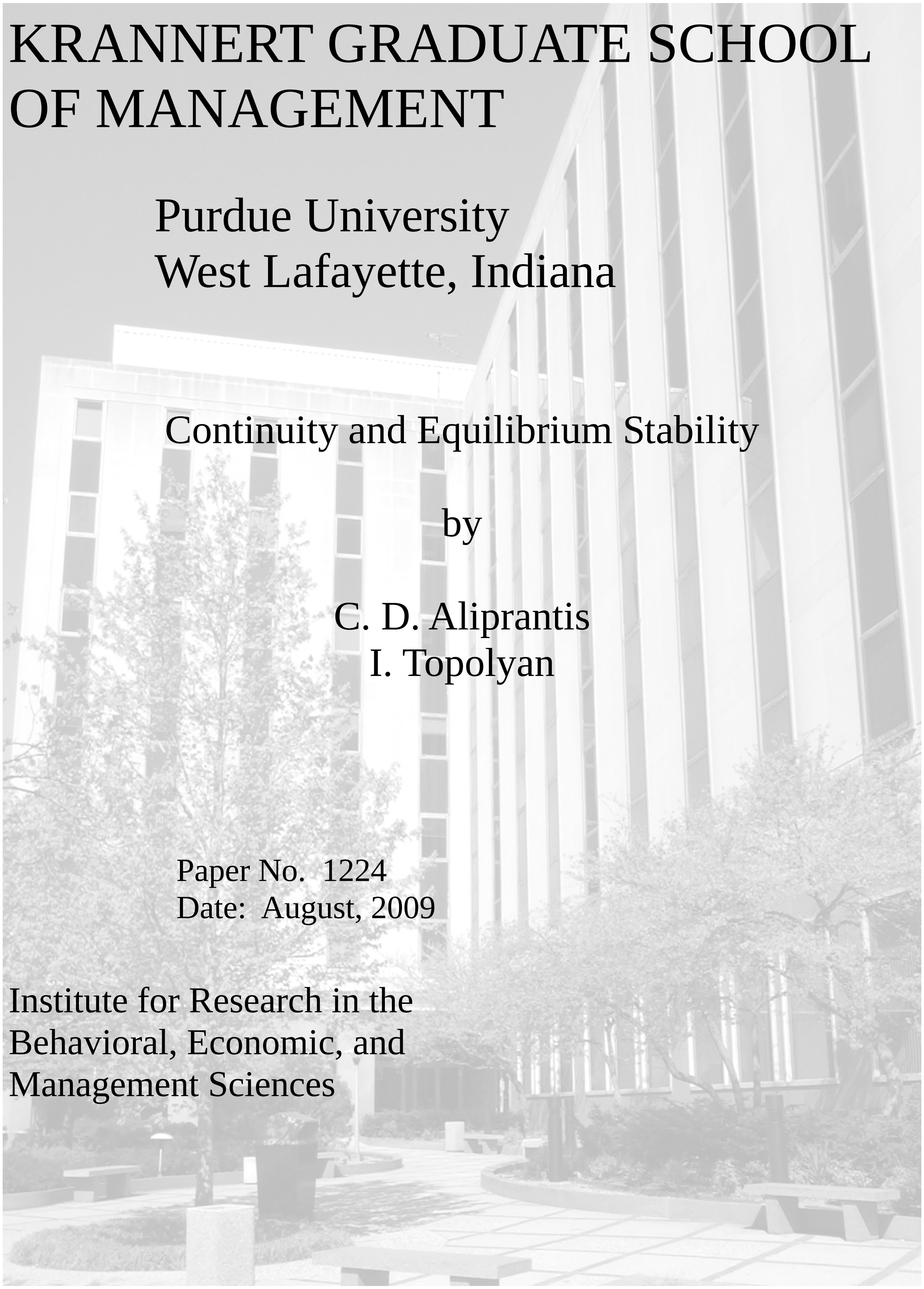

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Continuity and Equilibrium Stability

by

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CONTINUITY AND EQUILIBRIUM STABILITY★

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Abstract. This paper discusses the problem of stability of equilibrium points in normal form games in the tremling-hand framework. An equilibrium point is called perffect if it is stable against at least one seqence of trembles approaching zero. A strictly perfect equilibrium point is stable against every such sequence.

