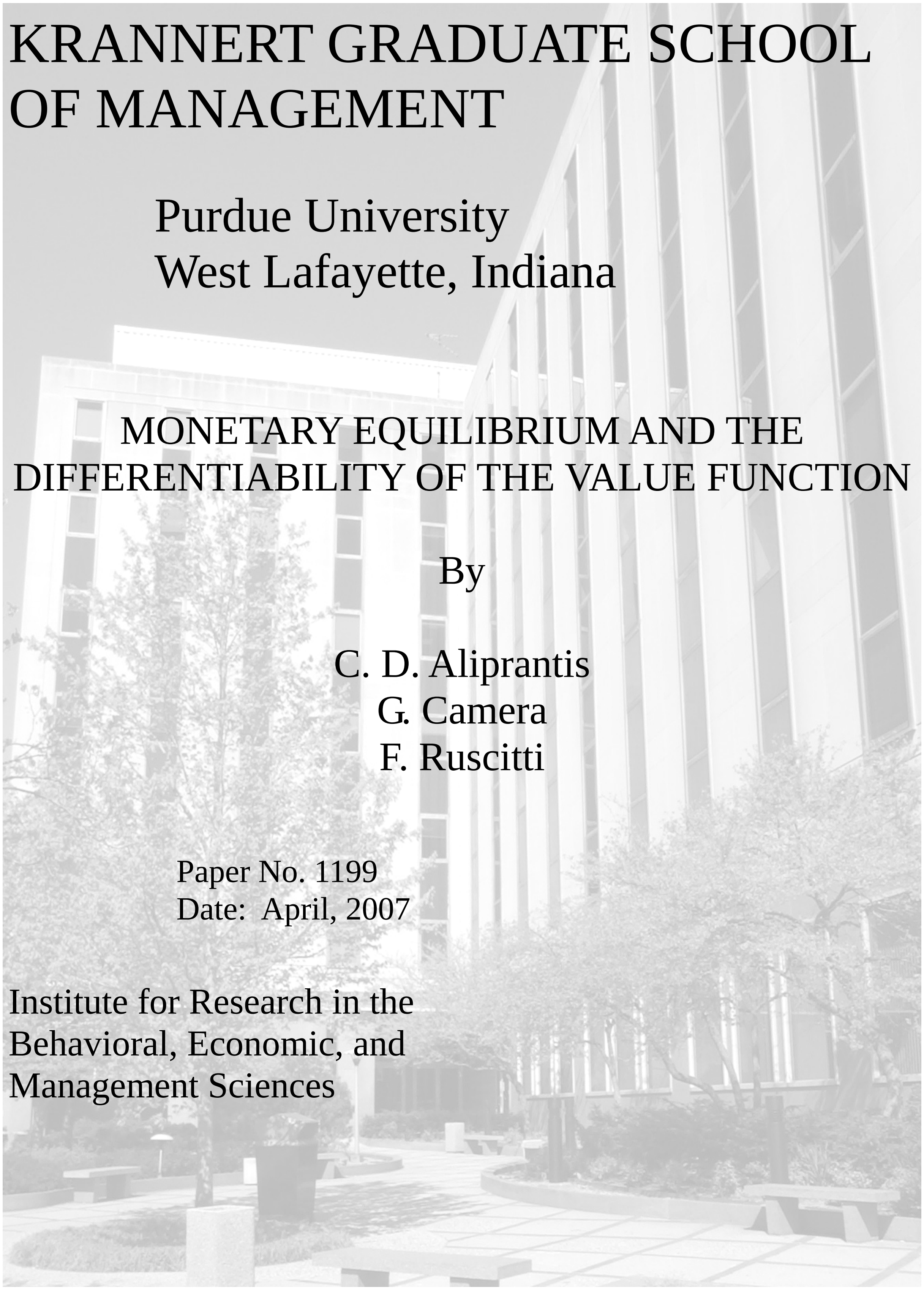

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MONETARY EQUILIBRIUM AND THE DIFFERENTIABILITY OF THE VALUE FUNCTION

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MONETARY EQUILIBRIUM AND THE DIFFERENTIABILITY OF THE VALUE FUNCTION★

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Abstract. In this study we offer a new approach to proving the differentiability of the value function, which complements and extends the literature on dynamic programming. This result is then applied to the analysis of equilibrium in the recent class of monetary economies developed in [13]. For this type of environments we demonstrate that the value function is differentiable and this guarantees that the marginal value of money balances is well defined.

JEL classification: E00, C61

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1. Introduction

This study develops two complementary objectives. First, it presents a new approach to proving the differentiability of the value function from dynamic programming. This initial result is then applied to the study of equilibrium for the recent class of monetary economies developed from the work in [13]. These economies present a natural example, but our differentiability result is of general applicability so our study contributes to a much wider literature in dynamic programming and macroeconomics.

To understand the thrust of our contribution, one should recall that in the typical dynamic economy the optimum lifetime utility of an agent is known as the value function. This function satisfies the classical Bellman equation and, in order to characterize and compute optima, it is often important to establish its differentiability. The common reference on this issue is the classical monograph [19], where Theorem 4.11 presents a set of conditions for the differentiability of the

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value function.[1] One of the conditions in this theorem is that, given the initial state x_0, the value of the policy function g must be in the interior of the upper section of the constraint correspondence Γ, i.e., $g(x_0) \in \operatorname{int}\Gamma(x_0)$. One can imagine situations in which the value of the policy function is on the boundary of Γ. For instance, this may occur when agents are constrained in their ability to save or to borrow, as it happens in several monetary models. An additional difficulty is associated to the statement, in the proof of the theorem, that "since $g(x_0) \in \operatorname{int}\Gamma(x_0)$ and Γ is continuous, it follows that $g(x_0) \in \operatorname{int}\Gamma(x)$, for all x in some neighborhood of x_0;" see [19, p. 85]. According to the assumptions imposed in [19], this claim does not seem to be substantiated. However, if the point $(x_0, g(x_0))$ is an interior point of the graph of the constraint correspondence Γ, then the claim is true. Indeed, in general, in most macroeconomic models this is the case.

