{44} = \min\{T_{42}, T_{43}\}$	> 0
5	0	T_{51}	$T_{52} (= T_{51})$	$T_{53} (= T_{51})$	$T_{54} (< T_{51})$	< 0
6	0	T_{61}	$T_{62} (< T_{61})$	$T_{63} (= T_{61})$	$T_{64} (<T_{62})$	< 0
7	0	T_{71}	$T_{72} (= T_{71})$	$T_{73} (< T_{71})$	$T_{74} (<T_{73})$	< 0
8	0	T_{81}	$T_{82} (< T_{81})$	$T_{83} (< T_{81})$	$T_{84} < \{T_{82}, T_{83}\}$	-
9	≥ 0	M	M	M	M	0
10	> 0	M	T_{102}	M	$T_{104} (= T_{102})$	0
11	> 0	M	M	T_{113}	$T_{114} (= T_{113})$	0
12	> 0	M	T_{122}	T_{123}	$T_{124} = \min\{T_{122}, T_{123}\}$	> 0
13	0	M	M	M	T_{134}	< 0
14	0	M	T_{142}	M	$T_{144} (<T_{142})$	< 0
15	0	M	M	T_{153}	$T_{154} (<T_{153})$	< 0
16	0	M	T_{162}	T_{163}	$T_{164} < \{T_{162}, T_{163}\}$	-

Table 2

The entries in columns 3, 4, 5 and 6 are possible values for the shortest path length from O to D in $\tilde{\omega}_{-e_1e_2}$. Since there is no path that contains e_1 and e_2, the effect of simultaneous addition of e_1 and e_2 to $\tilde{\omega}_{-e_1e_2}$ cannot be better than the best of effects due to addition of, e_1 or e_2. This explains the zero entry for sets, 5 to 8 and 13 to 16 under column 2. The greater than zero entry for sets 10, 11 and 12 under column 2 is due to Assumption 1 and the same is valid for set 12 under column 7 due to Assumption 2. Applying Lemma 6 the proof of the proposition is complete. □

$g_{e_1e_2}(y) > 0 \ \forall \ y \in H$ with $y_{e_1} = 0$ and $y_{e_2} = 0$ means that the interaction between simultaneous investment in the links e_1 and e_2, is not complementary for any investment decision in the remaining links. The following proposition specifies a sufficient condition for $g_{e_1e_2}(y)$ to be lesser than zero.

Proposition 5 *If e_1 and e_2 are consecutive links in a path from O to D in G then $\forall y \in H$ with $y_{e_1} = 0$ and $y_{e_2} = 0$ $g_{e_1e_2}(y) < 0$.*

Proof: The proof is by employing the same tabular approach as in Proposition 4, as reported in Table 3. We only justify the entries in column 2 for the sets 4, 12 and 13, and for those in column 7 for sets 8 and 16. The remaining entries are straightforward. The probability that set 13 is strictly greater than zero is due to Assumption 1. The justification is similar for the entries in column 2 for the sets 4 and 12, and so too in column 7 for sets 8 and 16. Hence we are required to justify only for the sets, 4 and 8 separately.

