Stock Return Extrapolation, Option Prices, and Variance Risk Premium

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Abstract

This paper presents an equilibrium model of stock return extrapolation in the presence of stochastic stock return volatility. The model is highly tractable, leading to closed-form solutions for the stock price, its dynamics, variance risk premium as well as providing analytic option pricing formulas that nest the Heston stochastic volatility formulation. The main novel predictions of the model are as follows. Consistent with survey evidence, following negative stock returns, in addition to expecting future stock returns to be lower, investors also expect future returns to be more volatile. The magnitude of the negative correlation between the stock return and its variance, the negative skewness, is increasing in the extrapolation degree. A more bearish investor sentiment, due to extrapolating bad recent stock performance, leads to a more negative variance risk premium, and higher option prices with the price increases being more pronounced for out-of-the-money options, as in data. The variance risk premium predicts future stock market returns negatively even after controlling for the realized, or conditional, variance, consistent with empirical evidence.

JEL Classifications: G12, G13.
Keywords: Extrapolation, sentiment, stochastic volatility, option prices, variance risk premium, predictability.

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1 Introduction

Much empirical evidence shows that investors' bullish and bearish sentiment and past stock market returns affect index option prices (e.g., Bakshi and Kapadia (2003), Amin, Coval, and Seyhun (2004), Han (2008)), and these findings cannot be reconciled with standard rational option pricing models. In fact, these findings are not reconciled with existing behavioral models either since currently there is no work provides well-founded option pricing formulas that incorporate the investor sentiment. In this paper, we develop a tractable equilibrium model of investor sentiment, where the sentiment is induced by extrapolating past stock market returns. Our model delivers analytic option pricing formulas that are driven by the sentiment, which allows us to illustrate that the bullish/bearish sentiment on the stock affects option prices in a way consistent with the empirical evidence. Our model also provides new insights on the effects of investor sentiment on the underlying stock returns as well as on the variance risk premium, a quantity which has attracted much attention recently.

In our model, there is a single investor whose subjective beliefs on future stock returns have an extrapolative belief component that is driven by past stock returns.[1] A key feature of our model is the presence of stochastic stock return volatility, which introduces a novel channel through which the investor sentiment that is induced by extrapolating past stock returns affects all asset prices, including option prices, variance swap rates, and hence the variance risk premium. To be more specific, in our model, the stock is a claim to a risky payoff, which is determined by a fundamental process with time-varying variance, the fundamental variance. The investor has correct beliefs on the fundamental process, but misperceives the endogenous stock mean returns with a bias proportional to the weighted average of past stock returns, referred to as *sentiment.* Central parameters of the model are the *relative weights*, which controls the relative weights of past returns in the sentiment construction, and the *extrapolation degree,* which controls the extent to which the investor's extrapolative bias is reflected on her future stock return expectation. Our model delivers closed-form expressions for the quantities of interest.

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[1]This belief structure is consistent with the growing survey evidence, which shows that most investors extrapolate past stock market returns while forming their subjective beliefs about future stock market returns (Vissing-Jorgensen (2003), Amromin and Sharpe (2014), Greenwood and Shleifer (2014), Kaplanski et al. (2016), Cassella and Gulen (2018)). Landier, Ma, and Thesmar (2017) present further evidence of extrapolative beliefs in an experimental setting.

We first solve our model and determine the equilibrium stock price and its dynamics. We show that in the presence of extrapolative beliefs, the stock price is driven by investor sentiment in addition to the fundamental and its variance. Following positive stock returns, either due to higher fundamental or lower fundamental variance, the investor sentiment rises and becomes more bullish as the investor expects future stock returns to be higher, which in return increases her stock demand, leading to a higher stock price. The extent to which the stock price responses to a change in sentiment (fundamental variance) is given by the sentiment (fundamental variance) elasticity of the stock price. A higher extrapolation degree directly increases the investor's future stock return expectation, and hence leads to a higher sentiment elasticity. Whereas, a higher relative weights makes the sentiment process less persistent as the sentiment is now driven mostly by the most recent returns, and hence leads to a lower sentiment elasticity. These effects are largely consistent with earlier theoretical works. However, by looking at the fundamental variance elasticity our model reveals new insights. We show that a higher extrapolation degree leads to a more negative fundamental variance elasticity by increasing its magnitude. Therefore, when the extrapolation degree is higher, the stock price becomes more sensitive to the fundamental variance shocks. This occurs because the fundamental variance shocks are also part of the stock returns that the investor extrapolates. Hence, following good recent performance due to negative fundamental variance shocks, the investor with higher extrapolation degree pushes up the stock price more.

Looking at the stock price dynamics, we first show that a higher investor sentiment leads to lower stock mean returns. This is because a higher sentiment due to good recent stock performance makes the investor to expect higher future returns, and hence she pushes up the stock price further. Due to this overvaluation with respect to fundamentals, the subsequent observed stock returns become lower on average. We also show that the stock returns exhibit stochastic volatility, and a higher extrapolation degree leads to a higher stock return volatility due to two channels. The first channel is already identified in earlier works, and is due to the fact that following positive (negative) stock returns, the investor expects future returns to be higher (lower), and hence pushes up (down) the stock price further, and this additional fluctuation in the stock price generates more volatile stock returns. The second channel is novel to our model, and is due to the fact that a higher extrapolation degree also increases the magnitude of the fundamental variance elasticity. This makes the stock price to be more sensitive to the fundamental variance shocks, leading to more volatile stock returns.

Our model also yields new insights on the correlation between the stock return and its variance. We show that a higher extrapolation degree leads to a more negative correlation by increasing its magnitude, and hence making the stock returns more negatively skewed. This occurs because a higher extrapolation degree leads to a more negative fundamental variance elasticity, as discussed earlier. Therefore, the stock price becomes more sensitive and moves in the opposite direction more, following the variance shocks. This correlation is important for our analysis, since it directly affects the persistence of the stock return variance under the risk-neutral measure, and hence significantly influences the derivative prices.

More notably, we show that due to her extrapolative bias in her expected stock return, the investor also misperceives the expected changes in the stock return variance. In particular, following positive stock returns, in addition to expecting future stock returns to be higher, the investor also expects future returns to be less volatile, consistent with empirical evidence (Amromin and Sharpe (2014), Kaplanski et al. (2016)). This occurs in our model because the investor's higher future return expectations when she has more bullish sentiment must be reconciled by either higher expected fundamental growth or lower expected fundamental variance. Since the investor has correct beliefs on the fundamental growth, she expects less volatile future fundamental, and consequently less volatile stock returns.

Given the much empirical evidence on the effects of investor sentiment on index option prices that are not reconciled by existing theories as discussed at the beginning, we next look at how the investor sentiment affects option prices in our model. Toward that, we first obtain new analytic option pricing formulas that are driven by investor sentiment in addition to the stock price and its return variance. The well-known stochastic volatility option pricing formulas of Heston (1993) arise as our special case when the investor has no extrapolative beliefs. With extrapolative beliefs, the investor sentiment affects option prices because it drives her variance bias, which is reflected in the risk-neutral stock return variance dynamics. We then illustrate that a more bearish sentiment leads to a higher price for both the call and the put options, and these effects are more pronounced for out-of-the-money options, consistent with empirical evidence (Bakshi and Kapadia (2003), Amin, Coval, and Seyhun (2004), Han (2008)). In our model, these results arise because a more bearish sentiment, due to bad recent stock returns, in addition to making the investor to expect lower future returns, also makes the investor to expect future returns to be more volatile, as discussed previously. Consequently, similar to

the usual vega effect, a higher expected volatility leads to higher option prices, irrespective of the option being a call or a put.

As discussed above, in our model, due to extrapolating past stock returns, the investor misperceives the expected changes in the stock return variance. Potentially, this mechanism has important implications for the risk premia that the investor is willing to accept for bearing the variance risk. Toward that, we next examine our model implications for the variance risk premium, the difference between the (average) realized variance and the variance swap rate. There, we first obtain the variance risk premium in closed-form, and show that a more bearish sentiment, due to bad recent stock performance, leads to a more negative variance risk premium. In our model, this occurs because a more bearish sentiment makes the investor to expect more volatile future returns, and hence the variance swap contract is feasible only if the variance swap rate is set higher, which in turn leads to a more negative variance risk premium. Conversely, a more bullish sentiment, due to good recent stock performance, leads to a higher (less negative) variance risk premium. The existing theoretical works attribute the fluctuations in the variance risk premium to various shocks, such as the time-variation in risk aversion (Bekaert and Engstrom (2010)), the time-variation in aggregate consumption variance (Bollerslev, Tauchen, and Zhou (2009)), Poisson shocks in the expected aggregate consumption mean and variance (Drechsler and Yaron (2011)), Knightian uncertainty shocks (Drechsler (2013), Miao, Wei, and Zhou (2019)). Therefore, one of our key contributions here is to complement this literature by presenting a new economic determinant of the variance risk premium, the extrapolative beliefs induced investor sentiment.

We also show that in the presence of extrapolative beliefs, the variance risk premium predicts future stock returns negatively, implying that a more negative variance risk premium leads to a higher future stock returns, consistent with empirical evidence (Bollerslev, Tauchen, and Zhou (2009), Drechsler and Yaron (2011), Bollerslev, Todorov, and Xu (2015), Kilic and Shaliastovich (2019), Pyun (2019)). We establish this result by first showing analytically that the slope coefficient in the univariate regression of future stock returns on the variance risk premium is negative, then through a simulation exercise, show that the slope coefficient is also statistically significant. In our model, the predictive power of the variance risk premium is due to the fact that it isolates the fluctuations in the investor sentiment. This is because the investor sentiment drives only the variance swap rate but not the realized variance, whereas the

stock return variance drives both the variance swap rate and the realized variance. Therefore much of the fluctuations in the variance risk premium come from the investor sentiment rather than the stock return variance, which is diminished by taking the difference in the definition of the variance risk premium. This means a more negative variance risk premium is mostly an indicator of a more bearish investor sentiment. Since the bearish sentiment times coincide with the undervaluation of the stock price relative to its fundamentals, the subsequent observed stock returns are higher on average.

We further show that in the presence of extrapolative beliefs, the variance risk premium predicts future stock returns negatively also in joint regressions, where the other regressor is the realized (or, conditional) variance, also consistent with the evidence in Bollerslev, Tauchen, and Zhou (2009). This stronger implication of our model enables us to demonstrate that the predictive power of variance risk premium in our model is indeed due to the investor sentiment. We again establish this result by first showing analytically that the slope coefficient of the variance risk premium in the joint regression is negative, then through a simulation exercise, show that the slope coefficient is also statistically significant. As in the case of the univariate regression, the predictive power of the variance risk premium is due to the fact that a more negative variance risk premium mostly indicates a more bearish sentiment, which drives the stock return predictability in our model. There, we further illustrate that the slope coefficient for the realized variance is not statistically significant, again consistent with the findings of Bollerslev, Tauchen, and Zhou (2009). The failure to find a statistically significant relation between the realized variance and the future stock returns is simply because the realized variance is driven only by the (conditional) stock return variance, which primarily drives the unpredictable component of stock returns. To the best of our knowledge, ours is the first work to demonstrate that the variance risk premium predicts stock returns not only in a univariate regression but also in a joint regression, which controls for the realized (or, conditional) variance.

Our paper adds to the growing theoretical literature on how investors' extrapolative beliefs affect asset prices. In this literature, investors' extrapolative beliefs are either based on endogenous past stock returns, as in this paper, (De Long et al. (1990), Hong and Stein (1999), Barberis et al. (2015, 2018), Adam, Marcet, and Beutel (2017), Jin and Sui (2018), Li and Liu (2018)), or based on exogenous past fundamentals (Barberis, Shleifer, and Vishny (1998),

Choi and Mertens (2013) (exchange economy), and Alti and Tetlock (2014), Hirshleifer, Li, and Yu (2015) (production economy)). Similar belief structure also arises in models in which investors' beliefs are based on their past experiences (Ehling, Graniero, and Heyerdahl-Larsen (2017), Malmendier, Pouzo, and Vanasco (2017), Nagel and Xu (2018)). Our paper is also related to the works that study the asset pricing implications of the investor sentiment arising from differences of opinion on fundamental signals (Scheinkman and Xiong (2003), Dumas, Kurshev, and Uppal (2009), Xiong and Yan (2010)). Our paper differs from all these works in various aspects with respect to its methodology, mechanism, and hence predictions. For instance, in none of these works investors misperceive the expected changes in the stock return variance as in our model, which is the main mechanism behind our novel predictions on option prices and variance risk premium.

In particular, in the key recent extrapolative beliefs model of Barberis et al. (2015), there are two types of investors, rational investors and extrapolators. They show that their model captures various features of actual stock market returns, while also being consistent with survey evidence on investor expectations. Our model implications for the stock market returns are also similar to theirs, since in both models the consensus belief on the stock is extrapolative. However, differently from theirs, our model is also able to capture various features of actual derivative prices. For example, even though they do not explicitly study derivatives in their paper, one could show that, due to their stock return volatility being constant, the extrapolation induced investor sentiment would not affect derivative prices in their model. This is because, both with or without extrapolators in their economy, under the risk-neutral measure, the stock price grows at a constant interest rate with a constant price change volatility, leading to similar derivative prices in both cases. Whereas in our model, by altering the risk-neutral dynamics of the stochastic stock return variance, the extrapolation induced investor sentiment does affect the prices of derivatives, which is one of the main messages of this paper.

The remainder of the paper is organized as follows. Section 2 presents our extrapolative beliefs model. Section 3 provides our results on the equilibrium stock price and its dynamics, Section 4 on option prices, and Section 5 on variance risk premium. Section 6 concludes. Appendix A contains all the proofs, and Appendix B discusses the parameter values employed in our tables and provides additional quantitative analysis.

2 Economy with Extrapolative Beliefs

In this section, we introduce a tractable security market economy in which the investor's subjective beliefs on future stock returns have an extrapolative belief component that is driven by past stock returns. A key feature of our model is the presence of stochastic volatility, which introduces a novel channel through which extrapolating past stock returns affects option prices and variance risk premium, as we demonstrate in subsequent sections.

2.1 Securities Market

We consider a continuous-time economy populated by a single investor with horizon T. There are two securities for trading, a risky stock representing the stock market and a riskless asset. The stock is in fixed supply of one unit and is a claim to the single risky payoff D_T, which is the time T realization of the fundamental (e.g. cash-flow news) process D with dynamics

$$dD_t = D_t[\mu dt + \sqrt{V_t}d\omega_{1t}], \tag{1}$$

$$dV_t = (\zeta - \kappa V_t)dt + \rho\sigma\sqrt{V_t}d\omega_{1t} + \sqrt{1-\rho^2}\sigma\sqrt{V_t}d\omega_{2t}, \tag{2}$$

where μ is the constant mean growth rate, V is the stochastic variance of the fundamental process, and ω_1, ω_2 are independent Brownian motions under the objective measure $\mathbb{P}$. The positive constants κ, σ, ζ/κ represent the mean reversion speed, volatility, and long-run mean of the fundamental variance V, respectively, and ρ is the correlation coefficient between the fundamental process and its variance, with parameter restrictions $2\zeta > \sigma^2$ and $-1 < \rho \leq 0$.[2] Without loss of generality, we set the initial value of the fundamental variance to its long-run mean, $V_0 = \zeta/\kappa$, for convenience. The stock price is denoted by S and is posited to follow dynamics

$$dS_t = S_t\left[\mu_{St}dt + \sigma_{S1t}d\omega_{1t} + \sigma_{S2t}d\omega_{2t}\right], \qquad S_T = D_T, \tag{3}$$

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[2]The first parameter restriction, $2\zeta > \sigma^2$, is well-known and guarantees the positivity of the fundamental variance, a square-root process. The second parameter restriction, $-1 < \rho \leq 0$, is the simplest sufficient (though not necessary) condition in our model to restrict our attention to the empirically relevant case of negative correlation between the stock returns and the changes in its variance, as observed in the data (French, Schwert, and Stambaugh (1987), Glosten, Jagannathan, and Runkle (1993)).

where the (conditional) stock mean return μ_S, and the diffusion terms σ_{S1} and σ_{S2} are determined endogenously in equilibrium. The riskless asset is in zero net supply with the exogenous constant rate of return, the interest rate, r.[3]

2.2 Investor's Extrapolative Beliefs and Optimization Problem

As discussed in Introduction, growing survey evidence indicates that most investors extrapolate past stock market returns while forming their subjective beliefs about future stock market returns. Consistent with this vast evidence, we model the investor's subjective beliefs on future stock returns to have an extrapolative bias component. That is, at all times t, the investor observes the stock price S_t, its diffusion terms σ_{S1t} and σ_{S2t}, but misperceives its mean return as $\mu_{St} + \theta X_t$ instead of the true one μ_{St}.[4] The process X in the subjective mean return is the (exponentially decaying) weighted average of past (log) stock returns, which in terms of its dynamics can be written as

$$dX_t = -\alpha X_t dt + \alpha d \ln S_t, \tag{4}$$

where the positive constant α, henceforth referred to as the *relative weights*, controls the relative weights of past returns in the construction of X, with a higher value assigning more weights to the most recent returns. Following Barberis et al. (2015), we also refer to the weighted past returns process X as the *sentiment* as it is the driving factor of the investor's bias, i.e., the difference between her subjective beliefs and the objective one.[5] Therefore,

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[3]In our setting, there is no intertemporal consumption, hence the interest rate can be taken as given. In our quantitative analysis, we set the interest rate to the reported average U.S. riskless rate of 0.72% as reported in Campbell (2018, Table 6.1). Other dynamic asset pricing models with no intertemporal consumption and exogenous interest rates include Kogan et al. (2006), Cvitanić and Malamud (2011), Pástor and Veronesi (2012), Basak and Pavlova (2013), Atmaz and Basak (2018).

[4]Alternatively, one could model the investor's perceived mean return as $(1 - \theta)\mu_{St} + \theta X_t$, for $\theta \in (0, 1)$, i.e., a convex combination of the true mean return μ_{St} and X_t. This specification also leads to similar results.

[5]Our sentiment measure differs slightly from the corresponding one in Barberis et al. (2015), since ours is in terms of (log) stock returns whereas theirs is in terms of stock price difference. On the other hand, referring the process X as the sentiment is consistent with the definition in Baker and Wurgler (2007), who define the sentiment broadly as the belief about future cash flows and investment risks that is not justified by the facts at hand. Moreover, referring the process X as the sentiment is also consistent with the widespread usage of the term in empirical studies. This is because the process X is driven by past stock market returns in our model, and as Brown and Cliff (2004) shows, the past stock market returns are important determinants of commonly employed sentiment measures in empirical studies. Finally, we note that the process X can also be equivalently specified in terms of its level as $X_t = \int_{-\infty}^{t} \alpha e^{-\alpha(t-u)} d \ln S_u$.

following positive stock returns, the sentiment rises and becomes more bullish as the investor expects future returns to be higher. Conversely, following negative stock returns, the investor sentiment falls and becomes more bearish. The positive constant θ, henceforth referred to as the *extrapolation degree*, controls the extent to which the investor's extrapolative bias is reflected on her subjective stock return expectation, with the rational benchmark economy, in which the investor has no bias, is obtained with $\theta = 0$.[6] To ensure the equilibrium stock price admits a real solution in the presence of extrapolative beliefs, we also impose the parameter restriction of $(\kappa + \rho\sigma(1+\theta))^2 > \sigma^2(1+\theta)(2+\theta)$, which turns out to be easily satisfied for economically plausible parameter values, as our subsequent quantitative analysis illustrates.

As the stock price dynamics (3) shows, there are possibly two sources of uncertainty in the stock returns, hence to fully characterize the investor's subjective probability measure, we need to specify her beliefs on one more process.[7] Toward that, consistent with the evidence in Greenwood and Shleifer (2014), who show that the average investor expectation of future returns is correlated with past stock returns but not with the past fundamentals, we simply assume that the investor has correct beliefs on the dynamics of the fundamental process (1). That is, at all times t, the investor observes the fundamental D_t, its variance V_t, as well as its mean growth rate μ. Hence, the investor, who has correct beliefs on the fundamental D but biased beliefs on the stock price S, perceives the stock price dynamics as

$$dS_t = S_t\left[(\mu_{St} + \theta X_t)\,dt + \sigma_{S1t}d\omega_{1t} + \sigma_{S2t}d\omega_{2t}^s\right], \tag{5}$$

where ω_2^s is the Brownian motion under the investor's subjective measure $\mathbb{P}^s$ and is given by $d\omega_{2t}^s = d\omega_{2t} - (\theta X_t/\sigma_{S2t})dt$, which follows immediately from equating (3) and (5).[8]

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[6]For intuition, one could also think of the investor's extrapolation degree θ as a parameter that controls the population size of investors with extrapolative beliefs in reality.

