Existence of Equilibrium and Comparative Statics in Differentiated Goods Cournot Oligopolies

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Abstract

We show that under horizontal differentiation pure symmetric Cournot equilibria exist if firms react to a rise in competitors' output in such a way that their market price does not rise. This condition is related to strategic complementarity, but not to convexity or differentiability. We rule out multiple equilibria under some additional conditions and discuss stability and regularity of equilibria with ordinal methods. Following entry equilibrium price decreases only if competitors' outputs enter inverse demand aggregated into a single number, otherwise it may increase even in stable equilibria. The comparative statics of quantities and profits for all equilibria are given.

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1 Introduction

Since Cournot's early contribution his model of oligopoly has received more and more attention, and nowadays is a basic building block of applied work on a wide range of topics involving imperfect competition. Its usefulness depends on two features: First, existence and uniqueness of equilibria at the market stage must be easily established, and second, comparative statics results should be readily available. In the context of homogeneous goods both these aspects have been treated extensively, whereas for differentiated goods there are much fewer results available. The existing literature on this subject is very rich, and to keep the introduction short we will discuss it below in a separate section 6.

We impose the condition that firms react to a rise in competitors' quantities by adjusting their own production in such a way that their market price does not rise (condition A). Doing so, output may increase or decrease, but must not decrease too strongly. This condition is formulated without making use of differentiability or convexity assumptions, rather is formulated in ordinal terms as a single-crossing condition.[1] If goods are substitutes, and under standard regularity conditions such as continuity and compact strategy spaces, we show that this condition implies the existence of symmetric pure Cournot equilibria. We also state a set of stronger sufficient conditions which in many applications may be easy to verify.

After establishing existence, one would also like to know when equilibrium is unique. We show that asymmetric equilibria can be ruled out if the additional weak condition that own market price reacts more to changes in own output than in competitors' outputs (condition B) is added. On the other hand, for competitor aggregation symmetric equilibrium is unique if each symmetric equilibrium is "diagonally stable" as defined below, and if there is a symmetric equilibrium that is unstable and "diagonally regular" then multiple symmetric equilibria exist. We define these notions in developing an ordinal approach to stability and regularity of symmetric equilibria. This approach does not need differential calculus, Jacobians and determinants, and turns out to be very intuitive.

One of the fundamental conclusions of the oligopoly literature is that stability of equilibrium is closely connected to "non-paradoxical" comparative statics results. Our analysis for differentiated goods, which is based on ordinal monotone comparative statics methods, makes precise comparative statics predictions for extremal (maximal and minimal) equilibria, which do not rely explicitly on stability conditions since extremal equilibria are generically stable. To deal with interior equilibria, we invoke our results on stability and regularity of equilibria. The comparative statics of entry using these methods does not rely on treating the number of firms as a continuous variable. Still, we show how each equilibrium, be it stable or not, can be

[1] The instruments of lattice theory were introduced into economics by Topkis (1979), Milgrom and Roberts (1990) and Vives (1990), and ordinal versions by Milgrom and Shannon (1994) and Milgrom and Roberts (1994).

associated in a natural way with a new "neighboring" stable equilibrium, and that a move to a new equilibrium resulting in counter-intuitive comparative statics means that the new equilibrium is unstable.[2]

We concentrate exclusively on the comparative statics of exogenous entry. Our main results are that if competitors' quantities enter inverse demand aggregated into one single number, then equilibrium prices do not increase as more firms enter the industry, and that the aggregate of competitors' outputs is strictly increasing. We also show by means of an example that this result is not extendable to general differentiated goods with aggregation into two or more numbers, i.e. equilibrium prices may rise even if there are no increasing returns to scale and equilibrium is stable.

As concerns quantities at equilibrium, total equilibrium output may rise or fall even if prices are decreasing, but will rise under the same condition that we already used to rule out asymmetric equilibria and an additional condition on the aggregator. We also confirm the known results that individual output rises or falls depending on whether goods are strategic complements or substitutes, while profits always decrease with entry.

The rest of our paper continues as follows: Section 2 sets out the model, section 3 introduces the main condition, and the existence results are presented in section 4. Section 5 presents our comparative statics results and the discussion of stability. Section 6 discusses the existing literature on existence and comparative statics of Cournot equilibria, and section 7 concludes. All proofs are relegated to the appendix.

2 The Model

There are n firms with identical production capacities K and identical production cost functions $c : [0, K] \rightarrow \mathbb{R}_+$. Denote firm i's output quantity by x_i, and by x_{-i} the vector of outputs of the other firms. Inverse demand of firm i ($i = 1, .., n$) is given by a function $p : [0, K]^n \rightarrow \mathbb{R}_+$, with $p_i = p(x_i, x_{-i})$,[3] which is symmetric in the other firms' outputs: Let $\tilde{x}_{-i}$ be any permutation of x_{-i}, then $p(x_i, \tilde{x}_{-i}) = p(x_i, x_{-i})$ for all $(x_i, x_{-i}) \in [0, K]^n$. That is, firms are completely symmetric in that, apart from identical production technologies, all demand functions have the same functional form and all competitors' goods enter each demand function symmetrically. Assume that p is nonincreasing in x_i and x_{-i}, i.e. in particular goods are *substitutes*, and strictly decreasing in x_i where inverse demand is positive. We impose the following regularity conditions:

Condition R (Regularity): 1. Production capacity K is limited: $0 < K < \infty$;[4]

[2] Echenique (2001) contains related work for games with complementarities.

[3] By $p(x_i, x_{-i})$ we mean that x_i is the first argument of p, i.e. that for all i own quantity x_i enters firm i's inverse demand differently from other firms' quantities x_j, $j \neq i$.

[4] Alternatively, as is often done, one may assume that inverse demand falls below marginal cost (given any output of rivals) or even becomes zero for outputs larger than a certain limit. All these

2. production cost $c(x_i)$ is continuous;[5]

3. inverse demand $p(x_i, x_{-i})$ is continuous in (x_i, x_{-i}).

Firm i's profits are, for $x_i \in [0, K]$ and given $x_{-i} \in [0, K]^{n-1}$,

$$\Pi(x_i, x_{-i}) = x_i p(x_i, x_{-i}) - c(x_i), \; i = 1 \ldots n. \quad (1)$$

We will now reformulate profits Π as a function of firm i's *market price* p_i and the other firms' outputs[6]. This is done by introducing the residual demand function χ.

Lemma 1 *There is a set $X \subset \mathbb{R}_+ \times [0, K]^{n-1}$ such that firm i's residual demand is given by a function $\chi : X \to [0, K]$ of own market price p_i and the other firms' output x_{-i} that is continuous and nonincreasing in $(p_i, x_{-i}) \in X$, and strictly decreasing in p_i. Also, there is a new constraint set $\pi(x_{-i})$ for firm i's maximization problem that is non-empty, closed, compact, and nonincreasing in x_{-i}.*

The very last fact follows from the assumption that goods are substitutes, and is necessary for the conclusion below that equilibrium prices are non-increasing.

Firm i's maximization problem can be expressed as

$$\max_{p_i \in \pi(x_{-i})} \tilde{\Pi}(p_i, x_{-i}) = \chi(p_i, x_{-i}) p_i - c(\chi(p_i, x_{-i})), \quad (2)$$

resulting in the price best reply[7] $P(x_{-i})$.

If we consider profits $\tilde{\Pi}(p_i, x_{-i})$ "on the diagonal" where all competitors produce the same amount $y \in [0, K]$, we write

$$\hat{\Pi}(p_i, y) = \tilde{\Pi}(p_i, y, \ldots, y). \quad (3)$$

Some special cases are, in order of increasing specialization, what we will call "*Competitor aggregation*", "*industry aggregation*",[8] and *homogeneous goods*. Under competitor aggregation there are functions $\hat{p} : \mathbb{R}_+^2 \to \mathbb{R}_+$ and $f : [0, K]^{n-1} \to \mathbb{R}_+$, where f is strictly increasing, such that

$$p(x_i, x_{-i}) = \hat{p}(x_i, f(x_{-i})), \quad (4)$$

where the competitors' quantities are aggregated into one number. One example is *additive aggregation* with $f(x_{-i}) = Y_i = \Sigma_{j \neq i} x_j$.

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assumptions ensure that firms' chosen outputs have a finite upper bound.

[5]Lower semi-continuity of c already guarantees that each firm's maximization problem has a solution, but condition A below rules out upward jumps in costs anyway.

[6]Note that the variable to be maximized over (output or price) is irrelevant as long as afterwards each firm 'commits' to a fixed production quantity or at least competitors believe that it is so.

[7]We adopt this formulation to avoid confusion with the standard (quantity) best response or reaction function $x_i = r(x_{-i})$.

[8]I would like to thank Karl Schlag for proposing these terms.

Under industry aggregation there exist functions $\bar{p}:\mathbb{R}_+\rightarrow\mathbb{R}_+$ and $f:[0,K]^{n-1}\rightarrow\mathbb{R}_+$ such that

$$p(x_i,x_{-i})=\bar{p}(x_i+f(x_{-i})), \tag{5}$$

and if goods are homogeneous then $f(x_{-i})=\Sigma_{j\neq i}x_j$. For industry aggregation it is easy to see that $\chi(p_i,x_{-i})=D(p_i)-f(x_{-i})$, where $D=\bar{p}^{-1}$ is the demand function, and the profit maximization problem becomes

$$\max_{p_i\in[\bar{p}(K+f(x_{-i})),\bar{p}(f(x_{-i}))]}\tilde{\Pi}(p_i,x_{-i})=(D(p_i)-f(x_{-i}))\,p_i-c(D(p_i)-f(x_{-i})).$$

In general it may not be possible to aggregate competitors' quantities into one single number as under competitor aggregation. In section 5 we state an example where quantities are aggregated in two numbers, and in section 6.3 of Hoernig (2000) it is shown that there are consumer's preferences giving rise to a set of inverse demand functions that do not allow for any aggregation of rivals' quantities.

