This note proves the equality in footnote 36.

Theorem

Let $\tilde{\Phi}(z_1, z_2, r)$ be the cumulative distribution function of a standard bivariate distribution with correlation coefficient r.

If $r \geq 0$

$$\sup \; -\frac{\frac{\partial^2}{\partial z_1 \partial z_2} \ln \tilde{\Phi}(z_1, z_2, r)}{\frac{\partial^2}{\partial z_2{}^2} \ln \tilde{\Phi}(z_1, z_2, r)} = r$$

and

if $r \leq 0$,

$$\inf \; -\frac{\frac{\partial^2}{\partial z_1 \partial z_2} \ln \tilde{\Phi}(z_1, z_2, r)}{\frac{\partial^2}{\partial z_1{}^2} \ln \tilde{\Phi}(z_1, z_2, r)} = r \, .$$

Proof.

Let the density function of $\tilde{\Phi}(z_1, z_2, r)$ be $\tilde{\phi}(z_1, z_2, r)$. Then $\tilde{\Phi}_{12} = \tilde{\phi}$. Let ϕ be the density function of the univariate standard normal distribution. Since

$$\tilde{\phi}(z_1, z_2, r) = \frac{1}{\sqrt{1-r^2}} \phi(z_1) \phi\left(\frac{z_2 - r z_1}{\sqrt{1-r^2}} \right) = \frac{1}{\sqrt{1-r^2}} \phi(z_2) \phi\left(\frac{z_1 - r z_2}{\sqrt{1-r^2}} \right), \tag{1}$$

by integrating it, we obtain

$$\tilde{\Phi}_1(z_1, z_2, r) = \phi(z_1) \Phi\left(\frac{z_2 - r z_1}{\sqrt{1-r^2}} \right) , \tag{2}$$

$$\tilde{\Phi}_2(z_1, z_2, r) = \phi(z_2) \Phi\left(\frac{z_1 - r z_2}{\sqrt{1-r^2}} \right) , \tag{3}$$

$$\tilde{\Phi}(z_1,z_2,r)=\int_{-\infty}^{z_1}\phi(s)\Phi\left(\frac{z_2-rs}{\sqrt{1-r^2}}\right)ds=\int_{-\infty}^{z_2}\phi(s)\Phi\left(\frac{z_1-rs}{\sqrt{1-r^2}}\right)ds, \tag{4}$$

and by differentiating (2), we obtain

$$\tilde{\Phi}_{11}(z_1,z_2,r)=-\frac{r}{\sqrt{1-r^2}}\phi(z_1)\phi\left(\frac{z_2-rz_1}{\sqrt{1-r^2}}\right)-z_1\phi(z_1)\Phi\left(\frac{z_2-rz_1}{\sqrt{1-r^2}}\right). \tag{5}$$

Putting these equations into the numerator of

$$\frac{\frac{\partial^2}{\partial z_1\partial z_2}\ln\tilde{\Phi}(z_1,z_2,r)}{\frac{\partial^2}{\partial z_1^2}\ln\tilde{\Phi}(z_1,z_2,r)}+r=\frac{\tilde{\Phi}_{12}\tilde{\Phi}-\tilde{\Phi}_1\tilde{\Phi}_2}{\tilde{\Phi}_{11}\tilde{\Phi}-\tilde{\Phi}_1^2}+r=\frac{(\tilde{\Phi}_{12}+r\tilde{\Phi}_{11})\tilde{\Phi}-(\tilde{\Phi}_2+r\tilde{\Phi}_1)\tilde{\Phi}_2}{\tilde{\Phi}_{11}\tilde{\Phi}-\tilde{\Phi}_1^2}, \tag{6}$$

we obtain

$$\begin{aligned}&(\tilde{\Phi}_{12}+r\tilde{\Phi}_{11})\tilde{\Phi}-(\tilde{\Phi}_2+r\tilde{\Phi}_1)\tilde{\Phi}_1\\&=\phi(z_1)\left\{\left(\sqrt{1-r^2}\,\phi\left(\frac{z_2-rz_1}{\sqrt{1-r^2}}\right)-rz_1\Phi\left(\frac{z_2-rz_1}{\sqrt{1-r^2}}\right)\right)\tilde{\Phi}\right.\\&\left.-\left(\phi(z_2)\Phi\left(\frac{z_1-rz_2}{\sqrt{1-r^2}}\right)+r\phi(z_1)\Phi\left(\frac{z_2-rz_1}{\sqrt{1-r^2}}\right)\right)\Phi\left(\frac{z_2-rz_1}{\sqrt{1-r^2}}\right)\right\}\end{aligned} \tag{7}$$