[7]This is in contrast to the models in which there is only one source of uncertainty in the stock returns, such as the one in Barberis et al. (2015). In these models extrapolating the endogenous stock returns alone is sufficient to fully characterize the investors' subjective measure. This case can also arise in our setting with $\rho = -1$, in which the stock price, sentiment, fundamental process and its variance are all driven by the single Brownian motion ω_1. Even though we omit the analysis for brevity, one can in fact show that our main economic mechanism and our most results also obtain in this polar case.

[8]We note that the last term of the subjective Brownian motion change $d\omega_{2t}^s$ contains two endogenous quantities X_t and σ_{S2t}, which complicate the analysis since solving for the equilibrium now necessarily involves conjecturing a stock price form and verifying this form in equilibrium. For this reason, we consider simple (logarithmic) preferences with no intertemporal consumption, which significantly simplify the verification of the conjectured stock price form while also leading to tractable closed-from solutions for the stock price as our

The investor's optimization problem is such that the investor chooses an admissible dynamic portfolio strategy ψ_S, the number of shares in the stock, so as to maximize her logarithmic preferences over W_T, the value of her portfolio at time T,

$$\max_{\psi_S} \mathrm{E}^s \left[\ln W_T \right], \tag{6}$$

subject to her dynamic budget constraint

$$dW_t = \left(W_t - \psi_{St} S_t \right) r dt + \psi_{St} dS_t, \tag{7}$$

where E^s denotes the expectation under the investor's subjective measure $\mathbb{P}^s$.

3   Extrapolative Beliefs Equilibrium

In this section, we investigate how the extrapolative beliefs induced investor sentiment affects equilibrium stock price and its dynamics in our model. In particular, we demonstrate that the magnitude of the fundamental variance elasticity of the stock price, the stock return variance, and the negative correlation between the stock return and its variance are all increasing in the extrapolation degree, but are not affected by the relative weights of past returns in the sentiment. We also show that following positive stock returns, in addition to expecting future stock returns to be higher, the investor also expects future returns to be less volatile, consistent with evidence.

Equilibrium in our economy is defined in a standard way. The economy is said to be in equilibrium if the stock price S and the investor's portfolio strategy ψ_S are such that the investor chooses her optimal portfolio strategy given the stock price, goods market clear at horizon ($D_T = W_T$), and stock and bond markets clear at all times t ($\psi_{St} = 1$ and $W_t - \psi_{St} S_t = 0$). In what follows, we denote the corresponding quantities in the rational benchmark economy in which the investor has no extrapolative beliefs with an upper bar ($\bar{}$).

Section 3 demonstrates. These issues do not arise in models in which investors' subjective beliefs are defined over exogenous fundamentals, which typically are studied tractably under more general (power) preferences with intertemporal consumption, and even in the presence of investor heterogeneity (see, for example, Dumas, Kurshev, and Uppal (2009)).

Proposition 1 presents the non-stationary equilibrium stock price S_t and its properties in the finite-horizon version of our economy.

Proposition 1 (**Non-stationary equilibrium stock price**)**.** *In the non-stationary economy with extrapolative beliefs, the equilibrium stock price is given by*

$$S_t = D_t e^{a(t) + b(t) V_t + c(t) X_t}, \tag{8}$$

where its sentiment elasticity $c(t)$ is given by

$$c(t) = \theta \frac{1 - e^{-\alpha(1+\theta)(T-t)}}{\alpha(1+\theta)}, \tag{9}$$

and its fundamental variance elasticity $b(t)$ solves the ordinary differential equation

$$\dot{b}(t) = \frac{1}{2} + \kappa b(t) + \frac{1}{2} \frac{1 + 2\rho\sigma b(t) + \sigma^2 b^2(t)}{1 - \alpha c(t)}, \tag{10}$$

with $b(T) = 0$, and $a(t)$ solves $\dot{a}(t) = r - \mu - \zeta b(t) - r\alpha c(t)$ with $a(T) = 0$.
In the non-stationary rational benchmark economy, the equilibrium stock price is given by $\bar{S}_t = D_t e^{\bar{a}(t) + \bar{b}(t) V_t}$, where its fundamental variance elasticity $\bar{b}(t)$ solves $\dot{\bar{b}}(t) = 1/2 + \kappa\bar{b}(t) + (1/2)(1 + 2\rho\sigma\bar{b}(t) + \sigma^2\bar{b}^2(t))$ with $\bar{b}(T) = 0$, and $\bar{a}(t)$ solves $\dot{\bar{a}}(t) = r - \mu - \zeta\bar{b}(t)$ with $\bar{a}(T) = 0$.

Consequently, in the non-stationary economy with extrapolative beliefs,

i) *the stock price is increasing in sentiment X_t, but is decreasing in fundamental variance V_t,*

ii) *the sentiment elasticity is increasing in the extrapolation degree θ, but is decreasing in the relative weights α.*

In the non-stationary rational benchmark economy, fluctuations in the stock price are driven by the fundamental D and its variance V. A higher current fundamental indicates a higher future stock payoff, hence leads to a higher stock price. On the other hand, a higher current fundamental variance indicates a more uncertain future stock payoff, and since the investor is risk averse, this leads to a lower stock price. In the presence of extrapolative beliefs, the economic roles of the fundamental and its variance are still as in the benchmark economy,

since the past stock return extrapolation does not alter the investor's risk attitude. However, the stock price is now additionally driven by the sentiment X as (8) shows. Following positive stock returns, either due to higher fundamental or lower fundamental variance, now the investor sentiment rises and becomes more bullish as the investor expects future returns to be higher, which in return increases her stock demand, and hence the stock price. Conversely, following negative stock returns, the investor sentiment falls and becomes more bearish, leading to a lower stock price (Property (i)).

The sentiment elasticity of the stock price $c(t)$ gives the percentage change in the stock price in response to a unit change in the sentiment, and it is positive as (9) shows. A higher extrapolation degree θ directly increases the investor's future stock return expectation by definition, and hence leads to a higher sentiment elasticity. On the other hand, a higher relative weights α makes the sentiment process less persistent as the sentiment is now driven mostly by the most recent returns. Therefore, the stock price becomes less sensitive to the changes in sentiment, since the latter is expected to revert back to its mean more rapidly. These results are summarized in Property (ii). Similarly, the fundamental variance elasticity of the stock price $b(t)$ gives the percentage change in the stock price in response to a unit change in the fundamental variance, and it is negative as in the rational benchmark economy discussed earlier. Even though we can determine the sign of the fundamental variance elasticity, a general methodology to obtain an analytical solution for it does not exist because of the time variation in the sentiment elasticity $c(t)$ in the ordinary differential equation (10). As we discuss below, this is one of the reasons we consider the stationary (infinite-horizon) version of our economy, which allows us to obtain the fundamental variance elasticity in closed-form as a constant and gain further insights on the effects of extrapolation.

In the remainder of the paper, we consider the stationary version of our economy by letting $T \to \infty$.[9] There are several benefits of considering a stationary equilibria. First, the stationarity removes the time-to-maturity effects and leads to constant stock price elasticities, which in return allows our model to be calibrated in a simpler way as our discussion in Appendix B illustrates. Second, as highlighted above, the stationary economy allows us to obtain the fundamental variance elasticity in closed-form, which in return allows us to carry

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[9]One recent example in the literature which also consider the infinite-horizon limit of their non-stationary economy to avoid time-to-maturity effects is Eraker and Wu (2017). Similar consideration is also present in Hong and Stein (1999).

out a simple comparative statics analysis, and therefore helps us determine and understand the full effects of the extrapolative beliefs on the stock price. Third, the stationary equilibria leads to a simpler stock price expression (while still keeping its key properties, see Proposition 2 below) and hence to simpler derivative prices. For instance, as we show in Section 4, the well-known stochastic volatility option pricing model of Heston (1993) arise as a special case of our stationary equilibria. Toward that, Proposition 2 presents the stationary equilibrium stock price S_t and summarizes its properties in the infinite-horizon version of our economy, henceforth simply referred to as the *economy with extrapolative beliefs.*

Proposition 2 (**Equilibrium stock price**). *In the economy with extrapolative beliefs, the equilibrium stock price is given by*

$$S_t = D_t e^{a+bV_t+cX_t}, \tag{11}$$

where its sentiment elasticity c and fundamental variance elasticity b are given by

$$c = \frac{\theta}{\alpha\,(1+\theta)}, \tag{12}$$

$$b = -\frac{2+\theta}{\kappa + \rho\sigma\,(1+\theta) + \sqrt{(\kappa + \rho\sigma\,(1+\theta))^2 - \sigma^2\,(1+\theta)\,(2+\theta)}}, \tag{13}$$

and a is some constant, with the parameter restriction of $\mu + \zeta b - r(1-\alpha c) = 0$.
In the rational benchmark economy, the equilibrium stock price is given by $\bar{S}_t = D_t e^{\bar{a}+\bar{b}V_t}$, where its fundamental variance elasticity $\bar{b}$ is given by $\bar{b} = -2/(\kappa + \rho\sigma + \sqrt{(\kappa + \rho\sigma)^2 - 2\sigma^2})$, and $\bar{a}$ is some constant, with the parameter restriction of $\mu + \zeta\bar{b} - \bar{r} = 0$.

Consequently, in the economy with extrapolative beliefs, in addition to Proposition 1 properties (i) and (ii), the fundamental variance elasticity is decreasing in the extrapolation degree θ, but is not affected by the relative weights α.

The stationarity equilibrium stock price (11) has a form similar to that in the non-stationarity equilibria of Proposition 1. However, now the sentiment elasticity $c > 0$ and the fundamental variance elasticity $b < 0$ are simple constants and given by (12)–(13) in closed-form. Proposition 2 also highlights that our earlier results of the stock price being increasing (decreasing) in sentiment (fundamental variance), and the sentiment elasticity be-

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ing increasing (decreasing) in the extrapolation degree (relative weights) continue to hold in the stationary economy. More notably, we are now able to show that a higher extrapolation degree leads to a more negative fundamental variance elasticity by increasing its magnitude. Therefore, when the extrapolation degree is higher, the stock price becomes more sensitive to the fundamental variance shocks. This is because the fundamental variance shocks are also part of the stock return that the investor's extrapolative bias is based on. Hence, following positive stock returns due to negative fundamental variance shocks, the investor expects higher future returns, and hence she pushes up the stock price. This effect is greater, the higher the extrapolation degree, which makes the stock price more sensitive to the changes in the fundamental variance. We also see that in the stationary economy the fundamental variance elasticity is not affected by the relative weights. This occurs for the same reason that the stock return variance itself is not affected by the relative weights, a result we discuss in detail in our next Proposition 3.[10]

Having determined the stock price in closed-form, we now look at its dynamics. Toward that, Proposition 3 presents the dynamics of the equilibrium stock price dS_t and sentiment dX_t, the stock return variance $\vartheta_t = \mathrm{Var}_t\left[dS_t/S_t\right]/dt$, and the correlation between the stock return and its variance $\rho_{S\vartheta} = \mathrm{Cov}_t\left[dS_t/S_t, d\vartheta_t\right]/\sqrt{\mathrm{Var}_t\left[dS_t/S_t\right]\mathrm{Var}_t\left[d\vartheta_t\right]}$, in closed-form, and summarizes their key properties in our stationary economy.

Proposition 3 (**Equilibrium stock price and sentiment dynamics, stock return variance, and correlation**)**.** *In the economy with extrapolative beliefs, the equilibrium stock price and sentiment dynamics are given by*

$$dS_t = S_t\left[\left(r+\frac{1+2\rho\sigma b+\sigma^2 b^2}{\left(1-\alpha c\right)^2}V_t-\theta X_t\right)dt+\frac{1+\rho\sigma b}{1-\alpha c}\sqrt{V_t}d\omega_{1t}+\frac{\sqrt{1-\rho^2}\sigma b}{1-\alpha c}\sqrt{V_t}d\omega_{2t}\right], \qquad (14)$$

$$dX_t = \alpha\left(r+\frac{1}{2}\frac{1+2\rho\sigma b+\sigma^2 b^2}{\left(1-\alpha c\right)^2}V_t-(1+\theta)X_t\right)dt+\alpha\frac{1+\rho\sigma b}{1-\alpha c}\sqrt{V_t}d\omega_{1t}+\alpha\frac{\sqrt{1-\rho^2}\sigma b}{1-\alpha c}\sqrt{V_t}d\omega_{2t}, \quad (15)$$

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[10]We also note that for the stock price in our infinite-horizon economy to be well-defined and finite-valued, we need to impose the parameter restriction of $\mu + \zeta b - r(1 - \alpha c) = 0$. As our discussion in Appendix B illustrates, for economically plausible parameter values of our model, this restriction is satisfied if, for example, we choose the mean growth rate of the fundamental as $\mu = 2.21\%$, an economically plausible value. Moreover, as the denominator of (13) illustrates, to ensure that the stationary equilibrium stock price admits a real solution we need to impose the parameter restriction $\left(\kappa + \rho\sigma\left(1+\theta\right)\right)^2 > \sigma^2\left(1+\theta\right)\left(2+\theta\right)$ that is introduced in Section 2.

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the equilibrium stock return variance is given by

$$\vartheta_t = \frac{1 + 2\rho\sigma b + \sigma^2 b^2}{(1 - \alpha c)^2} V_t, \tag{16}$$

and the equilibrium correlation between the stock return and its variance is given by

$$\rho_{S\vartheta} = \frac{\rho + \sigma b}{\sqrt{1 + 2\rho\sigma b + \sigma^2 b^2}}, \tag{17}$$

where the sentiment elasticity c and fundamental variance elasticity b are as in Proposition 2. In the rational benchmark economy, the equilibrium stock price dynamics is given by $d\bar{S}_t = \bar{S}_t\left[\left(\bar{r} + (1 + 2\rho\sigma\bar{b} + \sigma^2\bar{b}^2)V_t\right)dt + \left(1 + \rho\sigma\bar{b}\right)\sqrt{V_t}d\omega_{1t} + \sqrt{1 - \rho^2}\sigma\bar{b}\sqrt{V_t}d\omega_{2t}\right]$, the equilibrium stock return variance is given by $\bar{\vartheta}_t = \left(1 + 2\rho\sigma\bar{b} + \sigma^2\bar{b}^2\right)V_t$, and the equilibrium correlation between the stock return and its variance is given by $\bar{\rho}_{S\vartheta} = (\rho + \sigma\bar{b})/\sqrt{1 + 2\rho\sigma\bar{b} + \sigma^2\bar{b}^2}$, where the fundamental variance elasticity $\bar{b}$ is as in Proposition 2.

Consequently, in the economy with extrapolative beliefs,

> *i) the stock return variance is increasing in the extrapolation degree θ, but is not affected by the relative weights α,*
>
> *ii) the correlation between the stock return and its variance is decreasing in the extrapolation degree θ, but is not affected by the relative weights α.*

The drift term in (14) reveals that the stock mean return is positively related to the fundamental variance V but is negatively related to the sentiment X. These effects are intuitive. A higher fundamental variance indicates a more uncertain future stock payoff, and the risk averse investor requires a higher return to hold the stock in equilibrium. On the other hand, a higher sentiment, following positive stock returns, makes the investor to expect higher future returns, and hence she pushes up the stock price further. Due to this additional push, the stock price becomes overvalued relative to its fundamentals, therefore the subsequent observed stock returns become lower on average. Similarly, the dynamics in (15) reveals that the sentiment is a mean reverting process, with a higher relative weights α leading to a less persistent but a more volatile sentiment, as the sentiment is now driven largely by the most recent returns.[11]

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[11]Some elements of our stock price and the sentiment dynamics (14)–(15) are similar to the corresponding

In the presence of extrapolative beliefs, the equilibrium stock return variance (16) takes a simple form that is proportional to the fundamental variance. We find that a higher extrapolation degree θ leads to a higher stock return variance. In our model, this occurs due to two channels. The first channel is also present in Barberis et al. (2015) and is due to the amplification term $1/(1 - \alpha c) = 1 + \theta$ that arise from extrapolating endogenous stock returns. That is, following positive (negative) stock returns, the investor expects future returns to be higher (lower), and hence pushes up (down) the stock price further, and this additional fluctuation in the stock price leads to a higher return variance. On the other hand, the second channel is novel to our model, and is due to the fact that a higher extrapolation degree also increases the magnitude of the fundamental variance elasticity (Proposition 2). This makes the stock price to be more sensitive to the fundamental variance shocks, and hence leads to more volatile stock returns. On the other hand, somewhat surprisingly, we find that the stock return variance does not depend on the relative weights α. This occurs due to the two opposing effects of relative weights on the sentiment that we discussed earlier. That is, a higher relative weights leads to a less persistent sentiment process, and hence to a lower sentiment elasticity in (12). On the other hand, a higher relative weights leads to a more volatile sentiment process by increasing its volatility linearly. Since the investor sentiment contributes to the stock return volatility by an amount equals to its elasticity times its volatility, these two effects cancel each other out. These results are summarized in Property (i).

We also see that the equilibrium correlation between the stock return and its variance (17) is negative as in data (French, Schwert, and Stambaugh (1987), Glosten, Jagannathan, and Runkle (1993)). More importantly, we find that a higher extrapolation degree θ leads to a more negative correlation by increasing its magnitude, and hence leading to a more negatively skewed stock market returns (Property (ii)). This is because a higher extrapolation degree leads to a more negative fundamental variance elasticity by increasing its magnitude (Proposition 2). Therefore, the stock price becomes more sensitive and moves in the opposite direction more following the fundamental variance, or equivalently the stock return variance, shocks. We also see that this correlation is not affected by the relative weights. This happens

dynamics in Barberis et al. (2015). This is not surprising since our sentiment measure is constructed similarly to theirs. Therefore, in this section we do not focus much on the quantities or results that are common to both papers, and instead highlight the novel predictions that arise only in our model, which typically involve the stochastic stock return variance. Our results on option prices in Section 4, and variance risk premium in Section 5, however, are all novel, and hence we elaborate more on those.

because of the same reason that the stock return variance itself is not affected by the relative weights, as discussed above. As we show in Lemma 1 of Section 4, this correlation directly affects the persistence of the stock return variance under the risk-neutral measure, and hence significantly influences the derivative prices that we study in subsequent Sections 4 and 5.[12]

In addition to the evidence that following good stock performance investors expect future returns to be higher, the evidence also shows that following good performance investors also expect future returns to be less volatile. We now examine our model implications for the investor's expectation of the future stock return variance. Toward that, Proposition 4 presents the equilibrium stock return variance dynamics $d\vartheta_t$ both under the objective and the subjective measures, along with its key property in our stationary economy.

Proposition 4 (**Equilibrium stock return variance dynamics**)**.** *In the economy with extrapolative beliefs, the equilibrium stock return variance dynamics under the objective and subjective measures are given by*

$$d\vartheta_t = (\zeta_\vartheta - \kappa\vartheta_t)dt + \rho\sigma_\vartheta\sqrt{\vartheta_t}d\omega_{1t} + \sqrt{1-\rho^2}\sigma_\vartheta\sqrt{\vartheta_t}d\omega_{2t}, \tag{18}$$

$$d\vartheta_t = (\zeta_\vartheta - \kappa\vartheta_t + \eta_\vartheta X_t)dt + \rho\sigma_\vartheta\sqrt{\vartheta_t}d\omega_{1t} + \sqrt{1-\rho^2}\sigma_\vartheta\sqrt{\vartheta_t}d\omega_{2t}^s, \tag{19}$$

with the constants

$$\zeta_\vartheta = \zeta\frac{1 + 2\rho\sigma b + \sigma^2 b^2}{(1-\alpha c)^2}, \qquad \sigma_\vartheta = \sigma\sqrt{\frac{1 + 2\rho\sigma b + \sigma^2 b^2}{(1-\alpha c)^2}}, \qquad \eta_\vartheta = \frac{\theta}{b}\frac{1 + 2\rho\sigma b + \sigma^2 b^2}{1-\alpha c}, \tag{20}$$

where the sentiment elasticity c and fundamental variance elasticity b are as in Proposition 2. In the rational benchmark economy, the equilibrium stock return variance dynamics under the objective and subjective measures are the same and is given by $d\bar\vartheta_t = (\bar\zeta_\vartheta - \kappa\bar\vartheta_t)dt + \rho\bar\sigma_\vartheta\sqrt{\bar\vartheta_t}d\omega_{1t} + \sqrt{1-\rho^2}\bar\sigma_\vartheta\sqrt{\bar\vartheta_t}d\omega_{2t}$, with the constants $\bar\zeta_\vartheta = \zeta(1 + 2\rho\sigma\bar b + \sigma^2\bar b^2)$, and $\bar\sigma_\vartheta = \sigma\sqrt{1 + 2\rho\sigma\bar b + \sigma^2\bar b^2}$, where the fundamental variance elasticity $\bar b$ is as in Proposition 2.

