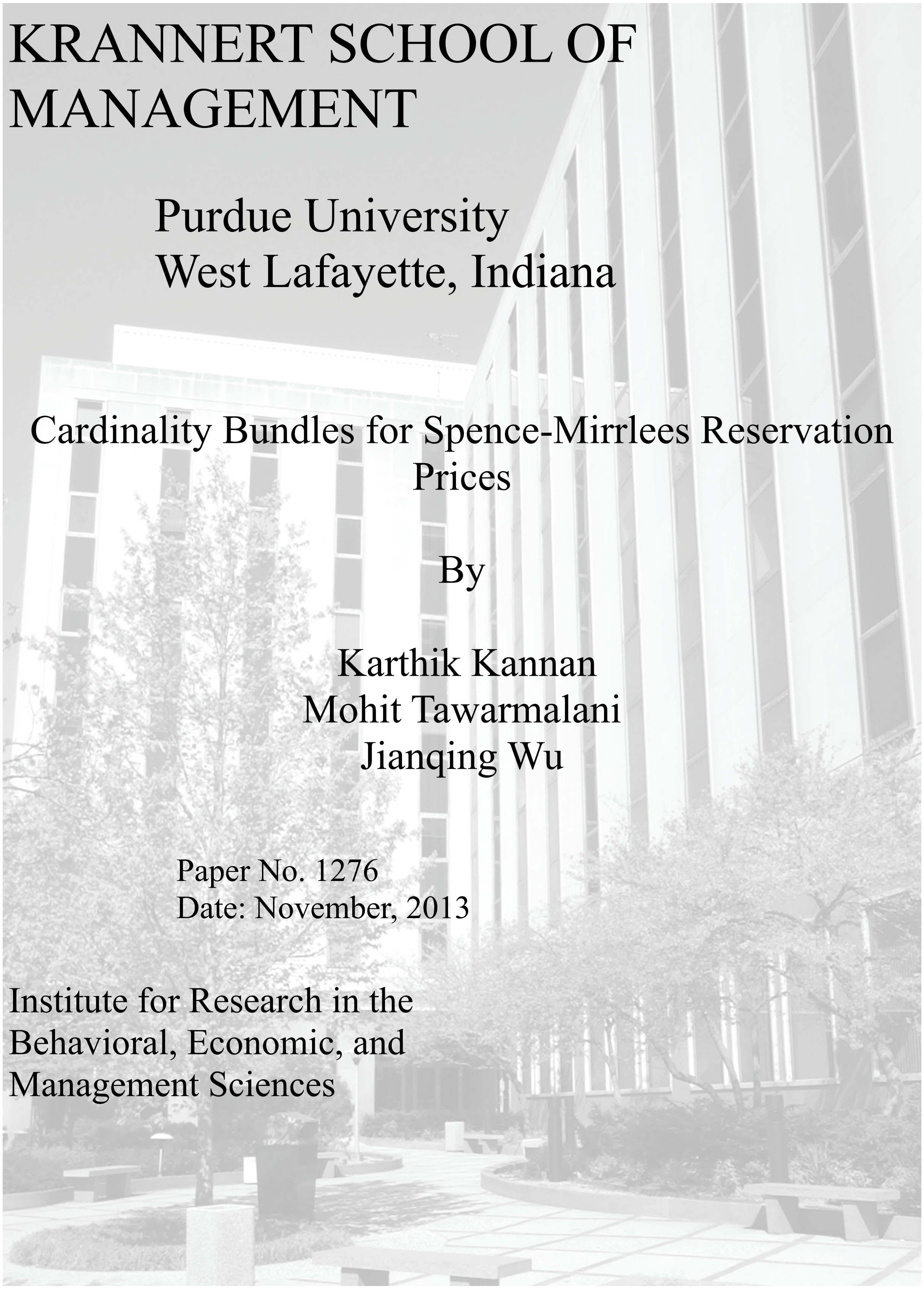

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Cardinality Bundles for Spence-Mirrlees Reservation Prices

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Cardinality Bundles for Spence-Mirrlees Reservation Prices

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Abstract

We study the pricing of cardinality bundles, where firms set prices that depend only on the size of the purchased bundle. The cardinality bundling (CB) problem we study was originally proposed by Hitt and Chen (2005) and it involves consumers having a specific preference structure called Spence-Mirrlees Single Crossing Property (SCP). We show that the optimal prices to the problem can be obtained, in strongly polynomial time, by solving a shortest-path problem. The network structure underlying the shortest path formulation is useful in developing an algorithm to solve the quantity-discount problem proposed by Spence (1980). Lastly, we also study the characteristics of the underlying problem that lead to similar strongly polynomial time solution approaches.

1 Introduction

Bundling and its benefits have been extensively studied in prior literature. Bakos and Brynjolfsson (1999) show that synergies among products can lead to more profitable opportunities when products are bundled than when they are sold separately. Initial work on bundling has studied the pricing strategies when all the possible combinations of bundles are available (Stigler, 1963, Adams and Yellen, 1976, McAfee et al., 1989). The problem with that scheme is that it is only computationally tractable for a small number of goods (Hanson and Martin, 1990).

With the emergence and rapid growth of low-cost reproduction and distribution technologies for information goods, researchers and information goods providers are more and more attracted to various forms of bundling. Other types of bundling that have been studied are: component bundling, where individual components are priced and not the bundles, and pure bundling, where only a bundle with all possible products is available. Of late, a new type of bundling, cardinality bundles (CB) has become popular. CB sets the same price for bundles of equal size. That is to say, for a firm selling J goods, CB offers one price for a bundle of one good, a second price for a bundle of any two goods, a third price for a bundle of any three goods and so on. Therefore, compared to mixed bundling, CB only requires J prices for J bundles. CB is implemented in reality. For example, Chu et al. (2011) note that CB is used to sell seasonal theater tickets.

Literature on CB is limited though. To the best of our knowledge, there have been three other papers that have studied CB. Hitt and Chen (2005) study the problem, assuming that the consumer can buy only one bundle. They explore the conditions under which a mixed bundling problem can be reduced to a CB problem. They also analyze optimal solutions for CB problem with an additional assumption about consumers' reservation price, known as the Spence-Mirrlees Single Crossing Property (SCP). Under the same assumption, i.e., on one bundle per consumer, Wu et al. (2008) explore the properties of the CB problem by using a nonlinear mixed-integer programming approach. They propose a Lagrangian relaxation and a subgradient method for solving the CB problems. The authors develop a heuristic solution strategy and provide an upper bound on the profit. However, there is a residual gap between their best upper and lower bounds at termination. Chu et al. (2011) consider a model where unit prices for bundles are decreasing with size. They computationally study how specifying prices for every bundle in this scenario leads to a profit that is close to offering every combination of bundles.

We begin by considering the model proposed by Hitt and Chen (2005), which derives prices for cardinality bundles assuming that reservation prices follow SCP. We show that the optimal prices can be obtained, in polynomial time, by solving a shortest-path problem. In contrast, the earlier techniques proposed in Hitt and Chen (2005) may generate non-optimal prices. The network structure underlying the shortest path formulation provides many insights into cardinality bundling. It paves the way for developing useful approximation schemes for the continuous case (see Spence (1980) and Section 3), reveals valid inequalities that help determine prices that disincentivize customers from purchasing more than one bundle (Kannan et al., 2013), and allows us to extend our analysis to costs that are not additively separable and include economies of scale as a special case (see Section 4).

The paper is structured as follows. In Section 2, we develop the shortest-path formulation for the discrete CBP model. In Section 3, we revisit the quantity discount problem and derive an approximation algorithm for its solution. Finally, in Section 4, we relax the assumption that costs are additively separable and develop efficient solution techniques for these problems.

2 CBP Discrete Case: Model & Analysis

A customized bundling problem models a situation where a vendor offers a menu of products that maybe purchased as bundles of various sizes and a price for each bundle size. The consumer who purchases a bundle of a particular size is free to choose any set of the products of the same cardinality. This model was originally proposed by Hitt and Chen (2005) where the authors assumed that the consumers could be ordered in a manner such that a consumer of higher type not only assigned a higher value to bundles of a given size but also derived higher marginal value from increasing the bundle size.

In this section, we consider the cardinality bundling problem where the vendor decides the prices and cardinalities of bundles offered in the market. The model, we treat in this section, was originally proposed in Hitt and Chen (2005). For completeness, we first review the basic model. Consider a vendor who sells J products and assume that there are I consumers in the market. In the following, we denote the bundle of size j as Bundle j. The vendor decides the prices p_j of each of the Bundles $j = 1, \ldots, J$. The bundle 0 is offered for free and is introduced to simplify notation, since it represents the case when a consumer chooses not to buy any product from the vendor. We assume that the cost of the Bundle j for vendor is c_j and that the total cost to the vendor is the sum of the costs for the bundle sold. It should be observed that there is no loss of generality in assuming that bundles of all sizes are offered in the market. If it is preferable for the vendor to not offer a bundle, then the price of a bundle of that size can be set to a value that is a slightly larger than the maximum any consumer would be willing to pay for that bundle ensuring that no consumer purchases a bundle of this size.

In the model of Hitt and Chen (2005), it is assumed that each consumer can purchase at most one bundle. Let $w_{ij} \geq 0$ denote the willingness-to-pay (WTP) of Consumer i for Bundle j. For every i, we set w_{i0} to zero to denote that consumers, who do not purchase anything, do not derive any value out of the vendor's products. Since the choice of the bundle rests with the consumer, if Consumer i purchases Bundle j_i, this bundle must maximize her consumer surplus, *i.e.*, $j_i \in \arg\max_j \{w_{ij} - p_j\}$. Let J_i be the set of bundles Consumer i prefers with price vector p. If $|J_i| > 1$, we assume that the consumer i purchases the bundle $j_i \in \arg\max\{p_j - c_j \mid j \in J_i\}$, *i.e.*, the bundle that yields the most profit to the vendor. This assumption is typical in the literature and is without loss of generality.[1]

Let x_{ij} be 1 if Consumer $i \in \{1, 2, \ldots, I\}$ buys Bundle $j \in \{0, 1, 2, \ldots, J\}$ and 0 otherwise. Then, CBP can be formulated as follows (see Wu et al., 2008):

$$\text{CBP1}: \quad \underset{x_{ij}, p_j}{\text{Max}} \quad \sum_{i=1}^{I} \sum_{j=0}^{J} x_{ij}(p_j - c_j)$$

$$\text{s.t.} \quad \sum_{j'=0}^{J} (w_{ij'} - p_{j'}) x_{ij'} \geq w_{ij} - p_j \ \forall i, \ \forall j \tag{1}$$

$$\sum_{j=0}^{J} x_{ij} = 1 \ \forall i \tag{2}$$

$$x_{ij} \in \{0, 1\} \ \forall i, \ \forall j. \tag{3}$$

[1]To see this, let $J'(j) = \{j' \mid p_{j'} - c_{j'} < p_j - c_j\}$ be the set of bundles that provide less profit to vendor than j. Observe that since the number of consumers and bundles is finite, there exists an $\epsilon > 0$ such that even if the price of a bundle that a consumer does not prefer is reduced by $J\epsilon$, the consumer continues to prefer the bundles in J_i after the change. Now, consider a new pricing scheme p', where the price of Bundle j is set to $p'_j = p_j - |J'(j)|\epsilon$. Then, it is easy to verify that, when the prices are p', Consumer i prefers the bundle $j_i \in \arg\max\{p_j - c_j \mid j \in J_i\}$ over other bundles in J_i and, since $|J'(j)| < J$, this preference is also over bundles not in J_i. Further, the vendor does not lose more than $JI\epsilon$ in the profit when he prices the bundles using p' instead of p. Since ϵ can be chosen to be arbitrarily small, this yields a sequence of solutions for which vendor's profit converges to the one obtained under our assumption.

Let (x^*, p^*) be a solution that generates the optimal profit. Constraints (1) enforce incentive compatability (IC) and individual rationality (IR) for Consumer i. The left hand side models the consumer surplus from the purchase decision and the right hand side models the consumer surplus from the purchase of alternate bundles. Setting $j = 0$ ensures that consumer only purchases bundles with non-negative surplus. Constraints (2) enforce that each consumer purchases only one bundle.

In Section 3.4 of Hitt and Chen (2005), in order to develop analytical insights into cardinality bundling, the authors focus on the case where consumer valuations satisfy the Spence-Mirrlees Single Crossing Property (SCP) (see Spence, 1980). We also assume throughout that consumer valuations satisfy this property and remark that this assumption is quite common in models of nonlinear pricing problems, as has also been observed by Hitt and Chen (2005). SCP imposes the following ordering on the consumers' WTP for the bundles:

$$w_{ij} \geq w_{i'j} \; \forall i > i', \tag{4}$$

$$w_{ij} - w_{ij'} \geq w_{i'j} - w_{i'j'} \; \forall i > i', \; \forall j > j'. \tag{5}$$

The interpretation of these conditions is straightforward. A consumer with a higher index has a (weakly) higher WTP for any bundle. Also, the WTP exhibits increasing differences, *i.e.*, as bundle size increases, the WTP for a higher-indexed consumer increases more rapidly than the WTP for a lower-indexed consumer. Essentially, this assumption states that consumers can be ordered by types, with higher type consumers valuing the products and marginal changes in bundle sizes more than the lower type consumers. Before we develop an efficient solution approach for this problem, we review the current approaches available via examples.