We give a sufficient condition for a Nash equilibrium point to be strictly perfect in terms of the primitive characteristics of the game (payoffs and strategies), which is new and not known in the literature. In particular, we show that continuity of the best response correspondence (which can be stated in terms of the primitives of the game) implies strict perfectness; we prove a number of other useful theorems regarding the structure of best responce correspondence in normal form games.

JEL classification: C7

Keywords: Strictly perfect equilibrium, best responce correspondence, unit simplex, face of a unit simplex.

1. Introduction

The idea of an equilibrium is central in the game theory, for it is an expression of an "ideal", or optimal way of playing the game. In his pioneering and revolutionary work Nash [**?**] formulated an equilibrium as follows: a strategy profile is called an equilibrium if the strategy of every player is a best response to other players'

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strategy. Alternatively, it is a strategy profile in which no player has a positive incentive to deviate, given the strategies of other players. Thus, we can view the Nash equilibrium as a fixed point of the best response correspondence.

Nash equilibrium is the most basic notion of an equilibrium, it expresses the idea of "non-regretting", i.e., none of the players is better-off deviating while other players stick to their strategies. However, Nash equilibrium concept is a broad equilibrium notion and appeared to be weak, for example it does not account for the possibility of mistakes, non-credible threats etc. The reason is that the elements of a Nash equilibrium profile are best responses to themselves, so there is a problem of mutual interdependence and "endogeneity", rather than dependence on the exogenous determinants. rather, we are looking for the solutions that are implied by the game form itself.

The question whether there is a way to determine which strategies are enforced by the game lead to development of numerous equilibrium refinements. Stronger criteria of equilibrium were needed in order to incorporate the possibility of mistakes and make sure that rationality break-up on some of the stages of the game does not throw the players away from the optimal path.

The non-optimality that arises via the unreached parts of the game was successfully resolved by Selten [**?**] in 1975, who introduced the perfect equilibrium concept as a refinement of subgame perfection. The perfect equilibrium point is stable against arbitrary small deviations from rationality, which is the basic intuition behind the satisfactory refinement of subgame perfection. In order to incorporate possibly unreahced parts of the game, Selten considered perturbed games, in which every strategy is played with positive probability, thus eliminating the possibility of having unreached information sets. He viewed the complete rationality as a limiting case of incomplete rationality, and thus proposed to define the perfect equilibrium profile as a limit point of the sequence of equilibria in perturbed games. I.e, b^* is perfect if for at least one sequence of perturbed games $\left\{\hat{\Gamma}^k\right\}$, b^* is a limit equilibrium point of $\left\{\hat{\Gamma}^k\right\}$.

Immediately the question arises whether the way incomplete rationality approaches complete rationality influences perfectness property. That is, can b^* be a limit equilibrium point for one test sequence and not be such for another sequence, both sequences approaching the game Γ. Okada [**?**] gives negative answer to this question, proving that actually not every perfect equilibrium point is stable against *arbitrary* perturbation of rationality. He thus introduced the idea of strictly perfect equilibrium point, which is limit equilibrium point for every sequence of perturbed games. Okada shows that if an equilibrum point is unique, or totally mixed, or strong, then it is strictly perfect. He also demonstrates that there are games with no perfect equilibria.

We come up with another sufficient condition for strict perfectness, which is the continuity of the best response correspondence at the equilibrium point, presented as a Theorem 3.6 in the section Results. Continuity of the best responce correspondence is something that is implied by the game form, so we formulate our the sufficient condition in terms of the primitives of the game (payoffs and strategies), making it more tractable and suitable for practical usage (see Theorem 3.13).

Also, it's not difficult to show that one of the conditions listed by Okada (strong equilibrium) implies continuity of the best response correspondence (see Lemma 3.7). However, there are cases when none of the conditions (1) – (3) hold, while the best responce correspondence is continuous (see Example in the Results section), which guarantees strict perfectness. Hence, continuity is a non-trivial condition among the sufficient conditions for the strict perfect equilibrium.