Consequently, in the economy with extrapolative beliefs, the expected change in the stock return variance under the subjective measure is decreasing in sentiment X_t.

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[12]In the option pricing literature it is well-known that the correlation between the stock return and its variance plays an important role in explaining the observed negative slopes in option implied volatilities, the so-called smile curves (see, for example, Duffie (2001)).

Proposition 4 shows that the investor also misperceives the expected changes in the stock return variance with her variance bias being $\eta_\vartheta X_t$. The key implication here is that a higher sentiment, due to good recent stock returns, in addition to making the investor to expect higher future returns, also makes the investor to expect future returns to be less volatile as $\eta_\vartheta < 0$. Conversely, a lower sentiment makes the investor to expect more volatile future returns. This is because investor's higher future return expectations when she has more bullish sentiment must be accompanied by either higher expected fundamental growth or lower expected fundamental variance. Since the investor has correct beliefs on the growth rate of the fundamental, she expects less volatile future fundamental, and hence less volatile stock returns. This result is consistent with the evidence in Amromin and Sharpe (2014) and Kaplanski et al. (2016), who show that following high stock returns, in addition to expecting the future stock returns to be higher, investors also expect them to be less volatile.[13] At this point we find it helpful to highlight that the investor's biased variance expectation is distinct from the correlation channel. That is, following high stock returns, the stock returns also tend to be lower due to the negative correlation between the stock return and its variance (negative skewness). Our finding here shows that even if this correlation were to be zero, the investor would still expect future returns to be less volatile. As we show in Sections 4 and 5, this result plays an important role in transmitting the effects of extrapolative beliefs to the prices of derivatives, in particular to the prices of options, variance swaps and hence to the variance risk premium.

4 Option Prices

In the previous section, we demonstrated how the investor's bullish/bearish sentiment due to extrapolating past stock returns affects the stock price and its dynamics. Given there is much empirical evidence showing that investors' bullish/bearish sentiment, and past stock market returns affect index option prices, in this section, we investigate our model implications for option prices. Toward that, we first obtain analytic option pricing formulas that are driven by

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[13]Goyal and Saretto (2009) provide a similar evidence of volatility bias in the cross-section by showing that investors overestimate (underestimate) future volatility of stocks that have experienced negative (positive) past returns. Adelino, Schoar, and Severino (2018) find a similar volatility bias in the housing market by showing that households extrapolate past house price changes when they form their expectations of future house price risk (variance) in a way that large recent price drops lead to more perceived risk.

investor sentiment in addition to the stock price and its stochastic return variance. We then illustrate that the investor sentiment affects option prices in our model in a way consistent with empirical evidence. That is, a more bearish sentiment leads to a higher price for both the call and the put options, and the price increases are more pronounced for the out-of-the-money options.

To ensure our option pricing formulas are comparable to those in the vast option pricing literature, we find it convenient to work under the risk-neutral measure with compact dynamics in terms of the correlated Brownian motions rather than the independent Brownian motions as we do so in the previous section. Lemma 1 presents the system for the equilibrium stock price dynamics $(dS_t, d\vartheta_t, dX_t)$ under the risk-neutral measure $\mathbb{P}^*$ in our stationary economy.

Lemma 1 (**Equilibrium risk-neutral dynamics**). *In the economy with extrapolative beliefs, the equilibrium risk-neutral dynamics of the stock price is given by the system*

$$dS_t = S_t\Big[rdt + \sqrt{\vartheta_t}d\omega^*_{St}\Big], \tag{21}$$

$$d\vartheta_t = \big(\zeta_\vartheta - \kappa^*_\vartheta\vartheta_t + \eta_\vartheta X_t\big)dt + \sigma_\vartheta\sqrt{\vartheta_t}d\omega^*_{\vartheta t}, \tag{22}$$

$$dX_t = \alpha\Big(r - \frac{1}{2}\vartheta_t - X_t\Big)dt + \alpha\sqrt{\vartheta_t}d\omega^*_{St}, \tag{23}$$

with the constant

$$\kappa^*_\vartheta = \kappa + \rho_{S\vartheta}\sigma_\vartheta, \tag{24}$$

where the constants ζ_ϑ, σ_ϑ, η_ϑ are as in Proposition 4, $\rho_{S\vartheta}$ is as in Proposition 3, and ω^*_S and ω^*_ϑ are correlated Brownian motions under the risk-neutral measure with $d\omega^*_{St}d\omega^*_{\vartheta t} = \rho_{S\vartheta}dt$. In the rational benchmark economy, the equilibrium risk-neutral dynamics of the stock price is characterized by the system $d\bar{S}_t = \bar{S}_t\Big[\bar{r}dt + \sqrt{\bar{\vartheta}_t}d\bar{\omega}^*_{St}\Big]$, $d\bar{\vartheta}_t = \Big(\bar{\zeta}_\vartheta - \bar{\kappa}^*_\vartheta\bar{\vartheta}_t\Big)dt + \bar{\sigma}_\vartheta\sqrt{\bar{\vartheta}_t}d\bar{\omega}^*_{\vartheta t}$, with the constant $\bar{\kappa}^*_\vartheta = \kappa + \bar{\rho}_{S\vartheta}\bar{\sigma}_\vartheta$, where the constants $\bar{\zeta}_\vartheta$, $\bar{\sigma}_\vartheta$ are as in Proposition 4, $\bar{\rho}_{S\vartheta}$ is as in Proposition 3, and $\bar{\omega}^*_S$ and $\bar{\omega}^*_\vartheta$ are correlated Brownian motions under the risk-neutral measure with $d\bar{\omega}^*_{St}d\bar{\omega}^*_{\vartheta t} = \bar{\rho}_{S\vartheta}dt$.

Lemma 1 shows that in the economy with extrapolative beliefs, under the risk-neutral measure the expected stock return is equal to the return of the riskless asset as usual. However, the expected change in the stock return variance now incorporates the investor's variance bias term $\eta_\vartheta X_t$, and hence depends on the investor sentiment. A simple intuitive way to see why

the investor's variance bias enters into the risk-neutral variance dynamics involves comparing it with the usual risk adjustment for the stock returns. The expected stock return under the risk-neutral measure is obtained by the true stock mean return, $r + \vartheta_t - \theta X_t$, minus the stock risk premium. In our model, the latter is given by $\vartheta_t - \theta X_t$, where the last term adjusts for the investor's bias on the stock return expectation. Similarly, the expected change in the stock return variance under the risk-neutral measure is obtained by the true expected change, $\zeta_\vartheta - \kappa \vartheta_t$, minus the corresponding risk premium for the variance. In our model, the latter is given by $\rho_{S\vartheta} \sigma_\vartheta \vartheta_t - \eta_\vartheta X_t$, where again the last term adjusts for the investor's bias on the stock return variance expectation.[14] The risk-neutral sentiment dynamics (23), on the other hand, is obtained by simply substituting the (log) stock price dynamics obtained via (21) into the dynamics (4). Moreover, as (24) shows, a more negative correlation between the stock return and its variance, say due to a higher extrapolation degree (Proposition 3 Property (ii)), leads to a more persistent stock return variance under the risk-neutral measure. Finally, we note that in the rational benchmark economy, the stock price dynamics under the risk-neutral measure is as in the well-known stochastic volatility model of Heston (1993).

We now consider standard European-style call and put options with a strike price K and a maturity date T_o written on the stock, whose price follows the dynamics (21) under the risk-neutral measure. At the maturity date, the payoff of the call option is $\max\{S_{T_o} - K, 0\}$, and the payoff of the put option is $\max\{K - S_{T_o}, 0\}$. The standard no-arbitrage arguments lead to the call and put option prices at time t as $C_t = e^{-r(T_o - t)} \mathrm{E}_t^* [\max\{S_{T_o} - K, 0\}]$, and $P_t = e^{-r(T_o - t)} \mathrm{E}_t^* [\max\{K - S_{T_o}, 0\}]$, respectively, where E^* denotes the expectation under the risk-neutral measure $\mathbb{P}^*$. Since the dynamics in (21)–(23) have an affine structure in the state processes ϑ and X, we determine the option prices in a tractable way using the so-called transform analysis (Duffie, Pan, and Singleton (2000)). Toward that, Proposition 5 provides the analytic formulas for the equilibrium call option price C_t and put option price P_t in our stationary economy.

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[14]The risk premium for the stock return variance follows from the market prices of risks given by (A.18) in the Appendix A, and the equilibrium stock return variance dynamics (18) in Proposition 4.

Proposition 5 (Equilibrium option prices). *In the economy with extrapolative beliefs, the equilibrium call and put option prices are given by*

$$C_t = S_t \Psi_1 \left(\ln S_t, \vartheta_t, X_t, t \right) - K e^{-r(T_o - t)} \Psi_2 \left(\ln S_t, \vartheta_t, X_{t,} t \right), \tag{25}$$

$$P_t = -S_t \left(1 - \Psi_1 \left(\ln S_t, \vartheta_t, X_t, t \right) \right) + K e^{-r(T_o - t)} \left(1 - \Psi_2 \left(\ln S_t, \vartheta_t, X_{t,} t \right) \right), \tag{26}$$

where, for $j = 1, 2$, the conditional probability function $\Psi_j(s, v, x, t)$ is given by

$$\Psi_j(s, v, x, t) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \mathrm{Re} \left[\frac{e^{-i\phi \ln K} \varphi_j(s, v, x, t; \phi)}{i\phi} \right] d\phi, \tag{27}$$

with i being the imaginary unit, $\mathrm{Re}[z]$ is the real part of a complex number z, and the characteristic function $\varphi_j(s, v, x, t; \phi)$ is given by

$$\varphi_j(s, v, x, t; \phi) = e^{\mathcal{A}_j(T_o - t) + \mathcal{B}_j(T_o - t)v + \mathcal{C}_j(T_o - t)x + i\phi s}, \tag{28}$$

where the deterministic functions $\mathcal{C}_j(\tau)$, $\mathcal{B}_j(\tau)$, $\mathcal{A}_j(\tau)$ solve the ordinary differential equations

$$\dot{\mathcal{C}}_j(\tau) = \eta_\vartheta \mathcal{B}_j(\tau) - \alpha \mathcal{C}_j(\tau), \tag{29}$$

$$\dot{\mathcal{B}}_j(\tau) = \frac{1}{2} \left(2u_j \phi i - \phi^2 \right) - \left(y_j - \rho_{S\vartheta} \sigma_\vartheta \phi i \right) \mathcal{B}_j(\tau) + \frac{1}{2} \sigma_\vartheta^2 \mathcal{B}_j^2(\tau)$$

$$+ \alpha \left(u_j + \phi i \right) \mathcal{C}_j(\tau) + \alpha \rho_{S\vartheta} \sigma_\vartheta \mathcal{B}_j(\tau) \mathcal{C}_j(\tau) + \frac{1}{2} \alpha^2 \mathcal{C}_j^2(\tau), \tag{30}$$

$$\dot{\mathcal{A}}_j(\tau) = r\phi i + \zeta_\vartheta \mathcal{B}_j(\tau) + \alpha r \mathcal{C}_j(\tau), \tag{31}$$

with $\mathcal{C}_j(0) = \mathcal{B}_j(0) = \mathcal{A}_j(0) = 0$, and the constants

$$u_1 = \frac{1}{2}, \qquad u_2 = -\frac{1}{2}, \qquad y_1 = \kappa, \qquad y_2 = \kappa_\vartheta^*, \tag{32}$$

where κ_ϑ^* is as in Lemma 1, ζ_ϑ, σ_ϑ, η_ϑ are as in Proposition 4, and $\rho_{S\vartheta}$ is as in Proposition 3. In the rational benchmark economy, the equilibrium call and put option prices are given by $\bar{C}_t = \bar{S}_t \bar{\Psi}_1 \left(\ln \bar{S}_t, \bar{\vartheta}_t, t \right) - K e^{-\bar{r}(T_o - t)} \bar{\Psi}_2 \left(\ln \bar{S}_t, \bar{\vartheta}_t, t \right)$, and $\bar{P}_t = -\bar{S}_t \left(1 - \bar{\Psi}_1 \left(\ln \bar{S}_t, \bar{\vartheta}_t, t \right) \right) + K e^{-\bar{r}(T_o - t)} \left(1 - \bar{\Psi}_2 \left(\ln \bar{S}_t, \bar{\vartheta}_t, t \right) \right)$, where, for $j = 1, 2$, the conditional probability function $\bar{\Psi}_j(s, v, t)$ is given by $\bar{\Psi}_j(s, v, t) = (1/2) + (1/\pi) \int_0^\infty \mathrm{Re}[e^{-i\phi \ln K} \bar{\varphi}_j(s, v, t; \phi) / i\phi] d\phi$, with the characteristic function $\bar{\varphi}_j(s, v, t; \phi)$ is given by $\bar{\varphi}_j(s, v, t; \phi) = e^{\bar{\mathcal{A}}_j(T_o - t) + \bar{\mathcal{B}}_j(T_o - t)v + i\phi s}$, where the determin-

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istic functions $\bar{\mathcal{B}}_j(\tau)$, $\bar{\mathcal{A}}_j(\tau)$ solve the ordinary differential equations $\dot{\bar{\mathcal{B}}}_j(\tau) = (1/2)(2u_j\phi i - \phi^2) - (\bar{y}_j - \bar{\rho}_{S\vartheta}\bar{\sigma}_\vartheta\phi i)\bar{\mathcal{B}}_j(\tau) + (1/2)\bar{\sigma}_\vartheta^2\bar{\mathcal{B}}_j^2(\tau)$, and $\dot{\bar{\mathcal{A}}}_j(\tau) = \bar{r}\phi i + \bar{\zeta}_\vartheta\bar{\mathcal{B}}_j(\tau)$, with $\bar{\mathcal{B}}_j(0) = \bar{\mathcal{A}}_j(0) = 0$, and the constants $\bar{u}_1 = 1/2$, $\bar{u}_2 = -1/2$, $\bar{y}_1 = \kappa$, $y_2 = \bar{\kappa}_\vartheta^*$, where $\bar{\kappa}_\vartheta^*$ is as in Lemma 1, $\bar{\zeta}_\vartheta$, $\bar{\sigma}_\vartheta$ are as in Proposition 4, and $\bar{\rho}_{S\vartheta}$ is as in Proposition 3.

In the rational benchmark economy, option prices are as in the stochastic volatility model of Heston (1993), and hence are driven by the stock price and its return variance. However, in the presence of extrapolative beliefs, option prices are additionally driven by the investor sentiment X, as (25)–(26) show. This is because, as we show in Lemma 1, under the risk-neutral measure, the expected change in the stock return variance incorporates the investor's variance bias term $\eta_\vartheta X_t$ in terms of the sentiment, which is reflected in option prices. There is much empirical evidence showing that bullish and bearish investor sentiment (or past stock market returns, which our sentiment measure is based on) affect index option prices (Bakshi and Kapadia (2003), Amin, Coval, and Seyhun (2004), Han (2008)). In particular, Han (2008) argues that his empirical findings cannot be reconciled with standard rational option pricing models. Therefore, by providing new, analytic, investor-sentiment driven option pricing formulas, we believe we take a significant step in the right direction to potentially reconcile these evidence.

In order to see how exactly the investor sentiment affects option prices in our model we rely on a simple numerical analysis, since obtaining tractable analytic comparative statics is typically not feasible when option prices take forms as those in Proposition 5. Toward that, we report the quantitative effects of investor sentiment on our option prices of Proposition 5 in Table 1, which considers three month maturity options when the stock price is $S_t = 100$ and the stock return volatility is at its steady-state value of $\sqrt{\vartheta}_t = 15.98\%$.[15] We consider three different option moneyness K/S_t levels, as well as four different extrapolation degrees, in which $\theta = 0$ corresponds to the rational benchmark economy yielding standard Heston prices, and $\theta = 0.25$ corresponding to our baseline extrapolation degree. We also consider three different sentiment levels, bullish ($X_t = 0.10$, which corresponds to the extrapolative bias of $\theta X_t = 2.5\%$ in our baseline model), bearish ($X_t = -0.10$, which corresponds to the extrapolative bias of $\theta X_t = -2.5\%$ in our baseline model), and no sentiment ($X_t = 0$). We report the percentage

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[15]The parameter values used in Table 1 follow from Table 3 of Appendix B, where we also discuss how we determine the baseline parameter values employed in our Tables. In Appendix B, we also provide the corresponding analysis for shorter and longer maturity options (Table 4–5), which show that our results here continue to hold for these options as well.

K/S_t	θ	Put option				Call option			
		X_t				X_t			
		-0.10	0	0.10	% Diff.	-0.10	0	0.10	% Diff.
	0	0.35	0.35	0.35	**0%**	10.51	10.51	10.51	**0%**
	0.25	0.38	0.37	0.35	**7.54%**	10.54	10.53	10.52	**0.25%**
0.9	0.50	0.42	0.39	0.37	**16.02%**	10.59	10.56	10.53	**0.56%**
	0.75	0.48	0.43	0.38	**24.75%**	10.64	10.59	10.55	**0.90%**
	0	3.08	3.08	3.08	**0%**	3.26	3.26	3.26	**0%**
	0.25	3.13	3.09	3.06	**2.25%**	3.31	3.27	3.24	**2.13%**
1	0.50	3.18	3.11	3.03	**4.95%**	3.36	3.29	3.21	**4.67%**
	0.75	3.25	3.13	3.01	**8.00%**	3.43	3.31	3.19	**7.55%**
	0	10.26	10.26	10.26	**0%**	0.45	0.45	0.45	**0%**
	0.25	10.26	10.24	10.22	**0.36%**	0.46	0.44	0.42	**8.64%**
1.1	0.50	10.27	10.23	10.19	**0.79%**	0.47	0.43	0.39	**20.80%**
	0.75	10.27	10.21	10.14	**1.28%**	0.47	0.41	0.34	**38.17%**

Table 1: **Option prices.** This table reports the put and call option prices for varying levels of option moneyness K/S_t, extrapolation degree θ, and sentiment X_t. The last columns report the percentage difference of option prices at the bearish sentiment ($X_t = -0.10$) from the bullish sentiment ($X_t = 0.10$). The option maturity is 3 months, the stock price is $S_t = 100$, the stock return volatility is 15.98% (steady-state value), and all the other parameter values are as in Table 3: $\alpha = 0.5$, $\kappa = 0.15$, $\zeta = 0.0018$, $\sigma = 0.0506$, $\rho = -0.15$, $\mu = 0.0221$.

difference of option prices at the bearish sentiment from the corresponding ones at the bullish sentiment in boldface in last columns (% Diff.).

Table 1 reveals that in the rational benchmark economy ($\theta = 0$), since option prices are as in Heston (1993), they do not vary with sentiment levels. However, in the presence of extrapolative beliefs ($\theta > 0$), both the put and call option prices are higher under the bearish sentiment as the uniform positivity of the terms in the % Diff. columns indicate. In our model this occurs because a more bearish sentiment, due to bad recent stock returns, in addition to making the investor to expect lower future returns, also makes the investor to expect future returns to be more volatile. A higher expected stock return volatility, similar to the usual vega effect, then leads to higher option prices irrespective of whether the option is a put

or call. Table 1 also reveals that the price increases under the bearish sentiment are more pronounced for out-of-the-money options ($K/S_t = 0.9$ for the put option, $K/S_t = 1.1$ for the call option). In particular, for our baseline value of $\theta = 0.25$, the out-of-the-money put price is 7.54%, while the out-of-the-money call is 8.64% higher under the bearish sentiment than those under the bullish sentiment. Our findings here are consistent with the empirical evidence. For example, Bakshi and Kapadia (2003) find that index options become more expensive after extreme negative past returns, whereas Amin, Coval, and Seyhun (2004) and Han (2008) provide evidence that bearish sentiment due to low past returns lead to steeper implied volatility smile due to increases in out-of-the-money index option prices.[16]

5 Variance Risk Premium

As we demonstrated in Section 3, due to extrapolating past stock returns, the investor misperceives the expected changes in the stock return variance. Potentially, this biased variance expectation has important implications for the risk premia the investor is willing to accept for bearing the variance risk in the stock market. Toward that, in this section, we examine our model implications for the variance risk premium, a quantity which has attracted much attention in the literature recently. We first obtain closed-form solution for the variance risk premium, and show that a more bearish sentiment, due to bad recent stock performance, leads to a more negative variance risk premium. We then show that the variance risk premium predicts future stock returns negatively, even after controlling for the realized (or, conditional) variance, consistent with empirical evidence.