Example 1 *Consider a scenario with $I = 4$ consumers, $J = 4$ bundle sizes, and costs $c_j = 0$ for all j. Suppose the WTP for the consumers are as given in Table 1(a). It can be verified easily that*

Table 1: WTP and left hand side values of Equation (6) for Example 1

Bundle size	Consumers' WTP			
	I_1	I_2	I_3	I_4
0	0	0	0	0
1	26	36	58	120
2	47	62	91	180
3	58	77	113	221
4	62	83	123	240

(a) Willingess to Pay ($[w_{ij}]$)

Bundle size	LHS values			
	I_1	I_2	I_3	I_4
0				
1	-4	-8	16	120
2	6	12	-14	60
3	-1	1	3	41
4	-2	-2	1	19

(b) Left hand side values of Equation (6)

they satisfy SCP. Observe that the CBP1 *is a mixed integer nonlinear program (MINLP) since the price vector* p_j *and consumer decisions* x_{ij} *are variables and their products appear in the objective and in Constraint (1). We use BARON (Tawarmalani and Sahinidis, 2002) to solve the MINLP formulation of* CBP1*. Note that this solver guarantees that it finds the global optimal solution at termination. The optimal solution thus found is to set* $p_1^* = p_2^* = 47$*,* $p_3^* = 62$*, and* $p_4^* = 72$*. It is easy to check that, with these prices, Consumer* 1, 2, 3, *and* 4 *buy Bundles* 2, 3, 4, *and* 4 *respectively. The optimal profit for the vendor is* 253.[2]

In the next example, we make a small change to the setting of Example 1 to illustrate that the assignment for a consumer depends on the WTP of all other consumers.

Example 2 *In the setting of Example 1, change* w_{41} *from* 120 *to* 100*, so that WTPs still satisfy SCP. If* CBP1 *is now solved using BARON, the optimal solution assigns Consumer* 1 *to Bundle* 0 *yielding a profit of* 256.[3] *Notwithstanding, if we restrict Consumer* 1 *to be allocated to Bundle* 2 *(by constraining* x_{12} *to* 1 *in* CBP1*) and solve the resulting problem using BARON, the optimal*

[2]Result 3 in Hitt and Chen (2005) proposes the following approach to optimally solve CBP1. Consumer i is assigned to the largest bundle size j that satisfies the following condition:

$$(I - i + 1)(w_{ij} - w_{i,j-1}) - (I - i)(w_{i+1,j} - w_{i+1,j-1}) \geq c_j - c_{j-1}. \tag{6}$$

Notice that, when Consumer i is assigned to a bundle, the WTP of Consumers other than i and $i+1$ are ignored. Here, the right hand side is 0 since we assume $c_{j'} = 0$ for all $1 \leq j' \leq J$. The left hand side values are shown in Table 1(b). For this instance, it can be easily verified that the approach of Hitt and Chen (2005) yields the same optimal solution as was found earlier using BARON.

[3]The optimal assignment of Consumer 1, 2, 3, and 4 is to Bundles 0, 0, 1, and 4 respectively. The corresponding prices are $p_1^* = 58$ and $p_2^* = p_3^* = p_4^* = 198$.

profit reduces to 253.[4] *This shows that the WTP of Consumer* 4 *must be considered if Consumer* 1 *is to be assigned optimally.*[5]

In the literature, there does not exist an approach to solve CBP1 optimally that does not use a global solver on the MINLP formuation directly (also, see Footnote 4). The MINLP-based approach is, however, not amenable to comparative statics because global optimality certificates are typically neither small nor easy to obtain. In this section, we develop an alternate solution approach that is efficient, guarantees optimality, and is amenable to comparative statics. First, we start by converting the MINLP formulation of CBP1 into a mixed integer program (MIP).

2.1 Properties of the Optimal Solution

Proposition 3 *There exists an optimal pricing scheme that is nondecreasing with bundle size.*

See Appendix A.1 for proof.

Proposition 4 *There exists an optimal solution to* CBP1 *that satisfies:*

$$\sum_{j'=j}^{J} x_{i+1j'} \geq \sum_{j'=j}^{J} x_{i,j'} \; i = 1, \ldots, I-1, \; \forall j. \tag{7}$$

That is, there exists an optimal solution where the mapping from consumer types to bundle sizes is non-decreasing, i.e., for any $i < I$, if Consumer i buys Bundle j, then Consumer $i+1$ buys a Bundle j' such that $j' \geq j$.

See Appendix A.2 for proof.

[4] Observe that Equation (6) is independent of w_{41} for the first Consumer (*i.e.*, $i = 1$). Therefore, as in Example 1, Hitt and Chen (2005) claims that it is optimal to assign Consumer 1 to Bundle 2. The solution obtained using this approach or using BARON with x_{12} forced to 1 is the same as that in Example 1 and yields a profit of 253.

[5] In the proof of Result 3, Hitt and Chen (2005) modify the above procedure slightly when the selected bundle sizes do not satisfy the property that higher type consumers buy larger sized bundles. However, this situation does not arise in the example we consider and we do not review this modification here.

Lemma 5 *Amongst the consumers purchasing a non-zero bundle size, the lowest indexed one is charged at her WTP in an optimal solution.*

See Appendix A.3 for proof.

Lemma 5, which was also discovered by Hitt and Chen (2005), defines the starting point to determine optimal prices for all the bundles. The more interesting is Proposition 4, which provides additional yet important constraints for converting CBP1 into an MIP. Note that results similar to Lemma 5 and Proposition 4 are crucial to solving the other cardinality bundling specifications.

2.2 A Polynomial Time Solution

In this section, we demonstrate that a solution to CBP1 can be obtained by solving a shortest path formulation. A key step in obtaining the formulation is that the vendor profit can be obtained by recursively decomposing CBP1 into subproblems.

2.2.1 Converting MINLP to MIP

Define $\mathrm{S}(i', j')$ as the cardinality bundling problem involving Consumers $i', \ldots, I$ and $j', \ldots, J$. Let $\Pi^{\mathrm{S}(i',j')}$ be the maximum vendor profit for the same problem. Consider the problem $\mathrm{S}(i, j)$ where Consumer i is allocated to purchase Bundle j in the problem, which we henceforth refer to as $\mathrm{S}(i, j|x_{ij} = 1)$. We claim that we can compute $\Pi^{\mathrm{S}(i,j|x_{ij}=1)}$ by focusing on the purchases of Consumers $i' > i$. Specifically, it is sufficient to consider the subproblem $\mathrm{S}(i + 1, j)$ since Bundles $j' < j$ will not be purchased by Consumers $i' > i$ (See Proposition 4). We show that $\Pi^{\mathrm{S}(i,j|x_{ij}=1)}$ is obtained by adding an increment to $\Pi^{\mathrm{S}(i+1,j)}$. The increment involves two parts – the cost c_j is incurred from allocating Bundle j to Consumer i; and the change in the revenue as we project the bundle allocations from the subproblem $\mathrm{S}(i + 1, j)$ to the original problem $\mathrm{S}(i, j|x_{ij} = 1)$. In the following paragraph, we illustrate how the revenue changes may be computed during the projection. Note that the process specified here can be recursively repeated to solve the overall vendor's problem $\mathrm{S}(1,0)$.

In Example 2, fix Consumer 1 to purchase Bundle 2. Since the costs are zero in this example, we focus only on the revenues. In order to obtain the revenue for the *higher-level* problem $\mathrm{S}(1,2|x_{12}=1)$, we consider the *subproblem* $\mathrm{S}(2,2)$. The w_{ij}s for the subproblem are shown in Table 2 (a). Two main questions arise. Are the optimal x_{ij} values from the subproblem retained in the higher-level problem also? Further, how can we recover the overall vendor revenue for the higher-level problem from the subproblem? Because we fixed Consumer 1's choice, we know that Bundle 2 will be priced at 47 via Lemma 5. That allocation leaves Consumer 2 with a surplus of $w_{22}-w_{12}=15$ for Bundle 2. In other words, by assigning Consumer 1 to Bundle 2, the vendor is leaving at least 15 for each of the higher type consumers as their additional surplus. So, the vendor sets the optimal x_{ij} as if he deals with consumers with w_{ij} for $i=\{2,3,4\}$ and $j=\{2,3,4\}$ that are less by 15 as seen in Table 2 (b). The "decreased" w_{ij}s and the original w_{ij}s in the subproblem are always different by the same constant and therefore, x^*_{ij} will not be different even if they are projected back to the higher-level problem. Using the vendor revenue obtained by solving the subproblem with the actual w_{ij}s, we recover the optimal revenue for the higher-level problem by first adding 47, which is Consumer 1's payment for Bundle 2, and then subtracting $15\times 3=45$, which is the additional surplus not extracted from Consumer 2, 3, and 4 because of assigning Consumer 1 to Bundle 2. Therefore, in this example, allocating Consumer 1 to purchase Bundle 2 leads to an increment of $w_{12}-3(w_{22}-w_{12})=2$ compared to the subproblem.

Table 2: Subproblems for Example 2

Bundle	Consumers' WTP		
size	I_2	I_3	I_4
2	62	91	180
3	77	113	221
4	83	123	240

(a) w_{ij} of the subproblem

Bundle	Consumers' WTP		
size	I_2	I_3	I_4
2	47	76	165
3	62	98	206
4	68	108	225

(b) Effective w_{ij} after $(w_{22}-w_{12})$ subtraction

The increment as we project the revenues from the subproblem $\mathrm{S}(i+1,j)$ to the higher-level problem $\mathrm{S}(i,j|x_{ij}=1)$ is $v_{ij} = w_{i,j} + (I-i)(w_{i,j} - w_{i+1,j})$. The term w_{ij}, in the first part of v_{ij}'s definition, captures the incremental revenue from offering Bundle j to Consumer i. The second part $(I-i)(w_{ij} - w_{i+1,j})$ captures the decrement in revenue to account for the vendor's inability to extract surplus from other consumers because of offering Bundle j to Consumer i. Recall that the cost c_j is incurred Bundle j is allocated. Therefore, the change in the vendor profit relative to the subproblem is $v_{ij} - c_j$. Formally, we use the definition of v_{ij} to convert the original MINLP problem CBP1 into an MIP.

Proposition 6 *The MINLP problem* CBP1 *can be reformulated as the following 0-1 integer linear problem* CBP2*:*

$$\begin{aligned} \text{CBP2}: \quad \underset{x_{ij}}{Max} \quad & \sum_{i=1}^{I}\sum_{j=0}^{J}(v_{ij} - c_j)x_{ij} \\ s.t. \quad & (2), (3), (7). \end{aligned}$$

See Appendix A.4 for proof.

Converting CBP1 into CBP2 is possible because $\sum_{i=1}^{I}\sum_{j=0}^{J} v_{ij}x_{ij}$ captures the total revenue for any feasible x_{ij}. We are able to ignore the prices since CBP1 can be recursively separated and v_{ij} is the incremental revenue relative to its corresponding subproblem. One may be able to recover the prices from the x_{ij}^* using the process specified in the proof.

In Example 2, v_{ij} is computed as shown in Table 3. So, to compute the profit, the appropriate v_{ij} values are summed up. For example, if a vendor tries to serve Consumer 1 with Bundle 1, Consumer 2 with Bundle 2, Consumer 3 with Bundle 3, and Consumer 4 with Bundle 4, then the total vendor profit is $v_{11} + v_{22} + v_{33} + v_{44} = 245$. The maximum profit is the summation of v_{ij} that yields the maximum value and also that has x_{ij} satisfy Constraints (2), (3), and (7). It is

$v_{1,0} + v_{2,0} + v_{3,1} + v_{4,4} = 256.$

Table 3: Computing v_{ij} for Example 2

Bundle size	v_{ij} I_1	I_2	I_3	I_4
0	0	0	0	0
1	-4	-8	16	100
2	2	4	2	180
3	1	5	5	221
4	-1	3	6	240

2.2.2 Shortest-Path Reformulation

The previous paragraph indicates a possible network flow structure to the problem. This subsection formally shows that CBP2 can be reduced into a shortest-path problem, which is polynomially solvable.

Theorem 7 CBP2 *is equivalent to the following shortest path problem on a graph which has* $2IJ + 2I + 2$ *nodes and* $(I+2)(J+1) + (I-1)(J+1)J/2$ *edges:*

$$\text{CBP3}: \quad \underset{x_{ij},\chi_{ijj'}}{Min} \quad -\sum_{i=1}^{I}\sum_{j=0}^{J}(v_{ij} - c_j)x_{ij}$$

$$s.t. \quad (3)$$

$$\sum_{j=0}^{J} \chi_{01j} = 1 \tag{8}$$

$$\sum_{j=0}^{J} \chi_{IJj} = 1 \tag{9}$$

$$x_{ij} = \sum_{j'=j}^{J} \chi_{ijj'} \ \forall i \forall j \ \forall j' \geq j \tag{10}$$

$$\sum_{j'=0}^{j} \chi_{i-1,j',j} = x_{ij} \ \forall i \ \forall j \ \forall j' \geq j \tag{11}$$

$$\chi_{ijj'} \in 0, 1 \ \forall i \ \forall j \ \forall j' \geq j. \tag{12}$$

See Appendix A.5 for proof. Since the shortest-path problem is an LP, we can relax 0-1 integer constraints on x_{ij} and $\chi_{ijj'}$, and let them be in between 0 and 1 for all i, j, and $j' \geq j$.

Figure 1 shows the shortest path problem structure for Example 2. The following describes the flow for any CBP1 formulation. One unit of flow starts from the top-left node, travels through the network, and finally arrives at the bottom-right node. The flow paths never tend upward to reflect the constraints on x_{ij}. The graph is multipartite with edges marked with dotted and solid lines. The flow must alternate between these types of edges to reach the destination. The variable x_{ij} indicates the flow on the solid edge, and $\chi_{ijj'}$ indicates the flow on the dotted one between the solid edges x_{ij} and $x_{i+1,j'}$. Everytime the flow passes through a solid edge, the cost incurred is $-v_{ij} + c_j$, whereas the cost of passing through the dotted edges is always zero. In Figure 1, the numbers above the solid lines show $-v_{ij} + c_j$ values corresponding to Example 2.

