On the Existence of a Unique Price Equilibrium for Models of Product Differentiation

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We prove the existence of a unique price equilibrium for models of product differentiation. These models are characterized by demand functions that depend on differences in prices and have log-increasing differences in own price and other prices, which implies that best response correspondences are increasing. Many discrete choice models satisfy these conditions, including logit models and nested logit models, and the existence theorem can be applied directly. With the logarithmic transformation, the existence theorem can be extended to models with homogeneous demand functions, which characterizes representative consumer models.

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1. Introduction

Most studies of differentiated product markets use discrete choice models, spatial models, or representative consumer models. Discrete choice models typically assume that a consumer purchases a unit of the brand that maximizes the difference between their reservation price and actual price, while spatial models typically assume that a consumer buys the brand that minimizes the sum of transportation costs and price. Since the demand for each brand depends only on differences in price, uniform increases in the price of all brands do not change the demand for any single brand. Representative consumer models assume that demand functions are obtained by the utility maximization of a representative consumer. Since demands depend on real income and relative prices, the demand for each brand is homogeneous of degree zero in all prices and income, so that demand functions depend only on differences in log income and in log prices.

This paper proves that such models have a unique price equilibrium if demand functions have log-increasing differences in all prices. The latter assumption is used by Milgrom and Roberts (1990), Vives (1990) and Milgrom and Shannon (1994) to construct increasing best response correspondences. Milgrom and Roberts (1990) and Vives (1990) apply supermodular games to price competition in a differentiated product market with increasing best response correspondences. Milgrom and Shannon (1994) generalize supermodular games to quasi-supermodular games and prove that non-decreasing best response correspondences exist if cost functions are convex and the log demand function has an increasing difference in own price and in the prices of other brands. Then Tarski's fixed-point theorem implies that a Bertrand equilibrium exists if the domain of the price vector is a complete lattice.

On the other hand, Caplin and Nalebuff (1991) use the Prékopa-Borel theorem on the aggregation of a class of unimodal density functions to prove the existence of convex-valued best-response correspondences for discrete choice and spatial models. Then Kakutani's fixed-point theorem guarantees the existence of a Bertrand equilibrium. Although these studies are path breaking, some problems remain in applying their results.

First, many applications assume an unbounded price vector domain. For example many applications in discrete choice models use either probit or logit models that are based on normal or double exponential distributions of tastes. These distributions have unbounded supports, and hence the domains of the demand functions are unbounded. Because the proofs of existence use fixed-point theorems on a compact convex set in Euclidean space or on a compact complete lattice, these proofs cannot be applied directly. Although there is some upper bound on the money that a consumer can pay for a unit of a differentiated product as assumed by Caplin and Nalebuff, most applications ignore boundedness for simplicity. Likewise, CES demand function, which is frequently used in applications, is strictly positive for an arbitrary positive price vector. The last part of Caplin and Nalebuff proves the existence of a price equilibrium in models with unbounded support like CES but their proof cannot be generalized systematically.

Second, although many applications implicitly assume the uniqueness of an equilibrium, uniqueness has only been proven for limited classes of models, and there is no systematic characterization of such classes. Milgrom and Roberts (1990) show that a dominant diagonal Jacobian matrix for the log payoff function is sufficient for uniqueness. For discrete choice models, however, direct conditions on the distribution of tastes is desired. Caplin and Nalebuff (1991) demonstrate a few cases in which the equilibrium is unique. They includes duopoly models with log concave distribution of tastes, oligopoly models in which the domain of tastes is the one-dimensional real line, and logit models. No sufficient condition is given that can be directly applied to the nested logit or probit models.

Furthermore, the relation between increasing best response correspondences and properties of the distribution of tastes has not been established. An increasing best response correspondence implies that a firm raises the price of its brand when the prices of other brands are raised. Although this is plausible, it does not hold for all well-behaved demand functions. On the other hand, Caplin and Nalebuff prove that some kind of concavity in the distribution function is sufficient for the quasi-concavity of a profit function.

This paper solves many of these problems for demand functions that are strictly positive, strictly decreasing, and depend on differences in prices. The Milgrom and Shannon sufficient condition for an increasing best response correspondence is that the demand function has log increasing differences in own price and the price of each other brand. We show that these two assumptions imply that the demand function of a brand is log concave in own price. These assumptions are required for the existence of increasing best response functions and for the existence of a unique Bertrand equilibrium. The method used by Caplin and Nalebuff is too generous, in that a class of concavity in all prices is obtained if applied to discrete choice models.

We show that independent log concave distributions of tastes are sufficient for a demand function that has log increasing differences. This includes the logit models , and these results extend to the nested logit model. We examine a correlated probit model with three brands, and show that the best response functions of all three brands are increasing only if the demand functions are consistent with independent distributions of reservation prices. We also examine a unidimensional spatial model and show that the demand function has log increasing differences if the transportation cost function is either linear or quadratic.

Another widely used differentiated product model is the representative consumer model of Dixit and Stiglitz (1977), in which a representative consumer is assumed to maximize a CES utility function subject to his budget constraint. Hence demand functions are homogeneous of degree zero in prices and income. With a logarithmic transformation, such models are homeomorphic to discrete choice models, and the existence of a unique price equilibrium is established with some additional assumptions.

Section 2 presents the model and proves the existence of a unique equilibrium when demand for each brand depends on differences in prices, and extends the basic theorem to other cases, including the case of representative consumer models. Section 3 applies these results to several discrete choice models, spatial models and representative consumer models. Section 4 concludes.

In this paper, increasing and decreasing mean non-decreasing and non-increasing, respectively; if we mean strictly increasing or decreasing we will say so. Similarly, positive and negative mean non-negative and non-positive, respectively.

2. Existence of a unique price equilibrium

This section presents the model and proves the main existence theorem. Subsection 2.1 presents the model and notation. Subsection 2.2 gives background mathematical concepts and theorems required for the proof of the theorem. Subsection 2.3 proves the theorem formally. Subsection 2.4 provides some extensions of the theorem.

2.1 Model

We assume that there are n brands of a differentiated product. The demand function for brand i is $D_i(\mathbf{p})$ or $D_i(p_i|\mathbf{p}_{-i})$, $(i=1,...,n)$, where $\mathbf{p}=(p_1,...,p_n)$ is the price vector of the differentiated products and $\mathbf{p}_{-i}=(p_1,..,p_{i-1},p_{i+1},.,p_n)$ is the price vector of all other brands. The cost function for brand i is $C_i(x_i)$. Each brand is produced by a single firm that maximizes its profit, $\Pi_i(p_i|\mathbf{p}_{-i})=p_iD_i(p_i|\mathbf{p}_{-i})-C_i(D_i(p_i|\mathbf{p}_{-i}))$, given the prices of other brands.

The set of maximizers of $\Pi_i(p_i|\mathbf{p}_{-i})$ with respect to p_i is a best response correspondence $r_i(\mathbf{p}_{-i})$. When $r_i(\mathbf{p}_{-i})$ is single valued, there exists a best response function, which we denote $r_i(\mathbf{p}_{-i})$, using the same notation.

We show in this section that the following five conditions are sufficient for the existence of a unique price equilibrium. For each i,

(i) $D_i(p_i|\mathbf{p}_{-i})$ is strictly positive and strictly decreasing in p_i on R^n,

(ii) $D_i(\mathbf{p})=D_i(\mathbf{p}+k\mathbf{u}^n)$ for all k, where $\mathbf{u}^n$ is the n vector whose elements are all unity.

(iii) $D_i\left(p_i^H \middle| \mathbf{p}_{-i}^H\right) D_i\left(p_i^L \middle| \mathbf{p}_{-i}^L\right) \geq D_i\left(p_i^H \middle| \mathbf{p}_{-i}^L\right) D_i\left(p_i^L \middle| \mathbf{p}_{-i}^H\right)$ for $\mathbf{p}_{-i}^H \geq \mathbf{p}_{-i}^L$ and $p_i^H \geq p_i^L$,

(iv) $C_i(x_i)$ is convex.

(v-a) $D_i(p_i|\mathbf{p}_{-i})$ is increasing in $\mathbf{p}_{-i}$ on R^n, or

(v-b) $C_i(x_i) = c_i x_i$ where $c_i \geq 0$.

Condition (ii) is equivalent to the assumption that $D_i(\mathbf{p})$ depends only on differences in prices, $p_1 - p_i, .., p_n - p_i$. It is satisfied for discrete choice models and spatial models that assume that a consumer buys a unit of brand i if and only if $v_i - p_i \geq v_j - p_j$ for $j = 1, ..., n$, where $v_k (k = 1, .., n)$ is the reservation price of brand k determined by the preference or position of each consumer.

The second part of condition (i) is the standard law of demand. The first part means that all brands are active at all prices. This introduces a difficulty in using fixed-point theorems to prove the existence of an equilibrium, since some upper bound must be constructed, but the problem of selecting active firms disappears. Condition (v-a) implies that any two brands are gross substitutes. As we see in the formal proof in subsection 2.3, (v-a) is required only when the cost function is strictly convex. In discrete choice models, any two brands are gross substitutes, but it may not be the case for representative consumer models, especially if we include outside goods. Since we extend the main existence theorem to representative consumer models, the case in which cost functions are linear but all brands are not gross substitutes may deserve separate consideration.

In economic terms, (iii) implies that the elasticity of demand is a decreasing function of the price of other brands.[2] Hence it implies an increasing best response correspondence if marginal cost is constant. This result extends to convex cost functions if any two brands are gross substitutes.

The outline of the proof of existence of a unique equilibrium for the case (v-a) holds is illustrated in Figure 1. (1) We show that (ii) and (iii), together with (i), imply that the demand function of each brand is log concave in its price. (2) This is a sufficient condition for the existence of a unique maximizer of the profit function if the cost function is convex and the demand function is strictly decreasing. (3) Then we use (iii) and (iv) to show that the best response functions are increasing. As a result, a price equilibrium exists by Tarski's fixed-point theorem if the domain and region of the mapping that consists of best response functions are the same compact rectangular set. (4) For this purpose, we construct an upper bound, and (5) establish the existence of a price equilibrium. (6) Then we show that if (i) and (ii) hold, and best response correspondences are single valued in addition, then a firm raises the price of its own brand by strictly less than one dollar when the prices of all other brands increase by one dollar. (7) The uniqueness of an equilibrium is an immediate consequence of this result.

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2 If demands are strictly positive, (iii) is equivalent to

$$\frac{D_i\left(p_i{}^H \middle| \mathbf{p}_{-i}{}^H\right) - D_i\left(p_i{}^L \middle| \mathbf{p}_{-i}{}^H\right)}{D_i\left(p_i{}^L \middle| \mathbf{p}_{-i}{}^H\right)} = \frac{D_i\left(p_i{}^H \middle| \mathbf{p}_{-i}{}^H\right)}{D_i\left(p_i{}^L \middle| \mathbf{p}_{-i}{}^H\right)} - 1 \geq \frac{D_i\left(p_i{}^H \middle| \mathbf{p}_{-i}{}^L\right)}{D_i\left(p_i{}^L \middle| \mathbf{p}_{-i}{}^L\right)} - 1 = \frac{D_i\left(p_i{}^H \middle| \mathbf{p}_{-i}{}^L\right) - D_i\left(p_i{}^L \middle| \mathbf{p}_{-i}{}^L\right)}{D_i\left(p_i{}^L \middle| \mathbf{p}_{-i}{}^L\right)}.$$

By dividing both side by $p_i{}^H - p_i{}^L$ and letting $p_i{}^H \to p_i{}^L$, both sides converge to the elasticities.

Each arrow in Figure 1 corresponds to a lemma in the proof of the main existence theorem in subsection 2.3. In subsection 2.2, we summarize the mathematical definitions and theorems that we use in the formal proof.

2.2 Mathematical Preliminaries

This subsection summarizes mathematical concepts and theorems that we use in the proof of the main theorem. Further concepts and theorems are introduced as required.

2.2.1 Log concave and TP₂ functions

A positive valued function $f(\mathbf{x})$ on R^n is *log concave* if $f(\lambda\mathbf{x}+(1-\lambda)\mathbf{y}) \geq f(\mathbf{x})^{\lambda} f(\mathbf{y})^{1-\lambda}$ for all $\lambda \in (0,1)$. The support of a log concave function f, $\{\mathbf{x} \mid f(\mathbf{x})>0\}$, is convex. A log concave function on R inherits the properties of concave functions on its support; for example it is increasing or decreasing or unimodal and it has right and left derivatives, $f'^{+}(x)$ and $f'^{-}(x)$. If $0<f(x^H)<f(x^L)$ for $x^H>x^L$, then for $x>x^H$, $\ln f(x) \leq \ln f(x^H) - \alpha(x-x^H)$, where $\alpha = -\dfrac{f(x^H)-f(x^L)}{x^H-x^L} > 0$, so that $f(x) \leq f(x^H)\exp(-\alpha(x-x^H))$. Hence $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} xf(x) = 0$, which we use in the proof of Lemma 2.

A positive valued function, $f(x,y)$ on R^2, is TP_2 (totally positive of order 2) if $x^H>x^L$ and $y^H>y^L$ imply $f(x^H,y^H)f(x^L,y^L) \geq f(x^H,y^L)f(x^L,y^H)$.[3] Obviously, *a*

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[3] Karlin (1968). A function is totally positive of order r (TP_r) if for all $1 \leq m \leq r$,

$$x_1<x_2<....<x_m \text{ and } y_1<y_2<....<y_m, \begin{vmatrix} f(x_1,y_1) & f(x_1,y_2) & \ldots & f(x_1,y_m) \\ f(x_2,y_1) & f(x_2,y_2) & \ldots & f(x_1,y_1) \\ \vdots & \vdots & & \vdots \\ f(x_m,y_1) & f(x_m,y_2) & \ldots & f(x_m,y_m) \end{vmatrix} \geq 0 .$$

function that depends on a single variable is TP_2 and the product of TP_2 functions is TP_2. There is a well-known duality that *a positive Lebesgue-measurable function, $f(x)$ on R, is log concave if and only if $f(x-y)$ is TP_2 in x and y.*[4] Since monotone functions and continuous functions are Lebesgue-measurable, this duality holds for these functions.

2.1.2 Lattices and Tarski's fixed point theorem

In this paper we use only a component-wise order on R^n where $\mathbf{x} = (x_1,...,x_n) \geq \mathbf{y} = (y_1,...,y_n)$ if and only if $x_i \geq y_i$ for all $i = 1,...,n$.[5] If the inequality is

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4 Karlin (1968), p. 159. If $x^H > x^L$ and $y^H > y^L$, then

$$\theta = \frac{x^H - x^L}{(x^H - x^L) + (y^H - y^L)} \in (0,1),\ x^H - y^H = \theta(x^H - y^L) + (1-\theta)(x^L - y^H) \text{ and}$$

$x^L - y^L = (1-\theta)(x^H - y^L) + \theta(x^L - y^H)$. Then if $f(x)$ is log concave,

$f(x^H - y^H) \geq f(x^H - y^L)^{\theta} f(x^L - y^H)^{1-\theta}$ and

$f(x^L - y^L) \geq f(x^H - y^L)^{1-\theta} f(x^L - y^H)^{\theta}$. Multiplying both sides, we obtain

$f(x^H, y^H) f(x^L, y^L) \geq f(x^H, y^L) f(x^L, y^H)$. To show necessity, we can choose $x^H > x^L$ and $y^H > y^L$ for any $u > v$ such that $u = x^H - y^L$ $v = x^L - y^H$ and $\frac{u+v}{2} = x^H - y^H = x^L - y^L$. Hence if $f(x-y)$ is TP_2 in x and y,

$f\left(\frac{u+v}{2}\right) \geq \sqrt{f(u) f(v)}$. This condition is sufficient for log concavity if f is Lesbesgue-measurable, but if not, there is a famous counter example (see G. Hardy, et al. (1952), p.96) .