Set No.	Probability of the set	$f(\tilde{\omega}_{-e_1e_2})$	$f(\tilde{\omega}_{-e_1e_2} \cup \{e_1\})$	$f(\tilde{\omega}_{-e_1e_2} \cup \{e_2\})$	$f(\tilde{\omega}_{-e_1e_2} \cup \{e_1, e_2\})$	$\Delta f_{e_1e_2}(\tilde{\omega}_{-e_1e_2})$
1	≥ 0	T_{11}	$T_{12} (= T_{11})$	$T_{13} (= T_{11})$	$T_{14} (= T_{11})$	0
2	≥ 0	T_{21}	$T_{22} (< T_{21})$	$T_{23} (= T_{21})$	$T_{24} (= T_{22})$	0
3	≥ 0	T_{31}	$T_{32} (= T_{31})$	$T_{33} (< T_{31})$	$T_{34} (= T_{33})$	0
4	0	T_{41}	$T_{42} (< T_{41})$	$T_{43} (< T_{41})$	$T_{44} = \min\{T_{42}, T_{43}\}$	> 0
5	≥ 0	T_{51}	$T_{52} (= T_{51})$	$T_{53} (= T_{51})$	$T_{54} (< T_{51})$	< 0
6	≥ 0	T_{61}	$T_{62} (< T_{61})$	$T_{63} (= T_{61})$	$T_{64} (< T_{62})$	< 0
7	≥ 0	T_{71}	$T_{72} (= T_{71})$	$T_{73} (< T_{71})$	$T_{74} (< T_{73})$	< 0
8	≥ 0	T_{81}	$T_{82} (< T_{81})$	$T_{83} (< T_{81})$	$T_{84} < \{T_{82}, T_{83}\}$	≤ 0
9	≥ 0	M	M	M	M	0
10	≥ 0	M	T_{102}	M	$T_{104} (= T_{102})$	0
11	≥ 0	M	M	T_{113}	$T_{114} (= T_{113})$	0
12	0	M	T_{122}	T_{123}	$T_{124} = \min\{T_{122}, T_{123}\}$	> 0
13	> 0	M	M	M	T_{134}	< 0
14	≥ 0	M	T_{142}	M	$T_{144} (< T_{142})$	< 0
15	≥ 0	M	M	T_{153}	$T_{154} (< T_{153})$	< 0
16	≥ 0	M	T_{162}	T_{163}	$T_{164} < \{T_{162}, T_{163}\}$	< 0

Table 3

(i) Justification that $P(\Xi_{-e_1e_2}(4)) = 0$

Assume on the contrary that $\Xi_{-e_1e_2}(4) \neq \phi$. So there exists a $\tilde{\omega}_{-e_1e_2} \in \Xi_{-e_1e_2}(4)$. Let n_0 be the node which e_1 enters and e_2 leaves (see Figure 2). Let T(O, n_1) and T(O, n_3) be the shortest distance in $\tilde{\omega}_{-e_1e_2}$, from O to, n_1 and n_3 respectively. Let T(n_2, D) and T(n_4, D) be the shortest distance in $\tilde{\omega}_{-e_1e_2}$, to D, n_2 and n_4 respectively. We then have the following expressions.

$T_{41} \leq \mathrm{T}(O, n_3) + t_{e_3} + t_{e_4} + \mathrm{T}(n_4, D)$

$T_{42} = \mathrm{T}(O, n_1) + t_{e_1} + t_{e_4} + \mathrm{T}(n_4, D)$

$T_{43} = \mathrm{T}(O, n_3) + t_{e_3} + t_{e_2} + \mathrm{T}(n_2, D)$

Assume w.l.o.g. that min { T_{42}, T_{43}} = T_{42} so that $T_{44} = \mathrm{T}(O, n_1) + t_{e_1} + t_{e_4} + \mathrm{T}(n_4, D)$. Using the fact that $T_{43} < T_{41}$ we have $t_{e_2} + \mathrm{T}(n_2, D) < t_{e_4} + \mathrm{T}(n_4, D)$ we find that there is a path in $\tilde{\omega}_{-e_1e_2} \cup \{e_1, e_2\}$ from O to D with distance $\mathrm{T}(O, n_1) + t_{e_1} + t_{e_2} + \mathrm{T}(n_2, D) < T_{44}$. This is a contradiction.

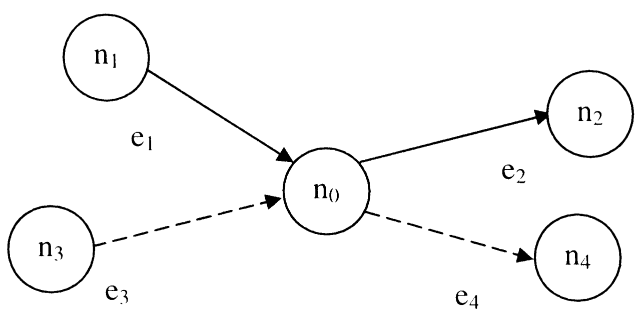

Figure 2

(ii) Justification that $\Delta f_{e_1e_2}(\tilde{\omega}_{-e_1e_2}) \leq 0 \ \forall \ \tilde{\omega}_{-e_1e_2} \in \Xi_{-e_1e_2}(8)$

Using the same notation as introduced in the previous part we have the following expressions.