The paper is organized as follows. In the section Definitions and Methodology we present our framework and give all the necessary definition of the concepts involved. In the section Results first we provide some technical findings concerning the topological structure of the best responce correspondence for a perturbed game. That is, we show how the best responce behaves when we are passing to a perturbed game (restrict the strategy space) from the initial, or unrestricted, game (Lemmas 3.1, 3.2, 3.3, 3.4). An important result which is central for proving our main theorem states that a best responce correspondence is continuous at the strategy profile a if and only if it is constant on some neighborhood of a (Lemma 3.5).

2. Definitions and Methodology

Definition 2.1. *A **normal form** Γ of a finite n-player game is a tuple $(\Pi_1, \cdots, \Pi_n, H)$, where Π_i is a finite set of pure strategies of player i, $\Pi = \prod_{i=1}^n \Pi_i$, and $H : \Pi \twoheadrightarrow \mathbb{R}^n$ is the payoff function that assigns to every $\pi \in \Pi$ the vector of payoffs $H(\pi) = (H_1(\pi), \cdots, H_n(\pi))$.*

Definition 2.2. *A **mixed strategy** a_i for player i is a probability distribution over Π_i. The set of all such probability distributions is denoted by $A_i \equiv \Delta_{c_i}$, where c_i is the cardinality of Π_i. The set of mixed strategies for the game Γ is $A = \prod_{i=1}^n A_i$.*

We can now define an expected payoff function h, which is an extension of the payoff function H to all of A.

Definition 2.3. *An **expected payoff function** is a function $h : A \twoheadrightarrow \mathbb{R}^n$ such that*

$$h(a) = \prod_{\pi \in \Pi} p_1(\pi)p_2(\pi)\cdots p_n(\pi)H(\pi),$$

where $p_i(\pi)$ is the probability that a assigns to the i^{th} component of π, i.e., the probability with which player i chooses π_i.

Mixed strategy a_i for player i is comletely mixed if $a_i \in \Delta^o_{p_i}$. Mixed strategy a is completely mixed if for all i, a_i is completely mixed.

Definition 2.4. *A **best responce correspondence** of player i is the correspondence $\mu_i : A_{-i} = \prod_{j \neq i} A_j \twoheadrightarrow A_i$ defined for each $a_{-i} \in A_{-i}$ as*

$$\mu_i(a_{-i}) = \{\tilde{a}_i \in A_i \,:\, h_i(\tilde{a}_i, a_{-i}) \geq h_i(a_i, a_{-i}) \ \forall a_i \in A_i\}.$$

Definition 2.5. *A **best responce correspondence** for N-person normal form game game is the correspondence $\mu : A \twoheadrightarrow A$ defined for each $a \in A$ as a tuple $(\mu_1(a_{-1}), \mu_2(a_{-2}), \ldots, \mu_N(a_{-N}))$, where for each i, $\mu_i(a_{-i})$ is player i's best responce correspondence defined as above.*

It follows by Berge's Maximum Theorem that best responce correspondence is upper semicontinuous, however at some points it may fail to be lower semicontinuous, which may disrupt equilibrium stability with respect to trembles.

Definition 2.6. *A **perturbed game** $\hat{\Gamma}$ of a normal form game Γ is a tuple (Γ, η), where $\eta = (\eta_1, \ldots, \eta_n)$ is a strictly positive vector of trembles satisfying for all i*

$$\sum_{k=1}^{c_i} \eta^i_k \leq 1,$$

such that for each $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, c_i\}$ we have

$$p_i(\pi^i_j) \geq \eta^i_j,$$

and for each i the following holds:

$$\sum_{j=1}^{c_i} p_i(\pi^i_j) = 1.$$

So vector η can be interpreted as a vector of minimum probabilities corresponding to each $\pi^i_j \in \Pi_i$. A perturbed game has the property that pure strategies are ruled out, that is the action set $\hat{A}_i$ for generic player i is a subset of $(A_i)^0$.

The latter restriction gives rise to the notion of maximum probability of the choice π^i_j. Observe that since no pure strategy can be played with zero probability, no pure strategy can be played with probability one either.

Definition 2.7. *A* ***maximum probability*** *of the choice π^i_j of player i is defined as*

$$\zeta(\pi^i_j) = 1 + \eta^i_j - \sum_{k=1}^{c_i} \eta^i_k < 1.$$

From the above we get immediately the first lemma in our results section.