5 For a general and comprehensive treatment of this subject, see Topkis (1998).

strict for some i, $\mathbf{x} > \mathbf{y}$. *Lattice operations* are defined by

$\mathbf{x} \vee \mathbf{y} = (\max(x_1, y_1), \ldots, \max(x_n, y_n))$ and $\mathbf{x} \wedge \mathbf{y} = (\min(x_1, y_1), \ldots, \min(x_n, y_n))$.[6] A set is a *lattice* if it is closed in lattice operations. Let $S \subseteq X \subseteq R^n$. Then the set of an upper bound of $S' \subseteq S$ with respect to S is $S'^{\,u}_S = \{\mathbf{y} \in S \mid \mathbf{x} \in S' \Rightarrow \mathbf{x} \le \mathbf{y}\}$, and $\bar{S}_S = \sup_S S'$ is the least upper bound of S' with respect to S if $\bar{S}_S \in S'^{\,u}_S$ and $\bar{S}_S \le \mathbf{x}$ for all $\mathbf{x} \in S'^{\,u}_S$. We can define $\inf_S S'$ analogously. If both $\sup_S S'$ and $\inf_S S'$ exist and belong to S for any nonempty subset S' of S, S is a complete lattice. If $\sup_S S' = \sup_X S'$ and $\inf_S S' = \inf_X S'$, in addition, S is a *subcomplete sublattice* of X.[7] A subcomplete sublattice of X is a complete lattice. *A lattice in R^n is a subcomplete sublattice of R^n if it is compact.*[8]

A function f from R^n to R^m is *increasing* if $\mathbf{x}^H \ge \mathbf{x}^L$ implies $f(\mathbf{x}^H) \ge f(\mathbf{x}^L)$. A correspondence f from a lattice to another lattice is increasing if $\mathbf{x}^H \ge \mathbf{x}^L$, $\mathbf{y}^H \in f(\mathbf{x}^H)$ and $\mathbf{y}^L \in f(\mathbf{x}^L)$ implies $\mathbf{y}^H \vee \mathbf{y}^L \in f(\mathbf{x}^H)$ and $\mathbf{y}^H \wedge \mathbf{y}^L \in f(\mathbf{x}^L)$. Tarski's fixed-point theorem states that *an increasing function (correspondence) from a complete lattice to itself has a fixed point*.[9]

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6 '$\mathbf{x} \vee \mathbf{y}$' is read as '$\mathbf{x}$ *meet* $\mathbf{y}$', and '$\mathbf{x} \wedge \mathbf{y}$' is read as '$\mathbf{x}$ *join* $\mathbf{y}$'.

7 Topkis (1998), p. 29. He gives an example that clarifies the difference between a complete lattice and subcomplete sublattice. $S' = [0,1) \cup \{2\}$ is a complete lattice, and $\sup_{S'} [0,1) = \{2\} \in S'$, but S' is not a subcomplete sublattice of R since $\sup_R [0,1) = \{1\} \ne \sup_X [0,1)$.

8 Topkis (1998), Theorem 2.3.1.

9 See, for example, Topkis (1998), Theorem 2.5.1.

2.1.3 Supermodularity and monotone comparative statics

A real function $f(\mathbf{x})$ on a lattice is *supermodular* (*submodular*) if $f(\mathbf{x} \vee \mathbf{y}) + f(\mathbf{x} \wedge \mathbf{y}) \geq (\leq) f(\mathbf{x}) + f(\mathbf{y})$. A continuously twice differentiable function is supermodular if and only if $\frac{\partial^2 f}{\partial x_i \partial x_j} \geq 0$ for $i \neq j$.

A *log supermodular* or MTP_2 (multivariate totally positive of order 2) function is similarly defined for positive functions by $f(\mathbf{x} \vee \mathbf{y}) f(\mathbf{x} \wedge \mathbf{y}) \geq f(\mathbf{x}) f(\mathbf{y})$. Hence a necessary and sufficient condition is $f \frac{\partial^2 f}{\partial x_i \partial x_j} - \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} \geq 0$ for $i \neq j$ if f is continuously twice differentiable, and $\frac{\partial^2 \ln f}{\partial x_i \partial x_j} \geq 0$ if it is strictly positive in addition. A positive function on R^2 is TP_2 if it is MTP_2. A function $f(\mathbf{x}, \mathbf{y})$ on $R^n \times R^m$ *has increasing differences* in $\mathbf{x}$ and $\mathbf{y}$ if $\mathbf{x}^H > \mathbf{x}^L$ and $\mathbf{y}^H > \mathbf{y}^L$ implies $f(\mathbf{x}^H, \mathbf{y}^H) f(\mathbf{x}^L, \mathbf{y}^L) \geq f(\mathbf{x}^H, \mathbf{y}^L) f(\mathbf{x}^L, \mathbf{y}^H)$.

Thus assumption (iii), $D_i(p_i^H | \mathbf{p}_{-i}^H) D_i(p_i^L | \mathbf{p}_{-i}^L) \geq D_i(p_i^H | \mathbf{p}_{-i}^L) D_i(p_i^L | \mathbf{p}_{-i}^H)$, holds if $D_i(p_i | \mathbf{p}_{-i}) > 0$ and $\ln(D_i(p_i | \mathbf{p}_{-i}))$ has an increasing difference in p_i and $\mathbf{p}_{-i}$. Then it is equivalent to a weaker condition that $D_i(p_i, p_j | \mathbf{p}_{-i,j})$ is TP_2 in p_i and p_j for each $j \neq i$, given $\mathbf{p}_{-i,j}$.[10] It is weaker than log supermodularity or MTP_2, $D_i(\mathbf{p} \vee \mathbf{p}') D_i(\mathbf{p} \wedge \mathbf{p}') \geq D_i(\mathbf{p}) D_i(\mathbf{p}')$, in that it does not require log supermodularity in $\mathbf{p}_{-i}$.

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[10] The proof of this equivalence is similar to that of Theorem 3.2 of Topkis (1978), Theorem 2.6.2 (page 45) of Topkis (1998) and Proposition 2.1 of Karlin and Rinnot

Ordinal monotone comparative statics are based on quasi-supermodularity and the single crossing property, which generalize supermodularity and increasing differences. A function f on a lattice is *quasi-supermodular* if $f(\mathbf{x}) \geq f(\mathbf{x} \wedge \mathbf{y})$ implies $f(\mathbf{x} \vee \mathbf{y}) \geq f(\mathbf{x})$ and $f(\mathbf{x}) > f(\mathbf{x} \wedge \mathbf{y})$ implies $f(\mathbf{x} \vee \mathbf{y}) > f(\mathbf{x})$ A function f on a product of two lattices has *the single crossing property* if $f(\mathbf{x}^H, \mathbf{y}^L) \geq f(\mathbf{x}^L, \mathbf{y}^L)$ implies $f(\mathbf{x}^H, \mathbf{y}^H) \geq f(\mathbf{x}^L, \mathbf{y}^H)$ and $f(\mathbf{x}^H, \mathbf{y}^L) > f(\mathbf{x}^L, \mathbf{y}^L)$ implies $f(\mathbf{x}^H, \mathbf{y}^H) > f(\mathbf{x}^L, \mathbf{y}^H)$ for $\mathbf{x}^H \geq \mathbf{x}^L$ and $\mathbf{y}^H \geq \mathbf{y}^L$. If a function does not have the single crossing property, there are $\mathbf{x}^H > \mathbf{x}^L$ and $\mathbf{y}^H \geq \mathbf{y}^L$ such that $f(\mathbf{x}^H, \mathbf{y}^L) \geq f(\mathbf{x}^H, \mathbf{y}^H)$ and $f(\mathbf{x}^L, \mathbf{y}^H) \geq f(\mathbf{x}^H, \mathbf{y}^H)$ hold with one inequality being a strict inequality.[11] The sufficiency part of *Milgrom and Shannon's monotonicity theorem* states that *if* $f(\mathbf{x}, \mathbf{y})$ *is quasi-supermodular in* $\mathbf{x}$ *and*

(1980). Let $(x_1, x_2, ..., x_n) \geq (y_1, y_2, ..., y_n)$. If $f(x_1, x_2, ..., x_n)$ is TP$_2$ in x_1 and x_j, $j \neq 1$, we obtain

$$f(x_1, x_2, ..., x_n) f(y_1, y_2, ..., x_n) \geq f(y_1, x_2, ..., x_n) f(x_1, y_2, x_3 ..., x_n),$$

$$f(x_1, y_2, x_3, x_4, ..., x_n) f(y_1, y_2, y_3, x_4, ..., x_n) \geq f(y_1, y_2, x_3, x_4 ..., x_n) f(x_1, y_2, y_3, x_4 ..., x_n),$$

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$$f(x_1, y_2, ..., x_{n-1}, x_n) f(y_1, y_2, ..., y_{n-1}, x_n) \geq f(y_1, y_2, ..., x_{n-1}, x_n) f(x_1, y_2, ..., y_{n-1}, x_n),$$

$$f(x_1, y_2, ..., y_{n-1}, x_n) f(y_1, y_2, ..., y_{n-1}, y_n) \geq f(y_1, y_2, ..., y_{n-1}, x_n) f(x_1, y_2, ..., y_{n-1}, y_n).$$

Multiplying sides by sides, and canceling out (which requires that all terms are strictly positive), we obtain

$$f(x_1, x_2, ..., x_{n-1}, x_n) f(y_1, y_2, ..., y_{n-1}, y_n) \geq f(x_1, y_2, ..., y_{n-1}, y_n) f(y_1, x_2, ..., x_{n-1}, x_n).$$

11 If $\mathbf{x}^H = \mathbf{x}^L$, both hold with equality.

has a single crossing property in $\mathbf{x}$ *and* $\mathbf{y}$*, then the set of maximizers of* $f(\mathbf{x},\mathbf{y})$ *with respect to* $\mathbf{x}$ *is an increasing correspondence of* $\mathbf{y}$.[12]

2.3 Proof of the main theorem

This section formally proves the main existence theorem.

Theorem 1: Suppose that (i) throgh (iv) and either (v-a) or (v-b) hold. Then there is a unique price equilibrium.

The outline of the proof for the case in which (v-a) holds is illustrated in Figure 1, where the seven arrows correspond to the seven lemmas that we prove below. The case of (v-b) is noted in the proof of Lemma 3.

Lemma 1: If $\ln\left(D_i\left(p_i|\mathbf{p}_{-i}\right)\right)$ has increasing difference in p_i and $\mathbf{p}_{-i}$, $D_i\left(\mathbf{p}+k\mathbf{u}^n\right)=D_i(\mathbf{p})$ for all k, and $D_i\left(p_i|\mathbf{p}_{-i}\right)$ is strictly decreasing in p_i then $D_i\left(p_i|\mathbf{p}_{-i}\right)$ is log concave in p_i.[13]

Proof

[12] Milgrom and Shannon (1994), Theorem 4, p. 162. If monotonicity is required with respect to the product set of the parameters and the set of domain, these conditions are necessary and sufficient.

[13] Letting the inverse demand function be $P(x)$, log concavity of the demand function is equivalent to $P''(x)x+P'(x)\leq 0$ if it is twice differentiable. Although its connection with log concavity has not been widely recognized, this condition is a well-known stability condition for a Cournot equilibrium that guarantees a decreasing best response correspondence. Novshek (1985) proved the existence of a Cournot equilibrium under this condition. Although he added $D(0)<\infty$ as an independent condition, it is an implication of the assumption.

The first assumption implies

$$D_i\left(p_i^H \middle| \mathbf{p}_{-i} + k^H \mathbf{u}^{n-1}\right) D_i\left(p_i^L \middle| \mathbf{p}_{-i} + k^L \mathbf{u}^{n-1}\right) \geq D_i\left(p_i^H \middle| \mathbf{p}_{-i} + k^L \mathbf{u}^{n-1}\right) D_i\left(p_i^L \middle| \mathbf{p}_{-i} + k^H \mathbf{u}^{n-1}\right) \text{ for}$$

$p_i^H \geq p_i^L$ and $k^H \geq k^L$. Since $D_i(\mathbf{p} + k\mathbf{u}^n) = D_i(\mathbf{p})$, we obtain

$D_i\left(p_i^H - k^H \middle| \mathbf{p}_{-i}\right) D_i\left(p_i^L - k^L \middle| \mathbf{p}_{-i}\right) \geq D_i\left(p_i^H - k^L \middle| \mathbf{p}_{-i}\right) D_i\left(p_i^L - k^H \middle| \mathbf{p}_{-i}\right)$, so that $D_i\left(p_i - k \middle| \mathbf{p}_{-i}\right)$ is TP_2 in p_i and k. Since $D_i\left(p_i \middle| \mathbf{p}_{-i}\right)$ is decreasing in p_i, it is Lebesgue-measurable, so that it is log concave in p_i by the duality between log concave functions and TP_2 functions. QED.

Lemma 2 shows that there is a unique profit-maximizing price if the demand is log concave, as obtained in Lemma 1.

Lemma 2: If $D_i\left(p_i \middle| \mathbf{p}_{-i}\right)$ is strictly decreasing and log concave in p_i and $C_i(\bullet)$ is convex and increasing, then for each $\mathbf{p}_{-i}$, $\Pi_i\left(p_i \middle| \mathbf{p}_{-i}\right) = p_i D_i\left(p_i \middle| \mathbf{p}_{-i}\right) - C\left(D_i\left(p_i \middle| \mathbf{p}_{-i}\right)\right)$ is continuous and strictly quasi concave, and there is a unique p_i that maximizes $\Pi_i\left(p_i \middle| \mathbf{p}_{-i}\right)$.

Proof

In the proof of this lemma, we suppress $\mathbf{p}_{-i}$ and brand subscripts. Continuity is immediate, since convex functions and log concave functions are continuous.

If $D(p)$ is strictly positive and log concave in p, $D(p)^{-1}$ is strictly convex in p, because $D(\lambda p + (1-\lambda)p') \geq D(p)^{\lambda} D(p')^{1-\lambda} > \left(\lambda D(p)^{-1} + (1-\lambda)D(p')^{-1}\right)^{-1}$ for $D(p) \neq D(p')$ and $\lambda \in (0,1)$.[14]

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[14] Suppose $\lambda \in (0,1)$ and $X > 0$ and $Y > 0$, and let $\mathrm{M}(m) = \left(\lambda X^m + (1-\lambda)Y^m\right)^{1/m}$ for $m \neq 0$ and $\mathrm{M}(0) = X^{\lambda} Y^{1-\lambda}$. Then $m > n$ implies $\mathrm{M}(m) \geq \mathrm{M}(n)$ and the equality

Since $D(p)^{-1}$ is strictly increasing and convex, it has a strictly increasing concave inverse $\psi(x)$, and $pD(p)=\frac{\psi(x)}{x}$. Let $z=\frac{1}{x}=\frac{1}{D(p)^{-1}}=D(p)$, then $pD(p)=\frac{\psi(x)}{x}=z\psi\left(\frac{1}{z}\right)$, so that $\psi\left(\frac{1}{z}\right)$ is the inverse demand function. Since $\psi(x)$ is strictly increasing and strictly concave, $x\psi\left(\frac{1}{x}\right)$ is strictly concave.[15] Hence $\Pi=pD(p)-C(D(p))=z\psi\left(\frac{1}{z}\right)-C(z)$ is strictly concave for any convex $C(\bullet)$, as the sum of a strictly concave function and a concave function is strictly concave. Since $D(p)$

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holds only if $X=Y$. See, for example, Hardy, Littlewood, and Polya (1952), Theorem16, p. 26.