Figure 1: A shortest-path problem structure

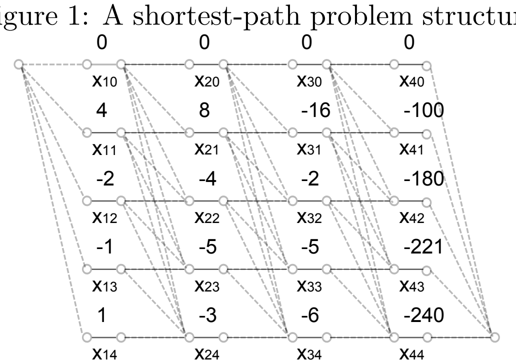
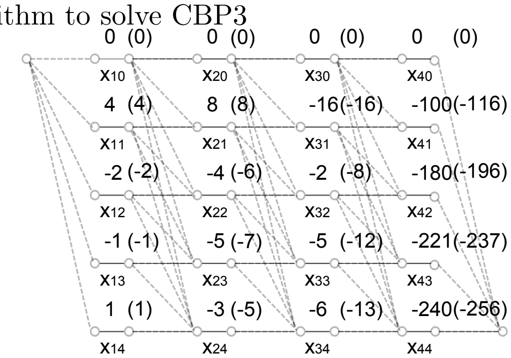


Figure 2: A dynamic programming algorithm to solve CBP3



A simple dynamic program can be used to solve this problem. For any given solid edge, we can record the cost of the shortest path from the top-left node until the edge under consideration. The corresponding numbers for Example 2 are shown in parenthesis in Figure 2. The shortest path for the entire problem is identified by the shortest path cost amongst the last set of solid lines prior to the end node. By tracing the path generated by the shortest path solution, we identify the optimal x_{ij} values. The distance corresponding to the shortest path in Figure 2 is -256. In this procedure, we only need one variable to keep track of the distance associated with the shortest path when considering Consumers 1 through I, and there are J computations for each consumer. Hence, the complexity of the dynamic programming is $O(IJ)$. Note from our solution approach that local differences of v_{ij} is not sufficient to guarantee optimality.

2.3 Additional Insights

This section further develops on the analyses thus far. Given the characterization of v_{ij}, Equation (6) is simply

$$\begin{aligned}
(I-i+1)(w_{ij}-w_{i,j-1}) &- (I-i)(w_{i+1,j}-w_{i+1,j-1}) \geq c_j - c_{j-1} \\
w_{i,j}+(I-i)(w_{i,j}-w_{i+1,j})-c_j &\geq w_{i,j-1}+(I-i)(w_{i,j-1}-w_{i+1,j-1})-c_{j-1} \\
v_{ij}-c_j &\geq v_{i,j-1}-c_{j-1}.
\end{aligned}$$

Note that any such solution approach involving local searches cannot attain the optima, which requires a dynamic programming approach, as we demonstrated earlier.

Second, even though our solution approach is different, we find the following result:

Lemma 8 *The bundle size that each consumer buys in optimal will weakly decrease if the marginal cost increases.*

See Appendix A.6 for proof.

Third, there exists a one-to-one relationship between w_{ij} and v_{ij} terms. Suppose $\boldsymbol{W}$ is the matrix comprised of all w_{ij} values and $\boldsymbol{V}$ of all v_{ij} values. Then, it is easy to show that

$$\boldsymbol{V} = \boldsymbol{W} \begin{pmatrix} I & 0 & \cdots & 0 \\ -(I-1) & (I-1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \text{ and } \boldsymbol{W} = \boldsymbol{V} \begin{pmatrix} \frac{1}{I} & 0 & \cdots & 0 \\ \frac{1}{I} & \frac{1}{I-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{I} & \frac{1}{I-1} & \cdots & 1 \end{pmatrix} \tag{13}$$

Fourth, the transformation between V and W will be useful for computational purposes since we can randomly generate v_{ij} values and use the transformation to generate w_{ij}s that automatically satisfy SCP.

Corollary 9 *For a given* v_{ij} $i \in \{1, 2, \ldots, I-1\}, j \in \{0, 1, \ldots, J\}$ *matrix, there always exists* w_{ij} $i \in \{1, 2, \ldots, I\}, j \in \{0, 1, \ldots, J\}$ *that satisfies SCP.*

See Appendix A.7 for proof. This result shows the flexibility of choosing v_{ij} $i \in \{1, 2, \ldots, I-1\}, j \in \{0, 1, \ldots, J\}$ values and generating w_{ij} satisfying SCP.

3 Continuous Case: Model and Analysis

Hitt and Chen (2005) builds extensively on Spence (1980), which can be construed as involving bundle sizes that are continuous. The approach employed by Spence (1980) to identify the prices is also valid only for highly restricted set of utility functions. As we will expand below, the continuous case is in general difficult to solve. However, the main insights from the discrete case carry over to this section as well, allowing us to arrive at the solution within a pre-specified gap. In the first subsection, we describe the model and the solution approach from Spence (1980) whereas the second subsection describes our solution approach.

3.1 Spence (1980): Model and Solution

The model specification used in this subsection is largely similar to that in the previous subsection except that we use a continuous variable $y \in \Re+$ to represent the bundle sizes, instead of the discrete one j. Naturally, the notations for the variables also change slightly: $p(y)$ represents the price for bundle size y; $c(y)$ represents the cost of bundle y; $w_i(y)$ the consumers' WTP for bundle size y; and y_i the bundle size that consumer i buys. The variable y_i is equivalent to $\sum_{j=0}^{J} jx_{ij}$ in the discrete case. Spence (1980) also assumes SCP and he states it as $w_i'(y) < w_{i+1}'(y)$ $\forall y$. Those conditions are equivalent to:[6]

$$0 = w_i(0) \leq w_i(y) \quad < \quad w_{i+1}(y)\ \forall y \tag{14}$$

$$w_i(y+d) - w_i(y) \quad < \quad w_{i+1}(y+d) - w_{i+1}(y)\ \forall y\ \forall d \geq 0. \tag{15}$$

The vendor's decision problem in the continuous case is as follows:

$$\text{CBPc1}: \quad \underset{y_i, p(y_i)}{\text{Max}} \quad \sum_{i=1}^{I}(p(y_i) - c(y_i)) \tag{16}$$

$$\text{s.t.} \quad w(y_i) - p(y_i) \geq w(y) - p(y)\ \forall i\ \forall y. \tag{17}$$

When solving the decision problem, the following constraint – which Spence (1980) finds the solution has to also satisfy – has to be included in CBPc1:

$$y_{i+1} \geq y_i\ \forall i \leq I - 1. \tag{18}$$

As a solution procedure, Spence (1980) shows that when WTP is differentiable, the optimal bundle size y_i for each consumer i can be obtained by solving the following equation:

$$(I - i + 1)w_i'(y_i) - (I - i)w_{i+1}'(y_i) = c'(y_i). \tag{19}$$

The equation is the first order condition $v_i'(y_i) = c'(y_i)$ if v_i, similar to the discrete case, is:

$$v_i(y_i) \quad = \quad (I - i + 1)w_i(y_i) - (I - i)w_{i+1}(y_i), \tag{20}$$

At first blush, solving the problem may not seem to be hard since each consumer's decision

[6] $w_i'(y) < w_{i+1}'(y)\ \forall y; \Rightarrow \int_y^d w_i'(y')dy' < \int_y^d w_{i+1}'(y')dy' \forall y\ \forall d; \Rightarrow w_i(y+d) - w_i(y) < w_{i+1}(y+d) - w_{i+1}(y) \forall y\ \forall d$

is independent of the other. However, the approach works only when the optimal solutions are not tight on any of the constraints in Equation (18) and those happen only under very restrictive cases. In most cases, the constraints are tight, and as a result the Lagrange multipliers are non-zero. Therefore, the optimality condition (19) no longer decomposes by consumer. Spence (1980) recognizes this limitation but does not propose an alternative approach.

There is yet another issue with Spence's approach. Condition (19) is only a local optimal condition. There may exist exponentially many solutions that satisfy Condition (19):

Example 10 *Consider* CBPc1 *with zero marginal cost, I consumers, and between* 0 *and J bundle sizes being available. Let*

$$w_i(y) = \frac{I}{I-i+1}(4Jy - y^2) - cos(y\pi)\ \forall i,$$

whice can be verified to satisfy SCP. As defined earlier,

$$v_i(y_i) = (I-i+1)w_i(y_i) - (I-i)w_{i+1}(y_i) = -cos(y_i\pi)\ \forall i.$$

The local optimal condition $v_i'(y_i)$ is satisfied for all $y_i \in \{1, 3, \cdots, \hat{J}\}$, where $\hat{J}$ is the largest odd number less than or equal to J. In other words, for every Consumer i, all odd values of y_i satisfy the local optimal condition and there are exponentially many different combinations of y_i^* that satisfy Equations (18) and (19). Choosing the optimal solution from the combinations remains a challenge.

Spence (1980) does not provide further detail on this calculation. To the best of our knowledge, this is not an easy problem to solve. Next, we discuss our solution approach.

3.2 Our proposed solution

Note that in Spence (1974), there is no explicit upper bound on the bundle-sizes provided by the vendor. To be consistent with the discrete case, we hereafter impose a restriction that the vendor only provides bundles of sizes Y or smaller. This assumption is not unreasonable because the vendor may be limited by production capacity constraints. So, in our formulation, CBPc1 will include both Equation (18) and the constraint $0 \leq y \leq Y$.

Solving CBPc1 is difficult for two reasons: (a) the decision variable $p(y)$ is a (continuous) function now, unlike in the discrete case when the corresponding variable is a set of point values; and (b) it is a non-convex problem because the feasible set defined by Constraint (17) is not convex. In the discrete case, it was perhaps easy to solve because the discretized version of Constraint (17) (which would have been $\sum_{j'=0}^{J} j' x_{i+1j'} \geq \sum_{j'=0}^{J} j' x_{i,j'}$) is equivalent to Constraint (7) (i.e., $\sum_{j'=j}^{J} x_{i+1j'} \geq \sum_{j'=j}^{J} x_{i,j'}$). So, we begin by exploring when the two conditions specified in the previous sentence are equivalent. For that, we redefine the formulation for the continuous case slightly differently.

Let $k_j, j \in \{0, 1, ...N\}$ be an arbitrary set of points corresponding to the y variable such that $k_0 = 0$, $k_N = Y$, and $k_{j+1} > k_j, j \in \{0, 1, ...N-1\}$. Instead of selecting y_i directly, the consumers choose $\tilde{x}_{ij}$, the weights placed on the discrete points – with restrictions $\sum_{j=0}^{N} \tilde{x}_{ij} = 1$ and $0 \leq \tilde{x}_{ij} \leq 1$. Thereby, they indirectly choose $y_i = \sum_{j=0}^{N} k_j \tilde{x}_{ij}$. In addition, we impose the *adjacency restriction* that at most two adjacent $\tilde{x}_{ij}$ variables are non-zero i.e., $\tilde{x}_{ij}\tilde{x}_{ij'} = 0 \ \forall j \leq N-2,\ \forall j' \geq j+2$.

Lemma 11 *For $i \in \{i_1, i_2\}$, if $\tilde{x}_{ij}\tilde{x}_{ij'} = 0 \ \forall j \leq N-2,\ \forall j' \geq j+2$, $\sum_{j'=0}^{N} \tilde{x}_{ij'} = 1$, and $0 \leq \tilde{x}_{ij} \leq 1 \ \forall j$, then $\sum_{j'=0}^{N} k_{j'}\tilde{x}_{i_2,j'} \geq \sum_{j'=0}^{N} k_{j'}\tilde{x}_{i_1,j'}$ is equivalent to $\sum_{j'=j}^{N} \tilde{x}_{i_2,j'} \geq \sum_{j'=j}^{N} \tilde{x}_{i_1,j'}$, $\forall j$.*

See Appendix B.1 for proof. It must be easy to realize why in the discrete case Lemma 11 is valid:

$k_j \in \{1, 2, 3, \ldots, J\}$ and the constraints specified in the lemma are naturally satisfied.

Using the lemma, we can transform Equation (18) in CBPc1. Also, similar to the discrete case, we recursively decompose the optimization problem to obtain the objective function $\sum_{i=1}^{I} v_j(y_i) - c_j(y_i)$. Both together transforms CBPc1 to yield:

$$\text{CBPc2}: \quad \underset{\tilde{x}_{ij}}{\text{Max}} \quad \sum_{i=1}^{I}\left(v_i\left(\sum_{j=0}^{N}\tilde{x}_{ij}k_j\right) - c\left(\sum_{j=0}^{N}\tilde{x}_{ij}k_j\right)\right) \tag{21}$$

$$\sum_{j=0}^{N}\tilde{x}_{ij} = 1 \;\forall i \tag{22}$$

$$\sum_{j'=j}^{N}\tilde{x}_{ij'} \geq \sum_{j'=j}^{N}\tilde{x}_{i-1,j'} \;\forall i \geq 2,\; \forall j \tag{23}$$

$$\tilde{x}_{ij}\tilde{x}_{ij'} = 0 \;\forall j \leq N-2,\; \forall j' \geq j+2 \tag{24}$$

$$0 \leq x_{ij} \leq 1 \;\forall i,\; \forall j. \tag{25}$$

Notice that CBPc2 is similar to CBP2, except mainly for the adjacency restriction. We next show that when the WTP and cost functions are piecewise linear, the adjacency restriction becomes irrelevant. So, specifically, consider a formulation where k_js correspond to the breakpoints of the WTP and the cost functions, *i.e.*, for any i, let $\{w_i(y), c_j(y)\}$

$$= \begin{cases} \{w_i(y), c_j(y)\} & \text{if } y = k_j \\ \left\{ \underbrace{\frac{y-k_j}{k_{j+1}-k_j}}_{=\tilde{x}_{ij}} w_i(k_{j+1}) + \underbrace{\frac{k_{j+1}-y}{k_{j+1}-k_j}}_{=\tilde{x}_{i,j+1}} w_i(k_j), \underbrace{\frac{y-k_j}{k_{j+1}-k_j}}_{=\tilde{x}_{ij}} c_j(k_{j+1}) + \underbrace{\frac{k_{j+1}-y}{k_{j+1}-k_j}}_{=\tilde{x}_{i,j+1}} c_j(k_j) \right\} & \text{if } k_{j-1} < y_i < k_j. \end{cases}$$

Correspondingly, define $v_{ij} = v_i(k_j) = w_i(k_j) + (I-i)(w_i(k_j) - w_{i+1}(k_j))$.

