Cournot versus Walras in Dynamic Oligopolies with Memory

Carlos Alós-Ferrer
Department of Economics, University of Vienna
Hohenstaufengasse, 9. A-1010 Vienna (Austria)

Abstract

This document contains the proofs of technical lemmata omitted from the print version.

A Proof of Theorem 1

Lemma A.2. *For any x, $M(x,\cdot)$ is a strictly concave function. Define $y^M(x) = argmax\{M(x,y)/0 \leq y \leq Q_{max}\}$. Then, $y^M(x)$ is continuous and decreasing. Moreover, there exists $Q^M > x^C$ such that $y^M(x)$ is differentiable and strictly decreasing in $(0, Q^M)$ and $y^M(x) = 0$ for all $x \in [Q^M, Q_{max}]$.*

Proof. For a given, fixed x, the function $M(x,y)$ is strictly concave in y:

$$\frac{\partial^2 M(x,y)}{\partial y^2} = P''((N-1)x+y)y + 2P'((N-1)x+y) - C''(y) < 0$$

Hence, the first order condition

$$\frac{\partial M(x,y)}{\partial y} = P'((N-1)x+y)y + P((N-1)x+y) - C'(y) = 0$$

is necessary and sufficient for an interior global maximum.

The function y^M is continuous by the Maximum Theorem. For all x, we have that $M(x, Q_{max}) = -C(Q_{max}) - P(N\cdot x)x + C(x) < -C(0) - P(N\cdot x)x + C(x) = M(x,0)$ for all x. It follows that $y^M(x) < Q_{max}$. That is, either $y^M(x) = 0$ or it is an interior solution $(0 < y^M(x) < Q_{max})$.

In the sub-domain where it is strictly positive, $y^M(x)$ is implicitly defined by the first order condition above. By the Implicit Function Theorem, this function is differentiable in this set and has first derivative

$$\frac{dy^M}{dx} = -\left(\frac{\partial^2 M(x,y)}{\partial y^2}\right)^{-1} [(N-1)P''((N-1)x+y)y + (N-1)P'((N-1)x+y)] < 0$$

Hence, $y^M(x)$ is strictly decreasing in this set, and equal to zero outside it. Continuity of $y^M(x)$ in its whole domain and the fact that $y^M(x^C) = x^C > 0$ complete the claim. ■

Lemma A.6. *No state* $\mathrm{mon}(x,K)$ *with $x^C \leq x \leq x^W$ can be destabilized with only one mutation.*

Proof. In the print version, it was left to the reader to show that, for $x^C \leq x \leq x^W$, $D(x, \cdot)$ is increasing in $[0, x]$. The first derivative is

$$\frac{\partial D}{\partial y} = P'((N-1)x + y)(y - x) + P((N-1)x + y) - C'(y)$$

and we will show that it is strictly positive in $[0, x)$, therefore proving the claim. Since $P'(\cdot) < 0$ and $y < x$, the first part of the expression is strictly positive. For the second part, notice that the function $P(N \cdot z) - C'(z)$ is strictly decreasing in z and has a zero at $z = x^W$.[1] Then, since $x \leq x^W$, we have that $P(N \cdot x) \geq C'(x)$. Since $y < x$, it follows that $P((N-1)x + y) > P(N \cdot x) \geq C'(x) > C'(y)$, completing the proof. ■

Lemma A.7. *For $p \in [0, P_{max}]$, the function $g_p(z) = p \cdot z - C(z)$ is strictly concave in z, and it attains a unique maximum $z(p)$. The function $z(p)$ is increasing and $z(P(N \cdot x^W)) = x^W$.*

Proof. The second derivative of g_p is simply $-C''(z) < 0$, proving strict concavity and the existence of a unique maximum in the compact set $[0, Q_{max}]$. Moreover, $z(p)$ is continuous by the Maximum Theorem, and, for every p, either $z(p)$ is an interior maximum, or $z(p) \in \{0, Q_{max}\}$. The first order condition for an interior maximum is $p = C'(z)$, i.e. $z(p) = [C']^{-1}(p)$. This function is strictly increasing because C' is strictly increasing. Hence, the function z must be increasing by continuity. Finally, $z(P(N \cdot x^W)) = x^W$ follows by definition. ■