15 Take any $0<x<z$, $\lambda\in(0,1)$ and let $y=\lambda x+(1-\lambda)z$. Then ψ is strictly concave if and only if $\left[\frac{\psi(x^{-1})-\psi(y^{-1})}{x^{-1}-y^{-1}}\right]<\left[\frac{\psi(y^{-1})-\psi(z^{-1})}{y^{-1}-z^{-1}}\right]$. Multiplying both sides by $\lambda x(x^{-1}-y^{-1})=(1-\lambda)z(y^{-1}-z^{-1})>0$, we obtain $\lambda x(\psi(x^{-1})-\psi(y^{-1}))<(1-\lambda)z(\psi(y^{-1})-\psi(z^{-1}))$, which is equivalent to $\lambda x\psi(x^{-1})+(1-\lambda)z\psi(z^{-1})<y\psi(y^{-1})$, or the strict concavity of $x\psi(x^{-1})$. Note 12 of Caplin and Nalebuff (1991) gives a more intuitive proof.

is strictly decreasing, if Π is strictly concave in demand, it is strictly quasi-concave in price.[16]

Hence a maximizer is unique if it exists. Since $D(p)$ is strictly decreasing and log concave, $\lim_{p\to\infty} D(p) = \lim_{p\to\infty} pD(p) = 0$, so that $\lim_{p\to\infty} \Pi(p) = -C(0)$. Because $D(p)$ is strictly positive and strictly decreasing, and C is strictly increasing and convex, $\frac{C(D(p)) - C(0)}{D(p)}$ is strictly decreasing. Hence

$\Pi(p) + C(0) = D(p)\left(p - \frac{C(D(p)) - C(0)}{D(p)}\right) > 0$ for a sufficiently large p, so that we can take a p_0 such that $\Pi(p_0) + C(0) = \varepsilon > 0$. Since $\lim_{p\to\infty} \Pi(p) = -C(0)$, there is a $\bar{p}$ such that $\Pi(p_0) + C(0) = \varepsilon > \Pi(p) + C(0)$ for all $p > \bar{p}$. Since $\Pi(0) > \Pi(p)$ for $0 > p$, any maximizer of $\Pi(p)$ is in $[0, \bar{p}]$, and it exists since $\Pi(p)$ is continuous and $[0, \bar{p}]$ is compact. QED.

Lemma 2 establishes the existence of a best response function, $r_i(\mathbf{p}_{-i})$. Lemma 3 shows that $r_i(\mathbf{p}_{-i})$ is increasing with an additional assumption that all brands are gross substitutes or that their marginal cost is constant.

16 This proof follows the argument in the text of Caplin and Nalebuff (1991). Their formal proof for constant marginal cost (APPENDIX, p.53) cannot be extended easily to convex cost functions.

Lemma 3: Suppose that $D_i(p_i|\mathbf{p}_{-i})$ is strictly positive, strictly decreasing in p_i, $C_i(\bullet)$ is increasing and convex, and $\ln D_i(p_i|\mathbf{p}_{-i})$ has increasing differences. Then if $D_i(p_i|\mathbf{p}_{-i})$ is increasing in $\mathbf{p}_{-i}$ or $C_i(\bullet)$ is linear, $r_i(\mathbf{p}_{-i})$ is increasing.[17]

Proof

Since any function on the real line is quasi-supermodular, it is sufficient to show that $D_i(p_i|\mathbf{p}_{-i})$ has the single crossing property in order to apply Milgrom and Shannon's monotonicity theorem. If $D_i(p_i|\mathbf{p}_{-i})$ does not have the single crossing property, there are $\mathbf{p}^H{}_{-i} \geq \mathbf{p}^L{}_{-i}$ and $p^H{}_i > p^L{}_i$ such that

(1) $$p^L{}_i D_i(p^L{}_i|\mathbf{p}^H{}_{-i}) - C_i(D_i(p^L{}_i|\mathbf{p}^H{}_{-i})) \geq p^H{}_i D_i(p^H{}_i|\mathbf{p}^H{}_{-i}) - C_i(D_i(p^H{}_i|\mathbf{p}^H{}_{-i}))$$

and

(2) $$p^H{}_i D_i(p^H{}_i|\mathbf{p}^L{}_{-i}) - C_i(D_i(p^H{}_i|\mathbf{p}^L{}_{-i})) \geq p^L{}_i D_i(p^L{}_i|\mathbf{p}^L{}_{-i}) - C_i(D_i(p^L{}_i|\mathbf{p}^L{}_{-i}))$$

hold with one strict inequality.

By multiplying (1) by $D_i(p^L{}_i|\mathbf{p}^L{}_{-i}) - D_i(p^H{}_i|\mathbf{p}^L{}_{-i}) > 0$ and (2) by $D_i(p^L{}_i|\mathbf{p}^H{}_{-i}) - D_i(p^H{}_i|\mathbf{p}^H{}_{-i}) > 0$, and adding up, we obtain

$$(p^H{}_i - p^L{}_i)(D_i(p^L{}_i|\mathbf{p}^H{}_{-i})D_i(p^H{}_i|\mathbf{p}^L{}_{-i}) - D_i(p^H{}_i|\mathbf{p}^H{}_{-i})D_i(p^L{}_i|\mathbf{p}^L{}_{-i}))$$

$$> (C(D_i(p^L{}_i|\mathbf{p}^H{}_{-i})) - C(D_i(p^H{}_i|\mathbf{p}^H{}_{-i})))((D_i(p^L{}_i|\mathbf{p}^L{}_{-i}) - D_i(p^H{}_i|\mathbf{p}^L{}_{-i})))$$

$$-(C(D_i(p^L{}_i|\mathbf{p}^L{}_{-i})) - C(D_i(p^H{}_i|\mathbf{p}^L{}_{-i})))(D_i(p^L{}_i|\mathbf{p}^H{}_{-i}) - D_i(p^H{}_i|\mathbf{p}^H{}_{-i}))$$

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[17] Milgrom and Shannon (1994), (p. 168) uses the relation between the single crossing property and the Spence-Mirrlees condition, while the following proof is direct, but the two proofs are essentially the same.

The left hand side is negative because $p^H{}_i > p^L{}_i$ and log supermodularity of demand implies that $D_i\left(p^H{}_i \middle| \mathbf{p}^H{}_{-i}\right)D_i\left(p^L{}_i \middle| \mathbf{p}^L{}_{-i}\right) \geq D_i\left(p^L{}_i \middle| \mathbf{p}^H{}_{-i}\right)D_i\left(p^H{}_i \middle| \mathbf{p}^L{}_{-i}\right)$. The right-hand side is positive, because if all brands are gross substitutes,

$D_i\left(p^L{}_i \middle| \mathbf{p}^H{}_{-i}\right) \geq D_i\left(p^L{}_i \middle| \mathbf{p}^L{}_{-i}\right), D_i\left(p^H{}_i \middle| \mathbf{p}^H{}_{-i}\right) \geq D_i\left(p^H{}_i \middle| \mathbf{p}^L{}_{-i}\right)$, and since $C(\bullet)$ is convex,

$$\frac{C\left(D_i\left(p^L{}_i \middle| \mathbf{p}^H{}_{-i}\right)\right) - C\left(D_i\left(p^H{}_i \middle| \mathbf{p}^H{}_{-i}\right)\right)}{D_i\left(p^L{}_i \middle| \mathbf{p}^H{}_{-i}\right) - D_i\left(p^H{}_i \middle| \mathbf{p}^H{}_{-i}\right)} \geq \frac{C\left(D_i\left(p^L{}_i \middle| \mathbf{p}^L{}_{-i}\right)\right) - C\left(D_i\left(p^H{}_i \middle| \mathbf{p}^L{}_{-i}\right)\right)}{D_i\left(p^L{}_i \middle| \mathbf{p}^L{}_{-i}\right) - D_i\left(p^H{}_i \middle| \mathbf{p}^L{}_{-i}\right)}.$$

Note that if cost functions are linear, the right-hand side is zero for all demand functions, so that we can omit the assumption of gross substitutes. QED

Because $r_i(\mathbf{p}_{-i})$ is increasing for each i, by Tarski's fixed-point theorem, the mapping $\mathbf{r}(\mathbf{p}) = \left(r_1(\mathbf{p}_{-1}), \ldots.., r_n(\mathbf{p}_{-n})\right)$ from R^n to itself has a fixed point if the domain and region of the mapping can be restricted to the same complete lattice of R^n. We will construct an upper bound for positive and strictly decreasing demand functions.

Lemma 4: Suppose that $\Pi_i\left(p_i \middle| \mathbf{p}_{-i}\right)$ is strictly quasi-concave, $D'^{+}_i\left(p_i \middle| \mathbf{p}_{-i}\right) < 0$ and $C_i'^{-}(x)$ exist and $D_i\left(\mathbf{p} + k\mathbf{u}^n\right) = D_i(\mathbf{p})$ for all k and $r_i(\mathbf{p}_{-i})$ is increasing, then there is a q_i^0 such that $q_i \geq q_i^0$ implies that $r_i(\mathbf{p}_{-i}) \leq q_i$ for all $q_i\mathbf{u}_{n-1} \geq \mathbf{p}_{-i} \geq 0$.

Proof

Because the profit function is strictly quasi-concave,

$\Pi'^{+}\left(p_i \middle| \mathbf{p}_{-i}\right) = D_i\left(p_i \middle| \mathbf{p}_{-i}\right) + \left(p_i - C^{-}\left(D\left(p_i \middle| \mathbf{p}_{-i}\right)\right)\right)D_i'^{+}\left(p_i \middle| \mathbf{p}_{-i}\right) < 0$

implies $p_i > r_i(\mathbf{p}_{-i})$. Since $D_i(p_i|\mathbf{p}_{-i})$ depends only on differences in prices, $D_i(q_i|q_i\mathbf{u}^{n-1})$, $D_i(q_i + \Delta|q_i\mathbf{u}^{n-1})$ and hence $D_i'^{+}(q_i|q_i\mathbf{u}^{n-1})$, and $C'^{-}(D(q_i|q_i\mathbf{u}^{n-1}))$ do not depend on q_i. Since $D_i'^{+}(q_i|q_i\mathbf{u}^{n-1}) < 0$, if we take a sufficiently large q_i^0 , $q \geq q_i^0$ implies

$$\Pi'^{+}(q_i|q\mathbf{u}^{n-1}) = D_i(q_i|q_i\mathbf{u}^{n-1}) + (q_i - C'^{-}(D(q_i|q_i\mathbf{u}^{n-1})))D_i'^{+}(q_i|q_i\mathbf{u}^{n-1}) < 0,$$

so that $q_i > r_i(q_i\mathbf{u}^{n-1})$. Since $r_i(\mathbf{p}_{-i})$ is increasing, then $q_i > r_i(q_i\mathbf{u}^{n-1}) \geq r_i(\mathbf{p}_{-i})$ for $q_i\mathbf{u}^{n-1} \geq \mathbf{p}_{-i}$. QED.

Now we can prove the existence of a price equilibrium.

Lemma 5: For each $i = 1,...,n$ suppose that (i) $D_i(p_i|\mathbf{p}_{-i})$ is strictly positive and strictly decreasing in p_i, (ii) $\ln(D_i(p_i|\mathbf{p}_{-i}))$ has an increasing difference in p_i and $\mathbf{p}_{-i}$, and (iii) $D_i(\mathbf{p} + k\mathbf{u}^n) = D_i(\mathbf{p})$ for all k , and (iv) C_i is convex and increasing. Then if (v-a) $D_i(p_i|\mathbf{p}_{-i})$ is increasing in $\mathbf{p}_{-i}$ or (v-b) $C_i(x_i) = c_i x_i$, there is a price equilibrium.

Proof

Lemmas 1, 2 and 3 imply that there is an increasing mapping, $\mathbf{r}(\mathbf{p}) = (r_1(\mathbf{p}_{-1}),.....,r_n(\mathbf{p}_{-n}))$, from R^n to R^{n+}. Since C_i is convex and $D_i(p_i|\mathbf{p}_{-i})$ is log concave by Lemma 1, then it has right and left derivatives so the assumptions of lemma 4 are satisfied. If we take $\bar{q} > \max(q_1^0,...,q_n^0)$, where q_i^0 is defined as in Lemma 4, $\mathbf{r}(\mathbf{p})$ maps $\times_{i=1}^{n}[0,\bar{q}]^n$ to itself. Since $[0,\bar{q}]^n$ is a compact sublattice of R^n, it is a subcomplete sublattice of R^n, and hence a complete sublattice. Thus a fixed point that satisfies $\mathbf{p}_0 = \mathbf{r}(\mathbf{p}_0)$ exists in $[0,\bar{q}]^n$. Hence a price equilibrium exists. QED.

To prove uniqueness, we show that a firm raises its price by strictly less than one dollar when all other firms raise their prices by one dollar.

Lemma 6: If $D_i(p_i|\mathbf{p}_{-i})$ is strictly decreasing in p_i, $D_i(\mathbf{p}+k\mathbf{u}^n)=D_i(\mathbf{p})$, and the best response function $r_i(\mathbf{p}_{-i})$ is single valued for each i, $r_i(\mathbf{p}_{-i}+k\mathbf{u}^{n-1})<r_i(\mathbf{p}_{-i})+k$ for $k>0$.