$T_{41} \leq \mathrm{T}(O, n_3) + t_{e_3} + t_{e_4} + \mathrm{T}(n_4, D)$

$T_{42} = \mathrm{T}(O, n_1) + t_{e_1} + t_{e_4} + \mathrm{T}(n_4, D)$

$T_{43} = \mathrm{T}(O, n_3) + t_{e_3} + t_{e_2} + \mathrm{T}(n_2, D)$

$T_{44} = \mathrm{T}(O, n_1) + t_{e_1} + t_{e_4} + \mathrm{T}(n_4, D)$

We now have $T_{41} - T_{42} - T_{43} + T_{44} \leq 0$.

Applying Lemma 6 the proof of the proposition is complete. □

$g_{e_1e_2}(y) < 0 \ \forall \ y \in H$ with $y_{e_1} = 0$ and $y_{e_2} = 0$ means that the interaction between simultaneous investment in the links e_1 and e_2, is complementary for any investment decision in the remaining links. The sign of the coefficient $g_{e_1e_2}(y)$ for links e_1 and e_2 that do not satisfy the condition in Proposition 4 nor that in Proposition 5 depend on the network structure and the parameters of its links. Establishing the sign of higher order ($\geq$ 3) coefficients is complicated as the number of partitions required to analyze it would be exponential in the order being analyzed. Also the number of effects that needs to be considered is also combinatorial in nature, hence we do not consider any higher order degree.

5 APPROXIMATE SOLUTION PROCEDURE

In this section we discuss an approximate solution procedure to **P**. In Subsection 5.1 we discuss the approach and basis for the procedure, in which we minimize a first order approximation of the multilinear function for F(y). It also includes Theorem 3 which is the principal reult of this paper. Subsequently we illustrate how this is useful in deriving a lower bound as well. In Subsection 5.2, we employ this approach in an iterative scheme akin to methods using stochastic subgradients.

5.1 First Order Approximation for F(*y*)

Problem **P** can be shown to be NP complete by transforming a $|E|$-SAT problem to an equivalent problem given by (16). Given the complex nature of the problem, the key idea in our solution procedure to **P** is to approximate F(y) using the first order terms of the multilinear function given by expression (15) and solve the resulting problem.

$$\mathrm{F}(y) \approx \mathrm{F}^{(1)}(y \mid y_0) = \mathrm{F}(y_0) + \sum_{e \in E} g_e(y_0)\,(\, y_e - y_{0e}\,) \tag{24}$$

By disregarding the second and other higher order terms we do not capture the effect of interactions, arising due to simultaneous investment in more than one link. The first order approximation makes the decision variables separable, thereby easing the complexity of **P**. The resulting objective function is linear in the components of y. Now let $y_0 = \mathbf{0}$ and denote $F^{(1)}(y \mid y_0 = \mathbf{0})$ by $F^{(1)}(y)$. Also let $\tilde{y} = \underset{\forall y \in Y}{\arg\min}\, F^{(1)}(y) = \underset{\forall y \in Y}{\arg\min} \sum_{e \in E} g_e(\mathbf{0})\; y_e$. Thus $\tilde{y}$ is easily obtainable by solving a 0-1 knapsack problem. Now we have the following property of $\tilde{y}$.

Proposition 6 *$\tilde{y}$ is a strict local optima of $F(y)$.*

Proof : $\tilde{y}$ is an extreme point of the unit hypercube H. On the contrary assume that this solution is not a local optima of **P**, then there is a feasible extreme point solution, y^a neighboring $\tilde{y}$, such that $F(y^a) < F(\tilde{y})$. The Hamming distance between $\tilde{y}$ and any of its neighboring solutions is 1 viz. $|y^a - \tilde{y}| = 1$. Since $F(y)$ is monotone decreasing with y, we can conclude that $y^a > \tilde{y}$. Since , $g_e(\mathbf{0}) < 0 \;\forall\; e \in E$ following Proposition 3, it then leads us to the conclusion that y^a is the optimal solution to the first order approximation problem as $F^{(1)}(y^a) < F^{(1)}(\tilde{y})$. This is a contradiction and the assumption that y^a is a feasible solution is incorrect. □

As a consequence of this proposition any traditional local search procedures starting with $\tilde{y}$ cannot be applied. In next section we show how this approximation idea can be extended to an iterative scheme to improve upon the solution, if possible.