3 The Condition

In this section we will introduce several conditions on profits which all guarantee the existence of equilibria. Even though the weakest one, condition A, seems somewhat abstract at first sight, it nicely uncovers the economic intuition behind our and previous existence results. Since condition A is in general not straightforward to verify, we then state the two strictly stronger conditions AD and D, which are more easily checked.

The main condition to be used for the proof of existence of equilibrium is of a type that has recently been shown to be of central importance in any exercise of comparative statics: In a lattice-theoretic context, Milgrom and Shannon (1994) have shown that the set of maximizers of the parametric maximization problem $\max_{x\in S}f(x,t)$, where $S\subset\mathbb{R}$, is nondecreasing in (t,S) *if f satisfies the weak single crossing property (WSCP) in (x,t)*: for all $x'>x$ and $t'>t$ we have that

$$f(x',t)-f(x,t)\geq(>)\,0\;\Rightarrow\;f(x',t')-f(x,t')\geq(>)\,0.$$

In the context of game theory this result can be applied to best reply maps, and we do so after our change of variables from own quantity to own price described above.

Underlying our results is the following condition (It is a "dual" single-crossing condition because the inequality signs in the definition are reversed):

Condition A: $\hat{\Pi}(p_i,y)$ satisfies the dual weak single crossing property in (p_i,y), i.e. for all $p_i'>p_i$ and $y'>y$ we have that

$$\hat{\Pi}(p_i',y)-\hat{\Pi}(p_i,y)\leq(<)\,0\;\Rightarrow\;\hat{\Pi}(p_i',y')-\hat{\Pi}(p_i,y')\leq(<)\,0. \tag{6}$$

Even though this condition seems to be extremely abstract, its interpretation is very simple and economically intuitive: Condition A means that, *starting from a situation where all other firms produce identical quantities, if these other firms all raise their outputs by the same amount, it will be advantageous for firm i to adjust its output only in such a way that the resulting market price is not higher than before.* Doing so, own output may increase or decrease, depending on whether goods are strategic substitutes or complements.

Note that condition A imposes the dual single crossing property only on the "diagonal", i.e. where competitors all produce the same quantity. This is sufficient for the existence result since we are only interested in symmetric equilibria, while we will have to state a condition covering the whole space of outputs to deal with asymmetric equilibria. In addition, condition A is formulated for identical *increases* in the outputs of all competitors. This is equivalent to formulating the corresponding condition in terms of an increase in just one competitor's quantity as long as inverse demand is symmetric in competitors' outputs, while it is more general if inverse demands are not symmetric.[9] Even though our condition (and the proof of existence of symmetric pure equilibrium) applies to these more general cases, in this paper we will concentrate on the symmetric case.

Condition A means that best reply price are non-increasing in the other firms' outputs. In the proofs of theorem 2 (existence of equilibrium) and lemma 4 (monotonicity of the aggregator) below we only make use of an even weaker requirement which follows from condition A:

Condition AW: Extremal best price replies "on the diagonal" are continuous but for downward jumps.

This means that downward jumps in own price are allowed but not upward jumps. Clearly this follows when the price best replies are non-increasing. Still, we need condition A for the comparative statics of prices, and also condition AW is less intuitive.

We have stated what condition A means; still we should say what it rules out. First, condition A and AW rule out strongly increasing returns to scale in production. The argument runs like this: Under increasing returns to scale, a reduction in own output raises marginal cost, therefore creating incentives for further output reduction. Therefore, if returns to scale are large, best reply output may be reduced so much that market price actually rises. In particular, note that condition A and AW rule out the existence of *avoidable fixed cost*, i.e. fixed costs that are not incurred if nothing is produced: Firm i may prefer to stop producing at all (possibly creating an upward jump in own price), instead of lowering its own price, if the other firms raise their outputs. Any other upward jump in production cost is

[9] Condition A therefore even applies to cases where firms are symmetric when all competitors produce identical outputs, i.e. for all $(x, y) \in [0, K]^2$, $p_i(x, y, .., y) = p_j(x, y, .., y)$, while demand functions may differ "off-diagonal". It also applies to the case where firms are symmetric, but are not affected equally by every competitor. One example of this last case are firms with only two direct neighbors in some kind of "circular city" model.

similarly excluded. Second, for general differentiated goods, demand may be such that condition A is violated even if marginal cost is constant. We show below that in this case condition A relates the slopes of the demand function with respect to own and others' quantities.

Condition A applies no matter whether inverse demand and production costs are differentiable or not, nor does it appeal to quasiconcavity of profits. Since it is an ordinal condition, it is not surprising that there is no *equivalent* condition in terms of derivatives even if demand or cost are differentiable. Using the *method of dissection* discussed in Milgrom and Shannon (1994, p. 167), we can find a sufficient differential condition that is slightly stronger than condition A.[10] If Π^{ii} and Π^{ij} are the second partial derivatives of the profit function of firm i with respect to outputs, and p^i and p^j the partial derivatives of the inverse demand of firm i with respect to x_i and x_j, condition A is implied by

Condition AD: For all i, $x_i \in [0, K]$, and $x_{-i} = (y, ..., y) \in [0, K]^{n-1}$,

$$\Pi^{ii}(x_i, x_{-i}) - \frac{p^i(x_i, x_{-i})}{p^j(x_i, x_{-i})} \Pi^{ij}(x_i, x_{-i}) \leq 0. \tag{7}$$

It is easily seen why condition A follows from condition AD. Consider a change dx_j in firm j's output, and a corresponding change dx_i in firm i's output such that firm i's market price p remains constant, leading to the marginal rate of substitution in demand $MRS = -dx_i/dx_j = p^j/p^i$. The total change in marginal profits Π^i is then

$$\frac{d}{dx_j} \Pi^i(x_i(x_j), x_{-i}) = -\frac{p^j}{p^i} \Pi^{ii} + \Pi^{ij}.$$

Condition AD means that this expression is non-negative which indicates that firm i must increase its quantity (or not change it) to return to an optimum. This in turn at most lowers its price, as demanded by condition A.

We can express condition AD in an equivalent form which makes the relation to existing results much clearer. Consider a change dx_j in firm j's output, and a corresponding change dx_i in firm i's output such that firm i's market price p remains constant, resulting in the marginal rate of substitution in demand $MRS = -dx_i/dx_j = p^j/p^i$. Let ε_{MRS} be the elasticity of this marginal rate of substitution with respect to own output x_i,

$$\varepsilon_{MRS} = \frac{\partial(p^j/p^i)}{\partial x_i} \frac{x_i}{p^j/p^i} = x_i \frac{p^{ij}p^i - p^{ii}p^j}{p^i p^j},$$

then condition AD can be restated as

$$\Pi^{ii} - \frac{p^i}{p^j} \Pi^{ij} = (1 - \varepsilon_{MRS}) p^i - c'' \leq 0.$$

[10] See Hoernig (2000), appendix 9.2.

This formulation yields two further interpretations: There are *at most "weakly" increasing returns to scale* (where "weakly" depends on the demand function), or *the profit margin* $p - c'$, "corrected for substitution", *is falling in own output.*[11] Condition AD is more likely to be violated if MRS is decreasing in x_i ($\varepsilon_{MRS} < 0$), and more likely to hold if MRS is increasing in x_i ($\varepsilon_{MRS} > 0$). It is easily verified that in the cases of industry aggregation or homogeneous goods $\varepsilon_{MRS} = 0$, and that condition AD then reduces to Amir and Lambson's (2000) homogeneous goods condition $p' - c'' \leq 0$. In fact, it can be shown that industry aggregation is the most general case of differentiated goods where $\varepsilon_{MRS} = 0$.[12] As for competitor aggregation, it can be shown that ε_{MRS} only depends on the properties of $p(x, f)$ and not on the properties of the aggregator $f(x_{-i})$ since the derivative of the latter cancels out.
Condition AD also implies a lower bound on the slope of quantity best replies $r_i(x_{-i})$,

$$\frac{\partial}{\partial x_j} r_i(x_{-i}) = -\frac{\Pi^{ij}}{\Pi^{ii}} \geq -\frac{p^j}{p^i} = \left.\frac{dx_i}{dx_j}\right|_{p=const}. \tag{8}$$

For homogeneous goods this reduces to the well-known condition that the slope of best replies must be at least -1. Therefore we have obtained a very insightful interpretation of this classical slope condition, coinciding with the one we gave in the previous paragraph: *The slope of best replies must not be smaller than the MRS of demand*, which again leads to the result that own price does not rise after a best reply to an increase in someone else's output.

Apart from condition AD there are various other conditions that imply condition A and that may sometimes be easier to verify. Some we will discuss in the next section, but the one most easily verified is the following:[13]

Condition D: $\tilde{\Pi}(p_i, x_{-i})$ *has (weakly) decreasing differences in* (p_i, x_{-i}), *i.e.*

$$\tilde{\Pi}(p_i', x_{-i}') - \tilde{\Pi}(p_i, x_{-i}') \leq \tilde{\Pi}(p_i', x_{-i}) - \tilde{\Pi}(p_i, x_{-i}) \tag{9}$$

for all $p_i' \geq p_i \geq 0$ *and* $x_{-i}' \geq x_{-i} \in [0, K]^{n-1}$.

This condition is stronger than condition A (Milgrom and Shannon 1994). If inverse demand and production costs are twice continuously differentiable, it is equivalent to

$$\frac{\partial^2}{\partial p_i \partial x_j} \tilde{\Pi}(p_i, x_{-i}) \leq 0, \tag{10}$$

[11] An equivalent interpretation, due to Amir and Lambson (2000) for homogenous goods, is that "inverse demand or price decreases faster (...) at any given output level than does marginal cost at all lower output levels."

[12] If $\varepsilon_{MRS} = 0$ the partial differential equations $p_i^j = h_j(x_{-i}) p_i^i$ hold, with solution $p_i(x_i, x_{-i}) = F(x_i + H(x_{-i}))$ for some functions F and H with $\partial H / \partial x_j = h_j$.