Definition 2.8. *A mixed strategy $a^* \in A$ is called a* ***perfect equilibrium point*** *of a normal form game Γ if a^* is a Nash equilibrium for Γ and for some sequence of perturbed games $\hat{\Gamma}^k = (\Gamma, \eta^k)$ with $\eta^k \to 0$, there exists a Nash equilibrium point a^k of $\hat{\Gamma}^k$ for each k such that $a^k \to a^*$ as $k \to \infty$.*

Definition 2.9. *A mixed strategy $a \in A$ is called a* ***strictly perfect equilibrium point*** *of a normal form game Γ if a^* is a Nash equilibrium for Γ and for any sequence of perturbed games $\hat{\Gamma}^k = (\Gamma, \eta^k)$ with $\eta^k \to 0$, there exists a Nash equilibrium point a^k of $\hat{\Gamma}^k$ for each k such that $a^k \to a^*$ as $k \to \infty$.*

Notice that we can relax the assumption that a^* is a Nash equilibrium in Definitions 2.7 and 2.8, for Selten showed that a limit of equilibrium points is itself an equilibrium.

Definition 2.10. *A Nash equilibrium point $a^* \in A$ is called* ***strong*** *if for each player i*

$$h_i(a^*) > h_i(a^*_{-i}, a_i) \text{ for all } a_i \in A_i \text{ such that } a_i \neq a^*_i.$$

3. Results

Here we present the results regarding the implications of continuity of the best response correspondence to the problem of stability of equilibrium points. First, we discuss some technical results regarding the structure of a perturbed game.

Every action in a perturbed game is restricted to be played with the probability no less than the corresponding η_j, that is, $\forall\, a_j \in A_j$: $P(a_j) \geq \eta_j$, $j \in \{1, \ldots, p_i\}$. Combined with the restriction that the probabilities in the strategy profile sum up to one, we get immediately the following lemma.

Lemma 3.1. *For all i, $\hat{A}_i$ is a unit $(p_i - 1)$-simplex with vertices:*

$$(\eta_1, \ldots, \eta_{p_i-1}, 1 - \eta_1 - \ldots - \eta_{p_i-1}),$$
$$(\eta_1, \ldots, \eta_{p_i-2}, 1 - \eta_1 - \ldots - \eta_{p_i-2} - \eta_{p_i}, \eta_{p_i}),$$
$$\vdots$$
$$(1 - \eta_2 - \ldots - \eta_{p_i}, \eta_2, \ldots, \eta_{p_i}).$$

Fix $j \in I$ and $a_{-j} \in \hat{A}_{-j}$, then $H_j(\hat{a}_j) = H_j(\hat{a}_j, \hat{a}_{-j})$ describes an equation of a multidimensional plane on $\hat{A}_{-j}$, hence the following conclusion holds.

Lemma 3.2. *For all j, $\mu_j(\hat{A}_{-j})$ is an m-face of the unit $(p_j - 1)$-simplex $\hat{A}_j$ for some $m \in \{0, \ldots, p_j\}$.*

It follows from Berge Maximum Theorem that for all j, a_{-j} the best response correspondence is upper hemicontinuous in a_{-j}. However, it may fail to be lower hemicontinuous at some points. For example, consider the 2-person game with $A_1 = A_2 = [0, 1]$, and let

$$\mu_2(a_1) = \begin{cases} [0,1] & \text{if } \ a_1 = 0 \\ 1 & \quad \text{otherwise.} \end{cases}$$

As can be seen, the best response correspondence is not lower hemicontinuous at $a_1 = 0$. The fact that best response correspondence is not continuous at some points can make the Nash equilibrium points unstable.

Lemma 3.3. *For the N-player normal form game the following statements are equivalent:*

1. *The best resposce correspondence μ is continuous at $(x_1, \dots, x_N)$.*
2. *For each $i = 1, \dots, N$ μ_i is continuous at x_{-i}.*
3. *For each $i = 1, \dots, N$ there exists U_i open neighborhood of x_{-i} such that μ_i is constant on U_i.*
4. *There exists U open neighborhood of x such that μ is constant on U.*

Proof. (1) $\Rightarrow$ (2) Assume μ is continuous at x^0. Let's show upper-hemicontinuiuty first. Since μ is continuous. it is also upper-hemicontinuous, then for every $U \subseteq A$ neighborhood of $\mu(x^0)$ there exists $V \subseteq A$ neighborhood of x^0 such that $z \in V$ implies $\mu(z) \in U$.