Since

$$\begin{aligned}&\frac{\partial}{\partial z_2}\left\{\left(\sqrt{1-r^2}\,\phi\left(\frac{z_2-rz_1}{\sqrt{1-r^2}}\right)-rz_1\Phi\left(\frac{z_2-rz_1}{\sqrt{1-r^2}}\right)\right)\right.\\&=\left\{\left(-\left(\frac{z_2-rz_1}{\sqrt{1-r^2}}\right)\phi\left(\frac{z_2-rz_1}{\sqrt{1-r^2}}\right)-\frac{rz_1}{\sqrt{1-r^2}}\phi\left(\frac{z_2-rz_1}{\sqrt{1-r^2}}\right)\right)\right.=-\frac{z_2}{\sqrt{1-r^2}}\phi\left(\frac{z_2-rz_1}{\sqrt{1-r^2}}\right),\end{aligned}$$

and

$$\frac{\partial}{\partial z_2}\left(\phi(z_2)\Phi\left(\frac{z_1-rz_2}{\sqrt{1-r^2}}\right)+r\phi(z_1)\Phi\left(\frac{z_2-rz_1}{\sqrt{1-r^2}}\right)\right)$$

$$= \left(-z_2 \phi(z_2) \Phi\left(\frac{z_1 - r z_2}{\sqrt{1-r^2}} \right) - \frac{r}{\sqrt{1-r^2}} \phi(z_2) \phi\left(\frac{z_1 - r z_2}{\sqrt{1-r^2}} \right) + \frac{r}{\sqrt{1-r^2}} r \phi(z_1) \phi\left(\frac{z_2 - r z_1}{\sqrt{1-r^2}} \right) \right)$$

$$= -z_2 \phi(z_2) \Phi\left(\frac{z_1 - r z_2}{\sqrt{1-r^2}} \right)$$

(7) is equal to

$$\left(\tilde{\Phi}_{12} + r \tilde{\Phi}_{11} \right) \tilde{\Phi} - \left(\tilde{\Phi}_2 + r \tilde{\Phi}_1 \right) \tilde{\Phi}_1$$

$$= \phi(z_1) \left\{ -\int_{-\infty}^{z_2} \frac{s}{\sqrt{1-r^2}} \phi\left(\frac{s - r z_1}{\sqrt{1-r^2}} \right) ds \int_{-\infty}^{z_2} \phi(s) \Phi\left(\frac{z_1 - r s}{\sqrt{1-r^2}} \right) ds \right.$$

$$\left. + \left(\int_{-\infty}^{z_2} s \phi(s) \Phi\left(\frac{z_1 - r s}{\sqrt{1-r^2}} \right) ds \right) \int_{-\infty}^{z_2} \frac{1}{\sqrt{1-r^2}} \phi\left(\frac{s - r z_1}{\sqrt{1-r^2}} \right) ds \right\}.$$

$$= \frac{1}{\sqrt{1-r^2}} \left(\int_{-\infty}^{z_2} \phi\left(\frac{s - r z_1}{\sqrt{1-r^2}} \right) ds \right)^2 \left\{ \frac{\int_{-\infty}^{z_2} s \phi(s) \Phi\left(\frac{z_1 - r s}{\sqrt{1-r^2}} \right) ds}{\int_{-\infty}^{z_2} \phi\left(\frac{s - r z_1}{\sqrt{1-r^2}} \right) ds} \right.$$

$$\left. - \left\{ \frac{\int_{-\infty}^{z_2} \frac{s}{\sqrt{1-r^2}} \phi\left(\frac{s - r z_1}{\sqrt{1-r^2}} \right) ds}{\int_{-\infty}^{z_2} \phi\left(\frac{s - r z_1}{\sqrt{1-r^2}} \right) ds} \right\} \left\{ \frac{\int_{-\infty}^{z_2} \phi(s) \Phi\left(\frac{z_1 - r s}{\sqrt{1-r^2}} \right) ds}{\int_{-\infty}^{z_2} \phi\left(\frac{s - r z_1}{\sqrt{1-r^2}} \right) ds} \right\} \right\}$$

$$= \frac{\phi(z_1)}{\sqrt{1-r^2}} \left(\int_{-\infty}^{z_2} \phi\left(\frac{s - r z_1}{\sqrt{1-r^2}} \right) ds \right)^2 \left\{ \frac{\int_{-\infty}^{z_2} s \left(\Phi\left(\frac{z_1 - r s}{\sqrt{1-r^2}} \right) \phi\left(\frac{z_1 - r s}{\sqrt{1-r^2}} \right)^{-1} \right) \phi\left(\frac{s - r z_1}{\sqrt{1-r^2}} \right) ds}{\int_{-\infty}^{z_2} \phi\left(\frac{s - r z_1}{\sqrt{1-r^2}} \right) ds} \right.$$