[13] Here $x_{-i}' \geq x_{-i}$ means that $x_j' \geq x_j$ for all $j \neq i$.

for all $j \neq i$, or (see Hoernig 2000 for the proof),

$$\left(\Pi^{ii} - \frac{p^i}{p^j}\Pi^{ij}\right) - \left(p_i + xp^i - c'\right)\frac{\varepsilon_{MRS}}{x_i} \leq 0. \tag{11}$$

The last term in (11) disappears if goods are industry aggregates ($\varepsilon_{MRS} = 0$), or at interior best replies ($p_i + xp^i - c' = 0$), leading to equivalence with condition AD in these cases.

4 Existence of Equilibria

We will now state our main result on the existence of symmetric pure Cournot equilibria. Multiple symmetric equilibria can be ranked according to equilibrium quantities (or prices). If there is a symmetric equilibrium where quantities are smaller (higher) than in any other symmetric equilibrium, this equilibrium is called *minimal* (*maximal*).

Theorem 2 *Assume that inverse demand is nonincreasing in all arguments (goods are substitutes), and strictly decreasing in own output while inverse demand is positive. Under conditions R and A there exist symmetric pure Cournot equilibria.*

The proof can be found in appendix 8.2. From a technical point of view, at the heart of theorem 2 lies the fact that under condition A the price best reply $P(x_{-i})$ has *nonincreasing* maximal and minimal selections, which allows for the construction of a function on $[0, K]$ into itself that is continuous but for upward jumps and therefore has maximal and minimal fixed points (Milgrom and Roberts 1994).[14] These result in maximal and minimal symmetric pure strategy Cournot equilibria.

The equilibrium is unique if and only if the maximal and minimal equilibria are identical, but this cannot be established without further assumptions. Under the assumption that best replies are single-valued, which follows for example from the strong assumption that profits are quasi-concave, in section 5 we show that there is at most one symmetric equilibrium if all symmetric equilibria are stable, and that there are several symmetric equilibria if one of them is unstable (see proposition 8). These results are deferred to the section on entry because they are preceded by an in-depth discussion of the stability issue using ordinal methods.

On the other hand, using stronger versions of condition A and a weak additional condition B, one can exclude the existence of *asymmetric* equilibria. Let us state

[14] In a previous version, we constructed a nondecreasing map from the space of prices into itself. Tarski's (1955) theorem can then be applied to show that maximal and minimal fixed points exist (see Hoernig 2000).

two conditions related to condition A, both of which are strictly stronger and involve the whole space of competitors' outputs $[0, K]^{n-1}$: For all $i = 1...n$,

Condition AS: $\tilde{\Pi}(p_i, x_{-i})$ *satisfies the dual strict single crossing property in* (p_i, x_{-i}) *for all* $x_{-i} \in [0, K]^{n-1}$*: For all* $p_i' > p_i$ *and* $x_{-i}' > x_{-i}$,

$$\tilde{\Pi}(p_i', x_{-i}) - \tilde{\Pi}(p_i, x_{-i}) \leq 0 \Rightarrow \tilde{\Pi}\left(p_i', x_{-i}'\right) - \tilde{\Pi}\left(p_i, x_{-i}'\right) < 0. \tag{12}$$

Condition ASD: $\partial\tilde{\Pi}/\partial p_i$ *exists and is strictly decreasing in* x_j *for all* $j \neq i$ *where* $p(x_i, x_{-i})$ *is positive and for all* $x_{-i} \in [0, K]^{n-1}$.

Condition AS means that a firm will not raise *any* best reply market price as a reaction to an increase in competitors' outputs (as opposed to just maximum and minimum best reply prices under condition A). Condition ASD implies that a firm will *strictly decrease* its market price as a reaction to an increase in competitors' outputs and is stronger than conditions A and AS, and even stronger than weakly or strictly decreasing differences of profits (see Edlin and Shannon 1998b).

Let x_{-ij} be the vector of outputs of firms $k \neq i, j$. The additional conditions on inverse demands are:

Condition BW (weak): For all $j \neq i$*, all* $(x_i, x_j, x_{-ij}) \in [0, K]^n$*, and all* $\varepsilon > 0$, $p(x_i + \varepsilon, x_j, x_{-ij}) \leq p(x_i, x_j + \varepsilon, x_{-ij})$.

Condition BS (strict): For all $j \neq i$*, all* $(x_i, x_j, x_{-ij}) \in [0, K]^n$*, and all* $\varepsilon > 0$, $p(x_i + \varepsilon, x_j, x_{-ij}) \leq p(x_i, x_j + \varepsilon, x_{-ij})$*, where the inequality is strict when* $p(x_i, x_j + \varepsilon, x_{-ij}) > 0$.

If inverse demand is differentiable these conditions correspond to $p^i \leq p^j$ and $p^i < p^j$ (almost everywhere), respectively. Conditions BW and BS mean that each firm's changes in quantity influence its own market price more than the same changes in other firms' quantities, which is a very reasonable assumption as firms are symmetric. Note that the case of homogeneous goods, where $p^i = p^j = p'$, falls under condition BW. In fact, both these conditions follow from utility maximization of a representative consumer: If inverse demands are derived from maximizing a (strictly) concave utility function U, where at the optimum $p_i = \partial U/\partial x_i$, then the condition $p^i \leq p^j$ ($p^i < p^j$) follows from the (strict) negative definiteness of the Hessian and the symmetry of the demand functions.

With these conditions, we have the following proposition:

Proposition 3 *Asymmetric equilibria do not exist if either conditions AS and BS hold, or if conditions ASD and BW hold.*

For homogeneous goods we must assume condition ASD (including the assumption that profits are differentiable) to rule out asymmetric equilibria, while for differentiated goods the weaker condition AS is enough. On the other hand, condition AS must be accompanied with the slightly stricter condition BS.

5 Stability and Effects of Entry

A long-standing point of interest has been the question whether Cournot equilibrium approaches competitive equilibrium as more firms enter the market.[15] It has become common to call a Cournot equilibrium *quasi-competitive* if in equilibrium total quantity is increasing or price is decreasing in the number of firms (independently of where price actually converges). It is easy to see that for differentiated goods there is not necessarily a strict inverse relation between total quantity and market prices: Outputs are not of the same kind, thus it is not really clear what the economic meaning of 'total output' is, while preference for variety may actually lead to higher prices. Therefore it makes most sense to define quasi-competitiveness through decreasing prices.

In our framework of horizontal differentiation, the entry of a new competitor raises the number of goods (and welfare if consumers value variety), which in general may have surprising effects. As we will see in the following, under *competitor aggregation* and condition A the conventional wisdom (equilibrium prices decrease after entry) prevails, while this is no longer true in general.

Assume that there is a countable number of identical firms that may enter the market[16]. Let the aggregator $f : [0, K]^{\infty} \to F \subset \mathbb{R}$ be continuous, strictly increasing, and symmetric in its arguments: Let $\tilde{x}_{-i}$ be a permutation of $x_{-i} \in [0, K]^{\infty}$, then $f(\tilde{x}_{-i}) = f(x_{-i})$. For $n \geq 1$ let $g(x, n) = f\left((x)_{n-1}, 0, 0, ...\right)$ be the value of the aggregator if $n-1$ firms produce x and the other firms nothing; this function is strictly increasing in both arguments while $x > 0$ and $n > 1$, and continuous in x. For all $1 \leq n < \infty$ let the maximum of the aggregator for a finite number of producing firms be $\bar{f}_n = g(K, n)$, and let $\underline{f} = f(0)$. Under competitor aggregation inverse demand is given by

$$p_i = p\left(x_i, f\left(x_{-i}\right)\right), \tag{13}$$

where $x_{-i} \in [0, K]^{\infty}$ and $p : [0, K] \times F \to \mathbb{R}_+$ is continuous and non-increasing in (x_i, f_i), and strictly decreasing in its first argument while inverse demand is positive. We can write residual demand given f as $\chi(p, f)$, and price and quantity best replies given f as $P(f)$ and $R(f)$. Then quantity best replies given x_{-i} are $r(x_{-i}) = R(f(x_{-i}))$, and we have $R(f) = \chi(P(f), f)$. In particular, since χ is continuous in (p, f) and strictly decreasing in $p > 0$, and the extremal selections of P are non-increasing in f (thus have no upward jumps), maximal and minimal selections of R exist and are continuous but for upward jumps.

Imagine that each of n firms faces the value f of the aggregator of the other firms' output and all firms choose the same best reply $x \in R(f)$. The value of the new aggregator is given by the following correspondence, which is fundamental for

[15] The most recent contribution for homogeneous goods, using the traditional differentiable methodology, is Dastidar (2000).

[16] Since we are not interested in determining an endogenous free entry equilibrium, fixed cost of entry are irrelevant for our purpose.

our study of comparative statics and stability below: Define the family of correspondences $\psi_n : [\underline{f}, \infty) \to [\underline{f}, \bar{f}_n]$, $n \geq 1$ by $\psi_n(f) = g(R(f), n)$.[17] We show in the proof of the following lemma that all fixed points of ψ_n lie in $[\underline{f}, \bar{f}_n]$, and that f^* is a fixed point of ψ_n if and only if there is a symmetric equilibrium with n firms and quantity y^*, and $f^* = g(y^*, n)$. Note that for $n = 1$ by definition ψ_1 is defined only on $\{\underline{f}\}$, with value $\psi_1(\underline{f}) = \underline{f}$. Therefore the properties of symmetric equilibria correspond to the properties of the fixed points of ψ_n.

Under condition A, the existence of symmetric Cournot equilibria already follows from theorem 2, therefore the following comparative statics conclusions are not empty. The following lemma states the comparative statics of the aggregator, and holds even under the weaker condition AW.