B Proof of Theorem 2

Lemma B.3. *For $x \in [x^C, x^W]$, define*

$$h(x) = max\{y \in [x, 2 \cdot x] | H(x, y) \geq 0\}$$

Then, h is continuous, and either $h(x) = 2x$, or $h(x) \in (x, 2 \cdot x)$. In the latter case, $h(x)$ is implicitly defined by $H(x, h(x)) = 0$, and it is strictly decreasing.[2]

Proof. Fix x and denote $H_x(y) = H(x, y)$. Then,

$$\begin{aligned} H'_x(y) &= P'(y + (N-2)x)(y - x) + P(y + (N-2)x) - C'(y) \\ H''_x(y) &= P''(y + (N-2)x)(y - x) + 2P'(y + (N-2)x) - C''(x) < 0 \end{aligned}$$

and hence H_x is strictly concave. It follows that the set

$$K_x = \{y \in [x, 2 \cdot x] | H(x, y) \geq 0\}$$

is convex, and the correspondence $x \mapsto K_x$ is continuous and compact-valued, so $h(x)$ is continuous (e.g. by an application of the Maximum Theorem to the identity function $y \mapsto y$), and $K_x = [x, h(x)]$ (by concavity). Moreover,

$$H'_x(x) = P((N-1)x) - C'(x) > P(N \cdot x) - C'(x) \geq 0$$

[1] The first derivative is $N \cdot P'(N \cdot z) - C''(z) < 0$. The walrasian quantity x^W is a zero of this function by definition.

[2] Hence, h is either decreasing or "tent-shaped," because $2x$ is increasing and the implicit function defined by $H(x, h(x)) = 0$ is decreasing.

since $x \leq x^W$, and hence $h(x) > x$. Hence, either $h(x)$ is in the interior of the interval (and then, by continuity, $H(x, h(x)) = 0$), or $h(x) = 2x$.

It remains to show that $h(x)$ is strictly decreasing whenever $H(x, h(x)) = 0$. By the Implicit Function Theorem, in this case the function $h(x)$ is differentiable and has first derivative

$$h'(x) = -\left(\frac{\partial H(x, h(x))}{\partial x}\right)\frac{1}{H_x'(h(x))}.$$

Since H_x is strictly concave and $H_x(x) = H_x(h(x)) = 0$, with $h(x) > x$, it follows that $H_x'(h(x)) < 0$. By direct computation,

$$\frac{\partial H(x,y)}{\partial x} = (N-2)P'((N-2)x+y)(y-x) - P((N-2)x+y) + C'(x)$$

$$\frac{\partial H(x,h(x))}{\partial x} =$$

$$= (N-2)P'((N-2)x+h(x))(h(x)-x) - P((N-2)x+h(x)) + C'(x)$$

We claim this last quantity to be negative. Notice that $P'(\cdot) < 0$ and $(h(x) - x) > 0$, hence the first part is negative. For the second part, notice that from $H(x, h(x)) = 0$ follows that

$$P((N-2)x + h(x)) = \frac{C(h(x)) - C(x)}{h(x) - x}$$

and this quantity is strictly larger than $C'(x)$ by strict convexity of C.[3] Hence, the second part is also negative. In summary, $h'(x) < 0$. ■

Lemma B.4. *For $x > x^W$, define*

$$f(x) = min\{y \in [0,x] | D(x,y) \geq 0\}.$$

Then, f is continuous, and either $f(x) = 0$ or $f(x) \in (0, x^W)$. In the latter case, f is implicitly defined by $D(x, f(x)) = 0$, and it is strictly decreasing. Moreover, $D(x,y) > 0$ for all $y \in (f(x), x)$.

Proof. Consider $D_x(y) = D(x,y) = P((N-1)x+y)(y-x)+C(x)-C(y)$. Note that $D_x(x) = 0$ and

$$D_x'(y) = P'((N-1)x+y)(y-x) + P((N-1)x+y) - C'(y)$$

and hence $D_x'(x) = P(Nx) - C'(x) < 0$ for $x > x^W$. Hence, $D_x(y) > 0$ for $y < x$ close enough to x.

If $D_x(y) \geq 0$ for all $y \in (0, x)$ then the claim is true with $f(x) = 0$. If not, then there exists $x' \in (0, x)$ such that $D(x, x') = 0$.