The observations above motivate the first objective of this study, i.e., to develop a novel formulation of the differentiability of the value function. Our formulation is based on a proof that bypasses the two difficulties noted above by making a less restrictive assumption. The key requisite is that the initial state x_0 be an interior point of the lower section of Γ at $g(x_0)$, i.e., $x_0 \in \operatorname{int}\Gamma^{-1}(g(x_0))$. We then present a application of this result to a model of money based on [13]. In the model, anonymous individuals face idiosyncratic trading shocks, sometimes being buyers and other times being sellers. This feature ensures an explicit role for money, but also causes individual money balances to evolve stochastically, with obvious drawbacks in analytical tractability (e.g., see [9, 11, 16]). These are avoided by assuming quasilinear preferences, which eliminate of wealth effects and open the door to an equilibrium with a degenerate distribution of money holdings. In equilibrium everyone chooses identical savings and the optimal savings plan corresponds to a boundary solution in which agents hold equal shares of the money stock.

The use of dynamic programming in such an economy presents two complications. First, quasilinearity leads to a period return function that is unbounded in the available real balances; this contrasts with the standard assumption in dynamic programming of a bounded return function (see [19]). Second, one cannot apply the standard proof for the differentiability of the value function from [19], because individual savings lie on the boundary of the graph of the constraint correspondence. Indeed, though this monetary literature normally supposes that the value function is differentiable, we are not aware of any specific paper establishing this differentiability. We solve the first complication by restricting the value function on bounded intervals. We solve the second problem by applying our differentiability result.

We structure the paper as follows. In Section 2 we start by describing a typical representative agent economy, which is the basic framework found in [19]. In Section 3 we introduce the value function and present simple proofs of two of its basic properties. Section 4 contains our main result

[1]There are other works on this subject, for example see [5, 7, 8, 10, 14, 15, 18].

concerning the differentiability of the value function. Finally, Section 5 presents an application of our proof of the differentiability of the value function to the model of money developed in [13].

2. The Basic Dynamic Framework

The standard dynamic framework consists of an infinite horizon representative agent economy, where the representative agent can consume, produce, and save by accumulating some assets. In what follows we consider a deterministic framework where time is discrete, i.e., there are countably many periods labeled as $t = 0, 1, 2, \ldots$. In each period t the agent must make a choice from a given opportunity set X, the elements of which define the **states** of the economy and can be interpreted as the stock of real assets or capital available in a given period. In making this choice the agent faces a constraint that is described by a nonempty-valued "constraint" correspondence $\Gamma\colon X \twoheadrightarrow X$. Given that the state at the beginning of a period is x, the set $\Gamma(x)$ is the set of feasible states at the end of the period.

As usual the graph of the correspondence Γ is denoted by G_Γ, i.e.,

$$G_\Gamma = \{(x, y) \in X \times X\colon\ y \in \Gamma(x)\}\,.$$

It is assumed that there exists a function $F\colon G_\Gamma \to \mathbb{R}$ known as the **return** (or utility) **function**. The value $F(x, y)$ is interpreted as the payoff to the agent in a period that starts in state x and ends in state y.

A **plan** starting with $x_0 \in X$ is a sequence $\boldsymbol{x} = (x_0, x_1, x_2, \ldots)$ such that $x_{t+1} \in \Gamma(x_t)$ holds for each $t = 0, 1, 2, \ldots$. The state of the economy at the beginning of period $t + 1$ is x_{t+1} which is chosen by the agent on period t given that the state at the beginning of period t was x_t. The collection of all plans starting with x_0 is denoted $\Pi(x_0)$. That is,

$$\Pi(x_0) = \big\{\boldsymbol{x} = (x_0, x_1, x_2, \ldots) \in X^{\{0,1,2,\ldots\}}\colon\ x_{t+1} \in \Gamma(x_t)\ \text{ for all }\ t = 0, 1, 2, \ldots\big\}\,.$$

Since Γ is nonempty-valued, notice that $\Pi(x_0)$ is a nonempty set for each $x_0 \in X$.

It is assumed that agents have time-separable preferences and they discount future utility by a factor β, where $0 < \beta < 1$. This means that given an initial state x_0, the agent's lifetime utility is given by

$$U(\boldsymbol{x}) = \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})\,.$$

In order to make $U(\boldsymbol{x})$ well-defined, we impose the following convergence condition.

For each plan $\boldsymbol{x}$ *the series* $\sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$ *converges in* $\mathbb{R}$. **(C)**

Note that for any given initial state x_0 the above formula defines a function $U\colon \Pi(x_0) \to \mathbb{R}$. Now given an initial state x_0, the agent must choose a plan $\boldsymbol{x} \in \Pi(x_0)$ that maximizes his lifetime utility. In other words, given $x_0 \in X$, the agent must solve the optimization problem:

$$\begin{aligned} &\textit{Maximize:} \quad U(\boldsymbol{x}) \\ &\textit{Subject to:} \quad \boldsymbol{x} \in \Pi(x_0) \end{aligned} \tag{$\mathbf{P}$}$$

Any plan that solves the optimization problem ($\mathbf{P}$) is called an **optimal plan**. When condition ($\mathbf{C}$) is satisfied, a function—known as the *value function*— can be defined, which is associated with the optimization problem ($\mathbf{P}$).