Lemma 4 *Under competitor aggregation and condition A the maximum and minimum equilibrium values $f_{(n)}$ of the aggregator are non-decreasing in the number of firms n, and strictly increasing if monopoly is strictly viable.*

This very insightful lemma is the generalization of the known proposition for homogeneous goods that if $r' \geq -1$ the sum of *competitors'* outputs $Y_i = \Sigma_{j \neq i} x_i$ increases as more firms enter, which underlies most comparative statics results. Clearly, for homogeneous goods total output must then be non-decreasing (and price non-increasing): From $r' \geq -1$ it follows that any increase in Y_i is accompanied by an equal or smaller decrease in x_i, such that $Q = x_i + Y_i$ does not decrease. This need no longer be true with differentiated goods, as the additional assumptions of proposition 11 below indicate.

Our main comparative statics result on prices follows almost trivially from this lemma:[18]

Theorem 5 *(Quasi-competitiveness) Under competitor aggregation the following hold:*

1. Under condition A maximum and minimum equilibrium prices are non-increasing in the number of firms n.

2. Under condition ASD, and if $p(x_i, x_{-i})$ is strictly decreasing in x_j for all $j \neq i$ while $p_i > 0$, then maximum and minimum equilibrium prices are strictly decreasing in n as long as they are positive and monopoly is strictly viable.

Two remarks are in order: First, as noted above, condition ASD, the condition that $\partial\tilde{\Pi}/\partial p_i$ exists and is strictly decreasing in x_j for all $j \neq i$, is strictly stronger than condition A or even strictly decreasing differences of $\tilde{\Pi}$ in (p_i, x_{-i}) (see Edlin and Shannon 1998b). Thus the conclusion that extremal equilibrium prices are

[17] The graph of ψ_n, $n \geq 2$, is a continuous transformation of the graph of the best response R and has essentially the same shape, i.e. is homeomorphic.

[18] The counter-examples to Seade (1980) of De Meza (1985) do not apply here since they do not satisfy our condition A, just as they did not satisfy Seade's condition A.

strictly decreasing (instead of non-increasing) needs rather more restrictive assumptions.

Second, both statements of theorem 5 do not rely on the implicit function theorem (IFT), neither do they treat n as a continuous variable; Both of these are common for deriving comparative-statics conclusions. For general types of aggregation treating n as a continuous variable is impossible, and this was one of the motivations for using the ordinal approach.[19] The second statement includes differentiability in its assumptions, which is necessary to obtain a statement about *strict* monotonicity, but still its proof does not make use of the IFT.

Our arguments so far can by their nature only produce conclusions about maximal or minimal equilibria, but not about equilibria "in between". Here it is useful to make a distinction between the comparative statics of the equilibrium *point*, i.e. "the direction in which the equilibrium point moves", and comparative statics of equilibrium, i.e. where starting at a given previous equilibrium point after a change in parameters the "out-of-equilibrium dynamics" will converge (if it does). Equilibria can be unstable, and in general counter-intuitive comparative statics of equilibrium points are associated to these unstable equilibria.[20] For instance, in our case it could happen that the unstable equilibrium point moves towards higher prices. Yet, as Dierker and Dierker (1999) and Echenique (2001) stress in the context of games with complementarity, out-of-equilibrium dynamics will not follow the (downward-) shifted unstable equilibrium. Rather, using lattice-theoretic arguments, Echenique shows that some general classes of dynamics converge to a higher stable equilibrium, providing an ordinal version of the correspondence principle. We can apply their ideas to Cournot oligopoly even though in general it is not a game with complementarities.

There are several problems to consider in the case of Cournot oligopoly. First of all, best replies may be at least locally decreasing, which creates the possibility of cycles. This latter phenomenon is made even more likely if best replies are not single-valued. To concentrate on the effect of slope, for the study of non-extremal equilibria we assume that R and therefore ψ_n are single-valued. Then since both are upper hemi-continuous correspondences by condition R, they become continuous functions.

A second problem is that stability must be defined with respect to a certain class of out-of-equilibrium dynamics. We will concentrate on the continuous best-reply dynamics (CD) $\dot{x} = k\left(r\left(x_{-i}\right) - x\right)$, $k > 0$,[21] but will consider this dynamics only for equal outputs of all firms, giving rise to the concept of *diagonal stability.* Considering the whole space of outputs, instability can arise for two reasons. For homogeneous

[19] See also Seade's (1980) arguments and De Meza's (1985) critique.

[20] For the traditional approach to stability using differential methods, see Seade (1980), Al-Nowaihi and Levine (1985) and Furth (1986).

[21] This is not a contradiction to our use of the ordinal approach since we do *not* assume any expression to be differentiable, and technically it is unproblematic since r is a continuous function of x_{-i} if it is single-valued.

goods, as Seade (1980) has shown, it is precisely the diagonal of equal outputs that constitutes the "unstable manifold" (along which outputs diverge) if $r' > 1/(n-1)$, or, as Furth (1986) has shown, the unstable manifold involves some outputs rising and others decreasing if $r' < -1$ holds.[22] We can show that in our case the results are qualitatively the same even without taking recourse to differentiability. Since our purpose in this section is to compare symmetric equilibria, and also for lack of space, we will deal exhaustively with these questions in a sequel to this paper.

Consider the following continuous aggregator dynamics (CAD) based on the maps ψ_n: Let $\dot{f} = k(\psi_n(f) - f) = k(g(R(f), n) - f)$ for $f \in [\underline{f}, \bar{f}_n]$. This means that if the aggregator f increases then all firms respond with an output higher than that implied by f. In other words, for equal outputs this is just a rewriting of the best-reply dynamics in quantities, and it is easy to see that for $n \geq 2$ we have $\dot{x} \gtreqqless 0$ if and only if $\dot{f} \gtreqqless 0$ at $f = g(x, n)$, i.e. the dynamics of x and f do have the same properties:

$$\dot{x} \gtreqqless 0 \Leftrightarrow R(g(x,n)) \gtreqqless x \Leftrightarrow g(R(g(x,n)), n) \gtreqqless g(x,n) \Leftrightarrow \psi_n(f) \gtreqqless f \Leftrightarrow \dot{f} \gtreqqless 0.$$

Since the ψ_n have useful properties we will use the dynamics of f instead of the dynamics of x. It is clear that $\dot{f} > (<)\ 0$ if and only if $\psi_n(f) > (<)\ f$, and that at equilibrium points $\dot{f} = 0$.

Now we introduce our definition of local diagonal stability (We will drop the word "locally" in the following if no confusion can arise).

Definition 6 *A Cournot equilibrium point $(x^*)_n$ is locally diagonally stable (d-stable) if there exists $\delta > 0$ such that for all $\varepsilon \in (-\delta, \delta) \setminus \{0\}$ the CD starting at $(x^* + \varepsilon)_n$ converges to $(x^*)_n$.*

Clearly d-stability is a necessary (but not sufficient) condition for local stability in the usual sense: If an equilibrium is not d-stable, then it is unstable.

It is easy to see that an equilibrium is d-stable if and only if $\psi_n(f) > f$ in a neighborhood below and $\psi_n(f) < f$ in a neighborhood above the equilibrium point, i.e. if ψ_n strictly cuts the diagonal from above: $\dot{x} > 0$ is equivalent to $\dot{f} > 0$ which in turn is equivalent to $\psi_n(f) > f$ since ψ_n is continuous in f, etc. We say that a fixed point f^* of ψ_n is *d-regular* if ψ_n at f^* *strictly* cuts the diagonal either from above or from below. By the above definition a d-stable equilibrium is also d-regular, while if the equilibrium is unstable then either ψ_n cuts the diagonal from below, or the equilibrium is not d-regular, i.e. ψ_n touches the diagonal and may or may not cut it. Again, d-regularity is necessary but not sufficient for regularity in the usual sense.

Sufficient conditions on profits for d-stability (instability) can easily be obtained (let $e = (1, .., 1) \in \mathbb{R}^n$):

[22] For this latter case also see Amir and Lambson (2000). It can be shown that a strict version of our condition A, together with condition BW and the assumption that f is element-wise convex (which both hold for homogeneous goods) rules out this type of instability.

Condition S (US): $\frac{\partial \Pi_i}{\partial x_i}$ *exists, and there exists* $\varepsilon > 0$ *such that* $\frac{\partial \Pi_i}{\partial x_i}(x + \delta e)$ *is strictly decreasing (increasing) in* δ *for all* $\delta \in (-\varepsilon, \varepsilon)$.

Condition S means that, starting from outputs x_j, $j = 1..n$, if all competitors raise (lower) output by the same amount δ to $x_j + \delta$ then the best reply of firm i is strictly smaller (higher) than $x_i + \delta$, ruling out explosive paths where all outputs diverge upwards or downwards. Condition US on the other hand precisely requires that outputs diverge in both directions. These conditions imply directly that at the equilibrium point ψ_n strictly cuts the diagonal from above (below), leading to d-stability or instability as explained above:

Proposition 7 *If condition S (US) holds at a symmetric equilibrium* x^* *and the equilibrium price is positive, then it is d-stable (d-regular and unstable).*

The assumption that profits Π are differentiable in own output is necessary because without it we cannot conclude that the equilibrium is d-regular in the first place (see also the differences between conditions A, AS and ASD).

Intuitively, if there are multiple symmetric equilibria, some of them should be unstable, at least if best replies do not have jumps. That is intuition is correct is shown by the following proposition:

Proposition 8 *Let best replies be single-valued.*

1. If all symmetric equilibria are d-stable then there is at most one symmetric equilibrium.

2. Under condition A, if condition US holds at some symmetric equilibrium then there exist multiple symmetric equilibria.

This proposition says nothing about asymmetric equilibria (for this see proposition 3), neither does it exclude multiple symmetric equilibria if best replies are set-valued.

Let us now turn to the comparative statics of symmetric equilibrium points, be they interior or not.