By Berge's maximum theorem μ_i is upper-hemicontinuous for each i.

For the lower-hemicontinuity of μ_i, fix $U_j \subseteq A_{-j}$ open such that $U_j \cap \mu_j(x^0_{-j}) \neq \emptyset$ for each $j \in I$. Let $W \equiv \prod_{j=1}^N U_j$, which meets $\mu(x^0)$. Then by the lower-hemicontinuity of μ there exists $V \subseteq A$ open neighborhood of x^0 such that $z \in V$ implies $\mu(z) \cap U \neq \emptyset$. Consequently $z_{-i} \in V_{-i}$ implies $\mu_i(z_{-i}) \cap U_i \neq \emptyset$. Hence μ_i is lower-hemicontinuous.

(2) $\Rightarrow$ (1) Assume each μ_i is continuous at x_{-i}, then $\varphi \equiv \prod_{i=1}^N \mu_i : \prod_{i=1}^N A_{-i} \twoheadrightarrow A$ is continuous at x being a finite product of continuous compact-valued correspondences. From this we can show that μ is continuous at x.

Since φ is upper-hemicontinuous at x, for every U open neighborhood of $\varphi(x)$, $U \subseteq A$ there exists V open neighborhood of $\prod_{i=1}^N x_{-i}$ such that $z \in V \Rightarrow \varphi(z) \in U$. Hence there exists W open neighborhood of x such that $z \in W \Rightarrow \mu(z) \in U$, i.e. μ is upper-hemicontinuous at$x$.

Due to the lower-hemicontinuity of φ at x, for every U open in A s.t. $U \cap \varphi(\prod_{i=1}^N x_{-i}) \neq \emptyset$ there exists V open neighborhood of $\prod_{i=1}^N x_{-i}$ such that $z \in V \Rightarrow \varphi(z) \cap U \neq \emptyset$. Hence there exists W open neighborhood of x such that $z \in W \Rightarrow \mu(z) \cap U \neq \emptyset$. Therefore μ is lower-hemicontinuous at x.

(2) $\Rightarrow$ (3) Fix $i \in I$. Assume $\mu_i(x^0_{-i})$ is continuous at x^0_{-i}. need to show there exists an open neighborhood of x^0_{-i} such that μ_i is constant on that neighborhood.

Suppose by contradiction for each U_i open neighborhood of x^0_{-i} μ_i is not constant on U_i. WLOG $\mu_i(x^0_{-i})$ is an M-face M of the unit p_i-simplex A_i. Hence there exists W open neighborhood of M that does not contain any other face of the unit p_i-simplex, or an open set that meets M, but does not meet any other face of A_i.

Apply upper-hemicontinuity of μ_i in the first case, and lower-hemicontinuity of μ_i in the second case to derive contradiction:

1) there does not exist V open neighborhood of x^0_{-i} such that $z \in V$ implies $\mu_i(z) \in W$;

2) there does not exist V open neighborhood of x^0_{-i} such that $z \in V$ implies $\mu_i(z) \cap W \emptyset$.

Both cases imply that μ_I is not continuous at x^0_{-i}, contradiction.

(3) $\Rightarrow$ (2) is obvious, and equivalence of (3) and (4) is immediate, which finishes the proof. ■

Lemma 3.4. *For every $i \in I$ the corresponding faces of A_i and $\hat{A}_i$ are collinear, when we consider A_i and $\hat{A}_i$ to be subsets of the vector space $\mathbb{R}^{p_i}$.*

Lemma 3.5. *For every $i \in I$ if $\mu_i(A_{-i})$ is an m-face of the unit $(p_i - 1)$-simplex A_i, then $\hat{\mu}_j(\hat{A}_{-j})$ is the corresponding m-face of the unit $(p_i - 1)$-simplex $\hat{A}_j$.*

The above lemma is an immediate consequence of Lemma 0.5 and the fact that for every player, keeping the strategies of all other players fixed, his payoff function is a linear in probabilities constituting that player's mixed strategy.