Our starting point for the variance risk premium is the variance swap, a financial contract that allows investors to directly trade and speculate on the future stock return variance. More specifically, a time-t initiated τ-period variance swap is a claim to the payoff $\left(\frac{1}{\tau}\int_t^{t+\tau}\vartheta_u du - \Upsilon_t^*(\tau)\right)N$ at its maturity date $t+\tau$. Here, N is the notional amount, and $\Upsilon_t^*(\tau)$ is the variance

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[16]Similarly, Lemmon and Ni (2014) show that the index option implied volatilities are affected by the spread between the bullish and bearish sentiment in a way consistent with the findings in Han (2008), though they do not find significant effect for the stock market's previous month return. Our findings are also in line with the cross-sectional evidence in Goyal and Saretto (2009), who find that stocks with bad recent performance have higher option prices, and attribute this findings to investors overestimation of future volatility due to the negative skewness.

swap rate that is set at the contract initiation time t so that the contract has zero value. The standard no-arbitrage arguments show that the variance swap rate satisfies

$$\Upsilon_t^* (\tau) = \mathrm{E}_t^* \left[\frac{1}{\tau} \int_t^{t+\tau} \vartheta_u du \right], \tag{33}$$

where E_t^* denotes the conditional expectation under the risk-neutral measure $\mathbb{P}^*$. Consistent with the literature (e.g., Carr and Wu (2009)), we then define the variance risk premium as the difference between the (average) realized variance and the variance swap rate, where we define the (average) realized variance (realized over the life of the variance swap rate) as

$$\Upsilon_t (\tau) = \mathrm{E}_t \left[\frac{1}{\tau} \int_t^{t+\tau} \vartheta_u du \right], \tag{34}$$

where now the conditional expectation is taken under the objective measure $\mathbb{P}$. Proposition 6 presents the equilibrium variance risk premium $\Pi_t (\tau)$ in closed-form with its key property in our stationary economy.

Proposition 6 (Equilibrium variance risk premium). *In the economy with extrapolative beliefs, the equilibrium variance risk premium is given by*

$$\Pi_t (\tau) = \Upsilon_t (\tau) - \Upsilon_t^* (\tau), \tag{35}$$

where the realized variance $\Upsilon_t (\tau)$ and the variance swap rate $\Upsilon_t^* (\tau)$ are given by

$$\Upsilon_t (\tau) = A (\tau) + B (\tau) \vartheta_t, \tag{36}$$

$$\Upsilon_t^* (\tau) = A^* (\tau) + B^* (\tau) \vartheta_t + C^* (\tau) X_t, \tag{37}$$

with the deterministic functions for the realized variance $B (\tau)$, $A (\tau)$ are given by

$$B (\tau) = \frac{1 - e^{-\kappa \tau}}{\kappa \tau}, \tag{38}$$

$$A (\tau) = \frac{\zeta_\vartheta}{\kappa} \left(1 - \frac{1 - e^{-\kappa \tau}}{\kappa \tau} \right), \tag{39}$$

and the deterministic functions for the variance swap rate $C^* (\tau)$, $B^* (\tau)$, $A^* (\tau)$ by

$$C^* (\tau) = \frac{\eta_\vartheta}{\kappa_2 - \kappa_1} \Big(\frac{1 - e^{-\kappa_1 \tau}}{\kappa_1 \tau} - \frac{1 - e^{-\kappa_2 \tau}}{\kappa_2 \tau} \Big), \tag{40}$$

$$B^* (\tau) = \frac{1 - e^{-\kappa_1 \tau}}{\kappa_1 \tau} - \frac{\kappa_\vartheta^* - \kappa_1}{\kappa_2 - \kappa_1} \Big(\frac{1 - e^{-\kappa_1 \tau}}{\kappa_1 \tau} - \frac{1 - e^{-\kappa_2 \tau}}{\kappa_2 \tau} \Big), \tag{41}$$

$$A^* (\tau) = \frac{\zeta_\vartheta + r\eta_\vartheta}{\kappa_\vartheta^* + \frac{1}{2}\eta_\vartheta} \Big[1 - \frac{1 - e^{-\kappa_1 \tau}}{\kappa_1 \tau} - \frac{\kappa_1}{\kappa_2 - \kappa_1} \Big(\frac{1 - e^{-\kappa_1 \tau}}{\kappa_1 \tau} - \frac{1 - e^{-\kappa_2 \tau}}{\kappa_2 \tau} \Big) \Big]$$
$$+ \frac{\zeta_\vartheta}{\kappa_2 - \kappa_1} \Big(\frac{1 - e^{-\kappa_1 \tau}}{\kappa_1 \tau} - \frac{1 - e^{-\kappa_2 \tau}}{\kappa_2 \tau} \Big), \tag{42}$$

with the constants

$$\kappa_1 = \frac{(\kappa_\vartheta^* + \alpha) - \sqrt{(\kappa_\vartheta^* + \alpha)^2 - 4\alpha \left(\kappa_\vartheta^* + \frac{1}{2}\eta_\vartheta \right)}}{2}, \quad \kappa_2 = \frac{(\kappa_\vartheta^* + \alpha) + \sqrt{(\kappa_\vartheta^* + \alpha)^2 - 4\alpha \left(\kappa_\vartheta^* + \frac{1}{2}\eta_\vartheta \right)}}{2}, \tag{43}$$

such that $\kappa_1 < \kappa_\vartheta^* < \kappa_2$, where κ_ϑ^* is as in Lemma 1, and ζ_ϑ, η_ϑ are as in Proposition 4.
In the rational benchmark economy, the equilibrium variance risk premium is given by $\bar{\Pi}_t (\tau) = \bar{\Upsilon}_t (\tau) - \bar{\Upsilon}_t^* (\tau)$, where the realized variance $\bar{\Upsilon}_t (\tau)$ and the variance swap rate $\bar{\Upsilon}_t^* (\tau)$ are given by $\bar{\Upsilon}_t (\tau) = \bar{A} (\tau) + \bar{B} (\tau) \bar{\vartheta}_t$, and $\bar{\Upsilon}_t^* (\tau) = \bar{A}^* (\tau) + \bar{B}^* (\tau) \bar{\vartheta}_t$, with the deterministic functions are given by $\bar{B} (\tau) = (1 - e^{-\kappa \tau})/\kappa \tau$, $\bar{A} (\tau) = (\bar{\zeta}_\vartheta/\kappa)(1 - (1 - e^{-\kappa \tau})/\kappa \tau)$, and $\bar{B}^* (\tau) = (1 - e^{-\bar{\kappa}_\vartheta^* \tau})/\bar{\kappa}_\vartheta^* \tau$, $\bar{A}^* (\tau) = (\bar{\zeta}_\vartheta/\bar{\kappa}_\vartheta^*)(1 - (1 - e^{-\bar{\kappa}_\vartheta^* \tau})/\bar{\kappa}_\vartheta^* \tau)$, where $\bar{\kappa}_\vartheta^*$ is as in Lemma 1, and $\bar{\zeta}_\vartheta$ is as in Proposition 4.

Consequently, in the economy with extrapolative beliefs, the variance risk premium is increasing in sentiment X_t, but is decreasing in the stock return variance ϑ_t.

In the rational benchmark economy, the fluctuations in the realized variance and variance swap rate, and hence, in the variance risk premium, are all due to the fluctuations in the stock return variance. In this economy, the variance risk premium is negative, and a higher stock return variance leads to a more negative variance risk premium. Negativity of the variance risk premium is intuitive as the investor is risk averse, she dislikes higher stock return variance, and hence, she is willing to accept a negative stock return on average to hedge away increases in variance. In the presence of extrapolative beliefs, the fluctuations in the realized variance is still due to stock return variance, but the fluctuations in the variance swap rate, and hence in the variance risk premium, is now additionally driven by the sentiment X, as (37) shows.

Now, a more bearish sentiment, due to bad recent stock performance, makes the investor to expect more volatile future returns, and hence, the variance swap contract is feasible only if the variance swap rate is set higher, leading to a more negative variance risk premium. Conversely, a more bullish sentiment leads to a higher (less negative) variance risk premium.[17]

Negativity of the average variance risk premium is well-documented in the empirical literature (e.g., Bakshi and Kapadia (2003), Bakshi and Madan (2006), Carr and Wu (2009), Bollerslev, Tauchen, and Zhou (2009), Barras and Malkhozov (2016)). Moreover, as discussed in the Introduction, the existing theoretical works attribute the fluctuations in the variance risk premium to various shocks, such as the time-variation in risk aversion (Bekaert and Engstrom (2010)), the time-variation in aggregate consumption variance (Bollerslev, Tauchen, and Zhou (2009)), Poisson shocks in the expected aggregate consumption growth and variance (Drechsler and Yaron (2011)), Knightian uncertainty shocks (Drechsler (2013), Miao, Wei, and Zhou (2019)). Therefore, one of our key contributions here is to complement this literature by demonstrating a new economic determinant for the fluctuations in the variance risk premium, the investor sentiment.

Recently, the variance risk premium has attracted much attention in the literature mostly because of the growing evidence that it predicts future stock market returns. We now look at the predictive power of the variance risk premium in our model. Toward that, Proposition 7 reports the equilibrium slope coefficient β_Π in the univariate regression of (h-period) (annualized) future stock returns $\ln(S_{t+h}/S_t)/h$ on the (τ-horizon) variance risk premium $\Pi_t(\tau)$ in the steady-state, and its key property in our stationary economy.

Proposition 7 (**Predictive power of variance risk premium: Univariate regression**).
In the economy with extrapolative beliefs, the equilibrium slope coefficient in the univariate regression of future stock returns $\ln(S_{t+h}/S_t)/h$ on variance risk premium $\Pi_t(\tau)$ is given by

$$\beta_\Pi = \frac{\text{Cov}\left[\Pi_t(\tau), \ln(S_{t+h}/S_t)/h\right]}{\text{Var}\left[\Pi_t(\tau)\right]}, \tag{44}$$

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[17]To the best of our knowledge, currently there is no formal empirical study showing how the investor sentiment affects variance risk premium. However, our model prediction is in line with the evidence in Barras and Malkhozov (2016), who find that the magnitude of the variance risk premium increases in recessions, since one could argue that investors are more likely to have a bearish sentiment during recessions.

where

$$\text{Var}\left[\Pi_t\left(\tau\right)\right] = (B(\tau)-B^*(\tau))^2\text{Var}\left[\vartheta_t\right]+C^*(\tau)^2\,\text{Var}\left[X_t\right]-2(B(\tau)-B^*(\tau))C^*(\tau)\,\text{Cov}\left[\vartheta_t,X_t\right],\tag{45}$$

and

$$\begin{aligned}\text{Cov}\left[\Pi_t\left(\tau\right),\ln\left(S_{t+h}/S_t\right)/h\right] &= (B(\tau)-B^*(\tau))\Big(h_1\text{Var}\left[\vartheta_t\right]-h_2\text{Cov}\left[\vartheta_t,X_t\right]\Big)\\&\quad+C^*(\tau)\Big(h_2\text{Var}\left[X_t\right]-h_1\text{Cov}\left[\vartheta_t,X_t\right]\Big),\end{aligned}\tag{46}$$

with the constants

$$\text{Var}[\vartheta_t]=\frac{\sigma_\vartheta^2}{2\kappa}\frac{\zeta_\vartheta}{\kappa},\ \ \text{Var}[X_t]=\frac{\alpha}{2(1+\theta)}\frac{\zeta_\vartheta}{\kappa}\left(1+\frac{\frac{\sigma_\vartheta^2}{4\kappa}+\rho_{S\vartheta}\sigma_\vartheta}{\alpha(1+\theta)+\kappa}\right),\ \ \text{Cov}[\vartheta_t,X_t]=\alpha\frac{\zeta_\vartheta}{\kappa}\frac{\frac{\sigma_\vartheta^2}{4\kappa}+\rho_{S\vartheta}\sigma_\vartheta}{\alpha(1+\theta)+\kappa},\tag{47}$$

$$h_1=\frac{1}{2}\left(\frac{1}{1+\theta}\frac{1-e^{-\kappa h}}{\kappa h}+\frac{\theta}{1+\theta}\frac{e^{-\kappa h}-e^{-\alpha(1+\theta)h}}{\alpha(1+\theta)h-\kappa h}\right),\quad h_2=\theta\frac{1-e^{-\alpha(1+\theta)h}}{\alpha(1+\theta)h},\tag{48}$$

and $B\left(\tau\right)$, $B^*\left(\tau\right)$, $C^*\left(\tau\right)$ are as in Proposition 6, ζ_ϑ, σ_ϑ are as in Proposition 4, and $\rho_{S\vartheta}$ is as in Proposition 3.

In the rational benchmark economy, the equilibrium slope coefficient in the univariate regression of future stock returns $\ln(\bar{S}_{t+h}/\bar{S}_t)/h$ on the variance risk premium $\bar{\Pi}_t\left(\tau\right)$ is given by $\bar{\beta}_\Pi=\text{Cov}\left[\bar{\Pi}_t\left(\tau\right),\ln\left(\bar{S}_{t+h}/\bar{S}_t\right)/h\right]/\text{Var}\left[\bar{\Pi}_t\left(\tau\right)\right]$, where $\text{Var}\left[\bar{\Pi}_t\left(\tau\right)\right]=(\bar{B}(\tau)-\bar{B}^*(\tau))^2\text{Var}\left[\bar{\vartheta}_t\right]$, $\text{Cov}\left[\bar{\Pi}_t\left(\tau\right),\ln\left(\bar{S}_{t+h}/\bar{S}_t\right)/h\right]=(\bar{B}(\tau)-\bar{B}^*(\tau))((1-e^{-\kappa h})/2\kappa h)\text{Var}\left[\bar{\vartheta}_t\right]$, with the constant $\text{Var}\left[\bar{\vartheta}_t\right]=(\bar{\sigma}_\vartheta^2/2\kappa)(\bar{\zeta}_\vartheta/\kappa)$, and $\bar{B}\left(\tau\right)$, $\bar{B}^*\left(\tau\right)$ are as in Proposition 6, and $\bar{\zeta}_\vartheta$, $\bar{\sigma}_\vartheta$ are as in Proposition 4.

Consequently, in the economy with extrapolative beliefs, the slope coefficient of the variance risk premium in the univariate regression is negative.

In the rational benchmark economy, since all the fluctuations in the variance risk premium are due to the stock return variance, the slope coefficient takes a simple form that depends only on the distribution of the stock return variance. In the presence of extrapolative beliefs, the corresponding slope coefficient takes a richer form that also depends on the distribution of the sentiment as (44)–(48) show. The key implication of Proposition 7 is that the slope coefficient is negative, implying that a more negative variance risk premium leads to a higher future stock

returns, consistent with empirical evidence (Bollerslev, Tauchen, and Zhou (2009), Drechsler and Yaron (2011), Bollerslev, Todorov, and Xu (2015), Kilic and Shaliastovich (2019), Pyun (2019)). This is because a more negative variance risk premium mostly indicates a higher variance swap rate due to a more bearish investor sentiment (Proposition 6). Since these times coincide with the stock being undervalued relative to its fundamentals, the subsequent observed stock returns become higher on average.

Naturally, the slope coefficient only indicates the direction and the economic significance of the predictive power of the variance risk premium. To be able to conclude that the variance risk premium indeed predicts future stock returns, we also look at the statistical significance of the slope coefficient by carrying out a regression analysis based on our model's simulated data. Toward that, Table 2, Panel A, reports the average slope coefficients along with their Newey-West (HAC) corrected t-statistics in our simulation exercise.[18]

Table 2, Panel A, reveals that, in the univariate predictive regressions, the slope coefficient for the variance risk premium is statistically significant in the presence of extrapolative beliefs ($\theta > 0$), but not in the rational benchmark economy ($\theta = 0$). The predictive power of the variance risk premium in the presence of extrapolative beliefs is due to the fact that it isolates investor sentiment. This is because the investor sentiment drives only the variance swap rate but not the realized variance, whereas the stock return variance drives both the variance swap rate and the realized variance (Proposition 6). Hence, the variance risk premium fluctuations mostly come from the fluctuations in the investor sentiment rather than the fluctuations in the stock return variance, which is diminished by taking the difference in (35). This leads to more volatile variance risk premium (independent variable), and hence leads to more precise slope coefficient estimates and statistically significant $t-$statistics. Since the predictive power of the variance risk premium is due to the extrapolative beliefs induced sentiment, naturally we also see that a higher extrapolation degree leads to a higher magnitude for the t-statistics and

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[18]Based on our baseline model parameters in Table 3 of Appendix B, we simulate our model for varying levels of extrapolation degree θ at a daily frequency and extract stock returns and variance risk premium at the end of each month. We then run the predictive regressions with 12,000 monthly observations. In our regressions we set the return horizon to 3 months and the variance horizon to 1 month, consistent with the empirical tests (e.g., Bollerslev, Tauchen, and Zhou (2009), Drechsler and Yaron (2011)). We repeat our simulation 1,000 times and report the average slope coefficient, Newey-West (HAC) corrected t-statistics, and the adjusted R^2.

	Panel A: Univariate regression			
		θ		
	0	0.25	0.50	0.75
β_Π	-509.63	-146.54	-121.20	-107.07
	(-1.31)	(-3.60)	(-5.97)	(-8.09)
Adj. R^2	0.12%	0.49%	1.29%	2.34%

	Panel B: Joint regression			
		θ		
	0	0.25	0.50	0.75
β_Π	–	-124.65	-110.02	-99.81
	–	(-2.71)	(-4.85)	(-6.69)
β_Υ	–	0.35	0.29	0.22
	–	(0.92)	(0.99)	(0.94)
Adj. R^2	–	0.57%	1.38%	2.42%

Table 2: **Predictive power of variance risk premium for future returns.** In this table, Panel A reports the average estimates of the slope coefficient β_Π in the univariate regression $\ln(S_{t+h}/S_t)/h = \beta_0 + \beta_\Pi \Pi_t(\tau) + \epsilon_{t+h}$, and Panel B reports the average estimates of the slope coefficients β_Π and β_Υ in the joint regression $\ln(S_{t+h}/S_t)/h = \beta_0 + \beta_\Pi \Pi_t(\tau) + \beta_\Upsilon \Upsilon_t(\tau) + \epsilon_{t+h}$, along with their Newey-West (HAC) corrected t-statistics in parentheses for varying levels of extrapolation degree θ across the $1,000$ simulated paths of our economy with each path containing $12,000$ monthly observations. The return horizon is 3 months ($h = 0.25$), variance horizon is 1 month ($\tau = 0.083$), and all the other parameter values in our simulations are as in Table 3: $\alpha = 0.5$, $\kappa = 0.15$, $\zeta = 0.0018$, $\sigma = 0.0506$, $\rho = -0.15$, $\mu = 0.0221$.

adjusted R^2s, again due to even more volatile variance risk premium. In terms of magnitudes, the slope coefficients in Table 2 are also economically plausible and in line with empirical evidence. For example, the evidence in Bollerslev, Tauchen, and Zhou (2009) suggests that a one standard deviation increase in variance risk premium leads to a 0.15 standard deviation decrease in future stock returns. The corresponding numbers in our model are 0.03, 0.07, 0.11, and 0.15 standard deviation decreases in future stock returns for $\theta = 0$, 0.25, 0.50, and 0.75, respectively.[19]

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[19]More specifically, Bollerslev, Tauchen, and Zhou (2009, Table 2) shows the relevant slope coefficient for the

As Table 2, Panel A, also indicates, in the rational benchmark economy, $\theta = 0$, we fail to find a statistically significant relation between the variance risk premium and future stock returns even though the magnitude of the slope coefficient is higher. This is because in this economy all the fluctuations in the variance risk premium are due to the fluctuations in the stock return variance, which are diminished by the difference in the definition of the variance risk premium. This leads to less volatile variance risk premium relative to stock returns, and hence leads to less precise slope coefficient estimates and statistically insignificant $t-$statistics.