Theorem 12 *In the case with piecewise linear functions, Constraints (24) may be relaxed without* *any change to the optimal solution. The resulting decision problem is similar to* CBP2*:*

$$\text{CBPpl}: \quad \underset{\tilde{x}_{ij}}{Max} \quad \sum_{i=1}^{I}\sum_{j=0}^{N}(v_{ij}-c_j)\tilde{x}_{ij}$$
$$s.t. \quad \sum_{j=0}^{N}\tilde{x}_{ij} = 1 \ \forall i$$
$$\sum_{j'=j}^{N}\tilde{x}_{ij'} \geq \sum_{j'=j}^{N}\tilde{x}_{i-1,j'} \ \forall i \geq 2, \ \forall j$$
$$0 \leq \tilde{x}_{ij} \leq 1 \ \forall i, \ \forall j.$$

See Appendix B.2 for proof. The case with piecewise linear functions now can borrow the shortest path solution method we developed for the discrete case.

Even if the WTP and cost functions are not necessarily piecewise linear, we can use the previous approach by approximating the WTP and cost functions. For the approximation, $w_i(y)$ and $c(y)$ must be Lipschitz continuous, with β as the Lipschitz constant, *i.e.*, $\max\{\max_y c'(y), \max_{i,y} w_i{}'(y)\} \leq \beta < \infty$. We define variables $w_i^{pl}(y)$ and $c_j^{pl}(y)$ as piecewise linear appoximations of the WTP and cost functions respectively. So, if k_js correspond to the breakpoints, $w_i^{pl}(k_j) = w_i(k_j)$ and $c_j^{pl}(k_j) = c_j(k_j)$; whereas, for $k_j < y < k_{j+1}$, $w_i^{pl}(y)$ and $c_j^{pl}(k_j)$ are simply affine combinations of the respective functional values at k_j and k_{j+1}. Further, we let $k_j = jk, \ j \in \{0, ...N\}$, where k is an arbitrary constant (such that, as before, $k_0 = 0$ and $k_N = Y$). Let Π^{pla*} be the optimal vendor profit when using the approximate piecewise linear functions and Π^{c*} be the optimal profit for the original problem without the approximation.

Theorem 13 *If* $\epsilon_t = 2I(I+2)k\beta$*,*

$$\Pi^{pla*} \leq \Pi^{c*} \leq \Pi^{pla*} + \epsilon_t. \tag{26}$$

The solution approach to generate Π^{pla*} *for a continuous problem is polynomial in both the problem* *size $I \times J$ and $\frac{1}{\epsilon_t}$. So, it is a fully polynomial-time approximation scheme or FPTAS.*

See Appendix B.3 for proof. Given the error tolerance, ϵ_t, the vendor can choose k such that $k = \frac{\epsilon_t}{2I(I+2)\beta}$ and use the approximation functions to solve the decision problem. Also, notice that the problem which initially seemed difficult to solve, can now be approximated as a shortest path problem, solvable in $O(I^3 J/\epsilon_t)$. Since the solution approach is polynomial in both the problem size $I \times J$ and $1/\epsilon_t$, our approach provides a FPTAS solution, *i.e.*, we can solve the problem efficiently.

4 Submodular Cost Function

Thus far, the costs were assumed to be separable in the bundles, much like in Hitt and Chen (2005) and Spence (1980). Our solution approaches, which are strongly polynomial time algorithms, are highly dependent on the separability of the cost function. However, in reality, this may not be the case. For example, if the cost is concave in the total quantity across all the bundles sold (*i.e.*, economies of scale exist), then our solution approach will not be applicable. So, this section focuses on developing solution approaches to submodular cost functions. In the subsections below, we consider the discrete and the continuous cases separately.

4.1 Discrete Case

In this case, we model the problem similar to CBP1 except that we assume a submodular cost, $C\left(\sum_{j=0}^{J} x_{1j}j, \sum_{j=0}^{J} x_{2j}j, \ldots, \sum_{j=0}^{J} x_{Ij}j\right)$. The production cost is a function that is not necessarily separable in the bundles. Then, the vendor's decision problem is:

$$\begin{aligned} \text{CBPg}: \quad & \underset{x_{ij}}{\text{Max}} \quad \sum_{i=1}^{I}\sum_{j=0}^{J} p_{ij}x_{ij} - C\left(\sum_{j=0}^{J} x_{1j}j, \sum_{j=0}^{J} x_{2j}j, \ldots, \sum_{j=0}^{J} x_{Ij}j\right) \\ & \text{s.t.} \quad (1), (2), (3). \end{aligned}$$

The procedure in Section 2.2.1 to transform the original MINLP specification for the discrete case with separable costs to MIP is based on how consumer's purchasing decision affects the revenue for the vendor. Even in CBPg the revenue term is separable, and so the same definition of v_{ij} and the same procedure can be applied to transform CBPg into

$$\begin{aligned} \text{CBPg2}: \quad & \underset{x_{ij}}{\text{Max}} \quad \sum_{i=1}^{I}\sum_{j=0}^{J} v_{ij}x_{ij} - C\left(\sum_{j=0}^{J} x_{1j}j, \sum_{j=0}^{J} x_{2j}j, \ldots, \sum_{j=0}^{J} x_{Ij}j\right) \\ & \text{s.t.} \quad (2), (3), (7). \end{aligned}$$

The new formulation cannot be characterized as a shortest-path problem. Therefore, to solve the problem, we define a new decision variable based on the earlier one: $z_{ij} = \sum_{j'=j}^{J} x_{ij'}\ \forall i,\ \forall j$. Loosely speaking, z_{ij} captures decisions at the level of components within a bundle. For example, if some i has $x_{i5} = 1$, z_{ij} for all the components $j = \{0, 1, \ldots, 5\}$ that help achieve a bundle size of 5 is equal to 1. Our focus on the components is also consistent with submodularity, which is an assumption about the decreasing *marginal* values, of the cost.

Using the z_{ij} variables, CBPg2 can be converted to the following problem:

$$\text{CBPg3}: \quad \underset{z_{ij}}{\text{Max}} \quad \sum_{i=1}^{I}\sum_{j=0}^{J} v_{ij}(z_{ij} - z_{i,j+1}) - C\left(\sum_{j=0}^{J} z_{1j}, \sum_{j=0}^{J} z_{2j}, \ldots, \sum_{j=0}^{J} z_{Ij})\right) \tag{27}$$

$$\text{s.t.} \quad z_{i0} = 1\ \forall i \tag{28}$$

$$z_{ij} \leq z_{i,j+1}\ \forall i\ \forall j \leq J-1 \tag{29}$$

$$z_{ij} \geq z_{i+1,j}\ \forall i \leq I-1;\ \forall j \tag{30}$$

$$z_{ij} \in \{0,1\}\ \forall i;\ \forall j, \tag{31}$$

where $z_{i,J+1} = 0$. If the z_{ij} values are represented in a matrix, Z, the objective function is $\Pi(Z)$ and the cost is $C(Z)$. $\Pi(Z)$ is obtained directly from setting $x_{ij} = z_{ij} - z_{i,j+1}$ and also using

$\sum_{j=0}^{J} z_{ij}$ to capture the bundle size bought by consumer i. Constraints (30) replace the constraints $\sum_{j'=j}^{J} x_{ij'} \geq \sum_{j'=j}^{J} x_{i-1,j'}$. Constraints (28), (29), and (31) together replace $\sum_{j=0}^{J} x_{ij} = 1$.

Theorem 14 *Since the cost function $C(Z)$ is submodular in Z, then* CBPg3 *problem maximizes a supermodular objective function over a lattice. Hence, it is solvable in strongly polynomial time.*

The objective function, with the submodular cost function, is supermodular in Z. The feasible Z belongs to a lattice (see Appendix C.1 for additional explanation), something which was not obvious with x_{ij} values. The result that a supermodular function can be maximized over a lattice in strongly polynomial time is from Theorem 49.2 in Schrijver (2003).

Once we establish the problem as maximizing a supermodular function over a lattice, some properties in our formulation become corollaries of results established in the prior literature. For example, a result similar to Corollary 8 can be established using the same proof technique.

Corollary 15 *The bundle size that each consumer buys weakly decreases as the marginal cost increases.*

4.2 Continuous Case

Here again, similar to the discrete case, we include a submodular cost function, $C(y_1, y_2, \ldots, y_I)$. If $v_i(y_i) = (I - i + 1)w_i(y_i) - (I - i)w_{i+1}(y_i)\ \forall i$, as before, the vendor's decision problem is:

$$\begin{aligned} \text{CBPgc2}: \quad \underset{y_i}{\text{Max}} \quad & \sum_{i=1}^{I} v_i(y_i) - C(y_1, y_2, \ldots, y_I) \\ \text{s.t.} \quad & (18). \end{aligned}$$

For now, impose the restriction that the WTP functions are piecewise linear. Unlike in the separable cost case, we are unable to impose the piecewise linearity restriction on the cost function (because it is defined over I dimensions). So, the restriction needed on the cost in order to solve the vendor's problem in polynomial time is the focus of our analysis initially.

Let $k_j,\ j \in \{0, 1, ...N\}$ be the set of breakpoints of the WTP functions. Also, let the discrete k_j points correspond to the y variable such that $k_N = Y$, $k_0 = 0$, and $k_{j+1} > k_j, j \in \{0, 1, ...N-1\}$. Define $v_{ij} = v_i(k_j) = w_i(k_j) + (I - i)(w_i(k_j) - w_{i+1}(k_j))$, which corresponds to $v_i(y_i)$ values at the breakpoints. Similar to the discrete case, we consider a new decision variable instead of y_i, which relates to y_i as follows:

$$\tilde{z}_{ij} = \begin{cases} 1, & \text{if } k_j \leq y_i \\ y_i - k_j, & \text{if } k_j < y_i < k_{j+1} \\ 0, & \text{otherwise.} \end{cases} \quad \forall i \ \forall j$$

As before, we impose an the adjacency constraint to ensure that for each i, only one $\tilde{z}_{ij}$ can take value between 0 and 1. Therefore, CBPgc2 with piecewise linear w_i functions can be converted to:

$$\begin{aligned} \text{CBPgc3}: \quad \underset{\tilde{z}_{ij}}{\text{Max}} \quad & \sum_{i=1}^{I}\sum_{j=0}^{N} v_{ij}(\tilde{z}_{ij} - \tilde{z}_{i,j+1}) - C\left(\sum_{j=0}^{N}\tilde{z}_{1j}, \sum_{j=0}^{N}\tilde{z}_{2j}, \ldots, \sum_{j=0}^{N}\tilde{z}_{Ij})\right) \\ \text{s.t.} \quad & \tilde{z}_{i0} = 1 \ \forall i \\ & \tilde{z}_{ij} \leq \tilde{z}_{i,j+1} \ \forall i \ \forall j; \leq N-1 \\ & \tilde{z}_{ij} \geq \tilde{z}_{i+1,j} \ \forall i \leq I-1; \ \forall j \\ & (\tilde{z}_{ij} - \tilde{z}_{i,j-1})(\tilde{z}_{ij'} - \tilde{z}_{i,j'-1}) = 0 \ \forall i; \ \forall 1 \leq j \leq N-2; \ \forall j' \geq j+2 \qquad (32) \\ & 0 \leq \tilde{z}_{ij} \leq 1 \ \forall i; \ \forall j, \end{aligned}$$

where $z_{i,J+1} = 0$. CBPgc3 is similar to the discrete problem, CBPg3, except for the adjacency constraint (32) and $\tilde{z}_{ij}$ being a continuous as opposed to an integer variable. Those restrictions will naturally be satisfied if the optimal solution to CBPgc3 occurs at the N breakpoints. The condition on the cost when such a solution occurs is specified in the following lemma, where $\tilde{Z}$ represents z_{ij} values in a matrix form.

Lemma 16 *Let $\tilde{\mathbb{Z}}$ be the set of $\tilde{Z}$ matrices that are feasible to* CBPgc3*, and $\tilde{\mathbb{Z}}_{in} \subseteq \tilde{\mathbb{Z}} | z_{ij} \in \{0,1\}$. The optimal solution $\tilde{Z}^*$ for* CBPgc3 *is in $\tilde{\mathbb{Z}}_{in}$ if $\forall \tilde{Z} \in \tilde{\mathbb{Z}}$, $\exists$ $(\lambda_1, \lambda_2, \ldots, \lambda_M) | \sum_{m=1}^{M} \lambda_m = 1;\ \lambda_m \geq 0 \forall m$ such that $\tilde{Z}_m \in \tilde{\mathbb{Z}}_{in},\ m \in \{1, 2, \ldots, M\}, \tilde{Z} = \sum_{m=1}^{M} \lambda_m \tilde{Z}_m$ and $C(\tilde{Z}) \geq \sum_{m=0}^{M} \lambda_m C(\tilde{Z}_m)$.*

See Appendix C.2 for proof. The conditions specified in the lemma are: every feasible solution of CBPgc3 must be obtainable through an affine combination of feasible solutions occuring at the breakpoints, and the same affine combinations of the corresponding costs acts as an underestimator of the continuous cost. If the conditions are satisfied, then the optimal solution of CBPgc3 occurs at the breakpoints. Note that, even if the cost is not submodular but satisfies the conditions in Lemma 16, we can solve the problem in strongly polynomial time.