Proposition 9 *Let* ψ_n *be a continuous function for all* $n \geq 2$ *and monopoly be strictly viable. For some given* $n \geq 1$ *let* f_n^* *be a fixed point of* ψ_n*, i.e. an equilibrium aggregator with* n *firms:*

1. For $m > n$ *there is a smallest fixed point* f_m^* *of* ψ_m *above* f_n^**, with* $f_n^* < f_{n+1}^* < f_{n+2}^* < ....$ *Each* f_m^* *corresponds to a stable (or non-regular) equilibrium with* m *firms.*

2. If for $m > n$ *there is a highest fixed point* f_m^- *of* ψ_m *below* f_n^**, highest fixed points* f_l^- *below* f_n^* *exist for all* $n < l < m$*. These are ordered,* $f_n^* > f_{n+1}^- > ... > f_m^-$*, and for each* $k = n+1..m$ *the corresponding equilibrium with* k *firms is unstable (or non-regular).*

Note that both statements hold no matter whether the equilibrium at hand itself is stable or not, or even regular. It is purely based on the observation that after the transition from ψ_n to ψ_{n+1} we have $\psi_{n+1}(f_n^*) > f_n^*$, and on the assumption that ψ_{n+1} is continuous (but for upward jumps).

Statement 1 gives a precise meaning to the notion of "equilibrium is increasing in n", which is not at all clear when the number of firms is treated as a discrete variable. It also includes in a natural way the result about maximal and minimal equilibria in lemma 4, which were proved under less restrictive assumptions. Starting at f_n^*, the CAD dynamics under ψ_{n+1} clearly converges to the fixed point f_{n+1}^* since $\psi_{n+1}(f) > f$ on $\left[f_n^*, f_{n+1}^*\right)$ by definition of f_{n+1}^*. Thus in this sense there is necessarily a "convergence along the diagonal" to the "next-highest" equilibrium if the number of firms increases.

The second statement is related to corollary 3 in Echenique (2001), which made a similar statement for games with complementarities. It means that if some method of relating equilibria for different n to each other (for example by treating n as a continuous variable) leads to the counter-intuitive comparative statics result that "equilibrium is decreasing" then it must have picked unstable (or non-regular) equilibria as the number of firms increased.

Even so this does not mean that after entry of a new firm the new equilibrium will be at f_{n+1}^* since in general after entry the vector of outputs will *not* be on the diagonal. Still, we can show that the out-of-equilibrium dynamics of the incumbent firms starts with an upward trend: Consider an equilibrium with n firms and aggregator $f_n^* = g(x_n^*, n)$. A firm $n+1$ will enter if its best reaction $R(g(x^*, n+1))$ is positive, which directly leads to an *increase* in the aggregators faced by the incumbent firms, towards the next stable equilibrium, and away from any smaller unstable equilibrium. Furthermore, convergence to any unstable equilibrium will almost surely not occur. The question of where this dynamics actually converges is not one of local stability, and is complicated by the fact that overshooting or undershooting may occur, on the contrary to games with complementarities. A final answer must take into account the global properties of the out-of-equilibrium dynamics.

We will now show how our results are related to the differential approach. The above conditions are ordinal versions of the usual differential conditions as in Seade (1980), and we can even define some type of regularity without recourse to Jacobians and determinants. If profits are twice continuously differentiable, condition S (US) is equivalent to

$$\Pi^{ii} + (n-1)\,\Pi^{ij} < (>)\ 0,$$

which is precisely Seade's (1980) condition for stability on p. 23 (instability condition B2 or (12)). To make the relation to ψ_n clear in the differentiable case, assume that $f(.)$ is differentiable with component derivative f_x, and R is differentiable in f. Then from condition S it follows that $\partial\psi_n/\partial f < 1$, and from $\partial g/\partial x = (n-1)\,f_x$ we obtain

$$\frac{\partial \psi_n}{\partial f} = (n-1) f_x R' < 1 \text{ or } r' < \frac{1}{n-1} \text{ or } \Pi^{ii} + (n-1)\Pi^{ij} < 0,$$

with the corresponding inequalities for condition US. Furthermore, in the differential case the equilibrium is not d-regular if and only if $\Pi^{ii} + (n-1)\Pi^{ij} = 0$ or $r' = 1/(n-1)$.

Deliberately forcing the classical method based on the implicit function theorem to its limits, we assume that $g(x,n)$ is differentiable in n and treat n as a continuous variable. This is done only for illustrative purposes, and is the only time in this paper where we do this.

Lemma 10 *Under competitor aggregation, let inverse demand and production cost be twice continuously differentiable, and let $g(x,n)$ be differentiable in n. The derivatives with respect to n of the aggregator and price at an interior equilibrium are*

$$\frac{df_{(n)}}{dn} = \frac{x_{(n)} f_x \Pi^{ii}}{\Pi^{ii} + (n-1)\Pi^{ij}} \tag{14}$$

$$\frac{dp_{(n)}}{dn} = \frac{x_{(n)} p^j}{\Pi^{ii} + (n-1)\Pi^{ij}} \left(\Pi^{ii} - \frac{p^i}{p^j} \Pi^{ij} \right), \tag{15}$$

Let us first look at the comparative statics for the aggregator. Since f is increasing in x, and at an interior equilibrium necessarily $\Pi^{ii} \le 0$, the aggregator increases (decreases) in n if $\Pi^{ii} + (n-1)\Pi^{ij} < 0$ (> 0). These two conditions are precisely the differential conditions for d-stability and instability, respectively, as shown above.

Furthermore, if condition AD holds then an interior equilibrium *point* moves towards lower prices if and only if it is d-stable, and higher prices if and only if it is unstable.

Theorem 5 and corollary 10 do not extend to the case of non-aggregative demands, as the following example shows[23]. Note that the traditional comparative statics analysis based on the implicit function theorem is not applicable here since inverse demand cannot be written as a *differentiable* function of the number of firms. Assume there are n firms with production capacity $K \ge 1/2$ and zero production cost, and inverse demands are given by $p(x_i, x_{-i}) = \max\{0, 1 - x_i - \Sigma_{j \ne i}(1 - e^{-x_i x_j})\}$. Symmetric equilibrium outputs are given by the first-order condition (sufficient second-order conditions are satisfied)

$$2 - 2x - n + (n-1)\left(1 - x^2\right) e^{-x^2} = 0,$$

which for each value of $n \ge 1$ has exactly one solution $x_{(n)} \le 1/2$. Therefore for each n there is exactly one symmetric equilibrium which at the same time is minimal,

[23] In section 9.4.4 of Hoernig (2000) we give an intuitive explanation, using a fixed point map in prices, of why without aggregation into one number we cannot prove that prices are non-increasing.

maximal and interior. On the diagonal $x_j = \bar{x}$ $(j \neq i)$ condition AD is fulfilled for any number of firms since

$$\Pi^{ii} - \frac{p^i}{p^j}\Pi^{ij} = -x\bar{x} < 0,$$

and the equilibrium is d-stable according to the above definition since

$$\Pi^{ii} + (n-1)\,\Pi^{ij} = -2 - 2\,(n-1)\,x\left(2 - x^2\right)e^{-x^2} < 0.$$

Still, equilibrium prices fall until $n = 3$, and then rise:

<Figure 1>

One might ask: Up to which levels of aggregation does theorem 5 extend? The following example shows that it is "tight" in the sense that even if competitors' outputs are aggregated into *two* numbers then condition A does not imply non-increasing equilibrium prices. Consider the inverse demand function $p\,(x_i, x_{-i}) = \max\left\{0, 1 - x_i - \Sigma_{j\neq i}\left(x_i x_j^2 - x_i^2 x_j^3/2\right)\right\}$, and zero production cost. Condition A holds at the unique symmetric equilibrium for all $n \leq 189$, being stable for all n, but equilibrium market prices are increasing for all $n \geq 6$.

There are three other variables of interest whose equilibrium values vary with the number of firms: Total output, individual outputs, and profits. In supermodular games, i.e. games with complementarities, (smallest and largest) equilibrium strategies rise with the number of players, see Topkis (1998), theorem 4.2.3. In Cournot oligopoly the comparative statics of individual quantities, total quantities and profits each depend on a different condition.

Proposition 11 *Under competitor aggregation the following holds in the comparison between a symmetric equilibrium with n firms and one with $m > n$ firms:*

1. If the equilibrium aggregator increases, $f_{(m)} > f_{(n)}$, then
a) Individual outputs $x_{(n)}$ do not increase (decrease) if $xp\,(x, f)$ fulfills the DWSCP (WSCP) in (x, f) for all $j \neq i$, i.e. if goods are strategic substitutes (complements);
b) Individual profits $\Pi_{(n)}$ do not increase, and decrease strictly if $p\,(x, f)$ is strictly decreasing in f.

2. If the equilibrium price is positive and does not increase, then total output $Q_{(n)} = nx_{(n)}$ does not decrease (increases strictly) if f is quasi-convex and condition BW (BS) holds.

Finally, industry profits, i.e. the sum of profits of all firms in the industry, may be increasing or decreasing, as already described by Seade (1980) for homogeneous goods.

Proposition 11 does not depend on whether the equilibria in question are stable or even regular, thus is valid for all symmetric equilibria. Lemma 4 and theorem 5 show that for maximal and minimal equilibria the assumptions of proposition 11 hold, therefore their comparative statics are as expected. Proposition 9 indicates that one should expect them to hold as well for interior equilibria, because the aggregator tends to be increasing, and price decreasing, while we make no assumptions about *how* the equilibrium with more firms is actually established.

Lastly, we give an illustration of what quasi-convexity of f means: Two units of good 2 instead of one unit of good 2 and one unit of good 3 result in a (weakly) lower price of good 1, i.e. are a closer substitute for good 1. This implies a taste for variety, and is automatically fulfilled for homogeneous goods or if there are only two firms in the market.

6 Related literature

In this section we will give an overview of the existing literature on existence of Cournot equilibria and comparative statics of entry, and show how it relates to this paper.