Empirical evidence shows that the variance risk premium predicts future stock market returns also in the joint regression in which the other regressor is the realized variance. We now examine the predictive power of the variance risk premium in a joint regression in our model. Toward that, Proposition 8, reports the equilibrium slope coefficients β_Π and β_Υ in the joint regression of (h-period) (annualized) future stock returns $\ln(S_{t+h}/S_t)/h$ on the (τ-horizon) variance risk premium $\Pi_t(\tau)$ and realized variance $\Upsilon_t(\tau)$ and their key property in our stationary economy.

Proposition 8 (**Predictive power of variance risk premium: Joint regression**). *In the economy with extrapolative beliefs, the equilibrium slope coefficients in the joint regression of future stock returns $\ln(S_{t+h}/S_t)/h$ on the variance risk premium $\Pi_t(\tau)$ and the realized variance $\Upsilon_t(\tau)$ are respectively given by*

$$\beta_\Pi = \frac{\mathrm{Cov}[\Pi_t(\tau), \ln(S_{t+h}/S_t)/h]\mathrm{Var}[\Upsilon_t(\tau)] - \mathrm{Cov}[\Upsilon_t(\tau), \ln(S_{t+h}/S_t)/h]\mathrm{Cov}[\Pi_t(\tau), \Upsilon_t(\tau)]}{\mathrm{Var}[\Pi_t(\tau)]\,\mathrm{Var}[\Upsilon_t(\tau)] - \mathrm{Cov}[\Pi_t(\tau), \Upsilon_t(\tau)]^2}, \quad (49)$$

$$\beta_\Upsilon = \frac{\mathrm{Cov}[\Upsilon_t(\tau), \ln(S_{t+h}/S_t)/h]\mathrm{Var}[\Pi_t(\tau)] - \mathrm{Cov}[\Pi_t(\tau), \ln(S_{t+h}/S_t)/h]\mathrm{Cov}[\Pi_t(\tau), \Upsilon_t(\tau)]}{\mathrm{Var}[\Pi_t(\tau)]\,\mathrm{Var}[\Upsilon_t(\tau)] - \mathrm{Cov}[\Pi_t(\tau), \Upsilon_t(\tau)]^2}, \quad (50)$$

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variance risk premium for three months returns as 0.47. Multiplying this value with the standard deviation of the variance risk premium (0.1513) and dividing by the standard deviation of the stock market risk premium (0.4719) leads to 0.15, where the standard deviation values are obtained from their Table 1. The corresponding numbers in our model are computed similarly for each simulation path and the reported quantities are the averages across all paths. Moreover, in terms of % decrease in stock returns following a one standard deviation increase in variance risk premium, these quantities become $0.47 \times 0.1513 = 7.11\%$ in Bollerslev, Tauchen, and Zhou (2009), and the corresponding numbers in our model are 0.7%, 2.15%, 4.37%, and 7.18% decreases in future stock returns for $\theta = 0$, 0.25, 0.50, and 0.75, respectively. At this point, we would like to highlight that, our model generates these sizable stock return magnitudes with a low constant relative risk aversion efficient of one (logarithmic preferences).

where

$$\text{Var}\left[\Upsilon_t\left(\tau\right)\right] = B^2\left(\tau\right)\text{Var}\left[\vartheta_t\right], \tag{51}$$

$$\text{Cov}\left[\Pi_t\left(\tau\right),\Upsilon_t\left(\tau\right)\right] = B\left(\tau\right)\left((B(\tau)-B^*(\tau))\text{Var}\left[\vartheta_t\right]-C^*\left(\tau\right)\text{Cov}\left[\vartheta_t,X_t\right]\right), \tag{52}$$

$$\text{Cov}\left[\Upsilon_t\left(\tau\right),\ln\left(S_{t+h}/S_t\right)/h\right] = B\left(\tau\right)\left(h_1\text{Var}\left[\vartheta_t\right]-h_2\text{Cov}\left[\vartheta_t,X_t\right]\right), \tag{53}$$

and $\text{Var}\left[\Pi_t(\tau)\right]$, $\text{Cov}\left[\Pi_t(\tau),\ln(S_{t+h}/S_t)/h\right]$, $\text{Var}\left[\vartheta_t\right]$, $\text{Cov}\left[\vartheta_t,X_t\right]$, h_1, h_2 *are as in Proposition 7, and* $B\left(\tau\right)$, $B^*\left(\tau\right)$, $C^*\left(\tau\right)$ *are as in Proposition 6.*
In the rational benchmark economy, the joint regression of future stock returns $\ln(\bar{S}_{t+h}/\bar{S}_t)/h$ *on the variance risk premium* $\bar{\Pi}_t\left(\tau\right)$ *and the realized variance* $\bar{\Upsilon}_t\left(\tau\right)$ *is not well-defined since* $\bar{\Pi}_t\left(\tau\right)$ *and* $\bar{\Upsilon}_t\left(\tau\right)$ *are perfectly correlated.*

Consequently, in the economy with extrapolative beliefs, the slope coefficient of the variance risk premium in the joint regression is negative.

The key implication of Proposition 8 is that the slope coefficient for the variance risk premium is still negative even after controlling for the realized variance.[20] As in the case of the univariate regression, this is again due to the fact a more negative variance risk premium mostly indicating a more bearish sentiment, which is followed by higher subsequent stock returns on average. To be able to conclude that the variance risk premium indeed predicts future stock returns in the joint regressions, we again look at the statistical significance of its slope coefficient in our simulation exercise. Toward that, Table 2, Panel B, reveals that, in the joint predictive regressions, the slope coefficient for the variance risk premium is statistically significant, whereas the slope coefficient for the realized variance is not, consistent with the evidence in Bollerslev, Tauchen, and Zhou (2009). The predictive power of the variance risk premium is again due to the fact that by taking the difference in (35), the variance risk premium isolates the fluctuations in the investor sentiment, which is the driving force for the predictability in stock returns in our model. This in return leads to more volatile variance risk premium, and hence to more precise slope coefficient estimates. Whereas the failure to find a statistically significant coefficient for the realized variance is simply because, as (36) illustrates, the realized variance is driven only by the stock return (conditional) variance,

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[20]On the other hand, in the rational benchmark economy, $\theta = 0$, since all the fluctuations in the variance risk premium and the realized variance are due to the stock return variance, they are perfectly correlated, and hence the joint regression is not well-defined due to the exact multicollinearity.

which primarily scales the unpredictable component of stock returns. Therefore, a higher realized variance, or conditional variance in that matter, leads to a more volatile stock returns (dependent variable) relative to the realized variance (independent variable), and hence leads to less precise slope coefficient estimates and statistically insignificant $t-$statistics.

6   Conclusion

In this paper, we develop a dynamic equilibrium model of stock return extrapolation in the presence of stochastic stock return volatility. In our model, consistent with survey evidence, a higher (lower) sentiment, due to good (bad) recent stock performance, makes the investor to expect higher (lower) future returns. We demonstrate that through the stochastic stock return volatility channel, the investor sentiment affects the prices of all assets, including the derivatives such as options and variance swaps. Our model is tractable and delivers closed-form expressions for the quantities of interest, and in particular, provides analytic option pricing formulas that nest the well-known Heston formulation.

The main novel predictions of the model are as follows. The magnitude of the fundamental variance elasticity of the stock price, and the negative correlation between the stock return and its variance are both increasing in the extrapolation degree, but are not affected by the relative weights of past returns in the sentiment. Following positive (negative) stock returns, in addition to expecting future stock returns to be higher (lower), the investor also expects future returns to be less (more) volatile, consistent with survey evidence. A more bearish sentiment leads to a more negative variance risk premium, and a higher option prices with the price increases being more pronounced for out-of-the-money options, as in data. The variance risk premium predicts future stock returns negatively even after controlling for the realized (or, conditional) variance, consistent with empirical evidence.

To demonstrate our main economic mechanisms and results as clearly as possible in a tractable way, we consider an economy with a single investor, whose beliefs reflects the "consensus belief", rather than considering a more complex heterogeneous investor economy in which only some investors are extrapolators as in Barberis et al. (2015). This way we do not complicate our analysis with wealth-effects, which otherwise would lead to intractable non-stationarity equilibria in a constant relative risk aversion framework like ours. We leave the analysis incorporating these features for future research.

Appendix A: Proofs

Proof of Proposition 1. We first determine the equilibrium state price density in our non-stationary economy, and then obtain the stock price using the conjecture and verify method. The time-T marginal utility of the investor evaluated at the aggregate consumption (the stock payoff D_T) gives the equilibrium subjective state price density, denoted by ξ_T^s, that is,

$$\xi_T^s = D_T^{-1}. \tag{A.1}$$

On the other hand, the consistency relation, by equating the stock price dynamics under both measures (3) and (5), yields $d\omega_{2t}^s = d\omega_{2t} - (\theta X_t/\sigma_{S2t})dt$, where ω_2^s is the Brownian motion under the investor's subjective measure $\mathbb{P}^s$. In this case, the likelihood ratio between the investor's subjective measure $\mathbb{P}^s$ and the objective measure $\mathbb{P}$, denoted by L, becomes

$$L_T \equiv \frac{d\mathbb{P}^s}{d\mathbb{P}} = e^{\int_0^T \frac{\theta X_u}{\sigma_{S2u}} d\omega_{2u} - \frac{1}{2}\int_0^T \left(\frac{\theta X_u}{\sigma_{S2u}}\right)^2 du}, \tag{A.2}$$

with the dynamics $dL_t = L_t(\theta X_t/\sigma_{S2t})d\omega_{2t}$, where the endogenous stock diffusion term σ_{S2} and the sentiment process X are defined in (3) and (4), respectively. Next, we express (A.1) in terms of the equilibrium time-T objective state price density, denoted by ξ_T, as $\xi_T = L_T D_T^{-1}$, via the relation $\xi_T = L_T \xi_T^s$.[21] The time $t < T$ equilibrium state price density, ξ_t, in our economy satisfies $\xi_t = e^{r(T-t)}\mathrm{E}_t[\xi_T]$, which follows from the fact that the interest rate r is constant, and hence the time t price of the zero-coupon bond that pays 1 unit at time T is given by $e^{-r(T-t)}$, which by no-arbitrage must satisfy $\xi_t e^{-r(T-t)} = \mathrm{E}_t[\xi_T]$. We denote the dynamics of the equilibrium state price density as

$$d\xi_t = -\xi_t \left[rdt + m_{1t}d\omega_{1t} + m_{2t}d\omega_{2t}\right], \tag{A.3}$$

---

[21]This relation follows from the fact that the unique time-0 price of any security with a payoff X_T can be determined using either $\mathrm{E}^s[\xi_T^s X_T]$, where the expectation is taken under the subjective measure $\mathbb{P}^s$, or $\mathrm{E}[\xi_T X_T]$, where the expectation is taken under the objective measure $\mathbb{P}$. However, changing the measure from $\mathbb{P}^s$ to $\mathbb{P}$ using (A.2), we can always write the former expectation as $\mathrm{E}^s[\xi_T^s X_T] = \mathrm{E}[L_T \xi_T^s X_T]$, which after equated to the latter expectation immediately yields the relation $\xi_T = L_T \xi_T^s$.

where m_1 and m_2 are the equilibrium market prices of risk for the Brownian motions ω_1 and ω_2, respectively. Hence, an application of Itô's Lemma using the dynamics (3) and (A.3) yields

$$d\xi_t S_t = \xi_t S_t \left[\left(\mu_{St} - r - (\sigma_{S1t} m_{1t} + \sigma_{S2t} m_{2t}) \right) dt + (\sigma_{S1t} - m_{1t}) \, d\omega_{1t} + (\sigma_{S2t} - m_{2t}) \, d\omega_{2t} \right]. \quad (A.4)$$

On the other hand, by no arbitrage, the discounted stock price satisfies $\xi_t S_t = \mathrm{E}_t \left[\xi_T D_T \right] = \mathrm{E}_t \left[L_T \right] = L_t$, where the last equality follows from the fact that the likelihood ratio L is a $\mathbb{P}$ martingale. Therefore, the discounted stock price dynamics becomes

$$d\xi_t S_t = \xi_t S_t \left[\frac{\theta X_t}{\sigma_{S2t}} d\omega_{2t} \right]. \quad (A.5)$$

Matching the dynamics (A.4)–(A.5) gives the market prices of risk as $m_{1t} = \sigma_{S1t}$, and $m_{2t} = \sigma_{S2t} - (\theta X_t / \sigma_{S2t})$, and the risk premium as

$$\mu_{St} - r = \sigma_{S1t} m_{1t} + \sigma_{S2t} m_{2t} = \sigma_{S1t}^2 + \sigma_{S2t}^2 - \theta X_t, \quad (A.6)$$

in terms of the endogenous processes σ_{S1}, σ_{S2}, and X. We will use (A.6) to verify our conjecture later on.

To determine the stock price and its dynamics explicitly, we conjecture that the equilibrium stock price in our non-stationary economy takes the form

$$S_t = D_t e^{a(t) + b(t) V_t + c(t) X_t}, \quad (A.7)$$

for some deterministic functions of time $c(t)$, $b(t)$, and $a(t)$ with boundary conditions $c(T) = b(T) = a(T) = 0$. To verify our conjecture, we first note that (A.7) implies

$$d \ln S_t = d \ln D_t + \dot{a}(t) \, dt + b(t) \, dV_t + \dot{b}(t) \, V_t dt + c(t) \, dX_t + \dot{c}(t) \, X_t dt,$$

which after substituting $d \ln D_t = (\mu - V_t/2) \, dt + \sqrt{V_t} d\omega_{1t}$ as well as (2) and (4), becomes

$$\begin{aligned}
d \ln S_t &= \frac{1}{1 - \alpha c(t)} \left[\mu + \dot{a}(t) + \zeta b(t) + \left(\dot{b}(t) - \kappa b(t) - \frac{1}{2} \right) V_t + (\dot{c}(t) - \alpha c(t)) X_t \right] dt \\
&\quad + \frac{1 + \rho \sigma b(t)}{1 - \alpha c(t)} \sqrt{V_t} d\omega_{1t} + \frac{\sqrt{1 - \rho^2} \sigma b(t)}{1 - \alpha c(t)} \sqrt{V_t} d\omega_{2t},
\end{aligned} \quad (A.8)$$

with $1 - \alpha c(t) > 0$. Matching the diffusion terms in (3) and (A.8) immediately gives

$$\sigma_{S1t} = \frac{1 + \rho \sigma b(t)}{1 - \alpha c(t)} \sqrt{V_t}, \qquad \sigma_{S2t} = \frac{\sqrt{1 - \rho^2} \sigma b(t)}{1 - \alpha c(t)} \sqrt{V_t},$$

which in return gives the stock return variance as

$$\vartheta_t = \sigma_{S1t}^2 + \sigma_{S2t}^2 = \frac{1 + 2\rho\sigma b(t) + \sigma^2 b^2(t)}{(1 - \alpha c(t))^2} V_t. \tag{A.9}$$

Moreover, adding half the variance to the drift term in (A.8) gives the stock mean return as

$$
\begin{aligned}
\mu_{St} &= \frac{1}{1 - \alpha c(t)} \left[\mu + \dot{a}(t) + \zeta b(t) + \left(\dot{b}(t) - \kappa b(t) - \frac{1}{2} \right) V_t + (\dot{c}(t) - \alpha c(t)) X_t \right] \\
&\quad + \frac{1}{2} \frac{1 + 2\rho\sigma b(t) + \sigma^2 b^2(t)}{(1 - \alpha c(t))^2} V_t.
\end{aligned}
\tag{A.10}
$$

Therefore, our conjecture (A.7) is verified if the stock return variance (A.9) and stock mean return (A.10) satisfy the risk premium identity in (A.6), that is,

$$
\begin{aligned}
0 &= \left[\dot{c}(t) - (1 + \theta) \alpha c(t) + \theta \right] X_t + \left[\dot{b}(t) - \frac{1}{2} - \kappa b(t) - \frac{1}{2} \frac{1 + 2\rho\sigma b(t) + \sigma^2 b^2(t)}{1 - \alpha c(t)} \right] V_t \\
&\quad + \left[\dot{a}(t) - r + \mu + \zeta b(t) + r\alpha c(t) \right],
\end{aligned}
\tag{A.11}
$$

which is satisfied if the three square brackets terms are all zero by the method of undetermined coefficients. The first bracket term in (A.11) leads to the ordinary differential equation (ODE)

$$\dot{c}(t) = (1 + \theta) \alpha c(t) - \theta,$$

with the boundary condition $c(T) = 0$, whose solution is given readily by (9). We also note that $1 - \alpha c(t) > 0$ since

$$1 - \alpha c(t) = \frac{1 + \theta e^{-\alpha(1+\theta)(T-t)}}{1 + \theta}.$$

The second bracket term in (A.11) leads to the ordinary differential equation given by (10), which is a Riccati equation with *time-varying* coefficients (due to the time variation in $c(t)$), and in general a methodology for its analytical solution does not exist. Finally, the last bracket

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term in (A.11) leads to the ordinary differential equation $\dot{a}(t) = r - \mu - \zeta b(t) - r\alpha c(t)$ as reported in Proposition 1.

The equilibrium stock price in the non-stationary rational benchmark economy is obtained by setting $\theta = 0$ in (8), since $\theta = 0$ leads to $\bar{c}(t) = 0$ (see (9)), which in return leads to the ordinary differential equations for $\bar{b}(t)$ and $\bar{a}(t)$ as reported in Proposition 1.[22]

Property (i), which states that the stock price is increasing in sentiment X_t, but is decreasing in fundamental variance V_t, follows from the facts that $c(t) > 0$ and $b(t) < 0$. The positivity of $c(t)$ is immediate from (9). To show the negativity of $b(t)$, for contradiction we assume $b(t)$ is positive, and substitute (9) into (10), and rearrange to obtain

$$\dot{b}(t) = \frac{1}{2}\left(1 + \frac{(1+\theta)}{1+\theta e^{-\alpha(1+\theta)(T-t)}}\right) + \left(\kappa + \frac{\rho\sigma(1+\theta)}{1+\theta e^{-\alpha(1+\theta)(T-t)}}\right)b(t) + \frac{1}{2}\frac{\sigma^2(1+\theta)}{1+\theta e^{-\alpha(1+\theta)(T-t)}}b^2(t). \quad \text{(A.12)}$$

However, positivity of all the terms on the right hand side of (A.12) implies $b(t)$ is increasing over time, which along with the boundary condition $b(T) = 0$ contradicts the assumption of positive $b(t)$, and hence we conclude $b(t) < 0$. On the other hand, showing all the terms on the right hand side of (A.12) are positive is straightforward, as the positivity of the first and the last terms are immediate, as well as the positivity of the middle bracket term when $\rho \geq 0$. For $\rho < 0$ the middle bracket term is also positive as in this case this term is decreasing over time and takes its minimum value of $\kappa + \rho\sigma(1+\theta)$ as $T \to \infty$, which is always positive under our parameter restriction of $(\kappa + \rho\sigma(1+\theta))^2 > \sigma^2(1+\theta)(2+\theta)$ given in Section 2.

Property (ii), which states that the sentiment elasticity is increasing in the extrapolation degree θ, but is decreasing in the relative weights α, follows from

$$\frac{\partial}{\partial\theta}c(t) = \frac{1 - e^{-\alpha(1+\theta)(T-t)} + \theta\alpha(1+\theta)(T-t)e^{-\alpha(1+\theta)(T-t)}}{\alpha(1+\theta)^2} > 0,$$

$$\frac{\partial}{\partial\alpha}c(t) = \theta\frac{\alpha(1+\theta)(T-t)e^{-\alpha(1+\theta)(T-t)} - \left(1 - e^{-\alpha(1+\theta)(T-t)}\right)}{\alpha^2(1+\theta)} < 0,$$

where the last inequality follows from the fact that $xe^{-x} - (1 - e^{-x}) < 0$ for all x. $\qquad\square$

---

[22]In the non-stationary rational benchmark economy, one can in fact determine $\bar{b}(t)$ and $\bar{a}(t)$ explicitly, since the ordinary differential equation for $\bar{b}(t)$ is a Riccati equation with *constant* coefficients, for which a well-known analytical solution does exist.