5.2 Extension Using Stochastic Subgradients

Here we describe an iterative solution method to solve **P** by extending the first order approximation idea to the concept of stochastic subgradient. The assumption made is that the multilinear function can be extrapolated beyond H, even though it could be physically inappropriate. A stochastic subgradient of a function (stochastic and nondifferentiable) at a point provides a first order approximation to the function in a neighborhood of that point. The stochastic quasigradient (SQG) method is not applicable here as it requires that the feasible solution set be convex and compact, so as to solve the projection problem [Ermoliev, 1988]. The solution set Y, is compact but not convex. A continuous relaxation of the variables is also not useful as the solutions are not in the space of general integer variables. Using a rounding procedure to obtain a solution in the state space of binary variables from a fractional solution during the iterations may not even be feasible to the problem. We develop a solution by building upon the first order approximation for F(y), developed in Section 5.1. Our solution strategy is to use the first order derivative information in a search procedure. This method is quite similar to the stochastic linearization method which replaces a nonlinear stochastic objective function with a linear one.

The basic algorithm is outlined below.

Step 1. Set $j = 1$, $y^j = y_0$, a feasible solution and set $V = \{\varnothing\}$.

Step 2. Obtain $g(y^j)$.

Step 3. Solve for $y^{j+1} = \arg\min_{\forall y \in Y} F^{(1)}(y \mid y^j)$.

Step 4. If $y^j \notin V$ then set $V = V \cup \{y^j\}$ and, set $j = j + 1$, goto step 2.

Step 5. Output $\tilde{y} = \arg\min_{\forall y \in V} F(y)$

The final solution obtained from the algorithm is to a large extent dependent on the initial solution y_0. Again as before, it suffices to solve $\arg\min_{\forall y \in Y} \sum_{e \in E} g_e(y^j)\, y_e$ to solve for y^{j+1}, which is a knapsack problem. It is important to note that the final solution of this algorithm may bear no relation to even a local solution of the original objective function, F(y).

It is likely, that the exact evaluation of $g(y^k)$ in step 2 and F(y) in step 5 is computationally impractical due to the possibility of a large number of network realizations. So an alternative approach is to estimate these values by their sample mean. Now, let ξ^n $\forall\, n \in \{1, 2, ..., N\}$, be independent realizations of the random vector, ξ, and sampled from the probability distribution associated with investment decision y i.e. with probability $P(\xi = \xi^n \mid y)$. If N is reasonably large, we then have from the law of large numbers

$$g_e(y) \approx \overline{g_e(y)} = \{ \frac{1}{N(S)} \sum_{n=1}^{N} I(\xi^n) T(\pi_{k(\xi^n)}) \frac{\xi_e^n [\Delta p_e] + (1-\xi_e^n)[-\Delta p_e]}{P(\xi_e^n \mid y_e)} \} \{ \frac{1}{N} \sum_{n=1}^{N} I(\xi^n) \}$$

$$- M\{ \frac{1}{N} \sum_{n=1}^{N} I(\xi^n) \frac{\xi_e^n [\Delta p_e] + (1-\xi_e^n)[-\Delta p_e]}{P(\xi_e^n \mid y_e)} \} \{ \frac{1}{N} \sum_{n=1}^{N} I(\xi^n) \} \text{ and}$$

$F(y) \approx \overline{F(y)} = \frac{1}{N} \sum_{n=1}^{N} f(\xi^n)$. Here $I(\xi^n)$ is an indicator function that assumes the value 1, if $X(\xi^n) \neq \varnothing$, 0 otherwise and $k(\xi^n) = \arg_{k \in K} \{ I(k, \xi) = 1 \}$.