From the viewpoint of game theory, Cournot equilibria exist under the assumptions that profits are concave or quasi-concave and quantities bounded, using the general result that games with quasi-concave payoffs and compact strategy spaces possess pure equilibria (see Friedman 1991 or Dasgupta and Maskin 1986). These conditions involve strong indirect assumptions about demand and production costs, therefore there have been many attempts to identify those features in the Cournot oligopoly model that guarantee the existence of equilibria under weaker assumptions.

Most work has concentrated on economically meaningful conditions on demand or cost, or both. Interestingly, practically all proofs of existence are related with either one or the other of Hahn's (1962) pair of stability conditions,

$$p' + xp'' \leq 0 \Leftrightarrow \Pi^{ij} \leq 0, \tag{16}$$

$$p' - c'' \leq 0 \Leftrightarrow \Pi^{ii} \leq \Pi^{ij}, \tag{17}$$

which are thus seen to impose two different kinds of strong regularity on the model. The most intuitive way to characterize both strands of literature is to express all conditions used in terms of slopes of (quantity) reaction functions $r(x_{-i})$: Assuming that profits are twice differentiable, we obtain $r'(x_{-i}) = -\Pi^{ij}/\Pi^{ii}$. The second strand of literature effectively assumes that this slope is bounded below by -1, while strategic substitutes (complements) imply that $r' \leq (\geq) 0$. Our condition AD in general implies $r' \geq -p^j/p^i$, which is equal to -1 under homogeneous goods, and larger than -1 under condition BS.

Condition (16) is on demand only, and means that marginal revenue does not increase if competitors raise their outputs. This implies that goods are *strategic substitutes* in the terminology of Bulow *et al.* (1985), i.e. that best quantity replies are non-increasing. More generally, goods being strategic substitutes (complements) follows from profits $\Pi(x_i, x_{-i})$ having weakly decreasing (increasing) differences in outputs (x_i, x_j) for $j \neq i$, or $\Pi^{ij} \leq (\geq)\ 0$ under differentiability.

For homogeneous goods, Novshek (1985) shows that (possibly non-symmetric) Cournot equilibria exist if goods are strategic substitutes. For general aggregative games, i.e. "competitor aggregation", Corchón (1994 , 1996) effectively imposes both of Hahn's conditions (in my notation $\Pi^{ii} < \Pi^{ij} < 0$), and proves existence through the concavity of payoffs, while Kukushkin (1994) only assumes strategic substitutes. For strategic *complements* Vives (1990), shows existence of pure Cournot equilibria for differentiated goods (symmetric if firms are symmetric) in the general context of supermodular games.

The second strand, related to condition (17), initially imposed assumptions on costs only, and proves the existence of *symmetric* equilibria with homogeneous goods. McManus (1962, 1964) and Roberts and Sonnenschein (1976) show that symmetric pure Cournot equilibria exist assuming that production costs are convex or linear, while Szidarovsky and Yakowitz (1977) additionally assume that inverse demand is concave. Kukushkin (1993), assuming convex costs, shows existence of pure symmetric equilibria if outputs are discrete variables.[24] Amir and Lambson (2000), still for homogeneous goods, directly assume $p' - c'' < 0$, allowing for limited increasing returns to scale in production, and prove existence of pure symmetric Cournot equilibria. Their work is important in several respects: It shows that the assumption of convex costs can be relaxed, and that the relevant condition $p' - c'' < 0$ is lattice-theoretic in nature. As one can see from condition AD, our condition A is a direct generalization to differentiated goods of this condition.[25][26]

While condition A generalizes the second strand of the literature, its relation with the first strand is not straightforward. For homogeneous goods the assumption of strategic complements implies condition A if profits are (at least locally) concave. Since profits are locally concave at interior best replies, this result captures the fact that reaction functions certainly have slope larger than -1 if they are nondecreasing. For differentiated goods this relationship is not clear.

Most of the above authors have only covered the case of homogeneous goods. Kukushkin (1994) and Corchón (1994, 1996), assuming strategic substitutes, deal with additive aggregation, i.e. where the *sum* of competitors' outputs is relevant; Vives (1990) allows for general non-homogeneous goods and strategic complements,

[24] *Mixed* equilibria always exist if outputs are discrete.

[25] Amir (1996b) obtains new results in both strands of the literature, imposing various log-concavity or -convexity conditions on demand and cost, and using the ordinal approach of Milgrom and Shannon (1994).

[26] Spence (1976) presents a class of demand functions with a special functional structure where Cournot equilibria can be found maximizing a certain 'wrong' surplus function, later called a 'potential'. Here the question of existence of Nash equilibria reduces to the question of existence of maxima of this function. Slade (1994) finds a necessary and sufficient condition for the existence of a potential, e.g. that for homogeneous goods demand must be linear. The literature on 'potential games' is further developed in Monderer and Shapley (1996) and Kukushkin (1999); interestingly, the latter also uses an ordinal approach.

but if goods are strategic substitutes his approach works only if there are no more than two firms. Our work is the first to address the question of existence of equilibria with differentiated goods in a general context that does not make use of the assumption of strategic substitutes or complements. Rather, it is based on the second strand of literature and can be understood as a generalization of Amir and Lambson's (2000) work to differentiated goods.

Comparative statics on demand or cost variables for Cournot oligopoly have been analyzed by many authors, among them Frank (1965), Dixit (1986), Corchón (1994, 1996), while comparative statics with respect to entry have been discussed by Frank (1965), Ruffin (1971), Seade (1980b), Szidarovsky and Yakowitz (1982), Corchón (1994, 1996), and recently by Amir and Lambson (2000). The case of differentiated goods has been treated by Dixit (1986) for two firms, and by Corchón (1994, 1996) for additive aggregation, but they as most other authors have imposed the assumption of strategic substitutes from the outset, which is irrelevant for most comparative-statics conclusions, in particular for the question of quasi-competitiveness. In fact, Amir and Lambson have shown for homogeneous goods that the only relevant condition for decreasing equilibrium prices and increasing equilibrium total quantity after entry is that there are no strong increasing returns to scale or Hahn's condition (17). Our conditions AD and A are generalizations to differentiated goods of this condition.[27]

De Meza (1985) and Villanova, Paradis and Viader (1999) exhibit examples where for $n \geq 2$ equilibrium prices are decreasing (therefore quasi-competitive) but are higher than the monopoly price. This outcome is entirely due to the assumption of strongly increasing returns to scale in production that can only be exploited by monopolists. Our condition A rules this out (as did Seade's (1980) stability condition A), therefore theorem 5 holds also for the transition from monopoly to duopoly. De Meza also raises the interesting point that stability or "observability" of equilibrium does not imply decreasing equilibrium prices since the monopolist's output choice is by definition observable even if it lies on an "unstable" part of the reaction function. We agree with his point of view, but again theorem 5 is not subject to this critique since it does not depend on stability.

7 Conclusions

Using an ordinal approach, we established the existence of pure symmetric Cournot equilibria for homogeneous or differentiated goods under a simple condition that generalizes the condition of weakly increasing returns used in the literature for the case of homogeneous goods. We were able to rule out asymmetric equilibria using a

[27] An issue that is related but different from quasi-competitiveness is the issue of *convergence* of equilibrium to the 'competitive price'. Ruffin (1971), following McManus (1964) and Frank (1965), shows that the Cournot equilibrium price converges to the competitive price (defined as minimum average cost) if there are diseconomies of scale, and do not converge if there are increasing returns to scale.

weak additional condition, and after discussing stability and regularity of equilibria from an ordinal point of view, establish a strict relation between a weak notion of stability and multiplicity of symmetric equilibria under competitor aggregation.

Under the main condition (maximal and minimal) equilibrium prices are decreasing in the number of firms if competitors' quantities enter inverse demand as an aggregate, but may be increasing if competitors' quantities are aggregated into more than one number. For all equilibria, stable or unstable, there is sequence of neighboring stable equilibria involving a higher value of the aggregator and lower market prices. Total quantity increases with the number of firms under some additional conditions, while individual quantities increase (decrease) if goods are strategic complements (substitutes). Individual firm profits decrease after entry. Any decrease in the aggregator or increase in price must then be caused by the choice of an unstable equilibrium point, to which out-of-equilibrium dynamics (almost surely) does not converge.

One topic for further research is to extend our methods (and maybe some results) to cases where product differentiation is not symmetric, as e.g. in Hotelling models, or to models with exogenous or endogenous quality differences. We have indicated some possibilities for generalization in these directions. Second, similar results will be obtained by applying corresponding conditions to models of differentiated goods price competition.

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8 Appendix

8.1 Proof of Lemma 1

Proof. Let minimum and maximum prices be $p_K = p(K, .., K)$ and $p_0 = (0, .., 0)$, and the interval of possible prices with outputs by the other firms given

$$\pi(x_{-i}) = [p(K, x_{-i}), p(0, x_{-i})], \; x_{-i} \in [0, K]^{n-1}. \tag{18}$$

The set $\pi(x_{-i})$ is the new constraint set for the maximization over p_i, obviously non-empty, compact and convex, and is descending (nonincreasing) in x_{-i} (see Milgrom and Shannon 1994) since $p(., x_{-i})$ is nonincreasing in x_{-i}.