Proof

Since the profit maximization problem has a unique solution and $D_i(\mathbf{p}+k\mathbf{u}^n)=D_i(\mathbf{p})$, for $k>0$ we obtain

$$r_i(\mathbf{p}_{-i})D_i(r_i(\mathbf{p}_{-i})|\,\mathbf{p}_{-i})-C_i(D_i(r_i(\mathbf{p}_{-i})|\,\mathbf{p}_{-i}))$$

$$>\{r_i(\mathbf{p}_{-i}+k\mathbf{u}^{n-1})-k\}D_i(r_i(\mathbf{p}_{-i}+k\mathbf{u}^{n-1})-k|\,\mathbf{p}_{-i})-C_i(D_i(r_i(\mathbf{p}_{-i}+k\mathbf{u}^{n-1})-k|\,\mathbf{p}_{-i}))$$

$$=\{r_i(\mathbf{p}_{-i}+k\mathbf{u}^{n-1})-k\}D_i(r_i(\mathbf{p}_{-i}+k\mathbf{u}^{n-1})|\,\mathbf{p}_{-i}+k\mathbf{u}^{n-1})-C_i(D_i(r_i(\mathbf{p}_{-i}+k\mathbf{u}^{n-1})|\,\mathbf{p}_{-i}+k\mathbf{u}^{n-1}))$$

and

$$r_i(\mathbf{p}_{-i}+k\mathbf{u}^{n-1})D_i(r_i(\mathbf{p}_{-i}+k\mathbf{u}^{n-1})\,|\mathbf{p}_{-i}+k\mathbf{u}^{n-1})-C_i(D_i(r_i(\mathbf{p}_{-i}+k\mathbf{u}^{n-1})\,|\mathbf{p}_{-i}+k\mathbf{u}^{n-1}))$$

$$>\{r_i(\mathbf{p}_{-i})+k\}D_i(r_i(\mathbf{p}_{-i})+k|\mathbf{p}_{-i}+k\mathbf{u}^{n-1})-C_i(D_i(r_i(\mathbf{p}_{-i})+k|\mathbf{p}_{-i}+k\mathbf{u}^{n-1}))$$

$$=\{r_i(\mathbf{p}_{-i})+k\}D_i(r_i(\mathbf{p}_{-i})\,|\mathbf{p}_{-i})-C_i(D_i(r_i(\mathbf{p}_{-i})\,|\mathbf{p}_{-i}))$$

Adding up we obtain

$$0<k\{D_i(r_i(\mathbf{p}_{-i}+k\mathbf{u}^{n-1})\,|\mathbf{p}_{-i}+k\mathbf{u}^{n-1})-D_i(r_i(\mathbf{p}_{-i})\,|\mathbf{p}_{-i})\}$$

$$=k\{D_i(r_i(\mathbf{p}_{-i}+k\mathbf{u}^{n-1})-k|\mathbf{p}_{-i})-D_i(r_i(\mathbf{p}_{-i})\,|\mathbf{p}_{-i})\}$$

Since the demand function is strictly decreasing,

$D_i(r_i(\mathbf{p}_{-i}+k\mathbf{u}^{n-1})-k|\mathbf{p}_{-i})-D_i(r_i(\mathbf{p}_{-i})|\mathbf{p}_{-i})>0$ implies $r_i(\mathbf{p}_{-i}+k\mathbf{u}^{n-1})-k<r_i(\mathbf{p}_{-i})$. QED.

The uniqueness of the Bertrand equilibrium follows almost immediately from Lemma 6.

Lemma 7: If the best response function $r_i(\mathbf{p}_{-i})$ is increasing and single valued for each i and $r_i(\mathbf{p}_{-i} + k\mathbf{u}^{n-1}) < r_i(\mathbf{p}_{-i}) + k$ for $k > 0$, the price equilibrium is unique.

Proof

For two price vectors, $\mathbf{p} = (p_1, ..., p_n)$ and $\mathbf{q} = (q_1, ..., q_n)$, define $\|\mathbf{p} - \mathbf{q}\|$ by $\|\mathbf{p} - \mathbf{q}\| = \max_{j=1,...n} |p_j - q_j|$. If the equilibrium is not unique, $\mathbf{r}(\mathbf{p})$ has two fixed points and $\mathbf{p} \neq \mathbf{q}$. Let $m = \|\mathbf{p} - \mathbf{q}\| > 0$. Then from the definition of lattice operation,

$\mathbf{p}_{-i} \wedge \mathbf{q}_{-i} \leq \mathbf{p}_{-i}, \mathbf{q}_{-i} \leq \mathbf{p}_{-i} \vee \mathbf{q}_{-i} \leq (\mathbf{p}_{-i} \wedge \mathbf{q}_{-i}) + m\mathbf{u}^{n-1}$.

Since $r_i(\mathbf{p}_{-i} + m\mathbf{u}^{n-1}) < r_i(\mathbf{p}_{-i}) + m$ and $r_i(\mathbf{p}_{-i})$ are increasing, this inequality implies $|r_i(\mathbf{p}_{-i}) - r_i(\mathbf{q}_{-i})| \leq r_i((\mathbf{p}_{-i} \wedge \mathbf{q}_{-i}) + m\mathbf{u}^{n-1}) - r_i((\mathbf{p}_{-i} \wedge \mathbf{q}_{-i})) < m$. Hence $m > \|\mathbf{r}(\mathbf{p}) - \mathbf{r}(\mathbf{q})\|$. The last inequality contradicts the assumption that both $\mathbf{p}$ and $\mathbf{q}$ are fixed points, since $\|\mathbf{r}(\mathbf{p}) - \mathbf{r}(\mathbf{q})\| = \|\mathbf{p} - \mathbf{q}\| = m$ must hold. [18] QED.

These seven lemmas establish Theorem 1.

2.4 Extensions

This subsection provides some extensions to Theorem 1.

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[18] Both Milgrom and Roberts (1990) and Caplin and Nalebuff (1991) use the dominant diagonal property of a Hessian matrix of profit or log profit functions. Although their method may be extended to somewhat more general models, our method does not require twice differentiability.

2.4.1 Outside goods

Theorem 1 can be extended to the case in which price competition is restricted to a subset of brands and the prices of other brands do not change. To show this, note the following two points. First, since the best response functions are increasing, an upper bound can be established in Lemma 4 such that it is greater than the prices of these brands, and the same upper bound can be applied if the prices are reduced to the given prices. Second, Lemma 6 is robust to the existence of such brands if best response functions are increasing, because the changes in the value of the best response functions are smaller if the prices of a subset of brands do not change. Hence we have the following corollary.

Corollary 1: Suppose that the conditions of Theorem 1 are satisfied, and only $n_0 \geq 2$ firms set the prices of their brands to maximize profit while other prices are fixed. Then there is a unique price equilibrium.

A useful application of this result is a discrete choice model in which each consumer does not buy a brand if the gain from trade is not positive. This is equivalent to an outside good whose reservation price is zero.

2.4.2 General class of increasing differences

Lemma 1 can be extended to the relation between a general class of concave functions and a corresponding class of increasing difference functions. A function $f(x)$ is ρ concave if $f(x)^{\rho}$ is concave for $\rho > 0$, and convex for $\rho < 0$. By continuity, zero concavity is log concavity. A ρ concave function is ρ' concave if $\rho > \rho'$.[19] The theorem

[19] See footnote14.

of Caplin and Nalebuff (1991) provides the basis for the proof of Lemma 2. This proof shows that if the cost function is convex, -1 concavity of $D_i(p_i)$ is sufficient for the quasi-concavity of the profit function.

We can define ρ-increasing differences as follows. The function $f(\mathbf{x},\mathbf{y})$ has ρ-increasing differences if $f(\mathbf{x},\mathbf{y})^{\rho}$ has increasing differences for $\rho > 0$, decreasing differences for $\rho < 0$, and $\ln f(\mathbf{x},\mathbf{y})$ has increasing differences for $\rho = 0$. Generally, ρ-increasing differences do not imply ρ'-increasing differences for $\rho > \rho'$, but we can show that this is true if conditions (i) and (v-a) are satisfied.[20] Thus Lemma 1 and Lemma 2 can be extended to ρ-increasing differences for $\rho > -1$; hence we can show that a single-valued best correspondence exists for each firm. However, zero increasing differences are required for an increasing best response function, and Lemma 4 and Lemma 6 depend

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20 Then $D_i\left(p_i^H \middle| \mathbf{p}_{-i}^L\right) = X \le D_i\left(p_i^H \middle| \mathbf{p}_{-i}^H\right) = Y, D_i\left(p_i^L \middle| \mathbf{p}_{-i}^L\right) = Z \le D_i\left(p_i^L \middle| \mathbf{p}_{-i}^H\right) = U$. First consider the case $\rho \ge \rho' > 0$, and let $X^{\rho} + U^{\rho} \le Y^{\rho} + Z^{\rho}$. Then there are $\theta, \delta \in [0,1]$, $Y^{\rho} = (1-\theta)X^{\rho} + \theta U^{\rho}$, $Z^{\rho} = (1-\delta)X^{\rho} + \delta U^{\rho}$ and $\theta + \delta \ge 1$. Since $x^{\frac{\rho'}{\rho}}$ is concave, we obtain

$$X^{\rho'} + U^{\rho'} = X^{\rho\frac{\rho'}{\rho}} + U^{\rho\frac{\rho'}{\rho}} \le \left((1-\theta)X^{\rho} + \theta U^{\rho}\right)^{\frac{\rho'}{\rho}} + \left(\theta X^{\rho} + (1-\theta)U^{\rho}\right)^{\frac{\rho'}{\rho}}$$
$$\le \left((1-\theta)X^{\rho} + \theta U^{\rho}\right)^{\frac{\rho'}{\rho}} + \left((1-\delta)X^{\rho} + \delta U^{\rho}\right)^{\frac{\rho'}{\rho}} = Y^{\rho'} + Z^{\rho'}.$$

We can show the case for $0 > \rho \ge \rho'$ similarly, and the general result follows by continuity.

on this condition. Hence the extension in this direction would require an alternative method to establish the existence and the uniqueness of a price equilibrium.

2.4.3 Representative consumer models

Another extension involves the transformation of independent variables. Unlike quasi-supermodularity or the single crossing property, TP_2 is not ordinal in that even if $f(x,y)$ is TP_2 and ϕ is strictly increasing, $\phi(f(x,y))$ may not be TP_2. But *TP_2 is ordinal in independent variables* ; if $f(x,y)$ is TP_2 and ϕ is strictly increasing or strictly decreasing, then $f(\phi(x),\phi(y))$ is TP_2. Hence if the demand function for brand i depends on $\phi(p_i)-\phi(p_j)$, $j \neq i$, where ϕ is strictly increasing, and it is TP_2 in its own price and the price of each other brand, then it is log concave in $\phi(p_i)$.

We apply this property, together with Corollary 1, to representative consumer models. A representative consumer model assumes that the demand for each brand of a differentiated product is derived from the utility maximization of a representative consumer who is subject to a budget constraint. Suppose that there are n brands of a differentiated product and m other consumer goods. Let $\mathbf{p}$ and $\mathbf{q}$ be the price vectors of the differentiated product and other consumer goods, let Y be the income of the representative consumer, and let $u(\mathbf{x},\mathbf{y})$ be the utility function of a representative consumer. If the utility maximization problem has a unique solution for each $(\mathbf{p},\mathbf{q},Y)>0$, the demand functions are homogeneous of degree zero in $(\mathbf{p},\mathbf{q},Y)$. Hence they depend on the differences in $(\ln p_1,\ldots,\ln p_n,\ln q_1,\ldots,\ln q_m,\ln Y)$. Hence if $\ln D_i(p_i|\mathbf{p}_{-i},\mathbf{q},Y)$ is TP_2 in $\ln p_i$ and $\ln p_j$; $j \neq i$, TP_2 in $\ln p_i$ and $\ln q_k$, $k=1,\ldots,m$; and TP_2 in $\ln p_i$ and $\ln Y$ for each i; then it is log concave in $\ln p_i$. Since TP_2 is ordinal in independent variables, this condition is equivalent to the condition that $D_i\left(p_i \middle| \mathbf{p}_{-i}, \mathbf{q}, Y\right)$ is TP_2 in p_i and p_j, $j \neq i$, q_k, $k = 1,...,m$ and Y. This result corresponds to Lemma 1

Lemma 1': If $D_i\left(p_i \middle| \mathbf{p}_{-i}, \mathbf{q}, Y\right)$ has increasing differences in p_i and $\left(\mathbf{p}_{-i}, \mathbf{q}, Y\right)$ and is homogeneous of degree zero in $\left(\mathbf{p}, \mathbf{q}, Y\right)$, and $D_i\left(p_i \middle| \mathbf{p}_{-i}\right)$ is strictly decreasing in p_i then $\ln D_i\left(p_i \middle| \mathbf{p}_{-i}, \mathbf{q}, Y\right)$ is concave in $\ln p_i$.

Unlike the case of log concave demand, even if $\ln D(p)$ is concave in $\ln p$ and the cost function is increasing and convex, additional assumptions are required to obtain a single valued best response function. First, if $\ln p$ is to be defined, p must be strictly positive. Hence the lower bound of lemma 2 must be changed to $C'^{+}(0) > 0$, where it can be verified that charging a price strictly lower than $C'^{+}(0)$ is not profitable.[21] Second $\lim_{p \to \infty} pD(p) = 0$ may not hold; for example, when $D(p) = p^{-\theta}$ with $1 \geq \theta > 0$. Since we can prove strict quasi-concavity, we have a counterpart to Lemma 2.

Lemma 2': Suppose $D_i\left(p_i \middle| \mathbf{p}_{-i}\right)$ is strictly decreasing; $\lim_{p_i \to \infty} p_i D_i\left(p_i \middle| \mathbf{p}_{-i}\right) = 0$; $\ln D_i\left(p_i \middle| \mathbf{p}_{-i}, \mathbf{q}, Y\right)$ is concave in $\ln p_i$; and $C_i(x_i)$ is convex and increasing and

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21 Let $\underline{p} = C'^{+}(0)$ and $p < \underline{p}$. Then the convexity of the cost function and strictly decreasing demand imply

$$\left(pD(p) - C(D(p))\right) - \left(\underline{p}D(\underline{p}) - C(D(\underline{p}))\right) = \int_{D(\underline{p})}^{D(p)} \left(p - C_i'^{+}(x)\right) dx + \left(p - \underline{p}\right) D(\underline{p})$$

$$\leq \int_{D(\underline{p})}^{D(p)} \left(p - C_i'^{+}(0)\right) dx + \left(p - \underline{p}\right) D(\underline{p}) < 0.$$

$C_i'^{+}(0) > 0$. Then for each $\mathbf{p}_{-i}$, $\Pi_i\left(p_i \middle| \mathbf{p}_{-i}, \mathbf{q}, Y\right) = p_i D_i\left(p_i \middle| \mathbf{p}_{-i}, \mathbf{q}, Y\right) - C\left(D_i\left(p_i \middle| \mathbf{p}_{-i}, \mathbf{q}, Y\right)\right)$ is continuous and strictly quasi concave, and there is a unique p_i that maximizes

$\Pi_i\left(p_i \middle| \mathbf{p}_{-i}, \mathbf{q}, Y\right)$.

Proof

It remains to be shown that $\Pi = pD(p) - C(D(p))$ is strictly quasi-concave. Since $\ln D(p)$ is strictly decreasing and concave in $\ln p$, $\ln P(x)$ is strictly decreasing and concave in $\ln(x)$, where $P(x)$ is the inverse demand function of $D(p)$ and $\ln P(x)$ has a right derivative with respect to $\ln(x)$, and hence with respect to x. Then

$\dfrac{d^+\Pi}{dx} = P(x)\left\{1 + \dfrac{xP'^{+}(x)}{P(x)} - \dfrac{C'^{+}(x)}{P(x)}\right\} = P(x)\left\{1 + \dfrac{d^+ \ln P(x)}{d\ln(x)} - \dfrac{C'^{+}(x)}{P(x)}\right\}$. Since $\ln P(x)$ is strictly concave in $\ln(x)$, $\dfrac{d^+ \ln P(x)}{d\ln(x)}$ is strictly decreasing in x, and since $C'^{+}(x)$ is increasing by the convexity of C and $P(x)$ is strictly positive and strictly decreasing, $\dfrac{C'^{+}(x)}{P(x)}$ is strictly increasing. Hence $\left\{1 + \dfrac{d^+ \ln P(x)}{d\ln(x)} - \dfrac{C'^{+}(x)}{P(x)}\right\}$ is strictly decreasing and it can change sign at most once from plus to minus at a unique x, so that Π is strictly quasi-concave. Since the demand function is strictly decreasing if it is strictly quasi-concave in x, it is strictly quasi-concave in p. QED.[22]

The remaining Lemmas, Lemma 3 to Lemma 7, are robust to strictly increasing transformations of variables. Hence from Corollary 1, we obtain the following theorem.