Proof of Proposition 2. We conjecture that in the stationary economy a well-defined equilibrium stock price must take the form

$$S_t = D_t e^{a+bV_t+cX_t}, \tag{A.13}$$

for some constants c, b, and a.[23] In this case, (A.13) implies the (log) stock price dynamics as $d\ln S_t = d\ln D_t + bdV_t + cdX_t$, which after substituting $d\ln D_t = (\mu - V_t/2)\,dt + \sqrt{V_t}d\omega_{1t}$ as well as (2) and (4), becomes

$$d\ln S_t = \frac{1}{1-\alpha c}\left[\mu + \zeta b - \left(\kappa b + \frac{1}{2}\right)V_t - \alpha c X_t\right]dt + \frac{1+\rho\sigma b}{1-\alpha c}\sqrt{V_t}d\omega_{1t} + \frac{\sqrt{1-\rho^2}\sigma b}{1-\alpha c}\sqrt{V_t}d\omega_{2t}, \tag{A.14}$$

with $1 - \alpha c > 0$. Matching the diffusion terms in (3) and (A.14) immediately gives

$$\sigma_{S1t} = \frac{1+\rho\sigma b}{1-\alpha c}\sqrt{V_t}, \qquad \sigma_{S2t} = \frac{\sqrt{1-\rho^2}\sigma b}{1-\alpha c}\sqrt{V_t}, \tag{A.15}$$

which in return gives the stock return variance as

$$\vartheta_t = \sigma_{S1t}^2 + \sigma_{S2t}^2 = \frac{1+2\rho\sigma b + \sigma^2 b^2}{(1-\alpha c)^2}V_t. \tag{A.16}$$

Moreover, adding half the variance to the drift term in (A.14) gives the stock mean return as

$$\mu_{St} = \frac{1}{1-\alpha c}\left[\mu + \zeta b - \left(\kappa b + \frac{1}{2}\right)V_t - \alpha c X_t\right] + \frac{1}{2}\frac{1+2\rho\sigma b + \sigma^2 b^2}{(1-\alpha c)^2}V_t. \tag{A.17}$$

On the other hand, following similar steps to those for the non-stationary economy in the proof of Proposition 1, we again obtain the discounted stock price as $\xi_t S_t = L_t$, and hence obtain the similar expressions for the market prices of risk

$$m_{1t} = \frac{1+\rho\sigma b}{1-\alpha c}\sqrt{V_t}, \qquad m_{2t} = \frac{\sqrt{1-\rho^2}\sigma b}{1-\alpha c}\sqrt{V_t} - \frac{1-\alpha c}{\sqrt{1-\rho^2}\sigma b\sqrt{V_t}}\theta X_t, \tag{A.18}$$

---

[23]Even though, we here provide a detailed derivation paralleling the proof of Proposition 1 for the non-stationary (finite-horizon) economy, an alternative, much simpler derivation of the stationary (infinite-horizon) economy can be obtained by simply taking the limit $T \to \infty$ for the expression for $c(t)$ in (9) to obtain (12). Then, replacing the term $c(t)$ with the constant c in (10), while setting the equation to zero, leads to a simple quadratic equation for b, whose solution is given by (13).

and the risk premium

$$\mu_{St} - r = \vartheta_t - \theta X_t, \tag{A.19}$$

where ϑ_t is as in (A.16). Therefore, our conjecture (A.13) is verified if the stock return variance (A.16) and stock mean return (A.17) satisfy the risk premium identity in (A.19), that is,

$$0 = [(1+\theta)\,\alpha c - \theta]\, X_t + \left[\frac{1}{2} + \kappa b + \frac{1}{2}\frac{1 + 2\rho\sigma b + \sigma^2 b^2}{1 - \alpha c}\right] V_t + [r - \mu - \zeta b - r\alpha c], \tag{A.20}$$

which is satisfied if the square brackets terms are all zero by the method of undetermined coefficients. The first bracket term in (A.20) leads to the constant c given by (12). We also note that $1 - \alpha c = 1/(1+\theta) > 0$. After substituting c, the second bracket term in (A.20) leads to the quadratic equation

$$\frac{1}{2}(2+\theta) + (\kappa + \rho\sigma(1+\theta))\, b + \frac{1}{2}\sigma^2(1+\theta)\, b^2 = 0, \tag{A.21}$$

which has a real solution if

$$(\kappa + \rho\sigma(1+\theta))^2 > \sigma^2(1+\theta)(2+\theta), \tag{A.22}$$

as we assume throughout the paper (see also Section 2). Solving the quadratic equation (A.21) leads to two roots, but only one of the roots collapses to the corresponding rational benchmark economy unique root $(\lim_{T\to\infty} \bar{b}(t) = \bar{b} = -2/(\kappa + \rho\sigma + \sqrt{(\kappa + \rho\sigma)^2 - 2\sigma^2}))$ when $\theta = 0$, and we take that root to be the solution for b and report it in (13). Finally, the last bracket term in (A.20) leads to the parameter restriction of $\mu + \zeta b - r(1 - \alpha c) = 0$, for the stock price in our infinite-horizon economy to be well-defined.

The equilibrium stock price in the rational benchmark economy is obtained by setting $\theta = 0$ in (11), since $\theta = 0$ leads to $\bar{c} = 0$, and $\bar{b} = -2/(\kappa + \rho\sigma + \sqrt{(\kappa + \rho\sigma)^2 - 2\sigma^2})$.

Property, which states that the stock price is increasing in sentiment X_t, but is decreasing in fundamental variance V_t, follows from the facts that $c > 0$ and $b < 0$, which are immediate from (12)–(13). Property, which states that the sentiment elasticity is increasing in the

extrapolation degree θ, but is decreasing in the relative weights α, follows from

$$\frac{\partial}{\partial\theta}c = \frac{1}{\alpha\left(1+\theta\right)^2} > 0, \qquad \frac{\partial}{\partial\alpha}c = -\frac{\theta}{\alpha^2\left(1+\theta\right)} < 0.$$

Property, which states that the fundamental variance elasticity is decreasing in the extrapolation degree θ, follows from the fact that $\partial b/\partial\theta < 0$. To see this note that

$$\frac{\partial}{\partial\theta}b = -\frac{\Delta + \sqrt{\Delta^2 - \sigma^2\left(1+\theta\right)\left(2+\theta\right)} - \left(2+\theta\right)\left(\rho\sigma + \frac{2\Delta\rho\sigma - \sigma^2\left(3+2\theta\right)}{2\sqrt{\Delta^2 - \sigma^2\left(1+\theta\right)\left(2+\theta\right)}}\right)}{\left(\Delta + \sqrt{\Delta^2 - \sigma^2\left(1+\theta\right)\left(2+\theta\right)}\right)^2},$$

where we have defined $\Delta \equiv \kappa + \rho\sigma\left(1+\theta\right)$. Hence, $\partial b/\partial\theta < 0$ if and only if the numerator term above is positive, which after some algebra reduces to the equivalent condition

$$\left(\kappa - \rho\sigma\right)\left(\Delta + \sqrt{\Delta^2 - \sigma^2\left(1+\theta\right)\left(2+\theta\right)}\right) + \frac{1}{2}\sigma^2\left(2+\theta\right) > 0,$$

that always holds, since all the terms on the left hand side are positive, where the positivity of $\Delta = \kappa + \rho\sigma\left(1+\theta\right)$ is implied by (A.22) and the positivity of κ. $\qquad\square$

Proof of Proposition 3. The stock price dynamics (14) follows from substituting the mean return in (A.19), the diffusion terms in (A.15) along with (A.16) in the proof of Proposition 2 into (3). The sentiment dynamics (15) follows from substituting the log stock price dynamics obtained via (14) into (4). The equilibrium stock return variance is already determined and given by (A.16) in the proof of Proposition 2. The correlation between the stock return and its variance is determined by the stock price dynamics (14) and the return variance dynamics

$$d\vartheta_t = \frac{1 + 2\rho\sigma b + \sigma^2 b^2}{\left(1 - \alpha c\right)^2}dV_t, \tag{A.23}$$

which implies the variance

$$\mathrm{Var}_t\left[d\vartheta_t\right] = \left(\frac{1 + 2\rho\sigma b + \sigma^2 b^2}{\left(1 - \alpha c\right)^2}\right)^2 \mathrm{Var}_t\left[dV_t\right] = \left(\frac{1 + 2\rho\sigma b + \sigma^2 b^2}{\left(1 - \alpha c\right)^2}\right)^2 \sigma^2 V_t,$$

the covariance

$$\text{Cov}_t\left[\frac{dS_t}{S_t}, d\vartheta_t\right] = \frac{1 + 2\rho\sigma b + \sigma^2 b^2}{(1-\alpha c)^2}\text{Cov}_t\left[\frac{dS_t}{S_t}, dV_t\right] = \frac{1 + 2\rho\sigma b + \sigma^2 b^2}{(1-\alpha c)^2}\sigma\frac{\rho + \sigma b}{1-\alpha c}V_t,$$

and hence the correlation as in (17).

The equilibrium stock price dynamics, the stock return variance, and the correlation between the stock return and its variance in the rational benchmark economy are obtained by setting $\theta = 0$ in (14) and (16)–(17) as $\theta = 0$ leads to $\bar{c} = 0$ and $\bar{b} = -2/(\kappa + \rho\sigma + \sqrt{(\kappa + \rho\sigma)^2 - 2\sigma^2})$.

Property (i), which states that the stock return variance is increasing in the extrapolation degree θ, but is not affected by the relative weights α, follows from the facts that $\partial\vartheta_t/\partial\theta > 0$ and $\partial\vartheta_t/\partial\alpha = 0$. To see the first inequality holds, note that

$$\frac{\partial}{\partial\theta}\vartheta_t = V_t\frac{\partial}{\partial\theta}\frac{1 + 2\rho\sigma b + \sigma^2 b^2}{(1-\alpha c)^2} = V_t\frac{\partial}{\partial\theta}\left[(1+\theta)^2\left(1 + 2\rho\sigma b + \sigma^2 b^2\right)\right],$$

which is positive if $\frac{\partial}{\partial\theta}\left(1 + 2\rho\sigma b + \sigma^2 b^2\right)$ is positive, and this is always the case since

$$\frac{\partial}{\partial\theta}\left(1 + 2\rho\sigma b + \sigma^2 b^2\right) = 2\left(\rho\sigma + \sigma^2 b\right)\frac{\partial}{\partial\theta}b > 0,$$

as $\rho\sigma + \sigma^2 b < 0$ and $\partial b/\partial\theta < 0$. On the other hand the second equality holds since the amplification term $1/(1-\alpha c) = 1 + \theta$ does not depend on α, neither does the numerator term $1 + 2\rho\sigma b + \sigma^2 b^2$ in (16).

Property (ii), which states that the correlation between the stock return and its variance is decreasing in the extrapolation degree θ, but is not affected by the relative weights α, follows from the facts that $\partial\rho_{S\vartheta}/\partial\theta < 0$ and $\partial\rho_{S\vartheta}/\partial\theta = 0$. To see the first inequality holds, note that

$$\frac{\partial}{\partial\theta}\rho_{S\vartheta} = \frac{\partial}{\partial\theta}\frac{\rho + \sigma b}{\sqrt{1 + 2\rho\sigma b + \sigma^2 b^2}},$$

and this is always negative, since the numerator is decreasing $(\partial b/\partial\theta < 0)$, while the denominator is increasing in θ. On the other hand, the second equality holds since neither the denominator term $1 + 2\rho\sigma b + \sigma^2 b^2$, nor the numerator term $\rho + \sigma b$ depend on α in (17). $\square$

Proof of Proposition 4. The stock return variance dynamics under the objective measure (18) is given by substituting the fundamental variance dynamics (2) into (A.23). The stock return variance dynamics under the subjective measure (19) is given by substituting the investor's subjective Brownian motion $d\omega_{2t}^s = d\omega_{2t} - (\theta X_t/\sigma_{S2t})dt$ into the dynamics (18).

The equilibrium stock return variance dynamics under the objective measure in the rational benchmark economy is obtained by setting $\theta = 0$ in (18) as $\theta = 0$ leads to $\bar{c} = 0$ and $\bar{\eta}_\vartheta = 0$.

Property, which states that the expected change in the stock return variance under the subjective measure is decreasing in sentiment X_t, follows from the fact that $\eta_\vartheta < 0$ in (20). □

Proof of Lemma 1. The independent Brownian motions under the (objective) risk-neutral measure ω_1^* and ω_2^* are given by

$$d\omega_{1t}^* = d\omega_{1t} + m_{1t}dt, \qquad (A.24)$$

$$d\omega_{2t}^* = d\omega_{2t} + m_{2t}dt, \qquad (A.25)$$

where m_1 and m_2 are the stationary equilibrium market prices of risk, and given by (A.18) in the proof of Proposition 2. Substituting (A.24)–(A.25) into the dynamics (14) yields

$$dS_t = S_t\Big[(r + \vartheta_t - \theta X_t)\,dt + \frac{1 + \rho\sigma b}{1 - \alpha c}\sqrt{V_t}\,(d\omega_{1t}^* - m_{1t}dt) + \frac{\sqrt{1 - \rho^2}\sigma b}{1 - \alpha c}\sqrt{V_t}\,(d\omega_{2t}^* - m_{2t}dt)\Big],$$

which after employing (A.18) and rearranging becomes

$$dS_t = S_t\Big[rdt + \frac{1 + \rho\sigma b}{1 - \alpha c}\sqrt{V_t}d\omega_{1t}^* + \frac{\sqrt{1 - \rho^2}\sigma b}{1 - \alpha c}\sqrt{V_t}d\omega_{2t}^*\Big]. \qquad (A.26)$$

Similarly, substituting (A.24)–(A.25) into the stock return variance dynamics (18) yields

$$d\vartheta_t = (\zeta_\vartheta - \kappa\vartheta_t)dt + \rho\sigma_\vartheta\sqrt{\vartheta_t}\,(d\omega_{1t}^* - m_{1t}dt) + \sqrt{1 - \rho^2}\sigma_\vartheta\sqrt{\vartheta_t}\,(d\omega_{2t}^* - m_{2t}dt),$$

which after employing (A.18) and rearranging becomes

$$d\vartheta_t = \Big[\zeta_\vartheta - \sigma_\vartheta\sqrt{\vartheta_t}\Big(\rho m_{1t} + \sqrt{1 - \rho^2}m_{2t}\Big) - \kappa\vartheta_t\Big]dt + \sigma_\vartheta\sqrt{\vartheta_t}\Big(\rho d\omega_{1t}^* + \sqrt{1 - \rho^2}d\omega_{2t}^*\Big),$$

$$= \Big[\zeta_\vartheta - (\kappa + \rho_{S\vartheta}\sigma_\vartheta)\vartheta_t + \frac{\theta}{b}\frac{1 + 2\rho\sigma b + \sigma^2 b^2}{1 - \alpha c}X_t\Big]dt + \sigma_\vartheta\sqrt{\vartheta_t}\Big(\rho d\omega_{1t}^* + \sqrt{1 - \rho^2}d\omega_{2t}^*\Big). \qquad (A.27)$$

We note that the positivity of the mean reversion speed κ_ϑ^* follows from

$$\kappa_\vartheta^* = \kappa + \rho_{S\vartheta}\sigma_\vartheta = \kappa + \sigma\left(\rho + \sigma b\right)\left(1 + \theta\right) = \sqrt{\left(\kappa + \rho\sigma\left(1 + \theta\right)\right)^2 - \sigma^2\left(1 + \theta\right)\left(2 + \theta\right)} > 0,$$

where the last equality follows from substituting (13) and rearranging. Finally, for simplicity, we rewrite the risk-neutral dynamics of the stock price (A.26) and stock return variance (A.27) in terms of the correlated Brownian motions ω_S^* and ω_ϑ^* that are defined as

$$d\omega_{St}^* \equiv \frac{1 + \rho\sigma b}{\sqrt{1 + 2\rho\sigma b + \sigma^2 b^2}}d\omega_{1t}^* + \frac{\sqrt{1 - \rho^2}\sigma b}{\sqrt{1 + 2\rho\sigma b + \sigma^2 b^2}}d\omega_{2t}^*, \tag{A.28}$$

$$d\omega_{\vartheta t}^* \equiv \rho d\omega_{1t}^* + \sqrt{1 - \rho^2}d\omega_{2t}^*, \tag{A.29}$$

with $d\omega_{St}^* d\omega_{\vartheta t}^* = \rho_{S\vartheta}$, where $\rho_{S\vartheta}$ is as in (17). Substituting (A.28)–(A.29) into (A.26)–(A.27) simply gives (21)–(22).

The equilibrium risk-neutral dynamics of the sentiment (23) follows immediately after substituting $d\ln S_t = \left(r - \vartheta_t/2\right)dt + \sqrt{\vartheta_t}d\omega_{St}^*$ into (4).

The equilibrium risk-neutral dynamics in the rational benchmark economy are obtained by setting $\theta = 0$ in (21)–(22) as $\theta = 0$ leads to $\bar{c} = 0$ and $\bar{b} = -2/\left(\kappa + \rho\sigma + \sqrt{\left(\kappa + \rho\sigma\right)^2 - 2\sigma^2}\right)$, and hence to $\bar{\eta}_\vartheta = 0$, $\bar{\zeta}_\vartheta = \zeta\left(1 + 2\rho\sigma\bar{b} + \sigma^2\bar{b}^2\right)$, $\bar{\sigma}_\vartheta = \sigma\sqrt{1 + 2\rho\sigma\bar{b} + \sigma^2\bar{b}^2}$, and $\bar{\kappa}_\vartheta^* = \kappa + \bar{\rho}_{S\vartheta}\bar{\sigma}_\vartheta$, where $\bar{\rho}_{S\vartheta}$ is as in Proposition 3, and $d\bar{\omega}_{St}^* d\bar{\omega}_{\vartheta t}^* = \bar{\rho}_{S\vartheta}dt$. $\qquad\square$

Proof of Proposition 5. We determine the equilibrium call option price using the no-arbitrage formula $C_t = e^{-r(T_o - t)}\mathrm{E}_t^*\left[\max\left\{S_{T_o} - K, 0\right\}\right]$, which can be rewritten as

$$C_t = \mathrm{E}_t^*\left[e^{-r(T_o - t)}S_{T_o}\mathbf{1}_{\left\{s_{T_o} > k\right\}}\right] - Ke^{-r(T_o - t)}\mathrm{E}_t^*\left[\mathbf{1}_{\left\{s_{T_o} > k\right\}}\right], \tag{A.30}$$

where we have defined $s \equiv \ln S$ and $k \equiv \ln K$, and the expectations are taken under the risk-neutral measure, under which the stock price follows (21) in Lemma 1. Employing the change of measure techniques, the first expectation in (A.30) is equivalent to

$$\mathrm{E}_t^*\left[e^{-r(T_o - t)}S_{T_o}\mathbf{1}_{\left\{s_{T_o} > k\right\}}\right] = S_t\mathrm{E}_t^{\mathbb{Q}_1}\left[\mathbf{1}_{\left\{s_{T_o} > k\right\}}\right],$$

where the expectation on the right hand side is taken under the new measure $\mathbb{Q}_1$ which

is related to the risk-neutral measure $\mathbb{P}^*$ such that $d\mathbb{Q}_1/d\mathbb{P}^* = M_{T_o}$, where the likelihood process M has the dynamics $dM_t = M_t\sqrt{\vartheta_t}d\omega^*_{St}$ with the $\mathbb{Q}_1$ Brownian motion ω^1_S is given by $d\omega^1_{St} = d\omega^*_{St} - \sqrt{\vartheta_t}dt$. For the second expectation in (A.30), for convenience, we denote the risk neutral measure $\mathbb{Q}_2 \equiv \mathbb{P}^*$. Hence we rewrite (A.30) as

$$C_t = S_t \mathrm{E}^{\mathbb{Q}_1}_t \left[\mathbf{1}_{\left\{s_{T_o} > k\right\}}\right] - Ke^{-r(T_o-t)}\mathrm{E}^{\mathbb{Q}_2}_t \left[\mathbf{1}_{\left\{s_{T_o} > k\right\}}\right],$$

which can be written equivalently as $C_t = S_t\Psi_1\left(s_t, \vartheta_t, X_t, t\right) - Ke^{-r(T_o-t)}\Psi_2\left(s_t, \vartheta_t, X_t, t\right)$, where, for $j = 1, 2$, the conditional probability function $\Psi_j\left(s, v, x, t\right)$ is given by

$$\Psi_j\left(s, v, x, t\right) = \mathbb{Q}_j\left[s_{T_o} \geq k | s_t = s, \vartheta_t = v, X_t = x\right].$$

Moreover, when the characteristic function

$$\varphi_j\left(s, v, x, t; \phi\right) = \mathrm{E}^{\mathbb{Q}_j}\left[e^{i\phi s_{T_o}} | s_t = s, \vartheta_t = v, X_t = x\right], \tag{A.31}$$

subject to the terminal condition $\varphi_j\left(s, v, x, T; \phi\right) = e^{i\phi s}$ is well-defined, each conditional probability function Ψ_j can be recovered from the characteristic function via the Levy inversion theorem as in (27) (see, for example, Duffie (2001)).