The range of possible combinations between market price and the others' outputs is

$$X = \left\{ (p_i, x_{-i}) \in [p_K, p_0] \times [0, K]^{n-1} | p_i \in \pi(x_{-i}) \right\}. \tag{19}$$

Let $\bar{x}(x_{-i})$ be the maximum output that firm i will produce given that the other firms are already producing x_{-i}, either because this output is equal to capacity, or because inverse demand becomes zero:

$$\bar{x}(x_{-i}) = \min\{K, \min\{x \in [0, K] | p(x, x_{-i}) = 0\}\} \geq 0. \tag{20}$$

Then since p is strictly decreasing and continuous on $x_i \in [0, \bar{x}(x_{-i})]$, we can express firm i's output as a function $\chi : X \to [0, K]$ that is continuous and nonincreasing in $(p_i, x_{-i}) \in X$, and strictly decreasing in p_i with image $[0, \bar{x}(x_{-i})]$ for each $x_{-i} \in [0, K]^{n-1}$.[28]. ■

8.2 Proof of Theorem 2

Proof. First we show that price best replies are well-defined given the regularity conditions. Given any vector of outputs $x_{-i} \in [0, K]^{n-1}$ of the other firms, firm i's maximization problem is

$$\max_{p_i \in \pi(x_{-i})} \tilde{\Pi}(p_i, x_{-i}) = \chi(p_i, x_{-i}) p_i - c(\chi(p_i, x_{-i})),$$

where χ is continuous in p_i, c is continuous, and

$$\pi(x_{-i}) = [p(K, x_{-i}), p(0, x_{-i})]$$

is a non-empty compact set. Then $\tilde{\Pi}$ is a continuous function of p_i on the compact set $\pi(x_{-i})$ and therefore attains its maximum. Thus the price best reply $P(x_{-i})$ exists, where $P : [0, K]^{n-1} \to [p_K, p_0]$ is a correspondence that is symmetric in x_{-i}

[28]This is an application of a continuous version of the implicit function theorem.

since $p(x_i, x_{-i})$ is symmetric in x_{-i}. Now restrict P to the 'diagonal' $x_{-i} = (y, ..., y)$, $y \in [0, K]$, and define

$$\bar{P}(y) = P(y, .., y), \ y \in [0, K].$$

Secondly, maximal and minimal price best replies in $\bar{P}(y)$ are nonincreasing in y: The constraint set $\pi(x_{-i})$ is decreasing (Milgrom and Shannon 1994), since both $p(K, x_{-i})$ and $p(0, x_{-i})$ are nonincreasing in x_{-i}. This follows from the assumptions that goods are substitutes and that $p(x_i, x_{-i})$ is continuous in x_{-i}. Invoking this fact and condition A, by Milgrom and Shannon's (1994) monotonicity theorem the set of maximizers $\bar{P}(y)$ is decreasing in y. This implies in particular that maximum and minimum selections of $\bar{P}$ exist and are nonincreasing in y. Let $\tilde{P} : [0, K] \to [p_K, p_0]$ be a maximum or minimum selection, then $\tilde{P}$ is a nonincreasing function, and therefore continuous but for downward jumps (we only need the latter).

Finally, we construct fixed points of a suitable map. Given identical outputs $y \in [0, K]$ for all competitors, quantity best replies $\tilde{r} : [0, K] \to [0, K]$ "on the diagonal" are given by $\tilde{r}(y) = \chi\left(\tilde{P}(y), y, ..., y\right)$. Then $\tilde{r}$ is continuous but for upward jumps, since χ is continuous in all arguments, and decreasing in its first, while $P(x_{-i})$ is continuous but for downward jumps. Therefore $\tilde{r}$ has maximal and minimal fixed points (Milgrom and Roberts 1994, cor. 1), which are equilibrium outputs. ■

8.3 Proof of Proposition 3

Proof. Consider an asymmetric equilibrium $x = (x_1, \ldots, x_n)$, where w.l.o.g. $x_1 = y + \varepsilon > x_2 = y$. Then $x' = (x_2, x_1, x_3, \ldots, x_n)$ is also an equilibrium. For conciseness we now suppress the arguments $(x_3, \ldots, x_n)$. Equilibrium prices for firm 1 are $p = p(y + \varepsilon, y)$ and $p'(y, y + \varepsilon)$, with $p' > p$ by condition BS or $p' \geq p$ by condition BW.

First impose condition BS, leading to $p' > p$. Under condition AS, i.e. under the dual strict single-crossing property of firm i's profit $\tilde{\Pi}$ in (p_i, y) for all $i = 1..n$, all selections of best price replies are nonincreasing, i.e. $p' \leq p$ since $y + \varepsilon > y$, which is a contradiction to $p' > p$.

For the second statement impose condition BW, leading to $p' \geq p$. Since under condition ASD for all $i = 1..n$ the partial derivative $\partial\tilde{\Pi}/\partial p_i$ exists and is strictly decreasing in x_j $(j \neq i)$, price best replies are strictly decreasing in x_j (Theorem 2.8.5 in Topkis 1998). Therefore, since p is a best reply at y and p' at $y + \varepsilon > y$, we must have $p' < p$, and again arrive at a contradiction. ■

8.4 Proof of Lemma 4

Proof. Consider the family of correspondences ψ_n, $n \geq 1$, as defined in the text, and note that for all $f \in [\underline{f}, \infty)$

$$\underline{f} = g(0, n) \leq \psi_n(f) = g(R(f), n) \leq g(K, n) = \bar{f}_n$$

since $R(f) \in [0, K]$ for all f, therefore all fixed points of ψ_n lie in $\left[\underline{f}, \bar{f}_n\right]$. As g is continuous and strictly increasing in its first argument, the maximal and minimal selections of ψ_n, $\psi_n^{\max}$ and $\psi_n^{\min}$, are both continuous but for upward jumps. Clearly, for each f the value of $\psi_n(f)$ is non-decreasing in n, being constant only for f such that $R(f) = 0$. Also, since g is strictly increasing in its second argument if $x > 0$, $\psi_n(f) < \psi_{n+1}(f)$ for all $f \in \left[\underline{f}, \bar{f}_n\right]$ such that $\psi_n(f) > \underline{f}$, and $\psi_n(f) = \psi_{n+1}(f)$ if $\psi_n(f) = \underline{f}$ and $n \geq 2$ (because then $R(f) = 0$). For each $n \geq 1$, by theorem 1 in Milgrom and Roberts (1994) $\psi_n^{\min}$ has a minimal fixed point $f_n^{\min}$, and $\psi_n^{\max}$ has a maximal fixed point $f_n^{\max}$, which are non-decreasing in n. For $n = 1$ the unique fixed point is $\underline{f}$. If monopoly is strictly viable then $R\left(\underline{f}\right) > 0$, and $\psi_n\left(\underline{f}\right) > \underline{f}$ for all $n \geq 2$. In this case for $n \geq 2$ all fixed points are larger than $\underline{f}$, where ψ_n is strictly increasing in n, therefore the maximum and minimum ones are strictly increasing in n.

Last but no least, we show that f is the value of the aggregator at an equilibrium point with n firms if and only if it is a fixed point of ψ_n. Let y^* be quantity at a symmetric equilibrium with n firms, $f = g(y^*, n)$, then $\psi_n(f) = g(R(f), n) = g(y^*, n) = f$ because y^* is a best reply. For the converse, let f^* be a fixed point of ψ_n. For $n = 1$, $f^* = \underline{f}$, and any $y^* \in R\left(\underline{f}\right)$ fulfills $g(y^*, 1) = \underline{f}$. For $n \geq 2$, $g(x, n)$ is strictly increasing in x, therefore there is a unique $y \in [0, K]$ such that $f^* = g(y, n)$. Then $g(y, n) = f^* = \psi_n(f^*) = g(R(f^*), n)$, i.e. y is a best reply to f^* and therefore an equilibrium output. Therefore, the largest and smallest fixed points of ψ_n are the aggregators for the largest and smallest symmetric Cournot equilibria. ■

8.5 Proof of Theorem5

Proof. Under condition A, by lemma 4 the aggregator f in maximal and minimal equilibria is non-decreasing in n, and minimal and maximal best reply prices $P(f)$ (which are the new equilibrium prices) are non-increasing in f, thus extremal equilibrium prices are non-increasing in n.

As for statement 2, the maximal or minimal equilibrium aggregators are strictly increasing in n as long as equilibrium prices are positive and monopoly is strictly viable. If $\partial\tilde{\Pi}/\partial p_i$ exists and is strictly decreasing in x_j (and thus in f), by theorem 2.8.5 of Topkis (1998) extremal price best replies $P(f)$ are strictly decreasing in f while prices are positive. Thus extremal equilibrium prices are strictly decreasing in n. ■

8.6 Proof of Proposition 7

Proof. Consider a symmetric equilibrium with quantity x^* produced by each of the n firms. Let $f_\delta = g(x^* + \delta, n)$. Under condition S, by theorem 2.8.5 of Topkis (1998) the best quantity reply $R(f_\delta)$ to $x^* + \delta$ by all competitors is smaller (higher) than $x^* + \delta$ if $\delta > (<)\ 0$. Thus

$$f_\delta = g(x^* + \delta, n) > (<)\ g(R(f_\delta), n) = \psi_n(f_\delta),$$

and the equilibrium is d-stable. On the other hand, if condition US holds then d-regularity and instability follow along the same lines. ■

8.7 Proof of Proposition 8

Proof. Assume that all symmetric equilibria are d-stable, i.e. ψ_n strictly cuts the diagonal from above at each $f^* = g(x^*, n)$ where x^* is an equilibrium output. Since ψ_n is continuous this can only be true if there is exactly one symmetric equilibrium.

By condition A maximal and minimal equilibria exist. Assume now that at some symmetric equilibrium x^* condition US holds, then ψ_n strictly cuts the diagonal from below at $f^* = g(x^*, n)$. Since at the minimal equilibrium $x^*_{\min}$ we know that $\psi_n(f) \geq f$ from the left, and at the maximal equilibrium $x^*_{\max}$, $\psi_n(f) \leq f$ from the right, we have that $x^*_{\min} < x^* < x^*_{\max}$, and there are at least three different symmetric equilibria. ■

8.8 Proof of Proposition 9

Proof. Since monopoly is strictly viable, we have $\psi_m(f^*_n) > f^*_n$ for any $m > n$. Thus $\psi'_m(f) = \max\{f^*_n, \psi_m(f)\}$ is a continuous function on $\left[f^*_n, \bar{f}_m\right]$ and fulfills the assumptions of theorem 1 in Milgrom and Roberts (1994) and has a smallest fixed point $f^*_m > f^*_n$. This fixed point is the smallest fixed point of $\psi_m(f)$ above f^*_n. Since the same argument can be used on any $l > m > n$, there is a smallest fixed point f^*_l of ψ_l above f^*_m. Now since $\psi_l(f) > \psi_m(f) > f$ on $[f^*_n, f^*_m]$, f^*_l also is the smallest fixed point of ψ_l above f^*_n. Therefore the smallest fixed points above f^*_n exist and are ordered, $f^*_l > f^*_m > f^*_n$. The corresponding statement cannot be made about "highest fixed points below f^*_n", because they may not exist at all.[29] Still, if for $m > n$ a highest fixed point f^-_m below f^*_n exists then $f^-_m = \psi_m(f^-_m) > \psi_l(f^-_m)$ for all l between n and m. By continuity ψ_l must have a fixed point f^-_l somewhere in (f^-_m, f^*_n). By the same logic as above these fixed points are ordered.