The iterates y^j, obtained could be fluctuating and could change abruptly from iteration to iteration as we do not use any smoothing scheme for the stochastic subgradient.

6 COMPUTATIONAL STUDY AND INSIGHTS

In this section, we report our computational study and discuss the insights obtained. The goal of this study is two-fold 1) to evaluate the performance of the proposed solution procedure using a numerical example 2) gain insights based on sensitivity analysis with respect to problem parameters. The stochastic subgradient method is used because as mentioned before, the solution from this method is at least as good as the solution obtained from merely solving a single step of the first order approximation. Our study is based on a five link network as depicted in Figure 3. Instances were generated by varying the different problem parameters, such as link lengths, the link survival probabilities with and without investment, the value of the penalty cost, M. The parameters were chosen so as to reflect representative cases. The results of the study are reported in Tables 4a, 4b, 5a and 5b. The performance of the solution procedure is compared with the optimal solution obtained using brute force technique.

In the Tables the problem instance is described by the B, the p_e, q_e, c_e, t_e values $\forall$ e = 1, 2, 3, 4, 5 and the M value. The number of the problem instance is indicated by the first column. The following columns are also shown. y_{opt}, the optimal investment solution vector, $F(y_{opt})$ is the optimal objective function value which is the minimum expected shortest path length, $F_1(y_{opt})$ is the expected shortest path length over feasible realizations, corresponding to the optimal investment y_{opt}, and $Rel(y_{opt})$ is the O-D reliability corresponding to the optimal solution y_{opt}. Finally, the last four columns show the items just described, for the proposed solution procedure, indexed by s. For Tables 4a and 4b the initial survival probability vector, $(p_1\ \ p_2\ \ p_3\ \ p_4\ \ p_5)$ = (0.7, 0.7, 0.7, 0.7, 0.7). The following discussion pertains to Tables 4a and 4b, a similar discussion holds good when the initial survival probability vector, $(p_1\ \ p_2\ \ p_3\ \ p_4\ \ p_5)$ is (0.6, 0.6, 0.6, 0.6, 0.6) whose results are reported in Tables 5a and 5b. We set M as $T_{\max}$ + 1 in all of the instances except for those that are used for analyzing sensitivity w.r.t. the parameter M. As seen from the tables for all of the instances except two the solution procedure yielded the optimal solution. In fact, for almost all of the cases, the optimal solution was obtained in the first step of the stochastic subgradient method. This seems to suggest that the first order approximation can be used to obtain good solutions. Now we derive some insights based on the sensitivity analysis.

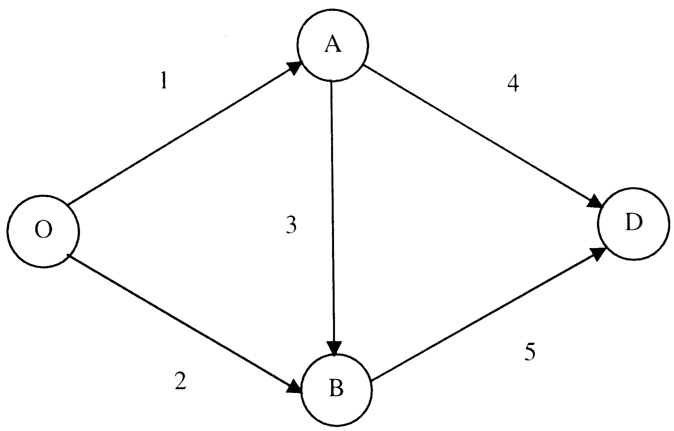

Figure 3 Five link test network

The general observation is that the optimal investment tended to make the paths of shorter length more reliable. Also links 1 and 5 appear in the optimal solution for most cases, this is because there are two paths that use them as opposed to just one path for the other 3 links. The remaining links appear in exactly 1 path.

(i) Effect of changing B

To understand this effect, we compare the results of No. 1 and No.2, which differ only by the parameter B. When the B is increased from 2 to 3, $F(y_{opt})$ decreased by 0.2806, in accordance with Theorem 2. Reliability increased by 0.0286 and $F_1(y_{opt})$ increased by 0.0295.