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[22] Caplin and Nalebuff (1991) consider demand functions that are log concave in $\ln p$ and obtain this result for strictly positive and constant marginal costs.

Theorem 1': Suppose that (i) $D_i\left(p_i \middle| \mathbf{p}_{-i}, \mathbf{q}, Y\right)$ is strictly positive and strictly decreasing in p_i, (ii) $D_i(\mathbf{p}, \mathbf{q}, Y)$ is homogeneous of degree zero in $(\mathbf{p}, \mathbf{q}, Y)$ (iii) $\ln D_i\left(p_i \middle| \mathbf{p}_{-i}, \mathbf{q}, Y\right)$ has increasing differences in p_i and $(\mathbf{p}_{-i}, \mathbf{q}, Y)$ and (iv) $C_i(x_i)$ is convex. Suppose also that $\lim_{p_i \to \infty} p_i D_i\left(p_i \middle| \mathbf{p}_{-i}, \mathbf{q}, Y\right) = 0$ and $C_i'^{+}(0) > 0$. Then if (v-a) $D_i\left(p_i \middle| \mathbf{p}_{-i}, \mathbf{q}, Y\right)$ is increasing in $(\mathbf{p}_{-i}, \mathbf{q}, Y)$ on R^n or (v-b) $C_i(x_i) = c_i x_i$, there is a unique price equilibrium for a given $\mathbf{q}$.

2.4.4 Uniqueness on a lattice

Strictly positive and strictly decreasing demand functions imply that competition is global, but Lemma 6 and Lemma 7 on uniqueness can be extended to localized markets. If the relevant conditions hold on a lattice in R^n, the equilibrium is unique on the lattice if it exists. Usually, the lattice is a competitive region with a given number of active firms.

Corollary 2: If $S = \left\{\mathbf{p} \middle| D_i(p) > 0, i = 1, \ldots, n\right\}$ is a lattice, $D_i\left(p_i \middle| \mathbf{p}_{-i}\right)$ is strictly positive and strictly decreasing in p_i on S, and the other assumptions of Theorem 1 hold, then a price equilibrium is unique on S.

Corollary 2 does not exclude the possibility that there is no equilibrium or there may be other equilibrium with some other subset of active firms. We shall apply this result to linear spatial models.

3. Applications and Examples

This section applies the results of the previous section to discrete choice models, spatial models, and representative consumer models. Many of the applications and examples of this section extend and generalize Section 6 and Section 8 of Caplin and

Nalebuff (1991), which prove the uniqueness of an equilibrium for duopoly models and uni-dimensional models of discrete choice with log concave density of tastes, logit models, and duopoly representative consumer models with CES utility functions . The results of this section are directly applicable to models used in industrial organization and other branches of microeconomics.

3.1 Linear random utility model.

A simple discrete choice model assumes that a consumer purchases brand i if and only if $v_i - p_i \geq v_j - p_j$ for all $j \neq i$, where $v_k (k = 1,.., n)$ is the reservation price of each brand. Anderson, de Palma and Thisse (1992) named this model a linear random utility model, since utility is linear both in the reservation price and in prices, and the reservation prices of each consumer are randomly distributed. We assume that the distribution function of these reservation prices has a density function $f(\mathbf{v})$. Then $D_i(\mathbf{p})$ is the integral of $f(\mathbf{v})$ over $A_i(\mathbf{p}) = \{\mathbf{v} | v_i - p_i \geq v_j - p_j, j \neq i\}$, $D_i(\mathbf{p}) = \int_{A_i(\mathbf{p})} f(\mathbf{v}) d\mathbf{v}$. All brands are gross substitutes in this class of model, so that Theorem 1 holds with convex cost functions. The density function $f(\mathbf{v})$ must be strictly positive on an unbounded domain for $D_i(\mathbf{p})$ to be strictly positive and strictly decreasing on R^n. A sufficient condition is that $f(\mathbf{v})$ is strictly positive on an unbounded rectangular domain, $(L, \infty)^n$, for some $L \in [-\infty, \infty)$. Most discrete choice models with unbounded support satisfy this assumption, so we assume that this condition holds.

Caplin and Nalebuff (1991) use the Prékopa-Borel theorem to obtain a sufficient condition for $D_i(\mathbf{p})$ to be -1 concave, which is sufficient for the quasi-concavity of the profit function. To present the Prékopa-Borel theorem, let f be a density function with a convex support and let A and B be non-empty subsets of the support. Then for $\lambda \in (0,1)$, $\lambda A + (1-\lambda)B$ is defined by $\lambda A + (1-\lambda)B = \{\mathbf{x} | \exists \mathbf{a} \in A, \ \mathbf{b} \in \mathbf{B}, \mathbf{x}, = \lambda \mathbf{a} + (1-\lambda)\mathbf{b}\}$.[23]

The *Prékopa-Borel theorem* states that if f is ρ-concave, then

$$\int_{\lambda A+(1-\lambda)B} f(\mathbf{v})d\mathbf{v} \geq \left\{ \left(\int_A f(\mathbf{v})d\mathbf{v}\right)^{\frac{\rho}{1+n\rho}} + \left(\int_B f(\mathbf{v})d\mathbf{v}\right)^{\frac{\rho}{1+n\rho}} \right\}^{\frac{1+n\rho}{\rho}} \text{ for } \rho \geq -\frac{1}{n}.$$ Since

$A_i(\lambda \mathbf{p} + (1-\lambda)\mathbf{q}) = \lambda A_i(\mathbf{p}) + (1-\lambda)A_i(\mathbf{q})$, if f is ρ-concave, then $D_i(\mathbf{p})$ is $\frac{\rho}{1+n\rho}$ concave. Hence a sufficient condition for $D_i(\mathbf{p})$ to be –1 concave is that f is $-\frac{1}{1+n}$ concave. In particular, if f is log concave, then $D_i(\mathbf{p})$ is also log concave.

An immediate consequence of the Prékopa-Borel theorem for our framework is that a duopoly model has a unique equilibrium if the density function of reservation prices is log concave and strictly positive. This result marginally extends Proposition 6 of

23 If $A, B \subset R^2$ and $A = \{(x,y) | x = 0, 0 \leq y \leq 2\}$ and $B = \{(x,y) | 0 \leq x \leq 2, y = 0\}$, $\frac{1}{2}A + \frac{1}{2}B = \{(x,y) | 0 \leq x \leq 1, 0 \leq y \leq 1\}$, since $(x,y) = \frac{1}{2}(0,2y) + \frac{1}{2}(2x,0)$. Although both A and B have Lebesgue measure zero, $\frac{1}{2}A + \frac{1}{2}B$ has Lebesgue measure one. This illustrates that $\frac{1}{2}A + \frac{1}{2}B$ is not the set in the middle of A and B, but may be a more bulky set. This is an example of the Prékopa-Borel theorem.

Caplin and Nalebuff (1991), in that no upper bound is assumed and marginal costs can be strictly convex.

Proposition 1: If $n = 2$ and f is log concave on R^2, then $D_i(\mathbf{p}) = \int_{A_i(\mathbf{p})} f(\mathbf{v})d\mathbf{v}$ is TP_2 in p_i and p_j, $j \neq i$. If f is strictly positive on $(L,\infty)^2$ for some $L \in [-\infty,\infty)$, a unique price equilibrium exists if cost functions are convex and marginal costs are positive.

Proof

If f is log concave, $D_i(\mathbf{p})$ is log concave by the Prékopa-Borel theorem. Since $D_i(\mathbf{p})$ depends only on the difference between prices we can write, $D_i(\mathbf{p}) = g(p_j - p_j)$, where the right-hand side is log concave. By the duality between log concave functions and TP_2 functions, the right-hand side is TP_2 in p_i and p_j. QED.

In Proposition 2, we show that if the marginal distributions of all reservation prices are independent and their density functions are log concave, their demand functions are log supermodular (MTP_2). This conclusion is stronger than the condition that the demand function of each brand is TP_2 in own price and another price. Although this assumption is more restrictive than that of Caplin and Nalebuff (1991), the proposition proves uniqueness as well and existence, and it can be directly applied to many models that do not have an upper bound on prices, such as the logit, probit and exponential models.

To prove this proposition, we use a theorem by Karlin and Rinnot (1980) on the aggregation of MTP_2 functions, which is somewhat similar to the Prékopa-Borel theorem.

They show that *if* $g_1(\mathbf{x} \vee \mathbf{y})g_2(\mathbf{x} \wedge \mathbf{y}) \geq g_3(\mathbf{x})g_4(\mathbf{y})$, *where* g_i $(i = 1, \ldots, 4)$ *are positive functions on* R^n, *then* $\left\{\int g_1(\mathbf{x})d\mathbf{x}\right\}\left\{\int g_2(\mathbf{x})d\mathbf{x}\right\} \geq \left\{\int g_3(\mathbf{x})d\mathbf{x}\right\}\left\{\int g_4(\mathbf{x})d\mathbf{x}\right\}$.[24]

Proposition 2: If the marginal distributions of all reservation prices are independent and their density functions are log concave, $D_i(\mathbf{p} \vee \mathbf{q})D_i(\mathbf{p} \wedge \mathbf{q}) \geq D_i(\mathbf{q})D_i(\mathbf{p})$. If, in addition, $f(\mathbf{v})$ is strictly positive on $(L, \infty)^n$ for some $L \in [-\infty, \infty)$, a unique price equilibrium exists if cost functions are convex and marginal costs are positive.

Proof

Remember that $D_i(\mathbf{p}) = \int_{A_i(\mathbf{p})} f(\mathbf{v})d\mathbf{v}$, where $A_i(\mathbf{p}) = \left\{\mathbf{v} \middle| v_i - p_i \geq v_j - p_j, j \neq i\right\}$. By putting $\mathbf{x} = (x_1, \ldots, x_n) = \mathbf{v} - \mathbf{p}$, and defining $I_i(\mathbf{x}) = \begin{cases} 1 & x_i \geq x_j, j = 1, \ldots, n \\ 0 & \textit{otherwise} \end{cases}$, we obtain $D_i(\mathbf{p}) = \int_{A_i(\mathbf{0})} f(\mathbf{x} + \mathbf{p})d\mathbf{x} = \int f(\mathbf{x} + \mathbf{p})I_i(\mathbf{x})d\mathbf{x}$. We will apply the Karlin and Rinnot theorem to $g_1(\mathbf{x}) = f(\mathbf{x} + \mathbf{p} \wedge \mathbf{q})I_i(\mathbf{x})$, $g_2(\mathbf{x}) = f(\mathbf{x} + \mathbf{p} \vee \mathbf{q})I_i(\mathbf{x})$, $g_3(\mathbf{x}) = f(\mathbf{x} + \mathbf{p})I_i(\mathbf{x})$, and $g_4(\mathbf{x}) = f(\mathbf{x} + \mathbf{q})I_i(\mathbf{x})$. If $x_i \geq x_j$ and $y_i \geq y_j$, then $\max(x_i, y_i) \geq \max(x_j, y_j)$ and $\min(x_i, y_i) \geq \min(x_j, y_j)$, so that $I_i(\mathbf{x})I_i(\mathbf{y}) = 1$ implies $I_i(\mathbf{x} \vee \mathbf{y})I_i(\mathbf{x} \wedge \mathbf{y}) = 1$, and we obtain $I_i(\mathbf{x} \vee \mathbf{y})I_i(\mathbf{x} \wedge \mathbf{y}) \geq I_i(\mathbf{x})I_i(\mathbf{y})$. Since the marginal distributions of reservation

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24 I had conjectured that the Karlin and Rinnot theorem implies that MTP_2 density leads to MTP_2 demand, since $D_i(\mathbf{p} \vee \mathbf{q}) = \int_{A_i(\mathbf{p} \vee \mathbf{q})} f(\mathbf{x})d\mathbf{x} = \int_{A_i(\mathbf{p}) \vee A_i(\mathbf{q})} f(\mathbf{x})d\mathbf{x}$, etc., where $A \vee B = \{z | \exists x \in A, y \in B, z = x \vee y\}$ etc., but found that the second equality does not hold if we examine the inequalities that define $A_i(\mathbf{p})$.

prices are independent, we can write $f(\mathbf{x}) = \prod_{i=1}^{n} f_i(x_i)$, and the duality between log concave functions and TP_2 functions implies

$f_i(x_i \vee y_i - (-(p_i \wedge q_i)))f_i(x_i \wedge y_i - (-(p_i \vee q_i))) \geq f_i(x_i - (p_i))f_i(y_i - (-q_i))$, so that

$f(\mathbf{x} \vee \mathbf{y} + \mathbf{p} \wedge \mathbf{q})f(\mathbf{x} \wedge \mathbf{y} + \mathbf{p} \vee \mathbf{q}) \geq f(\mathbf{x} + \mathbf{p})f(\mathbf{y} + \mathbf{q})$. Hence

$g_1(\mathbf{x} \vee \mathbf{y})g_2(\mathbf{x} \wedge \mathbf{y}) \geq g_3(\mathbf{x})g_4(\mathbf{y})$, and the Karlin and Rinnot theorem implies

$D_i(\mathbf{p} \vee \mathbf{q})D_i(\mathbf{p} \wedge \mathbf{q}) \geq D_i(\mathbf{q})D_i(\mathbf{p})$. QED.

The remaining part of this section gives some examples. First, Proposition 2 directly applies to logit models. It can be shown that Theorem 1 can be applied to nested logit models. For probit models with three brands, we show that Theorem 1 can be applied only to the case in which Proposition 2 essentially holds. Finally, we give an example in which we can apply Theorem 1, but demands for all brands are not quasi-concave in all prices.

Logit and Nested logit models

It is well known that the logit demand function, where the demand for a brand i is given by $D_i(\mathbf{p}) = \exp\left(\frac{a_i - p_i}{\mu}\right) \Big/ \sum_{j=1}^{n} \exp\left(\frac{a_j - p_j}{\mu}\right)$, is derived from the independent double exponential distribution of $(v_1, v_2, v_3, ..., v_n)$.[25] Since the density function of the double exponential distribution is log concave, the logit model of product differentiation has a unique price equilibrium if the cost function of each brand is increasing and convex.

[25] See Anderson, et al (1992), p. 39, for example.

Proposition 3: Suppose $D_i(\mathbf{p}) = \exp\left(\frac{a_i - p_i}{\mu}\right) \Big/ \sum_{j=1}^{n} \exp\left(\frac{a_j - p_j}{\mu}\right)$, $i = 1,...,n$. Then a unique price equilibrium exists if the cost function of each brand is increasing and convex.