3. The Value Function

In this section we first recall the definition of the value function and then state (with proofs) two of its basic properties.

Definition 3.1. *Assuming that condition* ($\mathbf{C}$) *holds true, the function* $v\colon X \to (-\infty, \infty]$ *defined for each* $x_0 \in X$ *by*

$$v(x_0) = \sup_{\boldsymbol{x} \in \Pi(x_0)} U(\boldsymbol{x})$$

is called the ***value function****.*

In what follows we establish two well known basic properties of the value function that are needed to carry out our work. Specifically, the value function satisfies the functional equation known as the Bellman equation (see [6]) and it is concave. We include the proofs of these statements both for completeness and because they differ from the standard proofs in [19].

Lemma 3.2. *The value function* v *is a solution of the Bellman equation. That is, for each* $x \in X$ *we have*

$$v(x) = \sup_{y \in \Gamma(x)} \big[F(x,y) + \beta v(y)\big]\,.$$

Proof. We must show that inequalities

$$v(x) \le \sup_{y \in \Gamma(x)} \big[F(x,y) + \beta v(y)\big] \quad \text{and} \quad v(x) \ge \sup_{y \in \Gamma(x)} \big[F(x,y) + \beta v(y)\big]\,.$$

are both true.

First, fix some $\boldsymbol{x} = (x_0, x_1, x_2, \ldots) \in \Pi(x_0)$ and note that the plan $\boldsymbol{z} = (x_1, x_2, x_3, \ldots)$ belongs to $\Pi(x_1)$. Now note that

$$\begin{aligned} U(\boldsymbol{x}) &= \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) = F(x_0, x_1) + \beta \sum_{t=1}^{\infty} \beta^{t-1} F(x_t, x_{t+1}) \\ &= F(x_0, x_1) + \beta \sum_{t=0}^{\infty} \beta^t F(x_{t+1}, x_{t+2}) \\ &= F(x_0, x_1) + \beta U(\boldsymbol{z}) \leq F(x_0, x_1) + \beta v(x_1) \\ &\leq \sup_{y \in \Gamma(x_0)} \big[F(x_0, y) + \beta v(y)\big]\,. \end{aligned}$$

This easily implies

$$v(x_0) = \sup_{\boldsymbol{x} \in \Pi(x_0)} U(\boldsymbol{x}) \leq \sup_{y \in \Gamma(x_0)} \big[F(x_0, y) + \beta v(y)\big]\,. \tag{3.1}$$

We distinguish two cases, $v(x_0) = \infty$ and $v(x_0) \in \mathbb{R}$. If $v(x_0) = \infty$, then it should be obvious that $v(x_0) = \sup_{y \in \Gamma(x_0)} \big[F(x_0, y) + \beta v(y)\big] = \infty$.

Now we assume $v(x_0) \in \mathbb{R}$. Fix some $y \in \Gamma(x_0)$. Let $z_0 = y$ and then pick an arbitrary plan $\boldsymbol{z} = (z_0, z_1, z_2, \ldots) \in \Pi(y)$. Then $\boldsymbol{x} = (x_0, y, z_1, z_2, \ldots) \in \Pi(x_0)$ and so

$$F(x_0, y) + \beta U(\boldsymbol{z}) = F(x_0, y) + \beta \sum_{t=0}^{\infty} \beta^t F(z_t, z_{t+1}) = F(x_0, y) + \sum_{t=1}^{\infty} \beta^t F(z_{t-1}, z_t) = U(\boldsymbol{x}) \leq v(x_0)\,.$$

This implies that

$$U(\boldsymbol{z}) \leq \tfrac{v(x_0) - F(x_0, y)}{\beta}$$

for all $\boldsymbol{z} \in \Pi(y)$. In other words, the real number $\frac{v(x_0) - F(x_0, y)}{\beta}$ is an upper bound of the set $\{U(\boldsymbol{z})\colon\ \boldsymbol{z} \in \Pi(y)\}$. Therefore, $v(y) := \sup_{\boldsymbol{z} \in \Pi(y)} U(\boldsymbol{z}) \leq \frac{v(x_0) - F(x_0, y)}{\beta}$, from which we get

$$F(x_0, y) + \beta v(y) \leq v(x_0)$$

for all $y \in \Gamma(x_0)$. Hence

$$\sup_{y \in \Gamma(x_0)} \big[F(x_0, y) + \beta v(y)\big] \leq v(x_0)\,. \tag{3.2}$$

From (3.1) and (3.2), we see that $v(x_0) = \sup_{y \in \Gamma(x_0)} \big[F(x_0, y) + \beta v(y)\big]$ in this case too. ■

The second property that must be established is the concavity of the value function.

Theorem 3.3. *If the return function $F\colon G_\Gamma \to \mathbb{R}$ is concave,*[2] *then the value function v is concave.*

[2]This guarantees (by the definition of a concave function) that the graph G_Γ is a convex subset of $X \times X$.