(ii) Effect of changing q_e

To understand this effect we compare the optimal solutions of No. 3 and No. 4. Let $y(3)$ = (1 1 0 0 1) and $y(4)$ = (0 1 1 0 1) respectively. This is because the increase in q_3 from 0.9 from 0.8 made link 3 more beneficial for investing in comparison with link 1. Note here that c_3 is unchanged. Using expression (15), $F(y(4)) - F(y(3)) = \{ g_2(0) + g_3(0) + g_5(0) +$

$g_{2,3}(0) + g_{2,5}(0) + g_{3,5}(0) + g_{2,3,5}(0)$ } - { $g_1(0) + g_2(0) + g_5(0) + g_{1,2}(0) + g_{1,5}(0) + g_{2,5}(0) + g_{1,2,5}(0)$ } = $G_1 \Delta p_3 - G_2$ (from Corollary 1), where G_1 (<0) and G_2 (<0) are appropriate constants. So when $\Delta p_3 = 0.2$ F(y(4)) - F(y(3)) > 0 and $\Delta p_3 = 0.3$ F(y(4)) - F(y(3)) < 0.

(iii) Effect of changing c_e

To understand this effect we compare No. 3 with Nos. 5, 6, 7, 8 and 9. For each of these instances c_e has been increased from 1 to 2 for e = 1, 2, 3, 4 and 5 respectively. As expected, increasing c_e for $e = 3$ or 4 does not alter the optimality of the original solution. For the remaining instances the optimal solution is different from that of No. 3 and this is because the original optimal solution would be infeasible, as it requires a budget of 4. We can conclude that link 2 and link 5 are very important as they have invested in, for all the cases. The reason being, these two links constitute the path with the least length and also because link 5 appears in two paths.

(iv) Effect of changing M

To understand this effect we compare No. 10 with No. 11 and No. 12. For $31 \leq M < 43.9$, the stochastic subgradient method outputs the optimal solution. For $43.9 \leq M < 57.3$, the solution from the stochastic subgradient method differs from the optimal solution, which stays the same. For $M \geq 57.3$, the optimal solution is altered and is now the same as that of the solution from the stochastic subgradient method. This solution maximizes the reliability for the given budget. As M is increased, the importance for minimizing unreliability is greater and hence investment decisions tend to maximize reliability. The transition in solution occurs in a discrete manner and occurs at breakpoint(s), as discussed above. This is valid for both the optimal solution as well as the solution from the subgradient method, although breakpoint could be different as seen here. In certain cases no such breakpoint might exist because the current M value yields a solution that already maximizes reliability.

(v) Effect of changing t_e

To understand this effect we compare No. 13 with No. 14. When t_3 is decreased from 5 to 1, effectively the investment of link 2 has been transferred to link 3. This is because with lower path length is usually favored for investment. Note that while the reliability worsened from 0.86848 to 0.85248, $F_1(y_{opt})$ improved from 25 to 22.5976, which means that the worsening of the former is outweighed by the improvement in the latter.

7 CONCLUDING COMMENTS

In this paper, we introduced the problem of investing in the links of a stochastic network in order to minimize the expected shortest path length for an O-D pair. A penalty cost was introduced to handle network realizations in the expectation, where the O-D pair is disconnected. Thus connectivity is considered in the objective function in addition to path length. The analysis of the problem was enabled by the explicit closed-form expression for the probability distribution of the network states in terms of the investment decisions (equation (9)).The problem is modeled as a two-stage stochastic program. The notable feature of the model being, the probability distribution of the random variables is dependent on the first-stage decision variables. The network structure was exploited to reformulate the problem as an equivalent deterministic program. By temporarily relaxing the investment decision variables and applying Taylor series expansion we are able to evaluate the coefficients of the multilinear objective function in an efficient manner. We illustrate some of the properties of these coefficients. We have also illustrated the monotone decreasing property of $F(y)$. Our proposed solution procedure approximates the objective function using only the first order terms of the multilinear function. Numerical experiments on a small-sized network show good performance on the problem instances solved.