We next use the results about cost decomposition to solve CBPgc2 when the WTP functions are not necessarily piecewise linear. We require $w_i(y)$ and $c(y_1, y_2, \ldots, y_I)$ to be Lipschitz continuous functions with gradients never greater than a constant β. Let us approximate the continuous function $w_i(y)$ to be piecewise linear $w_i^{pl}(y)$ with breakpoints at $k_j,\ j \in \{0, 1, ...N\}$. If the cost function is such that it inherently satisfies the conditions in Lemma 16 (*i.e.*, for any affine combination of feasible solutions, the corresponding affine combination of cost generates an under-estimator of the continuous cost function), then we can directly solve the approximate piecewise linear problem as a discrete problem. As an example, a concave submodular cost function $\sum_{i=0}^{I} C_i(y_i) + \beta C_t(y_1 + y_2 + \ldots + y_I)$ where $C_t()$ is concave in total demand, inherently satisfies the conditions in Lemma 16. Otherwise, we need to construct an approximation to the cost function. For that, we follow Tawarmalani et al. (2013) to decompose any submodular function over a lattice feasible set that yields the minimum weighted objective values in a way that also satisfies Lemma 16. With those approximate cost functions, we can obtain solutions at the breakpoints of the WTP functions using the strongly polynomial techniques described for the discrete case.

Suppose the approximate cost function is $C^a(\tilde{Z})$. Then, $C^a(\tilde{Z})$ must be such that $C^a(\tilde{Z}) =$

$C(\tilde{Z})$, $\forall \tilde{Z} \in \tilde{\mathbb{Z}}_{in}$ (*i.e.*, $C^a(\tilde{Z})$ and $C(\tilde{Z})$ take the same values at the breakpoints of the WTP functions). The decomposition generates λ_m such that the approximate function $C^a(\tilde{Z})$ as $C^a(\tilde{Z}) = \sum_{m=0}^{M} \lambda_m C(\tilde{Z}_m)$. Algorithm 1 describes the process to generate the decomposition $\sum_{m=1}^{M} \lambda_m \tilde{Z}_m$ for any given $\tilde{Z}$. An alternative explanation of the algorithm is available in Appendix C.3. We demonstrate the process using an example.

$\lambda_0 = 1$
for $m = 1 \to I + 1$ **do**
  $\tilde{z}_{max} = 0, \hat{i} = 0, \hat{j} = 0$
  $\tilde{Z}_m = \tilde{Z}$
  for $i = 1 \to I$ **do**
    for $j = 1 \to N$ **do**
      if $0 < \tilde{z}_{mij} < 1$ **then**
        if $\tilde{z}_{mij} > \tilde{z}_{max}$ **then**
          $\tilde{z}_{max} = \tilde{z}_{mij}, \hat{i} = i, \hat{j} = j$
        end if
        $\tilde{z}_{mij} = 0$
      end if
    end for
  end for
  $\lambda_m = \lambda_{m-1} - \tilde{z}_{max}$
  $\tilde{z}_{\hat{i}\hat{j}} = 1$
end for

Algorithm 1: Generating $\tilde{Z}_m$ and λ_m

Example 17 *Suppose there are three consumers. The continuous solution is that Consumer* 1 *purchases bundle* 0.4*,* 2 *purchases* 0.7*, and* 3 *purchases* 1.6*. The corresponding* $\tilde{Z}$ *is then* $\begin{pmatrix} 0.4 & 0.7 & 1 \\ 0 & 0 & 0.6 \end{pmatrix}$.

Then, using the algorithm, we get $\lambda = (0.3, 0.1, 0.2, 0.4)$*. Next, we have* $\tilde{Z}_1 = e(\tilde{Z}, 1) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

Similarly, we get $\tilde{Z}_2 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $\tilde{Z}_3 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$*, and* $\tilde{Z}_4 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$*. We can easily verify that the conditions* $\sum_{m=1}^{M} \lambda_m = 1$ *and* $\tilde{Z} = \sum_{m=1}^{M} \lambda_m \tilde{Z}_m$ *are satisfied.*

So, with the approximate piecewise linear WTP functions and the approximate cost function

$C^a(\tilde{Z})$, we can solve the problem using the techniques described for the discrete problem. The following theorem evaluates the gap between the approximate and the actual continuous solution. Let Π^{gpla*} be the optimal vendor profit when using the approximate piecewise linear functions and Π^{gc*} be the optimal profit for the original problem without the approximation.

Theorem 18 $\Pi^{gpla*} \leq \Pi^{gc*} \leq \Pi^{gpla*} + 2I(I+2)k\beta$.

See Appendix C.4 for proof.

5 Conclusion

Pricing of cardinality bundles has not been widely studied in literature although this bundling scheme is increasingly being adopted in industry. Our paper provides a comprehensive analysis of the problem when the consumer's willingness to pay satisfies Spence-Mirrlees condition and consumers are restricted to buy only one bundle. We contributed to this domain in three main regards. First, in the context of discrete bundle sizes, the problem first considered in Hitt and Chen (2005), we provide a solution approach, based on reformulating the problem as a shortest-path problem. Second, we revisit the quantity discount problem proposed in Spence (1980) and derive insights and solution approaches. Third, we extend the analyses to consider problems where the costs are not additively separable by bundle sizes.

Appendix

A Proofs for the Discrete Case

A.1 Proof of Proposition 3

Proof. We first show that there exists an optimal price vector that is non-decreasing. Assume p' is an optimal price vector that is not non-decreasing and is such that the smallest index k for which $p'_k > p'_{k+1}$ is the largest among all optimal price vectors. We claim that for every feasible solution to CBP1 $x_{ik} = 0$ for all i. Assume otherwise that $x_{ik} = 1$ for some i. Then, Constraint 2 implies that $x_{ik'} = 0$ for all $k' \neq k$. Since $w_{ik} \leq w_{ik+1}$ for all i, it follows that $w_{ik} - p'_k < w_{ik+1} - p'_{k+1}$ and Constraint 1 is violated. Therefore, we may assume that $x_{ik} = 0$. Then, consider an alternate price vector p such that $p_j = p'_j$ for all $j \neq k$ and $p_k = p_{k+1}$. Now, consider a solution (x, p') that is feasible to CBP1. First, observe that since $x_{ik} = 0$, the objective value corresponding to x is the same regardless of whether the price is p' or p. We claim that (x, p) is also feasible to CBP1 and therefore the optimal value with price p does not decrease. This follows from:

$$\sum_{j'=0}^{J} (w_{ij'} - p_{j'})x_{ij'} \geq \sum_{j'=0}^{J} (w_{ij'} - p'_{j'})x_{ij'} \geq w_{ik+1} - p'_{k+1} \geq w_{ik} - p_k,$$

where the first inequality follows since $p' \geq p$, the second because of feasibility of (x, p') and the last because $w_{ik+1} \geq w_{ik}$ and $p_k = p'_{k+1}$. If there is any other index k' such that $p_{k'} > p_{k'+1}$ it contradicts the choice of p' since $k' > k$. Therefore, p must be non-decreasing. ■

A.2 Proof of Proposition 4

Proof. Let $j_k(i')$ denote the bundle consumer i' buys in the k^{th} optimal solution to CBP1. Then, let $k' = \arg\max_k \min_i \{i \mid j_k(i) > j_k(i+1)\}$. This means that k' is the optimal solution where the first consumer that buys a higher type bundle than its immediate successor is of the largest type. Now, construct the solution $j(i')$ where $j(i') = j_{k'}(i')$ when $i' \neq i+1$ and $j(i+1) = j_{k'}(i)$. We show that $j(i')$ is a feasible assignment of bundles to consumers which achieves at least the same objective function value, thus deriving a contradiction to the choice of k'. Since we do not change the assignment for any $i' \neq i+1$, we only need to verify that $j(\cdot)$ satisfies $w_{i+1j(i+1)} - p_{j(i+1)} \geq w_{i+1j} - p_j$ for all j. Now, consider the following chain of inequalities:

$$0 \geq w_{i+1j_{k'}(i)} - p_{j_{k'}(i)} - w_{i+1j_{k'}(i+1)} + p_{j_{k'}(i+1)} \geq w_{ij_{k'}(i)} - p_{j_{k'}(i)} - w_{ij_{k'}(i+1)} + p_{j_{k'}(i+1)} \geq 0,$$

where the first inequality follows because $j_{k'}(i+1)$ is what $i+1$ chooses under $j_{k'}(\cdot)$, the second inequality because $j_{k'}(i) > j_{k'}(i+1)$ implies by SCP that $w_{i+1j_{k'}(i)} - w_{i+1j_{k'}(i+1)} \geq w_{ij_{k'}(i)} - w_{ij_{k'}(i+1)}$ and the last inequality because i chooses $j_{k'}(i)$ under $j_{k'}(\cdot)$. Therefore, equality holds throughout. Then, for any j, it follows that:

$$w_{i+1j(i+1)} - p_{j(i+1)} = w_{i+1j_{k'}(i)} - p_{j_{k'}(i)} = w_{i+1j_{k'}(i+1)} - p_{j_{k'}(i+1)} \geq w_{i+1j} - p_j,$$

where the first equality follows because $j(i+1) = j_{k'}(i)$, the second equality follows from the argument above and the third inequality because $i+1$ chooses $j_{k'}(i+1)$ under the feasible solution $j_{k'}(\cdot)$. Therefore, we have shown that $j(i')$ is a feasible assignment of bundles to consumers. Now, we show that the corresponding objective value does not decrease. This follows since

$$\sum_{i'} \left(p_{j(i')} - c_{j(i')}\right) = \sum_{i' \neq i+1} \left(p_{j_{k'}(i')} - c_{j_{k'}(i')}\right) + p_{j_{k'}(i)} - c_{j_{k'}(i)} \geq \sum_{i'} \left(p_{j_{k'}(i')} - c_{j_{k'}(i')}\right),$$

where the first equality follows by the definition of $j(i')$. To see the first inequality, observe that we have shown that $w_{ij_{k'}(i)} - p_{j_{k'}(i)} - w_{ij_{k'}(i+1)} + p_{j_{k'}(i+1)} = 0$. Therefore, $j_{k'}(i+1)$ yields the same surplus for i as does $j_{k'}(i)$. Then, the inequality follows since $p_{j_{k'}(i)} - c_{j_{k'}(i)} \geq p_{j_{k'}(i+1)} - c_{j_{k'}(i+1)}$. Otherwise, the solution $j'(i') = j_{k'}(i')$ for $i' \neq i$ and $j'(i) = j_{k'}(i+1)$ would be feasible and would yield a strictly higher objective value than $j_{k'}(\cdot)$, thereby contradicting the optimality of $j_{k'}(\cdot)$. ■

A.3 Proof of lemma 5

Proof. Let consumer i_1 be the lowest indexed one purchasing a non-zero sized bundle $j_1 > 0$. We know that $p^*_{j_1} \not> w_{i_1j_1}$ because the price cannot be higher than the WTP. We next claim that $p^*_{j_1} \not< w_{i_1j_1}$ and prove that using contradiction. Suppose the profit maximizing price is such that $p^*_{j_1} < w_{i_1j_1}$. Let $\Delta = w_{i_1j_1} - p^*_{j_1}$. In this case Δ is also the surplus obtained by Consumer i_1.

From Proposition 4, we know that any $i' > i_1$ buys a Bundle $j' \geq j_1$. So, $w_{i'j'} - p_{j'} \geq w_{i'j_1} - p_{j_1}$. Additionally SCP implies $w_{i'j'} \geq w_{i'j_1}$. Therefore, $w_{i'j'} - p_{j'} \geq w_{i'j_1} - p_{j_1} \geq w_{i_1j_1} - p_{j_1} = \Delta$. Consequently, if the prices are increased for all j by Δ, no consumer has an incentive to change their purchase decisions. Any $i < i_1$ will only observe a decreased surplus and will continue not to purchase any bundle $j \geq 1$. Every $i > i_1$ will continue to buy the same bundle even after the price increase. As a result, increasing the prices leads to higher profits counter to the assumption that $p^*_{j_1} < w_{i_1j_1}$. ■

A.4 Proof of Proposition 6

Proof. The key to the transformation is how the optimal prices can be determined for any given x_{ij} matrix satisfying Constraints (2) and (7). Given such x_{ij}s, the procedure for obtaining prices for CBP1 is as follows. Consider J' as the set of non-zero bundles allocated to at least one of the consumers. Bundles of size 0 are charged 0 by assumption. For $j \in J'$, the prices are set by solving CBP1a. For every $\{j|j \notin J', j \neq 0\}$, the prices are set to be the same as that of the next higher sized bundle purchased. If $J \notin J'$, the price for Bundle J is retained to be $w_{Ij} + \epsilon$.

Notice that CBP1a is only relevant if J' is not null. For the CBP1a formulation, ignore consumers buying zero-sized bundles and re-index the rest of consumers so that the lowest indexed consumer who buys a non-zero bundle is indexed at 1 and the rest sequentially numbered thereafter until some I'. Let $w_i^{i'}$ denote the WTP of consumer i for the bundle that consumer i' buys given the bundle allocation. Let p^i denote the price for the bundle i buys and c^i cost for the bundle i buys. Then, the prices are obtained by solving the following formulation:

$$\text{CBP1a}: \quad \underset{p^i}{\text{Max}} \quad \sum_{i=1}^{I'} (p^i - c^i) \tag{33}$$

$$\text{s.t.} \quad w_i^i - p^i \geq w_i^{i'} - p^{i'} \; \forall i, i' \tag{34}$$

$$w_1^1 - p^1 \geq 0. \tag{35}$$

Constraints (34) ensure that the surplus that Consumer i gets from purchasing her bundle is no less than the surplus she gets from bundles purchased by any other consumer. Constraint (38) enforces that the lowest-indexed consumer gets nonzero surplus. The objective equation (33) maximizes the total profit for the vendor.