Proof. Let $x_0, y_0 \in X$ and $0 < \alpha < 1$. The convexity of X implies $\alpha x_0 + (1-\alpha)y_0 \in X$. Fix $M, N \in \mathbb{R}$ with $v(x_0) > M$ and $v(y_0) > N$. Pick $\boldsymbol{x} \in \Pi(x_0)$ and $\boldsymbol{y} \in \Pi(y_0)$ such that

$$v(x_0) \geq \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) > M \quad \text{and} \quad v(y_0) \geq \sum_{t=0}^{\infty} \beta^t F(y_t, y_{t+1}) > N\,.$$

Since G_Γ is a convex set and $(x_t, x_{t+1}), (y_t, y_{t+1}) \in G_\Gamma$ for each $t = 1, 2, \ldots$, it follows that $\alpha x_t + (1-\alpha)y_t \in \Gamma(\alpha x_{t-1} + (1-\alpha)y_{t-1})$ holds for each $t = 1, 2, \ldots$. This implies

$$\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y} = (\alpha x_0 + (1-\alpha)y_0, \alpha x_1 + (1-\alpha)y_1, \ldots) \in \Pi(\alpha x_0 + (1-\alpha)y_0)\,.$$

Now note that the concavity of F implies

$$\begin{aligned}
v(\alpha x_0 + (1-\alpha)y_0) &\geq \sum_{t=0}^{\infty} \beta^t F(\alpha x_t + (1-\alpha)y_t, \alpha x_{t+1} + (1-\alpha)y_{t+1}) \\
&= \sum_{t=0}^{\infty} \beta^t F\big(\alpha(x_t, x_{t+1}) + (1-\alpha)(y_t, y_{t+1})\big) \\
&\geq \alpha \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) + (1-\alpha) \sum_{t=0}^{\infty} \beta^t F(y_t, y_{t+1}) \\
&> \alpha M + (1-\alpha)N\,.
\end{aligned}$$

Since this is true for all real numbers M and N with $v(x_0) > M$ and $v(y_0) > N$, it follows that $v(\alpha x_0 + (1-\alpha)y_0) \geq \alpha v(x_0) + (1-\alpha)v(y_0)$. This shows that v is concave. ■

Notice that the above proof is perhaps simpler than the one in [19], because it does not require the existence of an optimal plan. We now have the necessary background in order to present a new proof of the differentiability of the value function.

4. The Main Result

To develop the new proof of the differentiability of the value function, we start by letting $\xi = (\xi_1, \ldots, \xi_\ell)$ denote the arbitrary vector of the Euclidean space $\mathbb{R}^\ell$. Now consider a function $f\colon A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^\ell$. The function f is said to be differentiable at a point $a \in A$ if a is an interior point of A and there exist a neighborhood $V \subseteq A$ of a and a vector $q \in \mathbb{R}^\ell$ such that

$$f(\xi) = f(a) + q \cdot (\xi - a) + o(\xi - a)$$

holds for all $\xi \in V$, where as usual $\lim_{\xi \to a} \frac{o(\xi - a)}{\|\xi - a\|} = 0$.[3] It is well known from Calculus that the vector q (called the differential of f at a) is uniquely determined and coincides with the gradient of f at a, i.e.,

$$q = \nabla f(a) = \big(\tfrac{\partial f(a)}{\partial \xi_1}, \tfrac{\partial f(a)}{\partial \xi_2}, \ldots, \tfrac{\partial f(a)}{\partial \xi_\ell}\big)\,.$$

[3] For the o-notation and more on differentiability, see [3, p. 286].

A vector $p \in \mathbb{R}^\ell$ is said to be a **supergradient** of f at a point $a \in A$ if for each $\xi \in A$ we have

$$f(\xi) \leq f(a) + p \cdot (\xi - a)\,.$$

The (possibly empty) collection of all supergradients of f at a is denoted $\partial f(a)$ and called the **superdifferential** of f at a. That is,

$$\partial f(a) = \big\{p \in \mathbb{R}^\ell\colon\; f(\xi) \leq f(a) + p \cdot (\xi - a) \;\text{ for all }\; \xi \in A\big\}\,.$$

It turns out that if f is a concave function (and so by definition A is also a convex set), the superdifferential of f is nonempty at any interior point of A. (See also [1, Theorem 7.12, p. 265].) Moreover, we have the following theorem that we shall employ below; see [17, Theorem 25.1, p. 242] or [1, Theorem 7.25, p. 274].

Theorem 4.1. *Let A be an open convex subset of $\mathbb{R}^\ell$ and let $f\colon A \to \mathbb{R}$ be a concave function. Then f is differentiable at some $a \in A$ if and only if the superdifferential $\partial f(a)$ is a singleton, in which case the lone supergradient is in fact the differential of f at a.*

The following result that presents a sufficient condition of differentiability of convex and concave functions is due to Benveniste and Scheinkman [7].

Lemma 4.2 (Benveniste–Scheinkman)**.** *Let C be a convex subset of the Euclidean space $\mathbb{R}^\ell$ and let $f\colon C \to \mathbb{R}$ be a concave function. Assume that for some point $\overline{\xi}$ in the interior of C there exist an open ball $B \subseteq C$ of $\overline{\xi}$ and a concave function $g\colon B \to \mathbb{R}$ such that:*

(1) *f dominates g on B, i.e., $f(\xi) \geq g(\xi)$ for all $\xi \in B$, and*
(2) *$g(\overline{\xi}) = f(\overline{\xi})$.*

If g is differentiable at $\overline{\xi}$, then f is likewise differentiable at $\overline{\xi}$ and the differential of f coincides with the differential of g at $\overline{\xi}$. In particular, we have $\frac{\partial f(\overline{\xi})}{\partial \xi_i} = \frac{\partial g(\overline{\xi})}{\partial \xi_i}$ for each $i = 1, \dots, \ell$.