Potential applications domains for the framework presented in this paper apart from disaster management for transportation networks, include network protection against potential terrorist attacks. The network could be a rail network, power network or oil/gas pipeline network.

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$p_e = 0.7 \; \forall \; e \in E$

Table 4a Summary of computational results

B = 2

No.	q_1 q_2 q_3 q_4 q_5	c_1 c_2 c_3 c_4 c_5	t_1 t_2 t_3 t_4 t_5	M	y_{opt}	$F(y_{opt})$	$F_1(y_{opt})$	$Rel(y_{opt})$	y_s	$F(y_s)$	$F_1(y_s)$	$Rel(y_s)$
1	{0.8,0.8,0.8,0.8,0.8}	{1,1,1,1,1}	{10,10,10,10,10}	31	{01001}	21.9961	20.28	0.83992	{10001}	22.0893	20.4762	0.84672

Table 4b Summary of computational results

B = 3

No.	q_1 q_2 q_3 q_4 q_5	c_1 c_2 c_3 c_4 c_5	t_1 t_2 t_3 t_4 t_5	M	y_{opt}	$F(y_{opt})$	$F_1(y_{opt})$	$Rel(y_{opt})$	y_s	$F(y_s)$	$F_1(y_s)$	$Rel(y_s)$
2	{0.8,0.8,0.8,0.8,0.8}	{1,1,1,1,1}	{10,10,10,10,10}	31	{11001}	21.7155	20.3095	0.86848	{11001}	21.7155	20.3095	0.86848
3	{0.8,0.8,0.8,0.8,0.8}	{1,1,1,1,1}	{10,10,15,30,10}	41	{11001}	26.8835	24.7458	0.86848	{11001}	26.8835	24.7458	0.86848
4	{0.8,0.8,0.9,0.8,0.8}	{1,1,1,1,1}	{10,10,15,30,10}	41	{01101}	26.8494	24.2861	0.84664	{01101}	26.8494	24.2861	0.84664
5	{0.8,0.8,0.8,0.8,0.8}	{2,1,1,1,1}	{10,10,15,30,10}	41	{01101}	26.9087	24.2899	0.84328	{01101}	26.9087	24.2899	0.84328
6	{0.8,0.8,0.8,0.8,0.8}	{1,2,1,1,1}	{10,10,15,30,10}	41	{01001}	26.9681	24.2937	0.83992	{01001}	26.9681	24.2937	0.83992
7	{0.8,0.8,0.8,0.8,0.8}	{1,1,2,1,1}	{10,10,15,30,10}	41	{11001}	26.8835	24.7458	0.86848	{11001}	26.8835	24.7458	0.86848
8	{0.8,0.8,0.8,0.8,0.8}	{1,1,1,2,1}	{10,10,15,30,10}	41	{11001}	26.8835	24.7458	0.86848	{11001}	26.8835	24.7458	0.86848
9	{0.8,0.8,0.8,0.8,0.8}	{1,1,1,1,2}	{10,10,15,30,10}	41	{01001}	26.9681	24.2937	0.83992	{01001}	26.9681	24.2937	0.83992
10	{0.8,0.8,0.8,0.8,0.8}	{2,1,1,1,1}	{10,20,10,15,10}	31	{10010}	26.9601	26.1901	0.83992	{10010}	26.9601	26.1901	0.83992
11	{0.8,0.8,0.8,0.8,0.8}	{2,1,1,1,1}	{10,20,10,15,10}	43.9	{10010}	29.0251	26.1901	0.83992	{01011}	29.1838	26.7339	0.85728
12	{0.8,0.8,0.8,0.8,0.8}	{2,1,1,1,1}	{10,20,10,15,10}	57.3	{01011}	31.0963	26.7339	0.85728	{01011}	31.0963	26.7339	0.85728
13	{0.8,0.8,0.8,0.8,0.8}	{1,1,1,1,1}	{10,15,5,15,10}	26	{11001}	25.1315	25	0.86848	{11001}	25.1315	25	0.86848
14	{0.8,0.8,0.8,0.8,0.8}	{1,1,1,1,1}	{10,15,1,15,10}	26	{10101]	23.0995	22.5976	0.85248	{10101]	23.0995	22.5976	0.85248