Consider now for $m > n$ a smallest higher fixed point f^*_m (highest lower fixed point f^-_m). Since ψ_m is a continuous function and $\psi_m(f^*_n) > f^*_n$, it cuts the diagonal at f^*_m from the left (at f^-_m from the right) and from above, therefore the equilibrium at f^*_m (f^-_m) is either d-stable (unstable) or non-regular (We cannot say anything about how ψ_m cuts from the other side). ■

[29] Therefore the application of the "local converse" of theorem 1 in Milgrom and Roberts (1994), as they propose on p. 455, is somewhat problematic.

8.9 Proof of Lemma 10

Proof. Denote the partial derivatives of inverse demand $p(x,f)$ with respect to x and f as $p_1 < 0$ and $p_2 \leq 0$, of $f(x_{-i})$ at a symmetric equilibrium with respect to x_j $(j \neq i)$ as $f_x > 0$ (therefore $p^i = p_1$ and $p^j = p_2 f_x$), and of $g(x,n)$ with respect to x and n as $g_x = (n-1) f_x > 0$ and $g_n = x f_x > 0$, respectively. The first-order necessary condition for an interior best reply in terms of quantity, and the second derivatives of profits, are

$$
\begin{aligned}
\Pi^i &= \frac{\partial}{\partial x_i}\Pi(x_i, x_{-i}) = p(x_i, f(x_{-i})) + x_i p_1(x_i, f(x_{-i})) - c'(x_i) = 0, \\
\Pi^{ii} &= \frac{\partial^2}{\partial x_i^2}\Pi(x_i, x_{-i}) = 2p_1 + x_i p_{11} - c'', \\
\Pi^{ij} &= \frac{\partial^2}{\partial x_i \partial x_j}\Pi(x_i, x_{-i}) = (p_2 + x_i p_{12}) f_x.
\end{aligned}
$$

The partial derivatives of $x(f,n)$ are $x_f = 1/g_x$ and $x_n = -g_n/g_x$, and the first order condition can be expressed in terms of $f_{(n)}$ as

$$
\Pi^i = p\left(x\left(f_{(n)}, n\right), f_{(n)}\right) + x\left(f_{(n)}, n\right) p_1\left(x\left(f_{(n)}, n\right), f_{(n)}\right) - c'\left(x\left(f_{(n)}, n\right)\right) = 0,
$$

leading to

$$
\frac{df_{(n)}}{dn} = -\frac{\Pi^{ii} x_n}{\Pi^{ii} x_f + \left(p_2 + x_{(n)} p_{12}\right)} = \frac{x_{(n)} f_x \Pi^{ii}}{\Pi^{ii} + (n-1)\Pi^{ij}}.
$$

In terms of quantities,

$$
\Pi^i = p\left(x_{(n)}, g\left(x_{(n)}, n\right)\right) + x_{(n)} p_1\left(x_{(n)}, g\left(x_{(n)}, n\right)\right) - c'\left(x_{(n)}\right) = 0,
$$

then we obtain

$$
\frac{dx_{(n)}}{dn} = -\frac{x_{(n)} \Pi^{ij}}{\Pi^{ii} + (n-1)\Pi^{ij}}.
$$

The total derivative of equilibrium prices is

$$
\begin{aligned}
\frac{dp_{(n)}}{dn} &= \frac{d}{dn} p\left(x_{(n)}, f_{(n)}\right) = -p_1 \frac{x_{(n)} \Pi^{ij}}{\Pi^{ii} + (n-1)\Pi^{ij}} + p_2 \frac{x_{(n)} f_x \Pi^{ii}}{\Pi^{ii} + (n-1)\Pi^{ij}} \\
&= \frac{x_{(n)} p_2 f_x}{\Pi^{ii} + (n-1)\Pi^{ij}} \left(\Pi^{ii} - \frac{p_1}{p_2 f_x}\Pi^{ij}\right). \blacksquare
\end{aligned}
$$

8.10 Proof of Proposition 11

Proof. Since the equilibrium aggregator $f_{(n)}$ is increasing in n, thus by the DWSCP (WSCP) of $xp(x,f)$ in (x,f) equilibrium output $x_{(n)}$ is non-increasing (non-decreasing) in n. Since goods are substitutes, profits $\Pi(x,f)$ are non-increasing in f. Since $f_{(n)}$ is increasing in n,

$$\Pi\left(x_{(n)}, f_{(n)}\right) \geq \Pi\left(x_{(n+1)}, f_{(n)}\right) \geq \Pi\left(x_{(n+1)}, f_{(n+1)}\right),$$

where $x_{(n)}$ and $x_{(n+1)}$ are the corresponding equilibrium outputs, and the first inequality expresses the fact that $x_{(n)}$ maximizes profits given $f_{(n)}$. If $p(x,f)$ is strictly decreasing in f then the second inequality is strict.

As for the sum of outputs, assume that $Q_{(n+1)} = (n+1)\,x_{(n+1)} < Q_{(n)} = nx_{(n)}$, then $x_{(n+1)} < \frac{n}{n+1}x_{(n)}$. Denote by $(x)_m$ the m-dimensional vector with all elements equal to $x \in \mathbb{R}$. Then we have

$$\begin{aligned} p_{(n+1)} &= p\left(x_{(n+1)}, f\left[\left(x_{(n+1)}\right)_n\right]\right) \\ &> p\left(\frac{n}{n+1}x_{(n)}, f\left[\left(\frac{n}{n+1}x_{(n)}\right)_n\right]\right) \\ &\geq p\left(x_{(n)}, f\left[\left(\frac{n}{n+1}x_{(n)}\right)_{n-1}, \frac{n-1}{n+1}x_{(n)}\right]\right) \\ &\geq p\left(x_{(n)}, f\left[\left(x_{(n)}\right)_{n-1}, 0\right]\right) = p_{(n)} \end{aligned}$$

The first (strict) inequality follows from the fact that $p(x,f)$ is strictly decreasing in x while positive, non-decreasing in f, and $f(x)$ increasing in x; the second one from a transference of output $x_{(n)}/(n+1)$ from firm $n+1$ to firm 1 under condition BW, and the last one from the quasi-convexity of f, as we show below. Thus we have concluded that if $Q_{(n+1)} < Q_{(n)}$ then equilibrium price strictly increases. This is ruled out by assumption, and we have arrived at a contradiction. Under condition BS the second inequality is strict, and we can relax our counterfactual assumption to $Q_{(n+1)} \leq Q_{(n)}$ while arriving at the same contradiction.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is quasi-convex if $f\left(\Sigma_{i=1}^n \lambda_i a_i\right) \leq \max_{i=1..n} f(a_i)$ for $\lambda_i \geq 0$ for all i and $\Sigma_{i=1}^n \lambda_i = 1$. Let $e_i \in \mathbb{R}^n$, $i = 1..n$, be vectors of one's and equal to zero at element i, e.g. $e_1 = (0, 1, .., 1)$, and consider $a_i = x_{(n)} e_i$, $i = 1..n$. Because of symmetry, $f(a_i) = f(a_n) = f\left[\left(x_{(n)}\right)_{n-1}, 0\right]$ for all $i = 1..n$. Let $\lambda_1 = .. = \lambda_{n-1} = \frac{1}{n+1}$ and $\lambda_n = \frac{2}{n+1}$, then

$$\begin{aligned} f\left(\Sigma_{i=1}^n \lambda_i a_i\right) &= f\left(\left((n-2)\frac{1}{n+1} + \frac{2}{n+1}\right)_{n-1} x_{(n)}, (n-1)\frac{1}{n+1}x_{(n)}\right) \\ &= f\left(\left(\frac{n}{n+1}x_{(n)}\right)_{n-1}, \frac{n-1}{n+1}x_{(n)}\right) \\ &\leq f(a_n) = f\left[\left(x_{(n)}\right)_{n-1}, 0\right]. \end{aligned}$$

This proves the second statement. ■

Figure Legends

Figure 1: Rising equilibrium prices without aggregation.

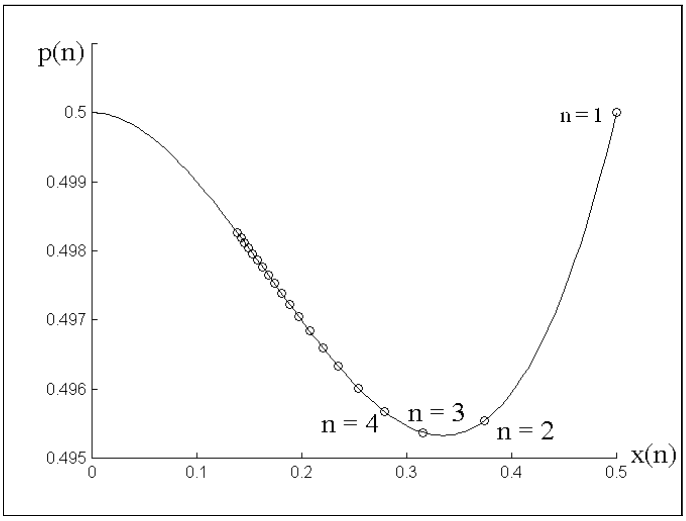

Figure 1