Proof. We also consider the functions f and g restricted to the open ball B of $\overline{\xi}$. We know that the superdifferential of f is nonempty at $\overline{\xi}$; see [1, Theorem 7.12, p. 265]. So, according to Theorem 4.1, it suffices to show that it is a singleton.

To this end, let p be a supergradient of f at $\overline{\xi}$. That is, for all $\xi \in B$ we have

$$f(\xi) \leq f(\overline{\xi}) + p \cdot (\xi - \overline{\xi})\,.$$

Since f dominates the function g and $f(\overline{\xi}) = g(\overline{\xi})$, it follows from the preceding inequality that

$$g(\xi) \leq g(\overline{\xi}) + p \cdot (\xi - \overline{\xi})$$

holds for all $\xi \in B$. That is, p is a supergradient of g at $\overline{\xi}$. Since g is differentiable at $\overline{\xi}$, we infer that $p = (\frac{\partial g(\overline{\xi})}{\partial \xi_1}, \dots, \frac{\partial g(\overline{\xi})}{\partial \xi_\ell})$. This shows that p is uniquely determined and that the differential of f coincides with that of g at $\overline{\xi}$. ■

We are now ready to present a result regarding the differentiability of the value function. Keep in mind here, that as mentioned in the introduction of this note, the usual assumption of the literature, $g(x_0) \in \mathrm{int}\Gamma(x_0)$, will be replaced with the condition $x_0 \in \mathrm{int}\Gamma^{-1}(g(x_0))$.

Theorem 4.3. *Assume that the set of state variables X is a convex subset of the Euclidean space $\mathbb{R}^\ell$ and that the return function $F\colon G_\Gamma \to \mathbb{R}$ is concave. Suppose that $(\overline{x}, \overline{y}) \in G_\Gamma$ is a point such that:*

(1) *$\overline{x}$ is an interior point of the lower section $\Gamma^{-1}(\overline{y}) = \{x \in X\colon\ \overline{y} \in \Gamma(x)\}$ of Γ at $\overline{y}$.*
(2) *There exists an open convex neighborhood N of $\overline{x}$ with $N \subseteq \Gamma^{-1}(\overline{y})$ such that $v(x) \in \mathbb{R}$ for each $x \in N$.*[4]
(3) *$v(\overline{x}) = F(\overline{x}, \overline{y}) + \beta v(\overline{y})$.*

If the function $H\colon \Gamma^{-1}(\overline{y}) \to \mathbb{R}$, defined by $H(x) = F(x, \overline{y})$, is differentiable at $\overline{x}$, then the value function v is differentiable at $\overline{x}$ and

$$\frac{\partial v(\overline{x})}{\partial \xi_i} = \frac{\partial H(\overline{x})}{\partial \xi_i} = \frac{\partial F(\overline{x}, \overline{y})}{\partial \xi_i}$$

holds for each $i = 1, \ldots, \ell$.

Proof. The concavity of F guarantees that the function H restricted to N (i.e., the function $H\colon N \to \mathbb{R}$) is also a concave function.

Next define the function $L\colon N \to \mathbb{R}$ by $L(x) = F(x, \overline{y}) + \beta v(\overline{y}) = H(x) + \beta v(\overline{y})$ and note that L is concave and differentiable at $\overline{x}$. Moreover, for each $x \in N$ we have $\overline{y} \in \Gamma(x)$, and so

$$v(x) = \sup_{y \in \Gamma(x)} [F(x, y) + \beta v(y)] \geq F(x, \overline{y}) + \beta v(\overline{y}) = L(x)\,.$$

Clearly, $v(\overline{x}) = L(\overline{x})$ and, by Theorem 3.3, the value function v is concave. But then, according to Lemma 4.2, v is differentiable at $\overline{x}$ and for each $i = 1, \ldots, \ell$ we have $\frac{\partial v(\overline{x})}{\partial \xi_i} = \frac{\partial H(\overline{x})}{\partial \xi_i}$. ■

Notice that if $(\overline{x}, \overline{y})$ satisfies property (3) of Theorem 4.3 and is an interior point of the graph G_Γ of the correspondence Γ, then properties (1) and (2) are also automatically satisfied. Therefore, Theorem 4.3 has wider applicability than the existing theorems, and of course encompasses the case when the value of the policy function is in the interior of the graph of the constraint correspondence Γ.

5. An Application to Monetary Economics

In this section we present a natural application of the results from the previous section, which is offered by the monetary economics literature. In the economy we present, the value function is defined on the real money balances held by an individual, and it is differentiable even if the policy

[4]This is certainly the case if the return function F is bounded.

function does not satisfy the standard interiority assumption. To be precise, we demonstrate the use of our differentiability result for the monetary economy studied in [13] where the value function is generally treated as being differentiable though the optimal plan is a boundary solution.

Time is discrete and the horizon is infinite. There is an infinite number of identical infinitely-lived agents who consume two perishable goods, good 1 and good 2, and discount future utility with a factor $\beta \in (0,1)$. The perishable goods are traded in competitive markets, 1 and 2. These two markets open sequentially in each period. Preferences differ across periods. At the beginning of each period every agent is either a buyer or a seller with equal probability. It is assumed that a shock process partitions the agent population into two "equal" parts, called *buyers* and *sellers*.[5] Output is sold on competitive spatially separated markets composed by many anonymous agents as in [2]. Buyers cannot produce good 1, but derive $u_1(c)$ utility from $c \geq 0$ consumption of good 1. Sellers do not wish to consume good 1 but can produce $y \geq 0$ of good 1 by suffering disutility y. Every agent derives $u_2(q)$ utility from consuming $q \geq 0$ of good 2 and can produce $h \geq 0$ of good 2 suffering disutility h. We assume that each utility function $u_i \colon [0,\infty) \to \mathbb{R}$ satisfies the Inada conditions.[6] Since each u_i' is a strictly decreasing continuous function, the Inada conditions guarantee that the equations $u_1'(c) = 1$ and $u_2'(q) = 1$ have unique solutions. That is, there exist unique quantities c^* and q^* such that $u_1'(c^*) = 1$ and $u_2'(q^*) = 1$.