$p_e = 0.6 \ \forall \ e \in E$

Table 5a Summary of computational results

B = 2

No.	q_1 q_2 q_3 q_4 q_5	c_1 c_2 c_3 c_4 c_5	t_1 t_2 t_3 t_4 t_5	M	y_{opt}	$F(y_{opt})$	$F_1(y_{opt})$	$Rel(y_{opt})$	y_s	$F(y_s)$	$F_1(y_s)$	$Rel(y_s)$
1	{0.8,0.8,0.8,0.8,0.8}	{1,1,1,1,1}	{10,10,10,10,10}	31	{10010}	22.5114	20.2907	0.79264	{10001}	22.913	20.7767	0.79104

Table 5b Summary of computational results

B = 3

No.	q_1 q_2 q_3 q_4 q_5	c_1 c_2 c_3 c_4 c_5	t_1 t_2 t_3 t_4 t_5	M	y_{opt}	$F(y_{opt})$	$F_1(y_{opt})$	$Rel(y_{opt})$	y_s	$F(y_s)$	$F_1(y_s)$	$Rel(y_s)$
2	{0.8,0.8,0.8,0.8,0.8}	{1,1,1,1,1}	{10,10,10,10,10}	31	{11001}	22.0285	20.3642	0.84352	{11001}	22.0285	20.3642	0.84352
3	{0.8,0.8,0.8,0.8,0.8}	{1,1,1,1,1}	{10,10,15,30,10}	41	{11001}	26.9725	24.3703	0.84352	{11001}	26.9725	24.3703	0.84352
4	{0.8,0.8,0.9,0.8,0.8}	{1,1,1,1,1}	{10,10,15,30,10}	41	{01101}	26.9638	23.5456	0.80416	{01101}	26.9638	23.5456	0.80416
5	{0.8,0.8,0.8,0.8,0.8}	{2,1,1,1,1}	{10,10,15,30,10}	41	{01101}	27.0157	23.5266	0.80032	{01101}	27.0157	23.5266	0.80032
6	{0.8,0.8,0.8,0.8,0.8}	{1,2,1,1,1}	{10,10,15,30,10}	41	{01001}	27.1194	23.4881	0.79264	{01001}	27.1194	23.4881	0.79264
7	{0.8,0.8,0.8,0.8,0.8}	{1,1,2,1,1}	{10,10,15,30,10}	41	{11001}	26.9725	24.3703	0.84352	{11001}	26.9725	24.3703	0.84352
8	{0.8,0.8,0.8,0.8,0.8}	{1,1,1,2,1}	{10,10,15,30,10}	41	{11001}	26.9725	24.3703	0.84352	{11001}	26.9725	24.3703	0.84352
9	{0.8,0.8,0.8,0.8,0.8}	{1,1,1,1,2}	{10,10,15,30,10}	41	{01001}	27.1194	23.4881	0.79264	{01001}	27.1194	23.4881	0.79264
10	{0.8,0.8,0.8,0.8,0.8}	{2,1,1,1,1}	{10,20,10,15,10}	31	{10010}	27.0074	25.9629	0.79264	{10010}	27.0074	25.9629	0.79264
11	{0.8,0.8,0.8,0.8,0.8}	{2,1,1,1,1}	{10,20,10,15,10}	40.1	{10010}	28.8943	25.9629	0.79264	{01011}	29.3744	27.0885	0.82432
12	{0.8,0.8,0.8,0.8,0.8}	{2,1,1,1,1}	{10,20,10,15,10}	55.3	{01011}	32.0447	27.0885	0.82432	{01011}	32.0447	27.0885	0.82432
13	{0.8,0.8,0.8,0.8,0.8}	{1,1,1,1,1}	{10,15,5,15,10}	26	{11001}	25.1565	25	0.84352	{11001}	25.1565	25	0.84352
14	{0.8,0.8,0.8,0.8,0.8}	{1,1,1,1,1}	{10,15,1,15,10}	26	{10101}	23.1405	22.4763	0.81152	{10101}	23.1405	22.4763	0.81152