Nested logit models are often used when there are several classes of brands and two brands in the same class are closer substitutes than two brands in different classes. When there are $g = 1,..,G$ classes and there are $j = 1,..,J_g$ brands in each class, the demand for brand i of class h is given by

$$D_{hi}(\mathbf{p}) = \left(\exp\left(\frac{a_{hi} - p_{hi}}{\mu_h}\right) \Big/ \sum_{j=1}^{J_h} \exp\left(\frac{a_{hj} - p_{hj}}{\mu_h}\right)\right) \bullet \left(\left(\sum_{j=1}^{J_h} \exp\left(\frac{a_{hj} - p_{hj}}{\mu_h}\right)\right)^{\mu_h/\mu} \Big/ \sum_{g=1}^{G} \left(\sum_{j=1}^{J_g} \exp\left(\frac{a_{gj} - p_{gj}}{\mu_g}\right)\right)^{\mu_g/\mu} \right),$$

where $0 < \mu_g \le \mu$. It is derived from a nested logit cumulative distribution function,

$$F\left(v_{11},...,v_{J_G G}\right) = \exp\left[-\sum_{g=1}^{G} \left(\sum_{j=1}^{J_g} \left(\exp\left(-\frac{a_{jg} + v_{jg}}{\mu_g} \right)\right)\right)^{\frac{\mu_g}{\mu}} \right],$$ (see Verboven (1996), p. 58, for example). Although the density function corresponding to this distribution function is difficult to obtain, it is easy to apply Theorem 1 directly to nested logit models.

Proposition 4: Suppose that there are $g = 1,..,G$ products and there are $j = 1,..,J_g$ brands in each product, and the demand for brand i of product h is

$$D_{hi}(\mathbf{p}) = \left(\exp\left(\frac{a_{hi} - p_{hi}}{\mu_h}\right) \Big/ \sum_{j=1}^{J_h} \exp\left(\frac{a_{hj} - p_{hj}}{\mu_h}\right)\right) \bullet \left(\left(\sum_{j=1}^{J_h} \exp\left(\frac{a_{hj} - p_{hj}}{\mu_h}\right)\right)^{\mu_h/\mu} \Big/ \sum_{g=1}^{G} \left(\sum_{j=1}^{J_g} \exp\left(\frac{a_{gj} - p_{gj}}{\mu_g}\right)\right)^{\mu_g/\mu} \right),$$

where $0 < \mu_g \le \mu$. Then there is a unique price equilibrium if the cost function of each brand is increasing and convex.

Proof

It is sufficient to show that $D_{hi}(\mathbf{p})$ is TP$_2$ in p_{hi} and $p_{h'i'}$ for $(h,i)\neq(h',i')$. Since TP$_2$ is robust to monotone transformations of independent variables, it is sufficient to show that $D_{hi} = P_{hi}\left(\tilde{P}_h^{(\mu_h-\mu)/\mu_h} \Big/ \sum_{g=1}^{G}\tilde{P}_g\right)$, where $P_{hi} = \exp\left(\dfrac{a_{hi}-p_{hi}}{\mu_h}\right)$ and $\tilde{P}_g = \left(\sum_{j=1}^{J_g} P_{gj}\right)^{\mu_g/\mu}$, is TP$_2$ in P_{hi} and $P_{hi'}$ for $(h,i)\neq(h',i')$. If $h = h'$,

$$\frac{\partial^2 \ln(D_{hi})}{\partial P_{hi}\partial P_{hi'}} = (1-\alpha_h)\tilde{P}_h^{2\alpha}\left\{1+\alpha_h\frac{\tilde{P}_h}{\sum_{g=1}^{G}\tilde{P}_g}\right\}+\left\{\alpha_h\frac{\tilde{P}_h^{(1-\alpha_h)}}{\sum_{g=1}^{G}\tilde{P}_g}\right\}^2 > 0,$$

where $\alpha_h = \dfrac{\mu_h}{\mu} \in (0,1)$. For $h = h'$, it is sufficient to show that D_{hi} is TP$_2$ in $\tilde{P}_h$ and $\tilde{P}_h'$. To verify, note that $\dfrac{\partial^2 \ln(D_{hi})}{\partial\tilde{P}_h\partial\tilde{P}_{h'}} = \left(\sum_{g=1}^{G}\tilde{P}_g\right)^{-2} > 0$. QED.

Probit models

Another model of product differentiation that is compatible with different degrees of substitutability is a probit model that assumes a multivariate normal distribution of tastes with non-zero covariances. If all covariances are zero and distributions of reservation prices are independent, the demand function of each brand is TP$_2$ in own price and another price, since the density function of the normal distribution is log concave, and we can apply Proposition 2.

Since Proposition 1 implies that demand functions are TP$_2$ in duopoly models, the simplest counter example includes three brands. Proposition 5 below states that the inverse in a sense of Proposition 2 holds for probit models with three brands.

Proposition 5: Suppose $n = 3$, and (v_1, v_2, v_3) has a trivariate normal distribution. Then the demand functions of all three brands are TP_2 in own price and each other's price for all $(p_1, p_2, p_3) \in R^3$ if and only if normally and independently distributed (u_1, u_2, u_3) can generate the same demands.

Proof

See Appendix.

We give two examples. The first example uses Proposition 5 to make a demand function that is log concave, and hence quasi-concave in all prices, but that does not have increasing differences. The second example gives a demand system that satisfies the condition of theorem 1, but all demands are not quasi-concave in all prices.

Example 1: Probit demand functions that do not have log increasing difference

As shown in the Appendix, when, $x_1 = v_2 - v_1$ and $x_2 = v_3 - v_1$ have a bivariate normal distribution, the variances of x_1 and x_2 are s_1^2 and s_2^2, and the correlation coefficient between x_1 and x_2 is r, $D_1 = \tilde{\Phi}\left(\frac{p_2 - p_1}{s_1}, \frac{p_3 - p_1}{s_2}, r\right)$, where $\tilde{\Phi}(z_1, z_2, r)$ is the cumulative distribution of a standard bivariate distribution with correlation coefficient r.

Then D_1 is TP_2 in p_1 and p_2 if and only if $\frac{\tilde{\Phi}_{12}\tilde{\Phi} - \tilde{\Phi}_1\tilde{\Phi}_2}{\tilde{\Phi}_1^{\,2} - \tilde{\Phi}_{11}\tilde{\Phi}} \leq \frac{s_2}{s_1}$. When $p_1 = p_2 = p_3 = 0$, we obtain the indefinite integral using polar transformation, and it can be shown that the condition is equivalent to $\frac{s_1 + s_2}{s_1 s_2 \pi} + \frac{r s_2 - s_1}{s_1^{\,2} s_2 \pi \sqrt{1 - r^2}\, \pi^2}\left\{2\sin^{-1}(r) + \pi\right\} \geq 0$.[26] This inequality

[26] Using $\tilde{\Phi}(0,0,r) = \frac{1}{4} + \frac{1}{2\pi}\sin^{-1}(r)$; see Johnson and Kotz (1972), p. 95.

does not hold if, for example, $s_1 = 4$, $s_2 = 1$ and $r = 0.9$, and then the left-hand side is about -0.1428. We can derive $s_1 = 4$, $s_2 = 1$ and $r = 0.9$ if the standard deviations and correlation coefficient for (v_1, v_2, v_3) are $\sigma_1 = 2.5$, $\sigma_2 = 2$, $\sigma_3 = 2$, $\rho_{12} = -0.575$, $\rho_{23} = -0.225$, $\rho_{13} = 0.925$, which are compatible with a positive-definite covariance matrix.

Example 2: Demand functions that have increasing differences but are not quasi concave.

There are 3 brands and the demands for each brand are,

$$D_1 = \frac{1}{2}\left\{\frac{P_1}{P_1 + P_2 + P_3} + \frac{P_1}{P_1 + \left(P_2/7\right) + 7P_3}\right\},\; D_2 = \frac{1}{2}\left\{\frac{P_2}{P_1 + P_2 + P_3} + \frac{\left(P_2/7\right)}{P_1 + \left(P_2/7\right) + 7P_3}\right\}, \text{ and}$$

$$D_3 = \frac{1}{2}\left\{\frac{P_3}{P_1 + P_2 + P_3} + \frac{7P_3}{P_1 + \left(P_2/7\right) + 7P_3}\right\},$$ where $P_i = \exp(-p_i)$. This demand system is obtained if the market of the differentiated product consists of two sub-markets with logit demand functions. Since TP_2 is robust to the monotone transformation of independent variables, it is sufficient to show that $A_{ij} = D_i \frac{\partial^2 D_i}{\partial P_i \partial P_i} - \frac{\partial D_i}{\partial P_i}\frac{\partial D_i}{\partial P_j} \geq 0$ for each $i \neq j$. With some manipulations, we obtain $A_{12} = \frac{7t^4 + 14t^3 - 8t^2 + 2t + 1}{28(P_1 + P_2 + P_3)^4 P_1^2 t^4}$,

$$A_{13} = \frac{t^4 + 2t^3 - 8t^2 + 14t + 7}{4(P_1 + P_2 + P_3)^4 P_1^2 t^4},\; A_{21} = \frac{(343t^4 + 98t^3 - 56t^2 + 14t + 1)P_2^2}{1372(P_1 + P_2 + P_3)^4 P_1^4 t^4},$$

$$A_{23} = \frac{(49t^4 + 14t^3 - 50t^2 + 14t + 1)P_2^2}{196(P_1 + P_2 + P_3)^4 P_1^4 t^4},\; A_{31} = \frac{(t^4 + 14t^3 - 56t^2 + 98t + 343)P_3^2}{4(P_1 + P_2 + P_3)^4 P_1^4 t^4}, \text{ and}$$

$$A_{32} = \frac{\left(4t^4 + 14t^3 - 50t^2 + 14t + 49\right)P_3^{\ 2}}{4\left(P_1 + P_2 + P_3\right)^4 P_1^2 t^4}$$, where $t = \frac{P_1 + \left(P_2/7\right) + 7P_3}{P_1 + P_2 + P_3} \in \left(\frac{1}{7}, 7\right)$. Since it can be shown that the six quadratic equations in the numerators are positive when $t \in \left(\frac{1}{7}, 7\right)$, this demand system satisfies the conditions of Theorem 1. Note that if we replace $\left(\frac{1}{7}, 7\right)$ by $\left(\frac{1}{8}, 8\right)$, we obtain $A_{32} = \frac{\left(u^4 + 16u^3 - 65u^2 + 16u + 64\right)P_3^{\ 2}}{4\left(P_1 + P_2 + P_3\right)^4 P_1^2 t^4}$, where

$u = \frac{P_1 + \left(P_2/8\right) + 8P_3}{P_1 + P_2 + P_3} \in \left(\frac{1}{8}, 8\right)$, and $u^4 + 16u^3 - 65u^2 + 16u + 64 = -20$ when $u = 2$. Hence when the demand systems of two markets are logistic, the aggregate demand functions may not satisfy the assumption of theorem 1.

If D_1 is quasi-concave on R^3, $\begin{vmatrix} 0 & \frac{\partial D_1}{\partial p_2} & \frac{\partial D_1}{\partial p_3} \\ \frac{\partial D_1}{\partial p_2} & \frac{\partial^2 D_1}{\partial p_2^{\ 2}} & \frac{\partial^2 D_1}{\partial p_2 \partial p_3} \\ \frac{\partial D_1}{\partial p_3} & \frac{\partial^2 D_1}{\partial p_2 \partial p_3} & \frac{\partial^2 D_1}{\partial p_3^{\ 2}} \end{vmatrix} \leq 0$ on $\left(p_1, p_2, p_3\right) \in R^3$.[27]

We can show that the value of the determinant is $-\frac{(x-9)(x-1)^3}{32x^7}$ when $P_2 = \frac{7}{8}(x-1)P_1$, $P_3 = \frac{1}{8}(x-1)P_1$. Since it is strictly positive for $x \in (1,9)$, D_1 is not quasi-concave on R^3.

The density function corresponding to this example is not unimodal. Since the Prékopa-Borel theorem is related to the correspondence between the unimodality of the density function and the quasi-concavity of the aggregate, it does not hold for non-

27 See Takayama (1985), p.127, for example.

unimodal densities. This example shows that demand functions that have log increasing differences globally are somewhat more robust to small perturbations, but are also vulnerable to a large perturbation.

3.2 Unidimensional Spatial Model[28]

This sub-section examines a spatial model in which all brands are aligned on a uni-dimensional line, where consumers are aligned on a line or a circle with unit density, and buy a unit of a brand that minimizes the sum of price and transportation cost. Suppose there are n brands at $(z_1,....,z_n)$ and a consumer at x buys a unit of brand i that minimizes $\varphi(|x - z_i|) + p_i$, where φ is a strictly increasing transportation cost. Since the demand of each firm is zero if its own price is sufficiently high, we cannot apply the results of the previous section directly. However, if the demand function of each firm is TP_2 on a competitive region where demands for all brands are strictly positive, the best response function is increasing and we can apply Corollary 2 to show that an equilibrium is unique on the domain if it is a lattice. Proposition 6 gives formal proof.

The first part of the proof uses Theorem 2.2.2. (e) of Topkis (1998), which states that *if* $f(\mathbf{x})$ *on* R^n *is the point-wise infimum of* $n(n-1)$ *bi-monotone functions and* n *univariate functions, it is a lattice-generating function*, *where bi-monotone function is increasing in one variable, decreasing in another variable and independent of all other variables*, and $f(\mathbf{x})$ *is a lattice-generating function if* $\{\mathbf{x} | f(\mathbf{x}) \geq 0\}$ *is a lattice*.[29]

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[28] The application to spatial models was suggested by an anonymous referees.

[29] Topkis (1998), page 24.

Proposition 6: The competitive region, $\{\mathbf{p} | \mathbf{p} \in R^n, D_i(p_i) > 0, i = 1,\ldots,n\}$, is a lattice. If $\varphi(x) = \tau x$ or $\varphi(x) = \tau x^2$, the price equilibrium is unique on the competitive domain if cost functions are positive and convex.[30]

Proof

Let $\varphi_{ij}(\mathbf{p}) = \max_x \{(\tau(|x - z_j|) + p_i) - (\tau(|x - z_j|) + p_i)\}$, then $D_i(\mathbf{p}) > 0$ if and only if $\min_{j \neq i} \varphi_{ij}(\mathbf{p}) > 0$, so that $\mathbf{p}$ is in the competitive region if and only if $\min_i \min_{j \neq i} \varphi_{ij}(\mathbf{p}) > 0$. Since $\varphi_{ij}(\mathbf{p})$ is bi-monotone, we can apply the theorem above by Topkis to show that $f(\mathbf{p}) = \min_i \min_{j \neq i} \varphi_{ij}(\mathbf{p})$ is a lattice-generating function by letting other n univariate functions to take any strictly positive valued functions that do not affect $\{\mathbf{x} | f(\mathbf{x}) > 0\}$. Then the closure of the competitive region, $\{\mathbf{x} | f(\mathbf{x}) \geq 0\}$, is a lattice and its interior is obviously a lattice.

Suppose that a brand has two neighboring brands, and the distance to the right brand is R and the distance to the left brand is L. Let the price of the brand be p and that of the left and right brands be q and r, respectively. Suppose that $\varphi(x) = \tau x$, demands for all three brands are strictly positive, and $D(p, q)$ is the demand of the firm for a given r.

Then

$$D(p^H, q^H)D(p^L, q^L) - D(p^H, q^L)D(p^L, q^H) = \frac{(p^H - p^L)(q^H - q^L)}{4\tau^2} \quad .$$

[30] $\varphi(x) = \tau x$ corresponds to the Hotelling problem. It is well known that an equilibrium may not exist in this case. Remember that Corollary 2 does not imply existence.