To obtain the characteristic function (A.31), we first compactly write the dynamics of processes under the measure $\mathbb{Q}_j$ for $j = 1, 2$ as

$$ds_t = \left(r + u_j\vartheta_t\right)dt + \sqrt{\vartheta_t}d\omega^j_{St}, \tag{A.32}$$

$$d\vartheta_t = \left(\zeta_\vartheta + \eta_\vartheta X_t - y_j\vartheta_t\right)dt + \sigma_\vartheta\sqrt{\vartheta_t}d\omega^j_{\vartheta t}, \tag{A.33}$$

$$dX_t = \alpha\left(r + u_j\vartheta_t - X_t\right)dt + \alpha\sqrt{\vartheta_t}d\omega^j_{St}, \tag{A.34}$$

where the constants $u_1 = 1/2$, $u_2 = -1/2$, $y_1 = \kappa$, $y_2 = \kappa^*_\vartheta$ and ω^1_ϑ is a $\mathbb{Q}_1$ Brownian motion with $d\omega^1_{St}d\omega^1_{\vartheta t} = \rho_{S\vartheta}dt$, where $\rho_{S\vartheta}$ is as in Proposition 3, and $\omega^2_S = \omega^*_S$ and $\omega^2_\vartheta = \omega^*_\vartheta$. We next employ the transform analysis (see, Duffie, Pan, and Singleton (2000)) and consider the function for $j = 1, 2$,

$$F\left(s, v, x, t; \phi, j\right) = e^{\mathcal{A}_j(T_o-t)+\mathcal{B}_j(T_o-t)v+\mathcal{C}_j(T_o-t)x+i\phi s}, \tag{A.35}$$

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with $\mathcal{C}_j(0) = \mathcal{B}_j(0) = \mathcal{A}_j(0) = 0$ so that $F(s, v, x, T_o; \phi, j) = e^{i\phi s}$, and partial derivatives

$$
\begin{aligned}
F_t &= -\left[\dot{\mathcal{A}}_j(\tau) + \dot{\mathcal{B}}_j(\tau)v + \dot{\mathcal{C}}_j(\tau)x\right]F, \quad F_v = \mathcal{B}_j(\tau)F, \quad F_{vv} = \mathcal{B}_j^2(\tau)F, \\
F_x &= \mathcal{C}_j(\tau)F, \quad F_{xx} = \mathcal{C}_j^2(\tau)F, \quad F_s = i\phi F, \quad F_{ss} = -\phi^2 F, \\
F_{vx} &= \mathcal{B}_j(\tau)\mathcal{C}_j(\tau)F, \quad F_{vs} = i\phi\mathcal{B}_j(\tau)F, \quad F_{xs} = i\phi\mathcal{C}_j(\tau)F,
\end{aligned}
$$

where we denoted by $\tau = T_o - t$. Then applying the Ito's Lemma to $F(s_t, \vartheta_t, X_t, t; \phi, j)$ yields

$$
\begin{aligned}
dF &= F_t dt + F_v d\vartheta_t + \frac{1}{2}F_{vv}d\vartheta_t d\vartheta_t + F_x dX_t + \frac{1}{2}F_{xx}dX_t dX_t \\
&\quad + F_s ds_t + \frac{1}{2}F_{ss}ds_t ds_t + F_{vx}d\vartheta_t dX_t + F_{vs}d\vartheta_t ds_t + F_{xs}dX_t ds_t,
\end{aligned}
$$

which after substituting the dynamics (A.32)–(A.34) and combing the same terms becomes

$$
\begin{aligned}
\frac{dF}{F} &= \left[-\dot{\mathcal{C}}_j(\tau) + \eta_\vartheta \mathcal{B}_j(\tau) - \alpha\mathcal{C}_j(\tau)\right]X_t dt \\
&\quad + \left[-\dot{\mathcal{B}}_j(\tau) + \frac{1}{2}\left(2u_j\phi i - \phi^2\right) - (y_j - \rho_{S\vartheta}\sigma_\vartheta\phi i)\mathcal{B}_j(\tau) + \frac{1}{2}\sigma_\vartheta^2\mathcal{B}_j^2(\tau)\right]\vartheta_t dt \\
&\quad + \left[\alpha(u_j + \phi i)\mathcal{C}_j(\tau) + \alpha\rho_{S\vartheta}\sigma_\vartheta\mathcal{B}_j(\tau)\mathcal{C}_j(\tau) + \frac{1}{2}\alpha^2\mathcal{C}_j^2(\tau)\right]\vartheta_t dt \\
&\quad + \left[-\dot{\mathcal{A}}_j(\tau) + \zeta_\vartheta\mathcal{B}_j(\tau) + \alpha r\mathcal{C}_j(\tau) + r\phi i\right]dt \\
&\quad + (i\phi + \alpha\mathcal{C}_j(\tau))\sqrt{\vartheta_t}d\omega_{St}^j + \sigma_\vartheta\mathcal{B}_j(\tau)\sqrt{\vartheta_t}d\omega_{\vartheta t}^j.
\end{aligned}
$$

For F to be a martingale its drift must be zero, so we use the method of undetermined coefficients, and set the drift term coefficients to zero. The term for the X_t gives (29), the term for the ϑ_t gives (30), and the remaining drift term gives (31). Since F is a $\mathbb{Q}_j$ martingale, we have

$$
F(s_t, \vartheta_t, X_t, t; \phi, j) = \mathrm{E}_t^{\mathbb{Q}_j}\left[F(s_{T_o}, \vartheta_{T_o}, X_{T_o}, T_o; \phi, j)\right] = \mathrm{E}_t^{\mathbb{Q}_j}\left[e^{i\phi s_{T_o}}\right] = \varphi_j(s_t, \vartheta_t, X_t, t; \phi),
$$

where the last equality follows from (A.31). Finally, using (A.35) we obtain (28).

On the other hand, given the call option price (25), we use the put-call parity $P_t = C_t - S_t + Ke^{-r(T_o-t)}$ and obtain the put option price as in (26).

The equilibrium call and put option prices in the rational benchmark economy are obtained

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by setting $\theta = 0$ in (25)–(26), since $\theta = 0$ leads to $\bar{\mathcal{C}}_j(\tau) = 0$, which in turn immediately leads to $\dot{\bar{\mathcal{B}}}_j(\tau) = (1/2)\left(2u_j\phi i - \phi^2\right) - \left(\bar{y}_j - \bar{\rho}_{S\vartheta}\bar{\sigma}_\vartheta\phi i\right)\bar{\mathcal{B}}_j(\tau) + (1/2)\bar{\sigma}_\vartheta^2\bar{\mathcal{B}}_j^2(\tau)$ due to (30), and $\dot{\bar{\mathcal{A}}}_j(\tau) = \bar{r}\phi i + \bar{\zeta}_\vartheta\bar{\mathcal{B}}_j(\tau)$ due to (31), with the constants $\bar{u}_j$ and $\bar{y}_j$ are given by $\bar{u}_1 = 1/2$, $\bar{u}_2 = -1/2$, $\bar{y}_1 = \kappa$, and $y_2 = \bar{\kappa}_\vartheta^*$, where $\bar{\kappa}_\vartheta^*$ is as in Lemma 1, $\bar{\zeta}_\vartheta$, $\bar{\sigma}_\vartheta$ are as in Proposition 4, and $\bar{\rho}_{S\vartheta}$ is as in Proposition 3. $\qquad\square$

Proof of Proposition 6. We first determine the (average) realized variance $\Upsilon_t(\tau)$. Using the stock return variance relation (16) and the dynamics (2), we obtain the stock return variance dynamics under the objective measure $\mathbb{P}$ as follows

$$d\vartheta_t = (\zeta_\vartheta - \kappa\vartheta_t)dt + \sigma_\vartheta\sqrt{\vartheta_t}(\rho d\omega_{1t} + \sqrt{1 - \rho^2}d\omega_{2t}), \tag{A.36}$$

where the positive constants ζ_ϑ and σ_ϑ are as in (24). The above dynamics leads to

$$\vartheta_u = \zeta_\vartheta\left(\frac{1 - e^{-\kappa(u-t)}}{\kappa}\right) + e^{-\kappa(u-t)}\vartheta_t + \sigma_\vartheta\int_t^u e^{-\kappa(u-s)}\sqrt{\vartheta_s}(\rho d\omega_{1s} + \sqrt{1 - \rho^2}d\omega_{2s}), \tag{A.37}$$

and hence to the expectation $\mathrm{E}_t[\vartheta_u] = \zeta_\vartheta\left(\frac{1-e^{-\kappa(u-t)}}{\kappa}\right) + e^{-\kappa(u-t)}\vartheta_t$, which implies the realized variance as in (36).

We next determine the equilibrium variance swap rate using (33), where the expectation is taken under the risk-neutral measure, under which the stock return variance follows (22) in Lemma 1. Since the drift terms of both the variance (22) and the sentiment (23) depend on one another, to be able to determine the expectation in a straightforward way we first define the auxiliary process

$$Z_t \equiv \zeta_\vartheta + \eta_\vartheta X_t - (\kappa_\vartheta^* - \kappa_1)\vartheta_t, \tag{A.38}$$

for some positive constant κ_1. The dynamics of Z then becomes $dZ_t = \eta_\vartheta dX_t - (\kappa_\vartheta^* - \kappa_1)d\vartheta_t$, which after substituting the dynamics (22)–(23) and employing (A.38) becomes

$$
\begin{aligned}
dZ_t &= \left[\alpha(\zeta_\vartheta + \eta_\vartheta r) - \kappa_2 Z_t + \left((\kappa_\vartheta^* - \kappa_1)(\kappa_1 - \alpha) - \frac{1}{2}\alpha\eta_\vartheta\right)\vartheta_t\right]dt \\
&\quad + \eta_\vartheta\alpha\sqrt{\vartheta_t}d\omega_{St}^* - (\kappa_\vartheta^* - \kappa_1)\sigma_\vartheta\sqrt{\vartheta_t}d\omega_{\vartheta t}^*,
\end{aligned}
$$

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where we have defined the constant κ_2 as

$$\kappa_2 \equiv \kappa_\vartheta^* - \kappa_1 + \alpha. \tag{A.39}$$

Thus, by taking κ_1 such that it solves the quadratic $(\kappa_\vartheta^* - \kappa_1)(\kappa_1 - \alpha) - \alpha\eta_\vartheta/2 = 0$, we obtain the dynamics as

$$dZ_t = \left(\alpha\left(\zeta_\vartheta + \eta_\vartheta r\right) - \kappa_2 Z_t\right) dt + \alpha\eta_\vartheta\sqrt{\vartheta_t}d\omega_{St}^* - \left(\kappa_\vartheta^* - \kappa_1\right)\sigma_\vartheta\sqrt{\vartheta_t}d\omega_{\vartheta t}^*, \tag{A.40}$$

whose drift term now does not depend on the variance process, which in return, due to (A.38), has the dynamics

$$d\vartheta_t = \left(Z_t - \kappa_1\vartheta_t\right) dt + \sigma_\vartheta\sqrt{\vartheta_t}d\omega_{\vartheta t}^*. \tag{A.41}$$

On the other hand, solving the quadratic equation for κ_1 leads to two roots, with both leading to same variance swap rates due to their symmetric form. Therefore, we take κ_1 to be the smaller root, which along with (A.39) also shows κ_2 is the larger root, which are given by (43) with the relation $\kappa_1 < \kappa_\vartheta^* < \kappa_2$.

The dynamics (A.40)–(A.41) lead to the solutions

$$
\begin{aligned}
\vartheta_u &= \vartheta_t e^{-\kappa_1(u-t)} + \int_t^u e^{-\kappa_1(u-s)} Z_s ds + \sigma_\vartheta \int_t^u e^{-\kappa_1(u-s)}\sqrt{\vartheta_s}d\omega_{\vartheta s}^*, \\
Z_s &= Z_t e^{-\kappa_2(s-t)} + \frac{\alpha\left(\zeta_\vartheta + \eta_\vartheta r\right)}{\kappa_2}\left(1 - e^{-\kappa_2(s-t)}\right) \\
&\quad + \alpha\eta_\vartheta \int_t^s e^{-\kappa_2(s-q)}\sqrt{\vartheta_q}d\omega_{Sq}^* - \left(\kappa_\vartheta^* - \kappa_1\right)\sigma_\vartheta \int_t^s e^{-\kappa_2(s-q)}\sqrt{\vartheta_q}d\omega_{\vartheta q}^*,
\end{aligned}
$$

and hence to the expectations $\mathrm{E}_t^*\left[\vartheta_u\right] = \vartheta_t e^{-\kappa_1(u-t)} + \int_t^u e^{-\kappa_1(u-s)}\mathrm{E}_t^*\left[Z_s\right] ds$, and $\mathrm{E}_t^*\left[Z_s\right] = Z_t e^{-\kappa_2(s-t)} + \alpha\left(\zeta_\vartheta + \eta_\vartheta r\right)\left(1 - e^{-\kappa_2(s-t)}\right)/\kappa_2$, which after substituting the latter into the former yields

$$\mathrm{E}_t^*\left[\vartheta_u\right] = \vartheta_t e^{-\kappa_1(u-t)} + \frac{\alpha\left(\zeta_\vartheta + \eta_\vartheta r\right)}{\kappa_2}\int_t^u e^{-\kappa_1(u-s)} ds + \left(Z_t - \frac{\alpha\left(\zeta_\vartheta + \eta_\vartheta r\right)}{\kappa_2}\right)\int_t^u e^{-\kappa_1(u-s)}e^{-\kappa_2(s-t)} ds.$$

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Therefore, using (33) we obtain the variance swap rate as

$$
\begin{aligned}
\Upsilon_t^*(\tau) &= \vartheta_t \frac{1}{\tau} \int_t^{t+\tau} e^{-\kappa_1(u-t)} du + \frac{\alpha(\zeta_\vartheta + \eta_\vartheta r)}{\kappa_2} \frac{1}{\tau} \int_t^{t+\tau} \int_t^u e^{-\kappa_1(u-s)} ds\, du \\
&\quad + \left(Z_t - \frac{\alpha(\zeta_\vartheta + \eta_\vartheta r)}{\kappa_2} \right) \frac{1}{\tau} \int_t^{t+\tau} \int_t^u e^{-\kappa_1(u-s)} e^{-\kappa_2(s-t)} ds\, du,
\end{aligned}
$$

and evaluating the simple integrals leads to

$$
\begin{aligned}
\Upsilon_t^*(\tau) &= \vartheta_t \frac{1 - e^{-\kappa_1 \tau}}{\kappa_1 \tau} + \frac{\alpha(\zeta_\vartheta + \eta_\vartheta r)}{\kappa_2} \frac{1}{\kappa_1} \left(1 - \frac{1 - e^{-\kappa_1 \tau}}{\kappa_1 \tau} \right) \\
&\quad + \left(Z_t - \frac{\alpha(\zeta_\vartheta + \eta_\vartheta r)}{\kappa_2} \right) \frac{1}{\kappa_2 - \kappa_1} \left(\frac{1 - e^{-\kappa_1 \tau}}{\kappa_1 \tau} - \frac{1 - e^{-\kappa_2 \tau}}{\kappa_2 \tau} \right),
\end{aligned}
$$

which after substituting (A.38), $\kappa_1 \kappa_2 = \alpha \left(\kappa_\vartheta^* + \frac{1}{2} \eta_\vartheta \right)$, and rearranging gives (37).

The equilibrium variance risk premium in the rational benchmark economy are obtained by setting $\theta = 0$ in (38)–(42), since $\theta = 0$ leads to $\bar{C}^*(\tau) = 0$ as $\bar{\eta}_\vartheta = 0$, $\bar{B}^*(\tau) = (1 - e^{-\bar{\kappa}_\vartheta^* \tau})/\bar{\kappa}_\vartheta^* \tau$, and $\bar{A}^*(\tau) = (\bar{\zeta}_\vartheta/\bar{\kappa}_\vartheta^*)(1 - (1 - e^{-\bar{\kappa}_\vartheta^* \tau})/\bar{\kappa}_\vartheta^* \tau)$ as either $\bar{\kappa}_1 = \bar{\kappa}_\vartheta^*$ along with $\bar{\kappa}_2 = \alpha$, or $\bar{\kappa}_1 = \alpha$ along with $\bar{\kappa}_2 = \bar{\kappa}_\vartheta^*$, depending on $\bar{\kappa}_\vartheta^*$ being greater or less than α but both cases leading to the same quantities, where $\bar{\kappa}_\vartheta^*$ is as in Lemma 1, and $\bar{\zeta}_\vartheta$ is as in Proposition 4.

Property, which states that the variance risk premium is increasing in sentiment X_t, but is decreasing in the stock return variance ϑ_t, follows from the facts that $C^*(\tau) < 0$ and $B(\tau) < B^*(\tau)$. The negativity of $C^*(\tau)$ is simply due to the fact that $\eta_\vartheta < 0$ in (40), whereas the second inequality holds if and only if

$$
\frac{1 - e^{-\kappa \tau}}{\kappa \tau} < \frac{1 - e^{-\kappa_1 \tau}}{\kappa_1 \tau} - \frac{\kappa_\vartheta^* - \kappa_1}{\kappa_2 - \kappa_1} \left(\frac{1 - e^{-\kappa_1 \tau}}{\kappa_1 \tau} - \frac{1 - e^{-\kappa_2 \tau}}{\kappa_2 \tau} \right), \tag{A.42}
$$

and since $\kappa_\vartheta^* < \kappa$ (see (24)) we have $(1 - e^{-\kappa \tau})/\kappa \tau < (1 - e^{-\kappa_\vartheta^* \tau})/\kappa_\vartheta^* \tau$, and hence a sufficient condition for (A.42) to hold is given by

$$
\frac{1 - e^{-\kappa_\vartheta^* \tau}}{\kappa_\vartheta^* \tau} < \frac{1 - e^{-\kappa_1 \tau}}{\kappa_1 \tau} - \frac{\kappa_\vartheta^* - \kappa_1}{\kappa_2 - \kappa_1} \left(\frac{1 - e^{-\kappa_1 \tau}}{\kappa_1 \tau} - \frac{1 - e^{-\kappa_2 \tau}}{\kappa_2 \tau} \right),
$$

which after rearranging becomes

$$\frac{1}{\kappa_\vartheta^* - \kappa_1}\left(\frac{1 - e^{-\kappa_1\tau}}{\kappa_1\tau} - \frac{1 - e^{-\kappa_\vartheta^*\tau}}{\kappa_\vartheta^*\tau}\right) > \frac{1}{\kappa_2 - \kappa_1}\left(\frac{1 - e^{-\kappa_1\tau}}{\kappa_1\tau} - \frac{1 - e^{-\kappa_2\tau}}{\kappa_2\tau}\right),$$

or, equivalently, in terms of simple integrals

$$\frac{1}{\tau}\int_t^{t+\tau}\int_t^u e^{-\kappa_1(u-s)}e^{-\kappa_\vartheta^*(s-t)}dsdu > \frac{1}{\tau}\int_t^{t+\tau}\int_t^u e^{-\kappa_1(u-s)}e^{-\kappa_2(s-t)}dsdu,$$

and the above inequality always holds since $\kappa_\vartheta^* < \kappa_2$. $\qquad\square$