Now, compare the problem specifications CBP1a and CBP1. The objective functions are the same given x_{ij} values. While Constraints (34) only include the surplus comparisons with bundles in J' and Constraint (35) compares with zero-sized bundles, they do not include comparisons with non-zero bundles not in J' that Constraints (1) includes. However, given the aforementioned pricing scheme, it is easy to verify that Constraints (1) also holds for the non-zero bundles not in J'.

Now, we solve CBP1. Decompose (34) into the following:

$$w_i^i - p^i \geq w_i^{i'} - p^{i'} \ \forall i > i' \tag{36}$$

$$w_i^i - p^i \geq w_i^{i'} - p^{i'} \ \forall i < i'. \tag{37}$$

We next establish the equivalence between Constraint (36) and

$$w_i^i - p^i \geq w_i^{i-1} - p^{i-1} \ \forall i. \tag{38}$$

It is obvious Constraint (36) implies (38). To prove the converse, consider the difference: $w_i^i - w_i^{i'} = \sum_{k=i'}^{i-1}(w_i^{k+1} - w_i^k) \geq \sum_{k=i'}^{i-1}(p^{k+1} - p^k) = p^i - p^{i'}$ The inequality is due to Constraint (38). Hence, we obtain $w_i^i - p^i \geq w_i^{i'} - p^{i'}$.

Next, we show that the optimal prices are obtained from (CBP1b)$\{Max \sum(p^i - c^i)|(38)(35)\}$ and that Constraint (37) is superfluous. In CBP1b, it is easy to realize Constraints (38) and (35) will be tight at optimality (because those constraints provide an upper bound for p^i). So, $p^{1*} = w_1^1$

and $p^{i*} = w_i^i - w_i^{i-1} + p^{i-1}$ for $i > 1$. Using the same prices, prices differences can be rewritten as

$$\begin{aligned}
p^{i'} - p^i &= ((w_{i'}^{i'} - w_{i'}^{i'-1}) + (w_{i'-1}^{i'-1} - w_{i'-1}^{i'-2}) + \ldots + (w_{i+1}^{i+1} - w_{i+1}^{i})) \\
&\geq ((w_i^{i'} - w_i^{i'-1}) + (w_i^{i'-1} - w_i^{i'-2}) + \ldots + (w_i^{i+1} - w_i^{i})) \\
&= w_i^{i'} - w_i^i.
\end{aligned}$$

The inequality follows from SCP and consequently, Constraint (37) is met.

Using the definition v_{ij} in Section 2.2.1, we can rewrite 33 as following:

$$\begin{aligned}
\sum_{i=1}^{I}(p^i - c^i) &= (w_I^I - c^I) + (2w_{I-1}^{I-1} - w_I^{I-1} - c^{I-1}) + \ldots \\
&\quad +((I - i + 1)w_i^i - (I - i)w_{i+1}^i - c^i) + \ldots + (Iw_1^1 + (I-1)w_2^1 - c^1) \\
&= \sum_{j=0}^{J}(v_{Ij} - c_j)x_{Ij} + \sum_{j=0}^{J}(v_{I-1,j} - c_j)x_{I-1,j} + \ldots + \sum_{j=0}^{J}(v_{1j} - c_j)x_{1j} \\
&= \sum_{i=1}^{I}\sum_{j=0}^{J}(v_{ij} - c_j)x_{ij}.
\end{aligned}$$

■

A.5 Proof of Theorem 7

Proof. We start with CBP2. First, let us define variables $\chi_{ijj'}$ as below:

$$\chi_{00j} = x_{1j} \ \forall j \tag{39}$$

$$\chi_{ijj'} = x_{ij}x_{i+1,j'} \ \forall j \ \forall j' \geq j \tag{40}$$

$$\chi_{IJj} = x_{Ij} \ \forall j. \tag{41}$$

Because these equations are simply definitions of new variables, adding them as constraints to CBP2 will not change the solution.

Then we show that the above three constraints together with Constraints (2) and (7) always imply Constraints (8) through (12). Constraint (12) is straightforward from the above definition. We can show that $\sum_{j=0}^{J} \chi_{01j} = \sum_{j=0}^{J} x_{1j} = 1$; the first inequality is due to Constraint (39) and the second due to Constraint (2). So, Constraint (8) is established. Along the same lines, Constraint (9) can also be established: $\sum_{j=0}^{J} \chi_{IJj} = \sum_{j=0}^{J} x_{Jj} = 1$; the first inequality is due to Constraint (41) and the second due to Constraint (2).

Next, we prove the validity of Constraints (10) and (11) by considering two cases: when $x_{ij} = 0$ and $x_{ij} = 1$. Using the definition for $\chi_{ijj'}$, we can show that when $x_{ij} = 0$, both $\sum_{j'=j}^{J} \chi_{ijj'}$ and $\sum_{j'=0}^{j} \chi_{i-1,j',j}$ are also equal to zero:

$$\sum_{j'=j}^{J} \chi_{ijj'} = \sum_{j'=j}^{J} x_{ij}x_{i+1,j'} = 0 = x_{ij}$$
$$\sum_{j'=0}^{j} \chi_{i-1,j',j} = \sum_{j'=0}^{j} x_{i-1,j'}x_{ij} = 0 = x_{ij}.$$

We next prove Constraints (10) and (11) for $x_{ij} = 1$. Under that condition,

$$\sum_{j'=j}^{J} \chi_{ijj'} = \sum_{j'=j}^{J} x_{ij}x_{i+1,j'} = \sum_{j'=j}^{J} x_{i+1,j'} \tag{42}$$
$$\sum_{j'=0}^{j} \chi_{i-1,j',j} = \sum_{j'=0}^{j} x_{i-1,j'}x_{ij} = \sum_{j'=0}^{j} x_{i-1,j'}. \tag{43}$$

Note that $\sum_{j'=j}^{J} x_{i+1,j'} \geq \sum_{j'=j}^{J} x_{i,j'} = 1$; the first inequality is due to Constraint (7) and the second one is because when $x_{ij} = 1$, $x_{ij''} = 0$ for all $j'' \neq j$. Furthermore, since $\sum_{j'=0}^{J} x_{i+1,j'} = 1$ and $x_{i+1,j'} \in \{0, 1\}$, $\sum_{j'=j}^{J} x_{i+1,j'} = 1$ and so, Equation (42) equals to 1. Constraint (7) also implies that $\sum_{j'=0}^{j} x_{i-1,j'} \geq \sum_{j'=0}^{j} x_{ij'}$, which along the same previous procedure results in Equation (43) equaling 1. Therefore, CPB_2 is equivalent to the following nonlinear mixed-integer problem CPB_{2a}:

$$
\begin{aligned}
\text{CBP2a}: \quad \underset{x_{ij}}{\text{Max}} \quad & \sum_{i=1}^{I}\sum_{j=0}^{J}(v_{ij}-c_j)x_{ij} \\
\text{s.t.} \quad & (2), (3), (7) \\
& (39), (40), (41) \\
& (8), (9), (10), (11), (12).
\end{aligned}
$$

Constraints (2), (7), and (40) are next shown to be redundant. To eliminate Constraint (40), we need to consider four cases, for $(x_{ij}, x_{i+1,j'}) \in \{(0,0),(0,1),(1,0),(1,1)\}$.

When $x_{ij} = 0$, i.e., the first two of the four cases, $0 = \sum_{j'=j}^{J}\chi_{ijj'} \geq \chi_{ijj'}$; the first equality is from Constraint (10) and the second is obvious. When $(x_{ij}, x_{i+1,j'}) = (1,0)$, $0 = \sum_{j'=0}^{j}\chi_{i-1,jj'} \geq \chi_{ijj'}$; the first equality is from Constraint (11) and the second is obvious. When $(x_{ij}, x_{i+1,j'}) = (1,1)$, we prove it by contradiction. Suppose $\chi_{ijj'} = 0$ under that condition. Since $\chi_{ijj'} = 0$ and $x_{ij} = \sum_{\hat{j}=j}^{J}\chi_{ij\hat{j}} = 1$, there must exist some $j'' \neq j'$ such that $\chi_{ijj''} = 1$. Therefore, $x_{i+1,j''} = \sum_{j'=0}^{j''}\chi_{i,j',j''} \geq 1$, which leads to $\sum_{j=0}^{J} x_{i+1,j} \geq x_{i+1,j'} + x_{i+1,j''} \geq 2$. It is contradictory to Constraints (2). Therefore, $\chi_{ijj'} = 1 = x_{ij}x_{i+1,j'}$. Next, we show that Constraint (2) is redundant.

$$
\begin{aligned}
\sum_{j=0}^{J} x_{ij} \quad & \underbrace{=}_{\text{by (11)}} \quad \sum_{j'=0}^{0}\chi_{i-1,j',0} + \sum_{j'=0}^{1}\chi_{i-1,j',1} + \ldots + \sum_{j'=0}^{J}\chi_{i-1,j',J} \\
& \underbrace{=}_{\text{by reorganization}} \quad \sum_{j'=0}^{J}\chi_{i-1,0,j'} + \sum_{j'=1}^{J}\chi_{i-1,1,j'} + \ldots + \sum_{j'=J}^{J}\chi_{i-1,J,j'} \\
& \underbrace{=}_{\text{by (10)}} \quad x_{i-1,0} + x_{i-1,1} + \ldots + x_{i-1,J} \\
& = \quad \sum_{j=0}^{J} x_{i-1,j} \underbrace{=}_{\text{by recursion}} \ldots = \sum_{j=0}^{J} x_{1,j} \underbrace{=}_{\text{by (8)}} 1.
\end{aligned}
$$

Finally, we show that Constraint (7) is redundant.

$$\begin{aligned}
\sum_{j'=j}^{J} x_{ij'} &\underbrace{=}_{11)} \sum_{j'=0}^{j} \chi_{i-1,j',j} + \sum_{j'=0}^{j+1} \chi_{i-1,j',j+1} + \ldots + \sum_{j'=0}^{J} \chi_{i-1,j',J} \\
&\underbrace{=}_{\text{by reorganization}} \sum_{j'=j}^{J} \chi_{i-1,0,j'} + \sum_{j'=j}^{J} \chi_{i-1,1,j'} + \ldots + \sum_{j'=j}^{J} \chi_{i-1,J,j'} \\
&\underbrace{\geq}_{\text{ignoring the first j-1 terms}} \sum_{j'=j}^{J} \chi_{i-1,j,j'} + \sum_{j'=j}^{J} \chi_{i-1,j+1,j'} + \ldots + \sum_{j'=j}^{J} \chi_{i-1,J,j'} \\
&\underbrace{=}_{\text{by (10)}} \sum_{j'=j}^{J} x_{i-1,j'}.
\end{aligned}$$

Constraints (2), (7), and (40) are redundant. Therefore, we can relax these three constraints without changing the problem. Therefore, after changing the objective function from max $\sum_{i=1}^{I}\sum_{j=0}^{J}(v_{ij} - c_j)x_{ij}$ to min $-\sum_{i=1}^{I}\sum_{j=0}^{J}(v_{ij} - c_j)x_{ij}$, CPB_{2a} is equivalent to CPB_3. ■

A.6 Proof of Lemma 8

Proof. Consider a CBP with I consumers, J bundles, $\mathbf{W}$ as the WTP matrix and its corresponding $\mathbf{V}$ as obtained using Equation (13). Define the problem as P_1 when the cost matrix is $\mathbf{C_1} = (0, c_1, c_2, \cdots, c_J)^T$. Let the optimal allocation for P_1 be given by $(\mathbf{X_1^*}, \mathbf{X_2^*}, \cdots, \mathbf{X_I^*})$. Similarly, define the problem as P_2 when the marginal cost is increased by δ and the cost matrix is $\mathbf{C_2} = (0, c_1 + \delta, c_2 + 2\delta, \ldots, c_J + J\delta)^T$. Call the corresponding optimal allocation as $(\mathbf{Y_1^*}, \mathbf{Y_2^*}, \ldots, \mathbf{Y_I^*})$. The lemma claims that no consumer purchases a higher-sized bundle in P_2 than what he purchases in P_1. We prove the result by contradiction. Assume consumers i_1 through i_2 purchase higher-sized bundles in P_2 than in P_1. Then,

$$\mathbf{X_i^{*T}} \cdot \mathbf{J} \geq \mathbf{Y_i^{*T}} \cdot \mathbf{J} \;\; \forall i \in \{1, 2, 3 \ldots i_1 - 1, i_2 + 1, \ldots, I\} \tag{44}$$

$$\mathbf{X_i^{*T}} \cdot \mathbf{J} < \mathbf{Y_i^{*T}} \cdot \mathbf{J} \;\; \forall i, i \in \{i_1, \ldots, i_2\}. \tag{45}$$

By Theorem 7, P_1 and P_2 are shortest-path problems. Therefore, $(\mathbf{X^*_{i_1}}, \mathbf{X^*_{i_1+1}}, \ldots, \mathbf{X^*_{i_2}})$ and $(\mathbf{Y^*_{i_1}}, \mathbf{Y^*_{i_1+1}}, \ldots, \mathbf{Y^*_{i_2}})$ are also the shortest paths in the respective problems. Hence,

$$\begin{aligned} \sum_{i=i_1}^{i_2} \mathbf{X_i^{*T}} \cdot (\mathbf{V_i} - \mathbf{C_1}) &\geq \sum_{i=i_1}^{i_2} \mathbf{Y_i^{*T}} \cdot (\mathbf{V_i} - \mathbf{C_1}) & (46) \\ \sum_{i=i_1}^{i_2} \mathbf{X_i^{*T}} \cdot (\mathbf{V_i} - \mathbf{C_2}) &\leq \sum_{i=i_1}^{i_2} \mathbf{Y_i^{*T}} \cdot (\mathbf{V_i} - \mathbf{C_2}). & (47) \end{aligned}$$