$$-\frac{\int_{-\infty}^{z_2} s\phi\left(\frac{s-rz_1}{\sqrt{1-r^2}}\right)ds}{\int_{-\infty}^{z_2}\phi\left(\frac{s-rz_1}{\sqrt{1-r^2}}\right)ds}\frac{\int_{-\infty}^{z_2}\left(\Phi\left(\frac{z_1-rs}{\sqrt{1-r^2}}\right)\phi\left(\frac{z_1-rs}{\sqrt{1-r^2}}\right)^{-1}\right)\phi\left(\frac{s-rz_1}{\sqrt{1-r^2}}\right)ds}{\int_{-\infty}^{z_2}\phi\left(\frac{s-rz_1}{\sqrt{1-r^2}}\right)ds}\Bigg\},$$

where we use (1). The terms in the brace is the covariance between s and

$\Phi\left(\frac{z_1-rs}{\sqrt{1-r^2}}\right)\phi\left(\frac{z_1-rs}{\sqrt{1-r^2}}\right)^{-1}$ evaluated with a density function proportional to $\phi\left(\frac{s-rz_1}{\sqrt{1-r^2}}\right)$.

Since Φ is log concave, $\frac{\Phi}{\phi}$ is increasing, so that $\Phi\left(\frac{z_1-rs}{\sqrt{1-r^2}}\right)\phi\left(\frac{z_1-rs}{\sqrt{1-r^2}}\right)^{-1}$ is decreasing or increasing according as $r \geq 0$ or $r \leq 0$. Hence the numerator of (6) is negative or positive according as $r \geq 0$ or $r \leq 0$. Since the denominator of (6) is strictly positive by the strict log concavity of Φ, (6) is positive or negative according as $r \geq 0$ or $r \leq 0$.

Hence

$$r \geq -\frac{\frac{\partial^2}{\partial z_1 \partial z_2}\ln\tilde{\Phi}(z_1,z_2,r)}{\frac{\partial^2}{\partial {z_2}^2}\ln\tilde{\Phi}(z_1,z_2,r)} \quad \text{if } r \geq 0$$

and

$$r \leq -\frac{\frac{\partial^2}{\partial z_1 \partial z_2}\ln\tilde{\Phi}(z_1,z_2,r)}{\frac{\partial^2}{\partial {z_2}^2}\ln\tilde{\Phi}(z_1,z_2,r)} \quad \text{if } r \leq 0.$$

It remains to show that

$$
-\frac{\dfrac{\partial^2}{\partial z_1 \partial z_2}\ln\tilde{\Phi}(z_1,z_2,r)}{\dfrac{\partial^2}{\partial {z_2}^2}\ln\tilde{\Phi}(z_1,z_2,r)} = \frac{\left(\dfrac{1}{\sqrt{1-r^2}}\phi(z_1)\phi(B)\right)\tilde{\Phi} - \big(\phi(z_1)\Phi(B)\big)\big(\phi(z_2)\Phi(A)\big)}{\big(\phi(z_1)\Phi(B)\big)^2 - \left\{-\dfrac{r}{\sqrt{1-r^2}}\phi(z_1)\phi(B) - z_1\phi(z_1)\Phi(B)\right\}\tilde{\Phi}}
$$

$$
= \frac{\dfrac{1}{\sqrt{1-r^2}}\left(\dfrac{\tilde{\Phi}}{\Phi(A)}\right) - \dfrac{\Phi(B)}{\phi(B)}\phi(z_2)}{\dfrac{\phi(z_1)\Phi(B)}{\Phi(A)} + \left\{\dfrac{r}{\sqrt{1-r^2}} + \dfrac{\sqrt{1-r^2}}{r}B\dfrac{\Phi(B)}{\phi(B)} + \dfrac{z_1}{r}\right\}\left(\dfrac{\tilde{\Phi}}{\Phi(A)}\right)} \to r
$$

as $z_1 \to \infty$ for given z_2, where $A = \frac{z_1 - rz_2}{\sqrt{1-r^2}}$ and $B = \frac{z_2 - rz_1}{\sqrt{1-r^2}}$. As $z_1 \to \infty$, $A \to \infty$,

$B \to -\infty$, $\phi(z_1) \to 0$, $\Phi(A) \to 1$, $\Phi(B) \to 0$, $\frac{\Phi(B)}{\phi(B)} \to 0$, $B\frac{\Phi(B)}{\phi(B)} \to -1$, which is easily

verified by L'hopital's rule, and

$$
\lim_{z_1\to\infty}\tilde{\Phi} = \lim_{z_1\to\infty}\int_{-\infty}^{z_2}\phi(s)\Phi\left(\frac{z_1 - rs}{\sqrt{1-r^2}}\right)ds
$$

$$
= \int_{-\infty}^{z_2}\phi(s)\left[\lim_{z_1\to\infty}\left(\Phi\left(\frac{z_1 - rs}{\sqrt{1-r^2}}\right)\right)\right]ds = \Phi(z_2),
$$

where $\left|\Phi\left(\frac{z_1 - rs}{\sqrt{1-r^2}}\right)\right| < 1$ and hence we can use the dominant convergent theorem. QED.