There is an authority that supplies fiat currency and sets monetary policy starting with the money stock $\hat{M}_0 > 0$. Let $\hat{M}_t$ denote the stock at the start of period t. Monetary policy is a time-invariant gross rate of growth $\gamma > \beta$ for the money stock, publicly announced. It is implemented via a deterministic per capita lump-sum cash transfer in market 2. So, $\hat{M}_{t+1} = \gamma \hat{M}_t$ is the cash available at the end of period t. Since sellers do not wish to consume, goods are non-storable and anonymity rules out credit, sellers in market 1 will demand cash compensation. Cash can also be used to trade in market 2. We let $p_{1,t}$ and $p_{2,t}$ denote the nominal prices of consumption in markets 1 and 2, respectively, on period t. Also, we work with real variables, defined to be the ratio of a nominal variable to $p_{2,t}$. Thus, we let $p_t = \frac{p_{1,t}}{p_{2,t}}$ denote the real price of goods in market 1 and let $\frac{\hat{M}_t}{p_{2,t}} = \hat{m}_t$ denote the real money stock available at the start of period t. We denote by $m_t \geq 0$ the real balances held by an arbitrary agent at the start of period t.[7]

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[5] If the population of agents is countable, then to each agent $j \in \mathbb{N}$ we assign a weight w_j so that $\sum_{j=1}^{\infty} w_{2j} = \sum_{j=1}^{\infty} w_{2j-1} = \frac{1}{2}$. So, we can consider "even" agents as buyers and "odd" ones as sellers, assuming that at the beginning of each period a shock reassigns agents to buyers and sellers, so that the above weight equation is valid. One way of doing this is by randomly exchanging the roles (buyer and seller) between each pair $(2j-1, 2j)$.

[6] A function $u \colon [0,\infty) \to \mathbb{R}$ is said to satisfy the Inada conditions, if (1) is twice differentiable on $(0,\infty)$, (2) $u'(x) > 0$ and $u''(x) < 0$ for each $0 < x < \infty$, and (3) $\lim_{x \to 0^+} u'(x) = \infty$ and $\lim_{x \to \infty} u'(x) = 0$; see [12, p. 120]. Replacing u by $u - u(0)$, we can assume without loss of generality that $u(0) = 0$.

[7] If we let $m_{t,j}$ be the real balances of agent $j \in \mathbb{N}$ in period t, then the aggregate money balances in period t are $\sum_{j=1}^{\infty} w_j m_{t,j} \leq \hat{m}_t$. It follows that if $m_{t,j} = m_t$ for each agent j, then $m_t \leq \hat{m}_t$ by money market clearing.

In what follows we consider allocations in which money is valued and aggregate real money balances are constant over time. Stationarity requires $\hat{m}_t = \hat{m}$ in each period t, and therefore $\frac{p_{2,t+1}}{p_{2,t}} = \frac{\hat{M}_{t+1}}{\hat{M}_t} = \gamma$, i.e., the gross inflation rate is deterministic and equal to γ. Finally, we let $\tau = (\gamma - 1)\hat{m}$ denote the real balance transfer in each period t.

The timing is as follows. The representative agent starts in market 1 with real balances m. Then, shocks are realized and the agent ends up in one of two states, buyer or seller. Trade and production takes place in market 1, and then market 2 opens. At this point in time, the agent has $m_{j,t}$ money balances, where $j \in \{b, s\}$ reflects his market 1 activities and b denotes a buyer and s a seller. Subsequently, the agent receives the money transfer and production and trade takes place in market 2. Then, a new period starts.

5.1. **The agent's problem.** The agent takes as given the sequence of nominal prices $(p_{1,t}, p_{2,t})_{t=0}^{\infty}$ in markets 1 and 2 and money supplies $(\hat{M}_t)_{t=0}^{\infty}$ that satisfy $\frac{\hat{M}_{t+1}}{p_{2,t+1}} = \frac{\hat{M}_t}{p_{2,t}} = \hat{m}$ for each $t = 1, 2 \ldots$. We will define by $X = \mathbb{R}_+$ the set of possible real balance holdings of the agent. At the start of a period t, the agent has $m_t \in X$ real balances available. During the course of the period, the agent will have to select savings $m_{t+1} \in \Gamma(m_t) = [0, \hat{m}]$ for the next period. We explain how this is done, below. For now, notice that the constraint correspondence $\Gamma\colon X \twoheadrightarrow X$ must satisfy $\Gamma(m_t) = [0, \hat{m}]$ for each $m_t \in X$ because there cannot be short selling of money, i.e., m_t cannot be negative and the supply of real balances is bounded above by $\hat{m}$.