However,

$$\begin{aligned} \sum_{i=i_1}^{i_2} \mathbf{X_i^{*T}} \cdot (\mathbf{V_i} - \mathbf{C_2}) \underbrace{=}_{\mathbf{C_2}=\mathbf{C_1}+\mathbf{J}\delta} & \sum_{i=i_1}^{i_2} \mathbf{X_i^{*T}} \cdot (\mathbf{V_i} - \mathbf{C_1}) - \delta \sum_{i=i_1}^{i_2} \mathbf{X_i^{*T}} \cdot \mathbf{J} \\ \underbrace{\geq}_{\text{by (46)}} & \sum_{i=i_1}^{i_2} \mathbf{Y_i^{*T}} \cdot (\mathbf{V_i} - \mathbf{C_1}) - \delta \sum_{i=i_1}^{i_2} \mathbf{X_i^{*T}} \cdot \mathbf{J} \\ \underbrace{>}_{\text{by (45)}} & \sum_{i=i_1}^{i_2} \mathbf{Y_i^{*T}} \cdot (\mathbf{V_i} - \mathbf{C_1}) - \delta \sum_{i=i_1}^{i_2} \mathbf{Y_i^{*T}} \cdot \mathbf{J} = \sum_{i=i_1}^{i_2} \mathbf{Y_i^{*T}} \cdot (\mathbf{V_i} - \mathbf{C_2}), \end{aligned}$$

which contradicts Equation (47). ■

A.7 Proof of Corollary 9

Proof. Let $\bar{v} = \max_{i\in\{1,2,\cdots,I-1\}, j\in\{1,2,\cdots,J\}} v_{ij}$ and $M = 2I^2\bar{v}$. Also, let $v_{Ij} = \frac{1}{2}(2I-j+1)jM$. To generate $\boldsymbol{W}$ from $\boldsymbol{V}$, we use Equation (13). So,

$$\begin{aligned} w_{Ij} &= v_{Ij} = \frac{1}{2}(2I-j+1)jM \\ w_{ij} &= \frac{I-i}{I-i+1} w_{i+1,j} + \frac{1}{I-i+1} v_{ij} \\ &= \frac{I-i}{I-i+1}\frac{I-(i+1)}{I-(i+1)+1} w_{i+2,j} + \frac{1}{I-(i+1)+1} v_{i+1,j} + \frac{1}{I-i+1} v_{ij} \\ &= \frac{1}{I-i+1} w_{I,j} + \sum_{i'=i}^{I} \frac{1}{I-i'+1} v_{i'j}. \end{aligned}$$

Therefore,

$$\begin{aligned} w_{i,j+1} - w_{ij} &= \frac{1}{I-i+1}(w_{I,j+1} - w_{I,j}) + \sum_{i'=i}^{I} \frac{1}{I-i'+1} v_{i',j+1} - \sum_{i'=i}^{I} \frac{1}{I-i'+1} v_{i'j} \\ &\geq \frac{1}{I-i+1}(J-j)M - 2I\bar{v} \geq \frac{1}{I}M - 2I\bar{v} \geq 0, \end{aligned}$$

which satisfies Equation (4). In addition, we have

$$\begin{aligned} (w_{i+1,j+1} - w_{i+1,j}) - (w_{i,j+1} - w_{i,j}) &= (\frac{1}{I-i} - \frac{1}{I-i})(J-j)M + \sum_{i'=i+1}^{I} \frac{1}{I-i'+1} v_{i',j+1} \\ &\quad - \sum_{i'=i+1}^{I} \frac{1}{I-i'+1} v_{i'j} - \sum_{i'=i}^{I} \frac{1}{I-i'+1} v_{i',j+1} \\ &\quad + \sum_{i'=i}^{I} \frac{1}{I-i'+1} v_{i'j} \\ &= \frac{(J-j)}{(I-i)(I-i+1)} M + \frac{1}{I-i} v_{i+1,j+1} - \frac{1}{I-i} v_{i+1,j} \\ &\geq \frac{1}{I^2} M - 2\bar{v} \geq 0, \end{aligned}$$

which satisfies Equation (5). Thus, SCP holds for $\boldsymbol{W}$. ■

B Proofs for the Continuous Case

B.1 Proof of Lemma 11

Proof. We first prove that $\sum_{j'=0}^{N} k_{j'} \tilde{x}_{i_2,j'} \geq \sum_{j'=0}^{N} k_{j'} \tilde{x}_{i_1,j'}$ implies $\sum_{j'=j}^{N} \tilde{x}_{i_2,j'} \geq \sum_{j'=j}^{N} \tilde{x}_{i_1,j'}$, $\forall j$ by contradiction. Assume that for some i_1, i_2 and j, the first condition holds but not the second:

$$\sum_{j'=j}^{N} \tilde{x}_{i_1,j'} > \sum_{j'=j}^{N} \tilde{x}_{i_2,j'}. \tag{48}$$

The *rhs* of the equation can only be 0, 1, or $\tilde{x}_{i_2,j} > 0$ because at most two and adjacent $\tilde{x}_{i_1,j}$ values can be non-zero by assumption. Obviously it cannot be 1 because the *lhs* is at most 1. If $rhs = 0$, then $\sum_{j'=0}^{j-1} \tilde{x}_{i_2,j'} = 1$. Since k_j is a strict increasing series, we have $\sum_{j'=0}^{j-1} \tilde{x}_{i_2,j'} k_j \leq k_j$. Therefore,

$$
\begin{aligned}
\sum_{j'=0}^{N} \tilde{x}_{i_1,j'} k_{j'} &= \underbrace{\sum_{j'=0}^{j-2} \tilde{x}_{i_1,j'} k_{j'}}_{\geq 0} + \underbrace{\tilde{x}_{i_1,j-1} k_{j-1} + \sum_{j'=j}^{N} \tilde{x}_{i_1,j'} k_{j'}}_{\text{Because } \sum_{j'=j}^{N} \tilde{x}_{i_1,j'} > 0 \text{ and only adjacent variables can be non-zero, it is } > k_{j-1}} \\
&> k_{j-1} \geq \sum_{j'=0}^{j-1} \tilde{x}_{i_2,j'} k_j \underbrace{=}_{rhs=0} \sum_{j'=0}^{N} \tilde{x}_{i_2,j'} k_{j'}.
\end{aligned}
$$

The last case is when the rhs takes the value of $\tilde{x}_{i_2,j} > 0$. It means that $\tilde{x}_{i_2,j}$ and possibly $\tilde{x}_{i_2,j-1}$ are non-zero; whereas the other $\tilde{x}_{i_2,j'} = 0 \ \ \forall j' \neq \{j, j-1\}$. Meanwhile, since $\sum_{j'=j}^{N} \tilde{x}_{i_1,j'} > \sum_{j'=j}^{N} \tilde{x}_{i_2,j'} > 0$, we also have $\sum_{j'=0}^{j-2} \tilde{x}_{i_1,j'} = 0$. Therefore,

$$
\begin{aligned}
\sum_{j'=0}^{N} \tilde{x}_{i_1,j'} k_{j'} &= \sum_{j'=j-1}^{N} \tilde{x}_{i_1,j'} k_{j'} \\
\geq k_{j-1} + \sum_{j'=j}^{N} \tilde{x}_{i_1,j'} (k_{j'} - k_{j-1}) &\geq k_{j-1} + (k_j - k_{j-1}) \sum_{j'=j}^{N} \tilde{x}_{i_1,j} \\
> k_{j-1} + (k_j - k_{j-1}) \tilde{x}_{i_2,j} &= \sum_{j'=0}^{N} \tilde{x}_{i_2,j'} k_{j'},
\end{aligned}
$$

which is also a contradiction. Thus, we proved $\sum_{j'=0}^{N} k_{j'} \tilde{x}_{i_2,j'} \geq \sum_{j'=0}^{N} k_{j'} \tilde{x}_{i_1,j'}$ implies $\sum_{j'=j}^{N} \tilde{x}_{i_2,j'} \geq \sum_{j'=j}^{N} \tilde{x}_{i_1,j'}$, $\forall j$. Next, we prove the reverse. Since $k_{j+1} > k_j$, we have $(k_j - k_{j-1}) \sum_{j'=j}^{N} \tilde{x}_{i_2,j'} \geq (k_j - k_{j-1}) \sum_{j'=j}^{N} \tilde{x}_{i_1,j'}$, $\forall j > 1$ and $k_0 \sum_{j'=0}^{N} \tilde{x}_{i_2,j'} \geq k_0 \sum_{j'=j}^{N} \tilde{x}_{i_1,j'}$. Explicitly writing the expressions in the previous sentence for $j' < j$ and summing up, $\sum_{j'=0}^{N} k_{j'} \tilde{x}_{i'j'} \geq \sum_{j'=0}^{N} k_{j'} \tilde{x}_{i,j'}$. ■

B.2 Proof of Theorem 12

Proof. In the piecewise linear case with the adjacency restriction, $w_i(\sum_{j'=0}^{N} \tilde{x}_{ij'} k_{j'}) = \sum_{j'=0}^{N} \tilde{x}_{ij'} w_i(k_{j'})$. Similar decompositions are also valid for c_j and v_{ij}. The resulting formulation, $\{\underset{\tilde{x}_{ij}}{\text{Max}} \sum_{i=1}^{I} \sum_{j=0}^{N} (v_{ij} - c_j)\tilde{x}_{ij} | (22), (23), (24), (25)\}$ is similar to CBP2, except for the adjacency restriction in Constraints (24) and the integrality constraint in CBP2. Notice that that integrality condition in CBP2 is redundant. We have already demonstrated that CBP2 is a shortest-path problem and so, the constraints are totally unimodular. It implies that, even without the integrality condition, the solutions will only take extremal points. For the same reasons, even without Constraints (24), the solutions will be integral but the resulting integral solutions satisfy the adjacency restriction naturally. ■

B.3 Proof of Theorem 13

Proof. $\Pi^{pla*} \leq \Pi^{c*}$ is easy to establish. Refer to the vendor's problem with the approximate piecewise linear functions as CBPpla. Let $p_j^*, j \in \{1, 2, \ldots, N\}$ be the optimal price of CBPpla. In the unapproximated problem, hereafter referred to as CBPc1, set the price function $p^c(y)$ to be: $p^c(y) = p_j^*,\ (j-1)k < y \leq jk, j \in \{1, 2, \ldots, N\}$. Then in CBPc1, every consumer i purchases the same bundles as in CBPpla. Only quantities y such that $y = jk, j \in \{0, 1, 2, \ldots, N\}$ are purchased because the surplus the consumer gets from any other $y, (j-1)k < y < jk$ is less. In that case, individual consumer surpluses are the same between CBPc1 and CBPpla. So, are the payments received by the vendor. Also, the cost for these two solutions are the same. Given a solution that generates the same profit as CBPpla is feasible to CBPc1, $\Pi^{pla*} \leq \Pi^{c*}$.

Before generating the upperbound on Π^{c*}, we first bound the gap between the original and the approximate functions

Lemma 19 $\max\{|w^{pl}(y) - w(y)|, |c^{pl}(y) - c(y)|\} \leq \epsilon\ \forall y$, *where* $\epsilon = k\beta$.

Proof. For any arbitrary y, $jk \leq y \leq (j+1)k$:

$$|w^{pl}(y) - w(y)| \leq |w((j+1)k) - w(jk)| = |\int_{jk}^{(j+1)k} w'(y)dy| \leq \int_{jk}^{(j+1)k} \beta dy = k\beta.$$

The gap is bounded because of Lipschitz continuity. For the same reason, $|c^{pl}(y) - c(y)| \leq k\beta$. ■

Fix the optimal decision variables for CBPc1 as given. Let $p^{c*}(y)$ be the optimal price, Π^{c^*} be the optimal profit, and $\{y_1^*, y_2^*, \ldots, y_{\hat{j}}, \ldots, y_{\hat{J}}\}$ be the ordered set of bundle sizes purchased by some consumer(s) in the optimal solution. Let $\alpha_{\hat{j}}$ be the number of consumers who purchase bundle $y^*_{\hat{j}}$. So, $\sum_{\hat{j}=1}^{\hat{J}} \alpha_{\hat{j}} \leq I$ and $\hat{J} \leq I$. Next, we prove that for $\delta \geq 2k\beta$, when prices are:

$$p^s(y) = p^{c*}(y^*_{\hat{j}}) - \hat{j}\delta,\ y^*_{\hat{j}-1} < y \leq y^*_{\hat{j}}, \hat{j} \in \{1, 2, \ldots, \hat{J}\} \tag{49}$$

a solution is feasible to CBPpla in a manner that no consumer purchases a lower-sized bundle than her purchase under CBPc1. Then, using the prices, we bound the profits generated.