Theorem 3.6. *Let $x^0 \in A$ be a Nash equilibrium for the normal form game Γ with N players. If for every $i = 1, \ldots, N$ $\mu_i : A_{-i} \twoheadrightarrow A_i$ is continuous at x^0_{-i}, then x^0 is strictly perfect equilibrium point.*

Proof. Claim: $\forall \epsilon > 0 \; \exists \bar{\eta} = (\bar{\eta}_i)_{i=1}^N$ such that for every $\hat{\eta} \geq \bar{\eta}$ the corresponding perturbed game $\hat{\Gamma}$ has a Nash equilibrium $\hat{x}$ such that $\| \hat{x} - x^0 \| \leq \epsilon$.

Since $\forall\, i\, \mu_i : A_{-i} \twoheadrightarrow A_i$ is continuous at x^0_{-i}, then by Lemma 0.4 $\forall\, i$ there exists U_{-i} open neighborhood of x^0_{-i} such that μ_i is constant on U_{-i}. Let $U = \bigcap_{i=1}^N (A_i \times U_{-i})$, then for every i the best response correspondence μ_i is constant on W_{-i}, also W is an open neighborhood of x^0 since all U_{-i}'s are.

WLOG $x^0 \in V_1 \times \ldots \times V_N \subseteq U$, where V_i is an open (in A_i) box around x^0_i. Therefore μ_i is constant on V_{-i} for every i.

Fix $\epsilon > 0$. Let $\xi > 0$ be such that ξ-neighborhood of x^0 in A in uniform topology is in V (we are able to do it since V is open and $x^0 \in V$). Let V' denote the ϵ'-neighborhood of x^0 (in the uniform topology), and define $W_i = V_i \cap V_i'$. Then $\forall\, i\; \forall x_{-i} \in W_{-i} : \;\; \mu_i(x_{-i}) = \mu_i(x^0_{-i})$.

Let $\epsilon' = \min\{\epsilon, \xi\} > 0$. Compose $\bar{\eta} = (\bar{\eta}_i)_{i=1}^N$ as follows: for each i, if x_i^0 is interior, then $\bar{\eta}_i$ is arbitrary, otherwise $\bar{\eta}_i$ is such that the corresponding $\hat{A}_i$ meets W_i. We claim that for every $\eta \geq \bar{\eta}$ the corresponding perturbed game $\hat{\Gamma}$ has a Nash equilibrium $\hat{x}$ such that $\|\, \hat{x} - x^0\| \leq \epsilon'$ (and consequently $\|\, \hat{x} - x^0\| \leq \epsilon$).

Indeed, fix $\hat{\eta} \geq \bar{\eta}$, and let M_i denote the best responce of player i to $x^0_- i$, which is an m_i-face of the unit p_i-simplex. Then for the perturbed game corresponding to $\hat{\eta}$ the best responce of player i is also the m_i-face, call it $\hat{M}_i$.

For each i, consider $Y_i = \hat{M}_i \cap W_i$. We claim that any point in $\cap_{i=1}^N Y_i$ is a Nash equilibrium of the perturbed game $\hat{\Gamma}$. Indeed, for every $y_i \in Y_i$ M_i is a best responce, so any point in Y is a Nash equilibrium. ∎

Lemma 3.7. *If $x^0 \in A$ is a strong equilibrium point, then μ_i is continuous at x^0_{-i} for every $i = 1, \ldots, N$.*

Proof. Since x^0 is a strong equilibrium point, μ_i is a singleton for every i. By the properties of best response correspondence μ_i is upper hemicontinuous, so for every i, μ_i is a singleton on some neighborhood of x^0_{-i}. Hence by Lemma 0.4, μ_i is continuous at x^0_{-i} for every $i = 1, \ldots, N$. ∎

So, strong equilibrium implies continuity of the best responces at that point, however the converse is not true. Also, there are examples (e.g., Matching pennies game) when the equilibrium point is unique and interior, but the best responces fail to be continuous at that equilibrium point. The following example shows that the continuity condition is essential in the sense that for certainn equilibrium points none of the three sufficient conditions provided by Okada hold, however the best response correspondence is continuous, which implies strict perfectness of the equilibrium point.