In market 1, if the agent is a buyer, i.e., $j = b$, then given the relative price p_t he can consume $c_t \in [0, \frac{m_t}{p_t}]$ goods. If he is a seller, i.e., $j = s$, then he can produce $y_t \geq 0$. The agent enters market 2 with $m_{b,t} = m_t - p_t c_t$ real balances if he was a buyer in market 1 and with $m_{s,t} = m_t + p_t y_t$ if he was a seller. Each agent also receives $\tau = \hat{m}(\gamma - 1)$, constant over time. Each agent in market 2 can consume $q_{j,t} \geq 0$ and produce $h_{j,t} \geq 0$. Therefore, the resource constraint in market 2 is

$$h_{j,t} = q_{j,t} + \gamma m_{t+1} - m_{j,t} - \tau \,, \tag{5.1}$$

and it holds with equality due to non-satiation. Since we need $h_{j,t} \geq 0$ we will work under the conjecture that the right-hand side of the equation above is non-negative. Conditions for this to occur, in an optimum, is $(u_2')^{-1}(1)$ sufficiently large. Notice also that we are conjecturing that m_{t+1} does not depend on j, which is without loss in generality because preferences are quasilinear. (For details about these claims see [13].)

We will concentrate on monetary competitive equilibria, in terms of the given price vector $(p_{1,t}, p_{2,t})_{t=0}^{\infty}$. We carry out the analysis under the assumption that everyone starts each period with identical real balances; for a justification of this assumption see the discussion in [13]. Within this context, we wish to define an optimal plan for the representative agent. Any such plan will involve sequences of consumption and production in markets 1 and 2 as well as choices of savings by means of money holdings. To simplify the discussion of the optimal plan, it is convenient to make four remarks all of which deal with the choices of consumption and production in a period for the representative agent. The key observation, at this point, is that these choices are *intratemporal* in nature.[8]

(1) At the start of each period t the expected utility of the agent is

$$\tfrac{1}{2}[u_1(c_t) - h_{bt}] + \tfrac{1}{2}(-y_t - h_{st}) + u_2(q_t)\,.$$

So, if we substitute the budget constraint from (5.1), then we get the following expression for the period expected utility:

$$m_t + \tau - \gamma m_{t+1} + u_2(q_t) - q_t + \tfrac{1}{2}[u_1(c_t) - p_t c_t] + \tfrac{1}{2}(-y_t + p_t y_t)\,.$$

(2) By market clearing, in an optimum we must have $y_t = c_t$ in market 1 of each period t. Indeed, half of the population buys and half sells goods in that market, agents are homogeneous, and we are operating under the conjecture that everyone starts each period with identical real balances.

(3) In a competitive equilibrium we must have $p_t = 1$ for each $t = 0, 1, \ldots$. Otherwise a seller in market 1 would set either $y_t = 0$ or $y_t = \infty$, choices which are inconsistent with an optimum.

(4) In an optimum we must have $c_t = \min\{\frac{m_t}{p_t}, c^*\}$ for each period $t = 0, 1, \ldots$. This is the case, since, given $p_t = 1$, the utility function $u(c_t) - p_t c_t$ is strictly concave and decreasing past the point c^*. It should also be clear that in an optimum $q_t = q^*$ for each t, because each agent can access unlimited resources (at constant marginal cost) in market 2, and preferences in that market are quasilinear.

Given the four observations above, we are left to deal with the *intertemporal* choices, i.e., the choices of money holdings. To this end, we define the *expected period utility* $F\colon X \times X \to \mathbb{R}$ by

$$F(m_t, m_{t+1}) := m_t - \gamma m_{t+1} + \tfrac{1}{2}[u_1(\min\{m_t, c^*\}) - \min\{m_t, c^*\}] + \tau + u_2(q^*) - q^*\,.$$

This plays the role of the period return function discussed earlier. It is not difficult to see that the real function $f(x) = u_1(\min\{x, c^*\}) - \min\{x, c^*\}$, $x \geq 0$, is concave, and from this it easily follows that the expected period utility function $F(x, y)$ satisfies the following four important properties:

(1) is a concave continuous function,

(2) is an unbounded function that satisfies the convergence condition (**C**),

(3) for each fixed savings choice $m_{t+1} \in [0, \infty)$ the function $F(x, m_{t+1})$ is differentiable at each $x > 0$, and

(4) for each $m_t > 0$ we have

$$\frac{\partial F(m_t, m_{t+1})}{\partial x} = \begin{cases} \tfrac{1}{2}[1 + u_1'(m_t)] & \text{if } 0 < m_t < c^* \\ 1 & \text{if } m_t \geq c^*\,. \end{cases}$$

[8] For more background and more details concerning the statements that follow, see the discussion in [4, 13].

Now, a plan starting with $m_0 \in X$ is a sequence $\boldsymbol{m} = (m_0, m_1, \ldots) \in X^{\{0,1,2,\ldots\}}$ such that $m_{t+1} \in \Gamma(m_t)$ holds for each $t = 0, 1, 2, \ldots$. The collection of all feasible plans starting with m_0 is denoted $\Pi(m_0)$, i.e.,

$$\Pi(m_0) = \{\boldsymbol{m} \in X^{\{0,1,2,\ldots\}} \colon \; m_{t+1} \in \Gamma(m_t) \; \text{ for all } \; t = 0, 1, 2 \ldots \}.$$

As before, we define the lifetime utility for an agent who starts with real balances m_0 by

$$U(\boldsymbol{m}) = \sum_{t=0}^{\infty} \beta^t F(m_t, m_{t+1}) \,.$$

The objective of the representative agent is to choose an optimal monetary savings plan $\boldsymbol{m}$ that solves our standard lifetime utility maximization problem:

$$\begin{aligned} &\textit{Maximize:} \quad U(\boldsymbol{m}) \\ &\textit{Subject to:} \quad \boldsymbol{m} \in \Pi(m_0) \end{aligned}$$

We discuss this maximization problem in detail, next.