Let an arbitrary Conusmer i purchases bundle $y^*_{\hat{j}_1}$ in CBPc1. Then, no bundle $y^*_{\hat{j}-1} < y < y^*_{\hat{j}}$ will be purchased under CBPpla given the prices in Equation (49): for $y^*_{\hat{j}_1-1} < y < y^*_{\hat{j}_1}$, $w^{pl}(y) - (p^{c*}(y^*_{\hat{j}_1}) - \hat{j}_1\delta) \leq w^{pl}(y^*_{\hat{j}_1}) - (p^{c*}(y^*_{\hat{j}_1}) - \hat{j}_1\delta)$. So, when considering the incentive of Consumer i to choose a lower-sized bundle, it is sufficient to consider her move to purchase $y^*_{\hat{j}_2} < y^*_{\hat{j}_1}$. In other words, δ should be such that $w^{pl}(y^*_{\hat{j}_1}) - (p^{c*}(y^*_{\hat{j}_1}) - \hat{j}_1\delta) \geq w^{pl}(y^*_{\hat{j}_2}) - (p^{c*}(y^*_{\hat{j}_2}) - \hat{j}_2\delta)$. Because of Lemma 19, the same condition is rewritten as $w_i(y^*_{\hat{j}_1}) - k\beta - (p^{c*}(y^*_{\hat{j}_1}) - \hat{j}_1\delta) \geq w_i(y^*_{\hat{j}_2}) + k\beta - (p^{c*}(y^*_{\hat{j}_2}) - \hat{j}_2\delta)$. Because $y^*_{\hat{j}_1}$ is optimally allocated to Consumer i in CBPc1, $w_i(y^*_{\hat{j}_1}) - p^{c*}(y^*_{\hat{j}_1}) \geq w_i(y^*_{\hat{j}_2}) - p^{c*}(y^*_{\hat{j}_2})$. So, it is sufficient for δ to satisfy $-k\beta + \hat{j}_1\delta \geq k\beta + \hat{j}_2\delta$. It then leads to $\delta \geq 2k\beta$. Therefore, if prices are as in Equation (49), each Consumer i in CBPpla purchases a bundle no smaller than the one she purchase in CBPc1.

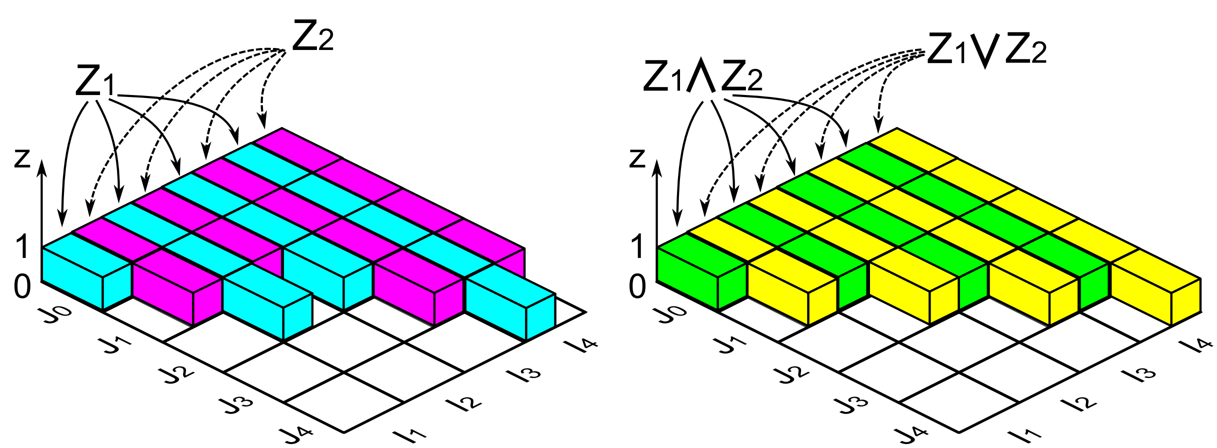
With the above prices, the total revenue in CBPpla is lower than in CBPc1 at most by $\sum_{\hat{j}=1}^{\hat{J}} \alpha_{\hat{j}}\hat{j}2k\beta \leq \sum_{i=1}^{I} 2ik\beta \leq 2I(I+1)k\beta$. Similarly, because of $\max_y\{|c_i^{pl}(y) - c(y)|\} \leq k\beta$, the vendor at most underestimates the bundle cost for each consumer by $k\beta$, leading to a total underestimation of $Ik\beta$ compared to CBPc1. If Π^{pla} is the profit in CBPpla with the prices above, $\Pi^{c^*} - \Pi^{pla} \leq 2I(I+1)k\beta + Ik\beta = 2I(I+2)k\beta$. So, $\Pi^{c^*} - 2I(I+2)k\beta \leq \Pi^{pla} \leq \Pi^{pla^*}$, where Π^{pla*} is the optimal profit for CBPpla. So, $\Pi^{c*} \leq \Pi^{pla*} + 2I(I+2)k\beta$.

Next, we consider the algorithm complexity. The maximum gap in the profits is $\epsilon_t \leq 2I(I+2)k\beta$. Therefore, for a given ϵ_t, we can set $k = \frac{\epsilon_t}{2I(I+2)\beta}$ and solve the approximate problem. The total bundle size in CBPc2 problem CBPpla is then $N = \frac{J}{k} = \frac{\beta 2I(I+2)J}{\epsilon_t}$. As we claimed in Section 2.2.2, the complexity of the dynamic programming is $O(IN)$. Thus, the complexity of this solution approach is $O(I^3 J/\epsilon_t)$. It is polynomial in both the problem size $I \times J$ and $1/\epsilon_t$. Therefore, it is a fully polynomial-time approximation scheme or FPTAS. ■

C Proofs for the Generalized Production Cost Section

C.1 A Demonstration for the Lattice Family Structure

Let $\mathbb{Z}$ be a set of matrices, whose elements satisfy Constraints (28) through (31). Suppose there are two matrices $\{Z^1, Z^2\} \in \mathbb{Z}$ representing two different feasible solutions to CBPg3. As an example, suppose Z^1 and Z^2 are the two feasible solutions each represented by a different color in Figure 3(a). Then we define $Z^1 \wedge Z^2$ as the upper envelope of Z^1 and Z^2 (shown in green color in Figure 3(b)) and $Z^1 \vee Z^2$ as the lower envelope (shown in yellow color in Figure 3(b)). It is easy to verify that both $Z^1 \wedge Z^2$ and $Z^1 \vee Z^2$ also satisfy Constraints (28) through (31),*i.e.*, $\{Z^1 \wedge Z^2, Z^1 \vee Z^2\} \in \mathbb{Z}$. Thus, the feasible solutions belong to a lattice family.



(a) Z^1 and Z^2 (b) $Z^1 \wedge Z^2$ and $Z^1 \vee Z^2$

Figure 3: Lattice Structure of S

C.2 Proof of Lemma 16

Proof. Define $R(\tilde{Z}) = \sum_{i=1}^{I}\sum_{j=0}^{N} v_{ij}(\tilde{z}_{ij} - \tilde{z}_{i,j+1})$ as the revenue function for $\tilde{Z}$. We can rewrite CBPgc3 as $\underset{\tilde{Z}}{\text{Max}} R(\tilde{Z}) - C(\tilde{Z}); \text{s.t.} \tilde{Z} \in \mathbb{Z}$. Similarly, CBPgc3b is:$\underset{\tilde{Z}}{\text{Max}} R(\tilde{Z}) - C(\tilde{Z}); \text{s.t. } \tilde{Z} \in \tilde{\mathbb{Z}}_{in}$. Call the optimal solution value of CBPgc3 as Π_{gc3} and that of CBPgc3b as Π_{gc3b}. Therefore, we need to prove that if $C(\tilde{Z})$ satisfies the conditions in the lemma, then $\Pi^*_{gc3} = \Pi^*_{gc3b}$.

Obviously, $\Pi^*_{gc3} \geq \Pi^*_{gc3b}$ always holds because the optimal solution of CBPgc3b is feasible to CBPgc3 and it generates the same optimal profits in both problems. We next prove $\Pi^*_{gc3} \leq \Pi^*_{gc3b}$. The conditions in the lemma mean, if $\tilde{Z}^*$ is the optimal to CBPgc3 we can always find $(\lambda_1, \lambda_2, \ldots, \lambda_M)| \sum_{m=1}^{M} \lambda_m = 1;\ \lambda_m \geq 0 \forall m$ and $\tilde{Z}_m \in \mathbb{Z}$, such that $\tilde{Z} = \sum_{m=1}^{M} \lambda_m \tilde{Z}_m$ and $C(\tilde{Z}^*) \geq \sum_{m=1}^{M} \lambda_m C(\tilde{Z}_m)$. Therefore,

$$\begin{aligned}
\Pi_{gc3}(\tilde{Z}^*) \quad & \underbrace{=}_{\tilde{Z}=\sum_{m=1}^{M}\lambda_m\tilde{Z}_m} R(\sum_{m=1}^{M}\lambda_m\tilde{Z}_m) - C(\sum_{m=1}^{M}\lambda_m\tilde{Z}_m) \underbrace{=}_{R()\text{is linear}} \sum_{m=1}^{M}\lambda_m R(\tilde{Z}_m) - C(\sum_{m=1}^{M}\lambda_m\tilde{Z}_m) \\
& \leq \sum_{m=1}^{M}\lambda_m R(\tilde{Z}_m) - \sum_{m=1}^{M}\lambda_m C(\tilde{Z}_m) = \sum_{m=1}^{M}\lambda_m \Pi_{gc3b}(\tilde{Z}_m) \\
& \underbrace{\leq}_{\lambda_m \geq 0} \sum_{m=1}^{M}\lambda_m \underset{m}{\text{Max}}\{\Pi_{gc3b}(\tilde{Z}_m)\} \underbrace{=}_{\sum_{m=1}^{M}\lambda_m=1} \underset{m}{\text{Max}}\{\Pi_{gc3b}(\tilde{Z}_m)\} \leq \Pi_{gc3b}(\tilde{Z}^*).
\end{aligned}$$

■

C.3 An Alternate Explanation for Algorithm 1

For a given $\tilde{Z} \in \tilde{\mathbb{Z}}$, sort all its elements $\{z_{ij}|0 < z_{ij} < 1\}$ with descending order and redefine these elements as $\pi_m,\ m \in \{1, 2, \ldots, M-1\}$. We can then generate one way of decomposing $\tilde{Z}$ into $\tilde{Z}_m \in \tilde{\mathbb{Z}}_{in}$ such that $\tilde{Z} = \sum_{m=1}^{M} \lambda_m \tilde{Z}_m$ where $\lambda \in \Lambda$. Let $\lambda_1 = 1 - \pi_1$ and $\lambda_m = \pi_{m+1} - \pi_m,\ \forall m > 1$. Define a function $e(Z, a)$ where Z is a matrix and a is a single value. Then $e(Z, a)$ returns a matrix with the same size of Z, setting all its elements $\geq a$ to 1 and those $< a$ to 0. Let $\tilde{Z}_1 = e(\tilde{Z}, 1)$ and $\tilde{Z}_m = e(\tilde{Z}, \pi_{m-1}),\ \forall m > 1$. Then we can easily verify that both $\lambda \in \Lambda$ and $\tilde{Z} = \sum_{m=1}^{M} \lambda_m \tilde{Z}_m$ hold.

C.4 Proof of Theorem 18

Proof. Since we use the same approach to generate the piecewise approximation for the WTP functions, it is straightforward that the same proof for Theorem 13 can also work here to prove $\Pi^{gpla*} \leq \Pi^{gc*}$ and the optimal revenue of CBPgc2 is no more than $2I(I+1)k\beta$ than that of CBPgpla. Therefore, we only need to prove that the cost is underestimated by no more than $Ik\beta$. When we decompose $C(\tilde{Z})$ to $\sum_{m=0}^{M} \lambda_m C(\tilde{Z}_m)$, each $\tilde{Z}_m$ is an extreme point on the grid box that contains $\tilde{Z}$ in the I dimensional space. Therefore, $|C(\tilde{Z}) - C^a(\tilde{Z})| \leq Ik\beta$. ■

References

Adams, W. J. and J. L. Yellen (1976). Commodity bundling and the burden of monopoly. *Quart. J. Econom 90*, 475–498.

Bakos, Y. and E. Brynjolfsson (1999). Bundling information goods: Pricing profits and efficiency. *Management Science 45*(12), 1613–1630.

Chu, C., P. Leslie, and A. Sorensen (2011). Bundle-size pricing as an approximation to mixed bundling. *The American Economic Review 101*(1), 263–303.

Hanson, W. and R. Martin (1990). Optimal bundle pricing. *Management Science 36*(2), 155–174.

Hitt, L. and P. Chen (2005). Bundling with customer self-selection: A simple approach to bundling low-marginal-cost goods. *Management Science 51*(10), 1481–1493.

Kannan, K. N., M. Tawarmalani, and J. Wu (2013). Cardinality bundles with constrained prices. working paper.

McAfee, R. P., J. McMillan, and M. D. Whinston (1989). Multiproduct monopoly, commodity bundling, and correlation of values. *Quart.J.Econom. 104*, 371–383.

Schrijver, A. (2003). *Combinatorial optimization, polyhedra and efficiency.* Springer-Verlag.

Spence, M. (1974). Competitive and optimal responses to signals: An analysis of efficiency and distribution. *J. Econ. Theory 7*, 296–332.

Spence, M. (1980). Multi-product quantity-dependent prices and profitability constraints. *Rev. Econom. Stud. 47*(5), 821–841.

Stigler, G. J. (1963). United states v. loew's inc.: A note on block booking. *Supreme Court Rev.*, 152–157.

Tawarmalani, M., J.-P. P. Richard, and C. Xiong (2013, 4). Explicit convex and concave envelopes through polyhedral subdivisions. *Mathematical Programming 138*, 531–577.

Tawarmalani, M. and N. V. Sahinidis (2002). *Convexification and Global Optimization in Continuous and Mixed-Integer Nonlinear Programming: Theory, Algorithms, Software and Applications.* Kluwer Academic Publishers.

Wu, S., L. Hitt, P. Chen, and G. Anandalingam (2008). Customized bundle pricing for information goods: A nonlinear mixed-integer programming approach. *Management Science 54*(3), 608–622.