If the transportation cost is $\varphi(x) = \tau x^2$, the right-hand side is

$\dfrac{(L+R)(p^H - p^L)(q^H - q^L)}{4L^2 R\tau^2}$. If there is no brand to the right, they are respectively

$\dfrac{(p^H - p^L)(q^H - q^L)}{4\tau^2}$ and $\dfrac{(p^H - p^L)(q^H - q^L)}{4L^2\tau^2}$. Hence, the demand function for each brand is TP_2 in own price and in the price of a neighboring brand on the competitive region, and Corollary 2 implies that a price equilibrium is unique on the competitive region. QED.

Anderson, de Palma and Thisse (1992)[31] show by numerical calculation that a Salop model in which finite brands are located at equal distance on a circle has a symmetric equilibrium if $\beta \leq 5.8$. It can be shown that demand for each brand is TP_2 in own price and the price of a neighboring brand on the competitive region if $\beta \leq 2.8$ by numerically calculating $\dfrac{\partial^2}{\partial p_i \partial p_{i+1}} \ln\left(D_i\left(p_{i-1}, p_i, p_{i+1}\right)\right)$. Hence the conditions required for best response functions that are globally increasing are much stronger than the conditions for the existence of a symmetric equilibrium.

3.3 Representative consumer models

The first five chapters of Anderson, de Palma, and Thisse (1992) reveal the close relation between a class of discrete choice models and a class of representative consumer models. The most concrete example is the relation between logit models and CES models. We use this relation in the first application, and the second application extends this relation to the relation between nested logit models and nested CES models. The last part gives a condition based on the elasticity of substitution.

31 Chapter 6. p. 176.

When a representative consumer maximizes a CES utility function

$U = \sum_{t=1}^{m} \left(x_i^{1-\frac{1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} x_i^{\alpha}$ under the budget constraint $\sum_{i=1}^{n} p_i x_i + x_0 = Y$, where x_0 is the consumption of the outside goods, $\sigma > 1$ and $0 < \alpha < 1$, $D_i = \frac{p_i^{-\sigma}}{\sum_{k=1}^{n} p_k^{-\sigma}} \frac{Y}{p(1+\alpha)}$, is obtained. We can apply Theorem 1' to this model.

Proposition 7: If the demand functions of brand i are $D_i = \frac{p_i^{-\sigma}}{\sum_{k=1}^{n} p_k^{-\sigma}} \frac{Y}{p_i(1+\alpha)}$ for $i = 1,...,n$, there is a unique price equilibrium if the cost function of each brand is convex and marginal cost is strictly positive.[32]

Proof

Because $\sigma > 1$, $\lim_{p_i \to \infty} p_i D_i (p_i | \mathbf{p}_{-i}, Y) = 0$. Since products of TP_2 functions are TP_2, and a function of a single variable is TP_2, it is sufficient to show that $\frac{p_i^{-\rho-1}}{\sum_{k=1}^{n} p_k^{-\rho}}$ is TP_2 in p_i and p_j. To verify this, note that $\frac{p_i^{-\rho-1}}{\sum_{k=1}^{n} p_k^{-\rho}}$ is logistic in $(\ln p_1,...,\ln p_n)$ and TP_2 is ordinal in independent variables. QED.

32 Theorem 14 of Caplin and Nalebuff (1991) establishes the existence for the case of constant marginal cost and the uniqueness for the case of duopoly. The assumption of this proposition is weaker and the conclusion is stronger.

The decomposition in the previous proof corresponds to the result of Anderson, de Palma and Thisse (1992) for the demand function obtained if a consumer buys a brand that maximizes $u = \ln(x_i) + \alpha \ln(x_0) + \varepsilon$ under the budget constraint $p_i x_i + x_0 = Y$, and ε is distributed independently among consumers as a double exponential distribution. Then the choice probability is logit and if a brand is chosen its demand is $\frac{1}{p_i}\frac{Y}{1+\alpha}$.

Similarly we can apply Theorem 1' to the demand function

$$D_{mn} = \frac{p_{mn}{}^{1-\sigma_m}}{\sum_{i=1}^{N_t} p_{mi}{}^{1-\sigma_m}} \frac{\left\{\left(\sum_{i=1}^{N_t} p_{mi}{}^{1-\sigma_m}\right)^{\frac{1}{1-\sigma_m}}\right\}^{1-\varepsilon}}{\sum_{t=1}^{M}\left\{\left(\sum_{i=1}^{N_t} p_{mi}{}^{1-\sigma_m}\right)^{\frac{1}{1-\sigma_m}}\right\}^{1-\varepsilon}} \frac{Y}{(1+\alpha)p_{mn}}$$

that maximizes a nested CES utility function

$$U = \left(\sum_{t=1}^{M}\left(\sum_{i=1}^{N_t} x_{ti}{}^{1-\frac{1}{\sigma_t}}\right)^{\frac{\sigma_t}{\sigma_t - 1}\left(1-\frac{1}{\varepsilon}\right)}\right)^{\frac{\varepsilon}{\varepsilon-1}} x_0{}^{\alpha}$$

subject to the budget constraint $\sum_{t=1}^{M}\sum_{i=1}^{N_t} p_{ti}x_{ti} + x_0 = Y$, where x_{ti} and p_{ti} $(i = 1,..,N_t, t = 1,..,M)$ are the consumption and price of brand i of commodity t, and $\sigma_m > \varepsilon > 1$. To show that the demand functions are TP_2 in own price and the price of another brand, it is sufficient to note that the first term is nested logit in log prices. This result corresponds to the extension by Verboven (1996) of the relation between logit models and CES models to that between nested logit models and nested CES models.

Proposition 8: If the demand function of each brand is derived by maximizing nested CES demand functions, there is a unique price equilibrium if the cost function of each brand is convex and marginal cost is strictly positive.

Proof

To apply Theorem 1', it remains to show that $\lim_{p_{mn}\to\infty} p_{mn}D_{mn} = 0$. This is the case, since $\sigma_{mn} > \varepsilon > 1$. QED.

Proposition 7 and Proposition 8 hold when the prices of some brands do not change. The CES utility function of Proposition 7 can be interpreted as a nested CES utility function with elasticity of substitution between a brand of the differentiated product and the outside good is unity, and a similar interpretation is possible for the nested CES utility function of Proposition 8.

More generally, when a representative consumer maximizes a linear homogeneous utility function $u(\mathbf{x})$ subject to budget constraint $\mathbf{px} = Y$, the demand function of brand i is $D_i = \frac{e_i(\mathbf{p})}{e(\mathbf{p})}Y$, where $e(\mathbf{p}) = \min_{u(\mathbf{x})\geq u} \mathbf{px}$ is the expenditure function corresponding to some utility level, and $e_i(\mathbf{p})$ is its partial derivative with respect to p_i. The Allen-partial elasticity of substitution, which generalizes the usual elasticity of substitution for two factor models, is $\eta_{ij} = \frac{\varepsilon_{ij}}{\theta_j}$, where ε_{ij} is the elasticity of the compensated demand function of brand i with respect to the price of brand j, and θ_j is the share of brand j in the expenditure of the consumer. When $D_i = \frac{e_i(\mathbf{p})}{e(\mathbf{p})}Y$, it can be shown to be equal to $\eta_{ij} = \frac{ee_{ij}}{e_i e_j}$.[33] By differentiating this equation with respect to p_i and comparing it to

33 Takayama (1985), p. 144.

$\frac{\partial^2 \ln D_i}{\partial p_i \partial p_j} \geq 0$, which is the condition for D_i to be TP_2 in p_i and p_j, we find that

$\frac{\partial^2 \ln D_i}{\partial p_i \partial p_j} \geq 0$ is equivalent to $(\eta_{ij} - 1)^2 \theta_i + p_i \frac{\partial \eta_{ij}}{\partial p_i} \geq 0$. Hence, by applying Theorem 1', we obtain the following proposition:

Proposition 9: If the utility function of the representative consumer is homogeneous and demand functions are single valued, there is a unique price equilibrium if $(\eta_{ij} - 1)^2 \theta_i + p_i \frac{\partial \eta_{ij}}{\partial p_i} \geq 0$ for each $i \neq j$, $\lim_{p_i \to \infty} p_i D_i = 0$ for each i, and the marginal cost of each brand is strictly positive and constant.[34]

Proposition 9 extends to competition that is restricted to a subset of brands. Proposition 9 holds if $\frac{\partial \eta_{ij}}{\partial p_i} \geq 0$. When demand functions are derived from CES utility functions, $\eta_{ij} = \frac{\alpha + \sigma - 1}{\alpha}$, and $\frac{\partial \eta_{ij}}{\partial p_i} = 0$. When they are derived from nested CES utility functions, $\eta_{ij} = \frac{\alpha + \varepsilon - 1}{\alpha}$ if two brands belong to different classes, and

$$\eta_{ij} = \frac{\alpha + \varepsilon - 1}{\alpha} + \frac{\sigma_m - \varepsilon}{\alpha} \frac{\sum_{t=1}^{M} \left\{ \left(\sum_{i=1}^{N_t} p_{mk}{}^{1-\sigma_m} \right)^{\frac{1}{1-\sigma_m}} \right\}^{1-\varepsilon}}{\left\{ \left(\sum_{k=1}^{N_t} p_{mk}{}^{1-\sigma_m} \right)^{\frac{1}{1-\sigma_m}} \right\}^{1-\varepsilon}}$$ if they belong to the same class m.

[34] All brands are gross substitutes if and only if the elasticity of substitution is greater than unity for each pair of brands. Since it seems that we cannot impose this assumption generally, the proposition is stated with constant marginal cost.

Since the last term is strictly increasing in the price of a brand in class m for $\sigma_m > \varepsilon > 1$, $\frac{\partial \eta_{ij}}{\partial p_i} \geq 0$ in both cases, but the last expression is fairly complicated. Proposition 9 is only a restatement of definitions, but it suggests that the curvatures of indifference surfaces cannot change abruptly in some direction to obtain globally increasing best response functions.

4. Concluding remarks

Since a model with increasing best response functions is a quasi-supermodular game, the comparative statics and other results obtained by Milgrom and Shannon (1994) hold. For example, the prices of all brands are increasing in the marginal cost of each brand.

One immediate extension of the present paper is to pure monopolistic competition models in which the number of brands is infinite. Most arguments of Section 2 do not depend on the finiteness of the number of brands. One simple exception is given in footnote 10, which shows the equivalence of the pair-wise definition and the product space definition of log increasing differences. Generally, the latter definition should be used. The other is more fundamental. In R^n, a bounded closed lattice is compact and complete. Hence extensions are limited to infinite dimensional brand spaces in which this property holds. The demand function that is derived from the CES utility function in integral form is likely to belong to this class, but measure theoretic complications may arise.

Section 2.4.2 suggests that the results of the present paper may be extended from log increasing difference demand functions to -1 increasing difference demand functions.

Then the game is not a quasi-supermodular game, and Kakutani's fixed-point theorem should be used to prove existence, though one difficulty would lie in the construction of the upper bound. It is also debatable whether the models of product differentiation often used in applications have demand functions that are between -1 increasing differences and zero increasing differences, but some examples in Section 3 suggest that the condition that all demand functions have log increasing differences globally may be very restrictive for some classes of models.

Although Section 3 includes many applications and examples, the conditions on the distribution of reservation prices or the conditions on the utility functions for demand functions that have log-increasing differences globally are far from complete. Proposition 5 and Example 1 and 2 suggest that some very subtle conditions are required, and Proposition 2 and Proposition 9 suggest that they include some kind of symmetry and smoothness.

Appendix: Proof of Proposition 5

Proposition 5: Suppose $n = 3$, and (v_1, v_2, v_3) has a trivariate normal distribution. Then the demands functions of all three brands are TP_2 in own price and each other's price for all $(p_1, p_2, p_3) \in R^3$ if and only if independently and normally distributed (u_1, u_2, u_3) can generate the same demands.

Proof

Sufficiency follows immediately from Proposition 1.

To prove the necessity, we examine the demand function of brand 1 first. A consumer buys brand 1 if $v_1 - p_1 \geq v_2 - p_2$ and $v_1 - p_1 \geq v_3 - p_3$, which is equivalent to

$p_2 - p_1 \geq v_2 - v_1$ and $p_3 - p_1 \geq v_3 - v_1$. If (v_1, v_2, v_3) have a trivariate normal distribution, then $x_1 = v_2 - v_1$ and $x_2 = v_3 - v_1$ have a bivariate normal distribution. Let the variances of x_1 and x_2 be s_1^2 and s_2^2 and the correlation coefficient between x_1 and x_2 be r. Then $D_1(p_1, p_2, p_3) = \tilde{\Phi}\left(\frac{p_2 - p_1}{s_1}, \frac{p_3 - p_1}{s_2}, r\right)$, where $\tilde{\Phi}(z_1, z_2, r)$ is the cumulative distribution function of a standard bivariate distribution with correlation coefficient r, let $\tilde{\phi}(z_1, z_2, r)$ be a corresponding density function. Then D_1 is TP$_2$ in p_1 and p_2 if and only if $\frac{\partial^2 \ln D_1}{\partial p_1 \partial p_2} \geq 0$, which is equivalent to $\frac{\tilde{\Phi}_{12}\tilde{\Phi} - \tilde{\Phi}_1\tilde{\Phi}_2}{\tilde{\Phi}_1^{\,2} - \tilde{\Phi}_{11}\tilde{\Phi}} \leq \frac{s_2}{s_1}$, where we use the fact that the strict log concavity of $\tilde{\Phi}$ implies that the denominator is strictly positive. It can be shown that $\frac{\tilde{\Phi}_{12}\tilde{\Phi} - \tilde{\Phi}_1\tilde{\Phi}_2}{\tilde{\Phi}_1^{\,2} - \tilde{\Phi}_{11}\tilde{\Phi}}$ and r have the same sign[35] so that $\frac{\tilde{\Phi}_{12}\tilde{\Phi} - \tilde{\Phi}_1\tilde{\Phi}_2}{\tilde{\Phi}_1^{\,2} - \tilde{\Phi}_{11}\tilde{\Phi}} \leq \frac{s_2}{s_1}$ holds for

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35 It is easy to verify that the density function of (z_1, z_2) is TP$_2$ if and only if $r \geq 0$. See Tong (1990), p. 75. Let $\varphi(z_1, z_2 | a, b)$ be the indicator function of $(z_1, z_2) \leq (a, b)$, so that it is unity if $a \geq z_1$ and $b \geq z_2$, and zero otherwise. By applying the Karlin and Rinnot theorem to $g_1(z_1, z_2) = \varphi(z_1, z_2 | a^H, b^H)\tilde{\phi}(z_1, z_2, r)$, $g_2(z_1, z_2) = \varphi(z_1, z_2 | a^L, b^L)\tilde{\phi}(z_1, z_2, r)$, $g_3(z_1, z_2) = \varphi(z_1, z_2 | a^H, b^L)\tilde{\phi}(z_1, z_2, r)$, and $g_4(z_1, z_2) = \varphi(z_1, z_2 | a^L, b^H)\tilde{\phi}(z_1, z_2, r)$, we can show that the cumulative distribution of a bivariate normal distribution is TP$_2$ if its density is TP$_2$, so that $\tilde{\Phi}_{12}\tilde{\Phi} - \tilde{\Phi}_1\tilde{\Phi}_2 \geq 0$ if $r \geq 0$. Similarly, the density function of $(z_1, -z_2)$ is TP$_2$ if and only if $r \leq 0$, so that $\tilde{\Phi}_{12}\tilde{\Phi} - \tilde{\Phi}_1\tilde{\Phi}_2 \leq 0$ if $r \leq 0$.