5.2. **The value function.** As shown in the previous section, the maximization problem of the representative agent has a corresponding value function $v \colon X \to \mathbb{R}$ defined for each $m_0 \in X$ by

$$v(m_0) = \sup_{\boldsymbol{m} \in \Pi(m_0)} U(\boldsymbol{m}) \,. \tag{5.2}$$

This value function satisfies the Bellman equation, that is, for each $x \in X$ we must have:

$$v(x) = \sup_{m \in \Gamma(x)} [F(x, m) + \beta v(m)].$$

Our objective is to show that v is continuous and differentiable. Since F is clearly a continuous function, we have that if v were continuous, then it would follow that for every $x > 0$ there exists at least one $\overline{y} \in \Gamma(x)$ such that $v(x) = F(x, \overline{y}) + \beta v(\overline{y})$. So, our task now is to prove that the value function v is indeed continuous.

In order to establish the continuity of v, it suffices to show that v is continuous on every interval of the form $[0, x]$, where $x > \hat{m}$. To this end, let $x > \hat{m}$ and fix $m_0 \in I = [0, x]$. Now for each plan $\boldsymbol{m} = (m_0, m_1, \ldots)$ notice that

$$|U(\boldsymbol{m})| = \Big| \sum_{t=0}^{\infty} \beta^t F(m_t, m_{t+1}) \Big| \leq \tfrac{k}{1-\beta} \,,$$

where (using the triangle inequality) the constant k is given by

$$k = (1 + \gamma)x + \tfrac{1}{2}[u_1(c^*) + c^*] + \tau + u_2(q^*) + q^* \,.$$

This yields $|v(m_0)| \leq \frac{k}{1-\beta}$, and from this we see that v is a bounded function on the interval I.

Next, let $B(I)$ and $C_b(I)$ denote the vector spaces of all bounded real functions and bounded continuous real functions defined on I, respectively. Clearly $C_b(I) \subseteq B(I)$ and both $B(I)$ and

$C_b(I)$ equipped with the metric induced by the sup norm are complete metric spaces, i.e., they are both Banach spaces. Next, we define the Bellman operator $T\colon B(I) \to B(I)$, defined for each function $f \in B(I)$ and $\alpha \in I$ by

$$T(f)(\alpha) = \sup_{m\in\Gamma(\alpha)} \left[F(\alpha, m) + \beta f(m)\right].$$

It is easy to see that T is a contraction with contraction constant β. This implies that T has a unique fixed point, which (according to Lemma 3.2) must be the value function v restricted to I. Now an easy application of the classical Berge's Maximum Theorem (see for instance [1, p. 570]) shows that T leaves $C_b(I)$ invariant, i.e., $T(C_b(I)) \subseteq C_b(I)$, which implies that T has also a fixed point in $C_b(I)$. But this fixed point of T in $C_b(I)$ must be the function v restricted to I. We conclude that v restricted to I is continuous, as desired.

We are now ready to discuss the differentiability of the value function. To start, observe that for each $0 \le m \le \hat{m}$ the lower section for m is $\Gamma^{-1}(m) = [0, \infty)$. This shows that for any real balances $x > 0$ and any real balances $\overline{y}$ that satisfy

$$v(x) = F(x, \overline{y}) + \beta v(\overline{y}),$$

the point $x > 0$ is an interior point of the lower section of Γ at $\overline{y}$, i.e., $x \in \mathrm{int}\Gamma^{-1}(\overline{y})$. Therefore, it follows from Theorem 4.3 that the value function given in (5.2) is differentiable at every real balance level $x > 0$ that the agent finds himself holding at the start of a period. It should be noted here that due to market clearing we have in general $\overline{y} = \hat{m}$. That is to say, agents hold all of the available money stock, in equal amounts. In this case, $(x, \overline{y}) \notin \mathrm{int}\ G_\Gamma$, and so it is not obvious that the differentiability theorem in [19] can be applied. However, our Theorem 4.3 shows not only that v is differentiable on $(0, \infty)$ but also that for each real balance level $x > 0$ we have:

$$v'(x) = \frac{\partial F(x, \overline{y})}{\partial x} = \begin{cases} \frac{1}{2}[1 + u_1'(x)] & \text{if } \ 0 < x < c^*, \\ 1 & \text{if } \ x \ge c^*. \end{cases}$$

That is to say, one can easily calculate the marginal value of real balances even if the optimal savings plan is a boundary solution.

6. Concluding Remarks

The study of macroeconomic equilibrium is often based on the use of dynamic programming techniques. In this context, if the value function is differentiable, then it is possible to characterize the equilibrium allocation and the policy function in a simple manner. The classical reference on establishing the differentiability of the value function is in [19], where Theorem 4.11 presents a proof based on the assumption that the policy function is in the interior of the upper section of the constraint correspondence. Two difficulties are associated to this approach. First, there are instances in which the optimal plan is a boundary solution, as in some monetary models where agents are constrained in their saving/borrowing abilities. Second, one must first establish that the policy function is *also* in the interior of the upper section of the constraint correspondence for all states that are in some neighborhood of the initial state. The new approach we have developed bypasses these two difficulties. It relies on an assumption for the initial state, and not for the policy function; one simply must ensure that the initial state is an interior point of the lower section of the constraint correspondence. The resulting proof of differentiability of the value function is simpler and of general applicability.

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