Proof of Proposition 7. The expression for the slope coefficient (44) in the univariate regression of future stock returns $\ln(S_{t+h}/S_t)/h$ on the variance risk premium $\Pi_t(\tau)$ is standard in econometric theory (see, for instance, Ruud (2000)). On the other hand, the variance of the variance risk premium (45) follows immediately from (35)–(37). Similarly, the covariance between the variance risk premium and the future stock returns, using (35)–(37), becomes

$$\begin{aligned}\mathrm{Cov}\left[\Pi_t(\tau), \ln(S_{t+h}/S_t)/h\right] &= (B(\tau) - B^*(\tau))\mathrm{Cov}\left[\vartheta_t, \ln(S_{t+h}/S_t)\right]/h \\ &\quad - C^*(\tau)\,\mathrm{Cov}\left[X_t, \ln(S_{t+h}/S_t)\right]/h,\end{aligned} \qquad (A.43)$$

where $\ln(S_{t+h}/S_t) = \ln(D_{t+h}/D_t) + b(V_{t+h} - V_t) + c(X_{t+h} - X_t)$, and $\ln(D_{t+h}/D_t) = \mu h - \frac{1}{2}\int_t^{t+h}V_u du + \int_t^{t+h}\sqrt{V_u}d\omega_{1u}$. To compute (A.43), we use (16) and obtain the first term as $\mathrm{Cov}\left[\vartheta_t, \ln(S_{t+h}/S_t)\right] = \mathrm{Cov}\left[V_t, \ln(S_{t+h}/S_t)\right](1 + 2\rho\sigma b + \sigma^2 b^2)/(1 - \alpha c)^2$, where

$$\mathrm{Cov}[V_t, \ln(S_{t+h}/S_t)] = \mathrm{Cov}[V_t, \ln(D_{t+h}/D_t)] + b\mathrm{Cov}[V_t, (V_{t+h} - V_t)] + c\mathrm{Cov}[V_t, (X_{t+h} - X_t)], \qquad (A.44)$$

with

$$\begin{aligned}\mathrm{Cov}\left[V_t, \ln(D_{t+h}/D_t)\right] &= -\frac{1}{2}\frac{1 - e^{-\kappa h}}{\kappa}\mathrm{Var}\left[V_t\right], &(A.45)\\[6pt] \mathrm{Cov}\left[V_t, (V_{t+h} - V_t)\right] &= -\left(1 - e^{-\kappa h}\right)\mathrm{Var}\left[V_t\right], &(A.46)\\[6pt] \mathrm{Cov}\left[V_t, (X_{t+h} - X_t)\right] &= \frac{\alpha}{2}\frac{1 + 2\rho\sigma b + \sigma^2 b^2}{(1 - \alpha c)^2}\frac{e^{-\kappa h} - e^{-\alpha(1+\theta)h}}{\alpha(1+\theta) - \kappa}\mathrm{Var}\left[V_t\right] \\ &\quad - \left(1 - e^{-\alpha(1+\theta)h}\right)\mathrm{Cov}\left[V_t, X_t\right], &(A.47)\end{aligned}$$

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where (A.45)–(A.46) follow from the well-known properties of the square-root process, whereas (A.47) follows from the last line of the strong solution for the sentiment process given by

$$
\begin{aligned}
X_u = {} & \alpha r \frac{1-e^{-\alpha(1+\theta)(u-t)}}{\alpha(1+\theta)} + \frac{\alpha\zeta}{2\kappa}\frac{1+2\rho\sigma b+\sigma^2 b^2}{(1-\alpha c)^2}\left[\frac{1-e^{-\alpha(1+\theta)(u-t)}}{\alpha(1+\theta)} - \frac{e^{-\kappa(u-t)}-e^{-\alpha(1+\theta)(u-t)}}{\alpha(1+\theta)-\kappa}\right] \\
& + \alpha\int_t^u\left(\frac{\rho\sigma}{2}\frac{1+2\rho\sigma b+\sigma^2 b^2}{(1-\alpha c)^2}\frac{e^{-\kappa(u-s)}-e^{-\alpha(1+\theta)(u-s)}}{\alpha(1+\theta)-\kappa} + \frac{1+\rho\sigma b}{1-\alpha c}e^{-\alpha(1+\theta)(u-s)}\right)\sqrt{V_s}d\omega_{1s} \\
& + \alpha\int_t^u\left(\frac{\sigma}{2}\frac{1+2\rho\sigma b+\sigma^2 b^2}{(1-\alpha c)^2}\frac{e^{-\kappa(u-s)}-e^{-\alpha(1+\theta)(u-s)}}{\alpha(1+\theta)-\kappa} + \frac{\sigma b}{1-\alpha c}e^{-\alpha(1+\theta)(u-s)}\right)\sqrt{1-\rho^2}\sqrt{V_s}d\omega_{2s} \\
& + \frac{\alpha}{2}\frac{1+2\rho\sigma b+\sigma^2 b^2}{(1-\alpha c)^2}\frac{e^{-\kappa(u-t)}-e^{-\alpha(1+\theta)(u-t)}}{\alpha(1+\theta)-\kappa}V_t + e^{-\alpha(1+\theta)(u-t)}X_t. \tag{A.48}
\end{aligned}
$$

Hence, substituting (A.45)–(A.47) into (A.44) and using the identity

$$
\frac{1}{2}\frac{1+2\rho\sigma b+\sigma^2 b^2}{1-\alpha c} = -\left(\frac{1}{2}+\kappa b\right), \tag{A.49}
$$

we obtain

$$
\mathrm{Cov}\left[\vartheta_t,\ln\left(S_{t+h}/S_t\right)\right]/h = h_1\mathrm{Var}\left[\vartheta_t\right] - h_2\mathrm{Cov}\left[\vartheta_t,X_t\right], \tag{A.50}
$$

where the positive constants h_1 and h_2 are as in (48). On the other hand, the second covariance term in (A.43) becomes

$$
\mathrm{Cov}[X_t,\ln(S_{t+h}/S_t)] = \mathrm{Cov}[X_t,\ln(D_{t+h}/D_t)] + b\mathrm{Cov}[X_t,(V_{t+h}-V_t)] + c\mathrm{Cov}[X_t,(X_{t+h}-X_t)], \tag{A.51}
$$

with

$$
\begin{aligned}
\mathrm{Cov}\left[X_t,\ln\left(D_{t+h}/D_t\right)\right] &= -\frac{1}{2}\frac{1-e^{-\kappa h}}{\kappa}\mathrm{Cov}\left[X_t,V_t\right], \tag{A.52} \\
\mathrm{Cov}\left[X_t,(V_{t+h}-V_t)\right] &= -\left(1-e^{-\kappa h}\right)\mathrm{Cov}\left[X_t,V_t\right], \tag{A.53} \\
\mathrm{Cov}\left[X_t,(X_{t+h}-X_t)\right] &= \frac{\alpha}{2}\frac{1+2\rho\sigma b+\sigma^2 b^2}{(1-\alpha c)^2}\frac{e^{-\kappa h}-e^{-\alpha(1+\theta)h}}{\alpha(1+\theta)-\kappa}\mathrm{Cov}\left[V_t,X_t\right] \\
&\quad -\left(1-e^{-\alpha(1+\theta)h}\right)\mathrm{Var}\left[X_t\right], \tag{A.54}
\end{aligned}
$$

where again the last equality follows from the last line of the strong solution for the sentiment process (A.48). Hence, substituting (A.52)–(A.54) into (A.51) and using the identity (A.49)

we obtain

$$\text{Cov}\left[X_t, \ln\left(S_{t+h}/S_t\right)\right]/h = h_1\text{Cov}\left[\vartheta_t, X_t\right] - h_2\text{Var}\left[X_t\right], \tag{A.55}$$

where the positive constants h_1 and h_2 are as in (48). Finally, substituting (A.50) and (A.55) into (A.43) yields (46).

The steady state variance of the stock return in (47) is immediate from the well-known properties of the square-root process ϑ, whose dynamics are given by (A.36). The steady state variance of the sentiment in (47) is obtained by using the diffusion terms in the strong form solution (A.48) and computing the variance $\text{Var}\left[X_t\right]$ through a straightforward but tedious algebra, then taking its limit $t \to \infty$. The steady state covariance between the stock return variance and the sentiment in (47) is obtained by using the strong form solutions (A.37) and (A.48), which after substituting the constants (24) and rearranging leads to

$$\text{Cov}\left[\vartheta_t, X_t\right] = \frac{1}{2}\alpha\sigma_\vartheta^2 \int_0^t \frac{e^{-\kappa(t-s)} - e^{-\alpha(1+\theta)(t-s)}}{\alpha\left(1+\theta\right) - \kappa}e^{-\kappa(t-s)}\text{E}\left[\vartheta_s\right]ds$$
$$+\alpha\rho_{S\vartheta}\sigma_\vartheta \int_0^t e^{-\alpha(1+\theta)(t-s)}e^{-\kappa(t-s)}\text{E}\left[\vartheta_s\right]ds,$$

substituting $\text{E}\left[\vartheta_s\right] = \zeta_\vartheta/\kappa$ and evaluating the integrals yields

$$\text{Cov}[\vartheta_t, X_t] = \frac{1}{2}\alpha\sigma_\vartheta^2\frac{\zeta_\vartheta}{\kappa}\frac{1}{\alpha\left(1+\theta\right)-\kappa}\left(\frac{1-e^{-2\kappa t}}{2\kappa} - \frac{1-e^{-(\alpha(1+\theta)+\kappa)t}}{\alpha\left(1+\theta\right)+\kappa}\right)+\alpha\rho_{S\vartheta}\sigma_\vartheta\frac{\zeta_\vartheta}{\kappa}\frac{1-e^{-(\alpha(1+\theta)+\kappa)t}}{\alpha\left(1+\theta\right)+\kappa},$$

which after taking the limit $t \to \infty$ gives the expression in (47).

The equilibrium slope coefficient in the univariate regression of stock returns $\ln(\bar{S}_{t+h}/\bar{S}_t)/h$ on the variance risk premium $\bar{\Pi}_t\left(\tau\right)$ in the rational benchmark economy are obtained by setting $\theta = 0$ in (45)–(46), since $\theta = 0$ leads to $\bar{C}^*\left(\tau\right) = 0$, $\bar{h}_2 = 0$, and $\bar{h}_1 = (1 - e^{-\kappa h})/2\kappa h$, we obtain $\text{Var}\left[\bar{\Pi}_t\left(\tau\right)\right] = (\bar{B}(\tau) - \bar{B}^*(\tau))^2\text{Var}\left[\bar{\vartheta}_t\right]$ and $\text{Cov}\left[\bar{\Pi}_t\left(\tau\right), \ln\left(\bar{S}_{t+h}/\bar{S}_t\right)/h\right] = (\bar{B}(\tau) - \bar{B}^*(\tau))((1 - e^{-\kappa h})/2\kappa h)\text{Var}\left[\bar{\vartheta}_t\right]$, with the steady state variance $\text{Var}\left[\bar{\vartheta}_t\right] = (\bar{\sigma}_\vartheta^2/2\kappa)(\bar{\zeta}_\vartheta/\kappa)$, and $\bar{B}\left(\tau\right)$, $\bar{B}^*\left(\tau\right)$ are as in Proposition 6, and $\bar{\zeta}_\vartheta$, $\bar{\sigma}_\vartheta$ are as in Proposition 4.

Property, which states that the slope coefficient of the variance risk premium in the univariate regression is negative, follows from the fact that $\text{Cov}[\Pi_t\left(\tau\right), \ln(S_{t+h}/S_t)/h] < 0$. To see this, note that in (46) the terms $B\left(\tau\right) - B^*\left(\tau\right)$ and $C^*\left(\tau\right)$ are negative, and the con-

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stants h_1 and h_2 are positive, hence a sufficient condition for this covariance to be negative is $\text{Cov} [\vartheta_t, X_t] < 0$. Using its expression in (47), we immediately see that $\text{Cov} [\vartheta_t, X_t] < 0$ if and only if $(\sigma_\vartheta^2/4\kappa) + \rho_{S\vartheta} \sigma_\vartheta < 0$. By substituting $\rho_{S\vartheta}$ and σ_ϑ this condition becomes

$$\frac{\sigma^2}{4\kappa} \frac{1 + 2\rho\sigma b + \sigma^2 b^2}{(1 - \alpha c)^2} + \sigma \frac{\rho + \sigma b}{1 - \alpha c} < 0.$$

Further substituting the identity (A.49) into the above inequality leads to $-\sigma (1/2 + \kappa b) /2\kappa + \rho + \sigma b < 0$, which after rearranging becomes $-\sigma/4\kappa + \rho + \sigma b/2 < 0$, and this inequality always holds as the left hand side is always negative. □

Proof of Proposition 8. The expressions for the slope coefficients (49)–(50) in the joint regression of future stock returns $\ln(S_{t+h}/S_t)/h$ on the variance risk premium $\Pi_t (\tau)$ and the realized variance $\Upsilon_t (\tau)$ are standard in econometric theory (see, for instance, Ruud (2000)). On the other hand, the variance of the realized variance (51), and the covariance between the variance risk premium and the realized variance (52) follow immediately from (36)–(37). Similarly, the covariance between the realized variance and the future stock returns, using (36), becomes $\text{Cov} [\Upsilon_t (\tau), \ln (S_{t+h}/S_t) /h] = B (\tau) \text{Cov} [\vartheta_t, \ln (S_{t+h}/S_t)] /h$, which along with (A.50) immediately gives (53).

The joint regression of future stock returns $\ln(\bar{S}_{t+h}/\bar{S}_t)/h$ on the variance risk premium $\bar{\Pi}_t (\tau)$ and the realized variance $\bar{\Upsilon}_t (\tau)$ in the rational benchmark economy is not well-defined since $\bar{\Pi}_t (\tau)$ and $\bar{\Upsilon}_t (\tau)$ are perfectly correlated as they are given by $\bar{\Pi}_t (\tau) = (\bar{A} (\tau) - \bar{A}^* (\tau)) + (\bar{B} (\tau) - \bar{B}^* (\tau))\bar{\vartheta}_t$ and $\bar{\Upsilon}_t (\tau) = \bar{A} (\tau) + \bar{B} (\tau) \bar{\vartheta}_t$, implying $\text{Corr} \left[\bar{\Pi}_t (\tau), \bar{\Upsilon}_t (\tau) \right] = 1$.

Property, which states that the slope coefficient of the variance risk premium in the joint regression is negative, follows from the fact that $\text{Cov} [\Pi_t(\tau), \ln(S_{t+h}/S_t) /h] \text{Var} [\Upsilon_t(\tau)] < \text{Cov} [\Upsilon_t(\tau), \ln(S_{t+h}/S_t) /h] \text{Cov} [\Pi_t(\tau), \Upsilon_t(\tau)]$. To see this, substituting (46) and (51)–(53) into this inequality leads to

$$\left\{ (B(\tau) - B^*(\tau))\left(h_1 \text{Var}[\vartheta_t] - h_2 \text{Cov}[\vartheta_t, X_t]\right) + C^*(\tau)\left(h_2 \text{Var}[X_t] - h_1 \text{Cov}[\vartheta_t, X_t]\right) \right\} \text{Var}[\vartheta_t]$$
$$< \left(h_1 \text{Var} [\vartheta_t] - h_2 \text{Cov} [\vartheta_t, X_t] \right) \left((B(\tau) - B^*(\tau)) \text{Var} [\vartheta_t] - C^* (\tau) \text{Cov} [\vartheta_t, X_t] \right),$$

which after canceling out the same terms becomes $C^*(\tau)\left(\text{Var} [X_t] \text{Var} [\vartheta_t] - \text{Cov} [\vartheta_t, X_t]^2 \right) < 0$, which always holds as $C^* (\tau)$ is negative while the term in the bracket is positive. □

Appendix B: Additional Quantitative Analysis

Parameter	Symbol	Value
Relative weights	α	0.5
Extrapolation degree	θ	$\{0, 0.25, 0.50, 0.75\}$
Fundamental variance mean reversion speed	κ	0.15
Fundamental variance long-run mean coefficient	ζ	0.0018
Fundamental variance volatility term	σ	0.0506
Correlation coefficient	ρ	-0.15
Fundamental process mean growth rate	μ	0.0221

Table 3: **Parameter values.** This table reports the parameter values used in our Tables. The derivation of these values is presented in the text.

In this section, we determine the baseline parameter values employed in our Tables and provide additional quantitative analysis for option prices. Consistent with its corresponding estimates in Barberis et al. (2015), we take the relative weights of most recent returns in the construction of the sentiment as $\alpha = 0.5$. We set the baseline extrapolation degree to $\theta = 0.25$, but to better understand the effect of the extrapolation degree we also consider higher degrees of $\theta = 0.50$ and $\theta = 0.75$, as well as the benchmark rational economy value of $\theta = 0$. We set the speed of mean reversion of the fundamental variance to $\kappa = 15\%$, which is also the speed of mean reversion of the stock return variance in our model, since this value lies roughly in the middle for the reported corresponding values for the realized stock variance in Bollerslev, Tauchen, and Zhou (2009) and Carr and Wu (2009). We match the long-run mean of the fundamental variance ζ/κ to the reported unconditional mean of the dividend growth rate variance of 0.1105^2 in Beeler and Campbell (2012), which immediately implies $\zeta = 0.183\%$. We set a small negative correlation of $\rho = -0.15$, and match the long-run mean of the stock return variance ζ_ϑ/κ to the reported corresponding value of 0.1598^2 in Campbell (2018, Table 6.1), and back out $\sigma = 5.06\%$. Finally, we match the interest rate r to the reported average riskless rate of 0.72% in Campbell (2018, Table 6.1), and back out an economically plausible parameter value for the mean growth rate of the fundamental of $\mu = 2.21\%$ from the relation $r = (\mu + \zeta b)/(1 - \alpha c)$. This procedure yields the parameter values in Table 3.

K/S_t	θ	Put option				Call option			
		X_t				X_t			
		-0.10	0	0.10	% Diff.	-0.10	0	0.10	% Diff.
0.9	0	0.15	0.15	0.15	**0%**	10.25	10.25	10.25	**0%**
	0.25	0.16	0.16	0.15	**6.80%**	10.27	10.26	10.26	**0.10%**
	0.50	0.18	0.17	0.16	**14.60%**	10.29	10.28	10.26	**0.22%**
	0.75	0.20	0.18	0.17	**22.89%**	10.31	10.29	10.27	**0.37%**
1	0	2.53	2.53	2.53	**0%**	2.65	2.65	2.65	**0%**
	0.25	2.55	2.53	2.51	**1.52%**	2.67	2.65	2.63	**1.45%**
	0.50	2.58	2.54	2.50	**3.33%**	2.70	2.66	2.62	**3.18%**
	0.75	2.62	2.55	2.49	**5.38%**	2.74	2.67	2.61	**5.13%**
1.1	0	10.07	10.07	10.07	**0%**	0.20	0.20	0.20	**0%**
	0.25	10.07	10.06	10.06	**0.13%**	0.20	0.20	0.19	**6.98%**
	0.50	10.07	10.06	10.04	**0.28%**	0.20	0.19	0.17	**16.40%**
	0.75	10.07	10.05	10.02	**0.45%**	0.20	0.18	0.16	**29.04%**

Table 4: **Shorter maturity option prices.** This table reports the put and call option prices for varying levels of option moneyness K/S_t, extrapolation degree θ, and sentiment X_t. The last columns report the percentage difference of option prices at the bearish sentiment ($X_t = -0.10$) from the bullish sentiment ($X_t = 0.10$). The option maturity is 2 months, the stock price is $S_t = 100$, the stock return volatility is 15.98% (steady-state value), and all the other parameter values are as in Table 3: $\alpha = 0.5$, $\kappa = 0.15$, $\zeta = 0.0018$, $\sigma = 0.0506$, $\rho = -0.15$, $\mu = 0.0221$.

| K/S_t | θ | Put option | | | | Call option | | | |
| | | X_t | | | | X_t | | | |
		-0.10	0	0.10	% Diff.	-0.10	0	0.10	% Diff.
0.9	0	1.01	1.01	1.01	**0%**	11.33	11.33	11.33	**0%**
	0.25	1.12	1.07	1.01	**10.53%**	11.45	11.39	11.34	**0.94%**
	0.50	1.26	1.14	1.03	**22.49%**	11.58	11.47	11.35	**2.03%**
	0.75	1.42	1.23	1.05	**34.94%**	11.74	11.56	11.37	**3.23%**
1	0	4.29	4.29	4.29	**0%**	4.65	4.65	4.65	**0%**
	0.25	4.41	4.32	4.22	**4.40%**	4.77	4.68	4.58	**4.05%**
	0.50	4.56	4.36	4.15	**9.72%**	4.92	4.72	4.51	**8.95%**
	0.75	4.73	4.42	4.08	**15.85%**	5.09	4.78	4.44	**14.57%**
1.1	0	10.89	10.89	10.89	**0%**	1.29	1.29	1.29	**0%**
	0.25	10.94	10.87	10.80	**1.34%**	1.34	1.27	1.19	**12.16%**
	0.50	11.00	10.85	10.68	**3.02%**	1.40	1.24	1.08	**29.87%**
	0.75	11.08	10.82	10.55	**5.05%**	1.48	1.22	0.94	**56.37%**

Table 5: **Longer maturity option prices.** This table reports the put and call option prices for varying levels of option moneyness K/S_t, extrapolation degree θ, and sentiment X_t. The last columns report the percentage difference of option prices at the bearish sentiment ($X_t = -0.10$) from the bullish sentiment ($X_t = 0.10$). The option maturity is 6 months, the stock price is $S_t = 100$, the stock return volatility is 15.98% (steady-state value), and all the other parameter values are as in Table 3: $\alpha = 0.5$, $\kappa = 0.15$, $\zeta = 0.0018$, $\sigma = 0.0506$, $\rho = -0.15$, $\mu = 0.0221$.

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