Example 3.8. (Essentiality of the continuity condition) Consider the two-person normal form game depicted in the table below. In this example equilibrium point (A, a) is neither interior nor unique, for (B, a) and (C, b) are Nash equilibria as well. It is also not strong equilibrium, however the best responce correspondence is continuous at (A, a).

1/2	a	b
A	1, 4	1, 0
B	1, 4	1, 0
C	0, 1	1, 1

Example 3.9. Consider the 2-person game with player 1 having choices $1, 2, \ldots, m$, and player 2 having n choices. If player 1 randomizes between his choices according to the vector of probabilities $p = (p_1, \ldots, p_m)$, and player 2 uses the strategy $q = (q_1, \ldots, q_n)$, then their utility functions are defined by

$$U_1 = \sum_{i=1}^{m} p_i(a_{i1}q_1 + \ldots + a_{in}q_n),$$

$$U_2 = \sum_{j=1}^{n} q_j(b_{j1}p_1 + \ldots + b_{jm}p_m),$$

where $(a_{ij})_{i=1,\ldots,m}^{j=1,\ldots,n}$ and $(b_{ij})_{i=1,\ldots,m}^{j=1,\ldots,n}$ are the payoff matrices for players 1 and 2, respectively.

Denote $a_{i1}q_1 + \ldots + a_{in}q_n$ by t_i^1, and $b_{j1}p_1 + \ldots + b_{jm}p_m$ by t_j^2. Then the above formulas simplify as

$$U_1 = \sum_{i=1}^{m} p_i t_i^1, \quad U_2 = \sum_{j=1}^{n} q_j t_j^2.$$

Player 1, given the strategy q of another player, optimezes w.r.t. p, given a constant vector $t^1 = (t_1^1, \ldots, t_m^1)$, which is completely determined by q. A version of Lemma 3.2 ensures that the best responce $p^*(t^1)$ is a face of the unit simplex Δ^m.

Example 3.10. Now consider the 3-person game with players 1, 2 and 3 having m, n and k choices, respectively, using the mixed strategies $p = (p_1, \ldots, p_m)$, $q = (q_1, \ldots, q_n)$, and $r = (r_1, \ldots, r_k)$. Then player 1's utility can be written (similarly for other players) as:

$$U_1 = \sum_{l=1}^{m} p_l \, (\sum_{i=1}^{n} \sum_{j=1}^{k} q_i r_j a_{lij}) = \sum_{l=1}^{m} p_l t_l^1,$$

where $t_l^1 = \sum_{i=1}^{n} \sum_{j=1}^{k} q_i r_j a_{lij}$.

The above examples make obvious the generalization to the N-player case. Every player j is maximizing the utility function that is linear in his own proabilities, taking the vector t^j as given.

Definition 3.11. *Given a vector $t = (t_1, \dots, t_k)$, an **order** of t is a vector of integers $s = (s_1, \dots, s_k)$ such that for each i, s_i is a rank of t_i among $\{t_1, \dots, t_k\}$ (assuming that $\max_{j \in \{1,\dots,k\}} t_j$ has the rank 1).*

Lemma 3.12. *Consider player i's utility function written in the form $U_i = \sum_{j=1}^n p_j t_j^i$, where p is the strategy of player i and t^i is a function of other players' strategies. Then the best responce correspondence $\mu_i(a_{-i})$ is constant in the neighborhood V of some a_{-i}^0 if and only if the order of t^i is constant on that neighborhood.*

Notice that we allow a_{-i}^0 to be on the boundary of A_{-i}, in this case the neighborhood of a_{-i}^0 is open in the relative topology with respect to A_{-i}. Combined with Lemma 3.3 and Theorem 3.6, this gives a sufficient condition for strict perfectness of a Nash equilibrium point, which is presented in the following theorem.

Theorem 3.13. *Let $a^* \in A$ be a Nash equilibrium for the normal form game Γ with N players. If there exists a neighborhhood $V = \prod_{i=1}^N V_i$ of a^* such that for every $i = 1, \dots, N$ the order of t^i is constant on V_{-i}, then a^* is strictly perfect equilibrium point.*

Finally, the following theorem demonstrates that continunity is a regular condition in the sense that the best responce ocrrespondence is continuous almost everywhere (with respect to the Lebesgue measure on A).

Theorem 3.14. *The best responce correspondence for any finite normal form game Γ is continuous almost everywhere.*

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