$r \le 0$. When $r \ge 0$, $\sup \frac{\tilde{\Phi}_{12}\tilde{\Phi} - \tilde{\Phi}_1\tilde{\Phi}_2}{\tilde{\Phi}_1^{\,2} - \tilde{\Phi}_{11}\tilde{\Phi}} = \sup \frac{\frac{\partial^2}{\partial z_1 \partial z_2} \ln \tilde{\Phi}}{\frac{\partial^2}{\partial z_1^{\,2}} \ln \tilde{\Phi}} \ge r$, as we will show in the last part of the proof. Hence if D_1 is TP$_2$ in p_1 and p_2 for all prices, $rs_1 \le s_2$, or $Cov(x_1, x_2) = rs_1 s_2 \le s_2^{\,2} = Var(x_2)$ must hold. Thus we obtain $Cov(v_2 - v_1, v_3 - v_1) \le Var(v_3 - v_1)$. By symmetry, if D_1 is TP$_2$ in p_1 and p_3, $Cov(x_1, x_2) = rs_1 s_2 \le s_1^{\,2} = Var(x_1)$. By the same argument, if D_2 is TP$_2$ in p_1 and p_3, $Cov(v_1 - v_2, v_3 - v_2) \le Var(v_2 - v_1)$, or $Cov(-x_1, x_2 - x_1) \le Var(-x_1)$, hence $Cov(x_1, x_2) = rs_1 s_2 \ge 0$, and $r \ge 0$.

Suppose that (u_1, u_2, u_3) are normally and independently distributed with $Var(u_1) = rs_1 s_2 \ge 0$, $Var(u_2) = s_1^{\,2} - rs_1 s_2 \ge 0$ and $Var(u_3) = s_2^{\,2} - rs_1 s_2 \ge 0$. Then $(u_2 - u_1, u_3 - u_1)$ have the same distribution as $(v_2 - v_1, v_3 - v_1)$, because $Var(u_2 - u_1) = Var(u_2) + Var(u_1) = s_1^{\,2}$, $Var(u_3 - u_1) = Var(u_3) + Var(u_1) = s_2^{\,2}$, and $Cov(u_3 - u_1, u_2 - u_1) = Var(u_1) = rs_1 s_2$.

It remains to show that $\sup \frac{\tilde{\Phi}_{12}\tilde{\Phi} - \tilde{\Phi}_1\tilde{\Phi}_2}{\tilde{\Phi}_1^{\,2} - \tilde{\Phi}_{11}\tilde{\Phi}} = \sup \frac{\frac{\partial^2}{\partial z_1 \partial z_2} \ln \tilde{\Phi}}{\frac{\partial^2}{\partial z_1^{\,2}} \ln \tilde{\Phi}} \ge r$. By definition, $\tilde{\Phi}_{12} = \tilde{\phi}$. Let ϕ be the density function of the univariate standard normal distribution. Then $\tilde{\phi}(z_1, z_2, r) = \frac{1}{\sqrt{1 - r^2}} \phi(z_1) \phi\left(\frac{z_2 - rz_1}{\sqrt{1 - r^2}}\right) = \frac{1}{\sqrt{1 - r^2}} \phi(z_2) \phi\left(\frac{z_1 - rz_2}{\sqrt{1 - r^2}}\right)$, and we obtain $\tilde{\Phi}_1(z_1, z_2, r) = \phi(z_1) \Phi\left(\frac{z_2 - rz_1}{\sqrt{1 - r^2}}\right)$, $\tilde{\Phi}_2(z_1, z_2, r) = \phi(z_2) \Phi\left(\frac{z_1 - rz_2}{\sqrt{1 - r^2}}\right)$,

$$\tilde{\Phi}(z_1,z_2,r) = \int_{-\infty}^{z_1} \phi(s)\Phi\left(\frac{z_2 - rs}{\sqrt{1-r^2}}\right)ds = \int_{-\infty}^{z_2} \phi(s)\Phi\left(\frac{z_1 - rs}{\sqrt{1-r^2}}\right)ds$$, and

$$\tilde{\Phi}_{11}(z_1, z_2, r) = -\frac{r}{\sqrt{1-r^2}}\phi(z_1)\phi\left(\frac{z_2 - rz_1}{\sqrt{1-r^2}}\right) - z_1\phi(z_1)\Phi\left(\frac{z_2 - rz_1}{\sqrt{1-r^2}}\right).$$

Putting these equations into $\frac{\tilde{\Phi}_{12}\tilde{\Phi} - \tilde{\Phi}_1\tilde{\Phi}_2}{\tilde{\Phi}_1{}^2 - \tilde{\Phi}_{11}\tilde{\Phi}}$, we obtain

$$\frac{\left(\frac{1}{\sqrt{1-r^2}}\phi(z_1)\phi(B)\right)\tilde{\Phi} - (\phi(z_1)\Phi(B))(\phi(z_2)\Phi(A))}{(\phi(z_1)\Phi(B))^2 - \left\{-\frac{r}{\sqrt{1-r^2}}\phi(z_1)\phi(B) - z_1\phi(z_1)\Phi(B)\right\}\tilde{\Phi}}, \tag{1}$$

where, $A = \frac{z_1 - rz_2}{\sqrt{1-r^2}}$ and $\frac{z_2 - rz_1}{\sqrt{1-r^2}}ds$.

We show that (1) converges to r as $z_1 \to \infty$ for given z_2. To show this, divide both the numerator and denominator of (1) by $\phi(z_1)\phi(z_2)\phi(B)\Phi(A)$ and rearrange to obtain

$$\frac{\frac{1}{\sqrt{1-r^2}}\left(\frac{\tilde{\Phi}}{\Phi(A)}\right) - \frac{\Phi(B)}{\phi(B)}\phi(z_2)}{\frac{\phi(z_1)\Phi(B)}{\Phi(A)} + \left\{\frac{r}{\sqrt{1-r^2}} + \frac{\sqrt{1-r^2}}{r}B\frac{\Phi(B)}{\phi(B)} + \frac{z_1}{r}\right\}\left(\frac{\tilde{\Phi}}{\Phi(A)}\right)}. \tag{2}$$

As $z_1 \to \infty$, $A \to \infty$, $B \to -\infty$, $\phi(z_1) \to 0$, $\Phi(A) \to 1$, $\Phi(B) \to 0$, $\frac{\Phi(B)}{\phi(B)} \to 0$, $B\frac{\Phi(B)}{\phi(B)} \to -1$, and

$$\lim_{z_1\to\infty}\frac{\tilde{\Phi}}{\Phi(A)} = \lim_{z_1\to\infty}\int_{-\infty}^{z_2}\phi(s)\Phi\left(\frac{z_1 - rs}{\sqrt{1-r^2}}\right)\Phi\left(\frac{z_1 - rz_2}{\sqrt{1-r^2}}\right)^{-1}ds$$

$$= \int_{-\infty}^{z_2} \phi(s) \left[\lim_{z_1 \to \infty} \left(\Phi\left(\frac{z_1 - rs}{\sqrt{1-r^2}} \right) \Phi\left(\frac{z_1 - rz_2}{\sqrt{1-r^2}} \right)^{-1} \right) \right] ds = \Phi(z_2)$$

Putting these limits to the left hand side of (3), we see that it converges to r. Hence $\sup \frac{\tilde{\Phi}_{12}\tilde{\Phi} - \tilde{\Phi}_1\tilde{\Phi}_2}{\tilde{\Phi}_1^2 - \tilde{\Phi}_{11}\tilde{\Phi}} \geq r$.[36] QED.

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REFERENCES

Anderson, S. P, A. de Palma, and J.F. Thisse, 1992. Discrete choice theory of product differentiation (MIT Press, Cambridge, MA and London).

Caplin, A. and B. Nalebuff, 1991. Aggregation and Imperfect Competition: On the Existence of Equilibrium, Econometrica 59, 25-59.

Dixit, A. K. and J. E. Stiglitz, 1977. Monopolistic Competition and Optimum Product Diversity, American Economic Review 67, 297-308.

36 It can be shown that $\frac{\tilde{\Phi}_{12}\tilde{\Phi} - \tilde{\Phi}_1\tilde{\Phi}_2}{\tilde{\Phi}_1^2 - \tilde{\Phi}_{11}\tilde{\Phi}} \leq r$, so that $\sup \frac{\tilde{\Phi}_{12}\tilde{\Phi} - \tilde{\Phi}_1\tilde{\Phi}_2}{\tilde{\Phi}_1^2 - \tilde{\Phi}_{11}\tilde{\Phi}} = r$. A proof is available at http://www.mgmt.purdue.edu/centers/ijio/.

Hardy, G, J.E. Littlewood, and G. Polya , 1952. Inequalities. Second edition (Cambridge University Press).

Johnson, N.L. and S. Kotz, 1972. Distributions in Statistics: Continuous Multivariate Distributions (John Wiley) .

Karlin, S., 1968. Total Positivity. Volume 1 (Stanford University Press).

Karlin, S. and Y. Rinnot, 1980. Classes of Orderings of Measures and Related Correlation Inequalities. 1. Multivariate Totally Positive Distributions, Journal of Multivariate Analysis 10, 467-498.

Milgrom, P. and J. Roberts, 1990. Rationalizability, Learning, and Equilibrium in Games with Strategic Complementarities, Econometrica 58, 1255-77.

Milgrom, P. and C. Shannon, 1994. Monotone Comparative Statics, Econometrica 62, 157-80.

Novshek, W. 1985.On the Existence of Cournot Equilibrium, Review of Economic Studies 52, 85-98,

Topkis, D. M., 1978. Minimizing a Submodular Function on a Lattice, Operation Research 26, 305-321.

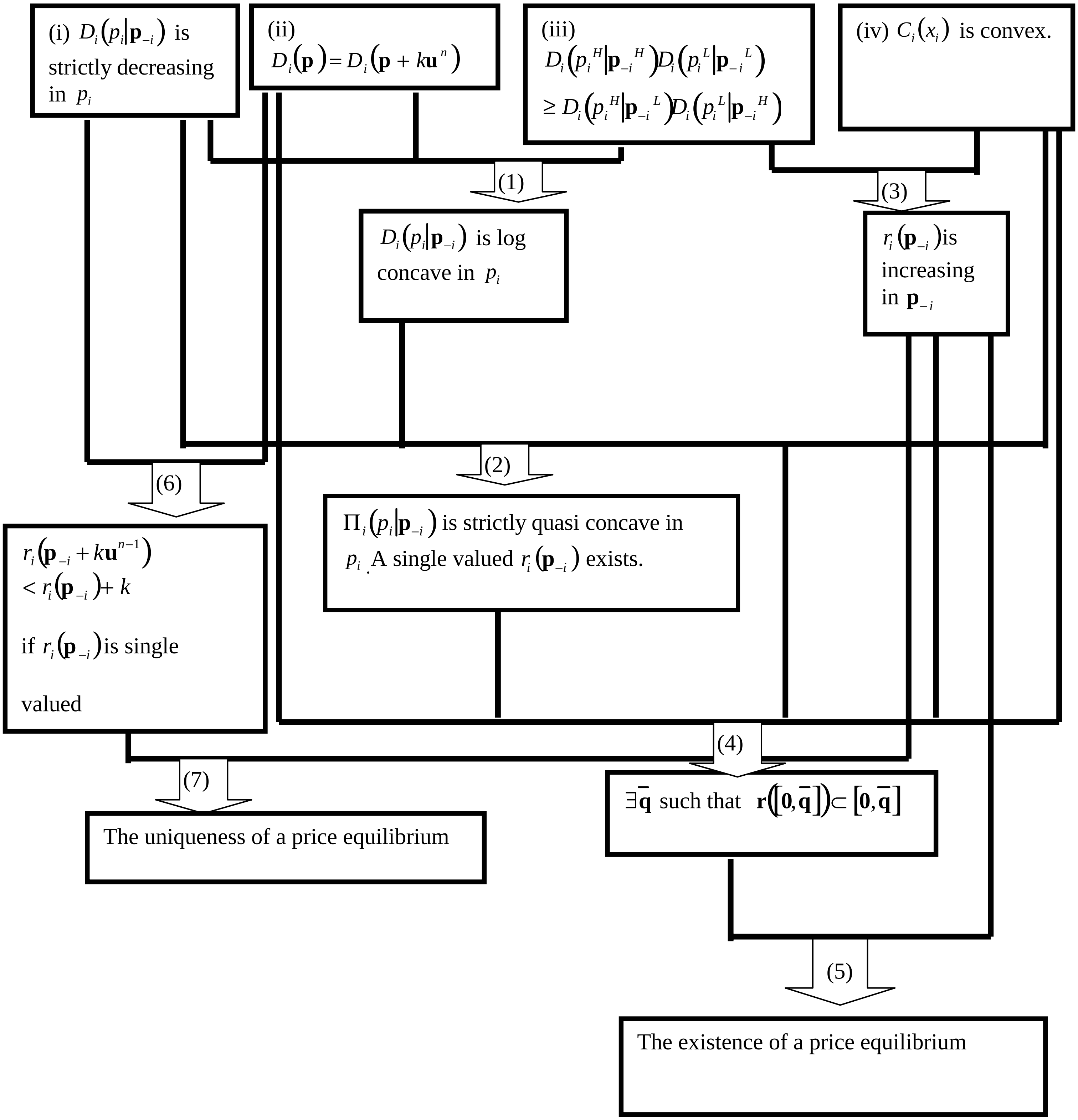
Topkis, D. M., 1998.Supermodularity and complementarity, Frontiers of Economic Research series(Princeton University Press, Princeton).

Tong, Y. L., 1990. The Multivariate Normal Distribution, (Springer-Verlag).

Verboven, F., 1996. The Nested Logit Model and Representative Consumer Theory. Economic Letters 50, 57-63.

Vives, X., 1990. Nash Equilibrium with Strategic Complementarities, Journal of Mathematical Economics 19, 305-21.

Figure 1


(i) $D_i(p_i|\mathbf{p}_{-i})$ is strictly decreasing in p_i
(ii) $D_i(\mathbf{p}) = D_i(\mathbf{p} + k\mathbf{u}^n)$
(iii) $D_i(p_i^H|\mathbf{p}_{-i}^H)D_i(p_i^L|\mathbf{p}_{-i}^L) \geq D_i(p_i^H|\mathbf{p}_{-i}^L)D_i(p_i^L|\mathbf{p}_{-i}^H)$
(iv) $C_i(x_i)$ is convex.
(1)
(3)
$D_i(p_i|\mathbf{p}_{-i})$ is log concave in p_i
$r_i(\mathbf{p}_{-i})$ is increasing in $\mathbf{p}_{-i}$
(6)
(2)
$r_i(\mathbf{p}_{-i} + k\mathbf{u}^{n-1}) < r_i(\mathbf{p}_{-i}) + k$ if $r_i(\mathbf{p}_{-i})$ is single valued
$\Pi_i(p_i|\mathbf{p}_{-i})$ is strictly quasi concave in p_i. A single valued $r_i(\mathbf{p}_{-i})$ exists.
(4)
(7)
$\exists \bar{\mathbf{q}}$ such that $\mathbf{r}([\mathbf{0},\bar{\mathbf{q}}]) \subset [\mathbf{0},\bar{\mathbf{q}}]$
The uniqueness of a price equilibrium
(5)
The existence of a price equilibrium