It is enough to show that, if $D(x, x') = 0$, then $D(x, y) > 0$ for all $y \in (x', x)$. This implies that there cannot exist two different quantities $x', x'' \in (0, x)$ with $D(x, x') = 0$ and $D(x, x'') = 0$ and, by continuity of D, proves the claim.

Suppose, then, $D(x, x') = 0$. Call $P' = P((N-1)x + x')$. We have that

$$P' \cdot x' - C(x') = P' \cdot x - C(x)$$

[3]Taking a Taylor expansion, $C(h(x)) = C(x) + C'(x)(h(x)-x) + \frac{1}{2}C''(\theta_x)(h(x)-x)^2$ and it follows that $\frac{C(h(x))-C(x)}{h(x)-x} > C'(x)$ since $C''(\cdot) > 0$.

Since the function given by $g_{P'}(z) = P' \cdot z - C(z)$ is strictly concave by Lemma A.7, it follows that $P' \cdot y - C(y) > P' \cdot x - C(x)$ for all $y \in (x', x)$.

Consider any $y \in (x', x)$. Since $y > x'$, we have that $P((N-1)x + y) < P((N-1)x + x') = P'$. Since $y < x$, it follows that $P((N-1)x + y)(y - x) > P' \cdot (y - x)$ and hence

$$D(x, y) = P((N-1)x+y)(y-x) - C(y) + C(x) > P' \cdot (y-x) - C(y) + C(x) > 0$$

as needed.

Note that, by Lemma 1, $D(x, x^W) > 0$, which implies that $f(x) < x^W$.

It remains to show that f is continuous for $x > x^W$. This follows from the Maximum Theorem since $f(x)$ is the argmin of the identity function in the set $K'_x = \{y / D(x, y) \geq 0\}$. We have shown that this set is a closed interval (compact and convex) for all $x > x^W$ (and we can define $K'_{x^W} = \{x^W\}$), and we know that $D(\cdot, \cdot)$ is a continuous function, hence continuity of the correspondence $x \mapsto K'_x$ follows.

In the sub-domain where it is strictly positive, $f(x)$ is implicitly defined by $D(x, f(x)) = 0$. By the Implicit Function Theorem, this function is differentiable in this set and has first derivative

$$f'(x) = -\left(\frac{\partial D(x, f(x))}{\partial x}\right) \frac{1}{D'_x(f(x))}.$$

By definition of $f(x)$ and continuity, it follows that $D'_x(f(x)) > 0$. By direct computation,

$$\begin{aligned}
\frac{\partial D(x, y)}{\partial x} &= (N-1)P'((N-1)x + y)(y - x) - P((N-1)x + y) + C'(x) \\
\frac{\partial D(x, f(x))}{\partial x} &= \\
&= (N-1)P'((N-1)x + f(x))(f(x) - x) - P((N-1)x + f(x)) + C'(x)
\end{aligned}$$

We claim this last quantity to be positive. Notice that $P'(\cdot) < 0$ and $(f(x) - x) < 0$, hence the first part is positive. For the second part, notice that from $D(x, f(x)) = 0$ follows that

$$P((N-1)x + f(x)) = \frac{C(x) - C(f(x))}{x - f(x)}$$

and this quantity is strictly lower than $C'(x)$ by strict convexity of C.[4] This implies the second part to be also positive.

In summary, f is a strictly decreasing function in the set where it is implicitly defined by $D(x, f(x)) = 0$, and zero outside it. By continuity, f is decreasing in its whole domain. ■

Lemma B.5. *The function $D(x, y)$ is strictly convex in x for $x > y$. For all $y < x^W$ there exists a unique $\phi(y) > x^W$ such that $D(\phi(y), y) = 0$ and $D(x, y) > 0 \; \forall \; x > \phi(y)$. Taking $\phi(x^W) = x^W$, the function ϕ is continuous in $(0, x^W]$.*[5]

In particular, if $x_2 > x_1 > y$ and $D(x_1, y) \geq 0$, then $D(x_2, y) > 0$.

[4] Taking a Taylor expansion, $C(f(x)) = C(x) + C'(x)(f(x) - x) + \frac{1}{2}C''(\theta_x)(f(x) - x)^2$ and it follows that $\frac{C(f(x)) - C(x)}{f(x) - x} < C'(x)$ since $C''(\cdot) > 0$ and $(f(x) - x) < 0$.

[5] It is easy to see that the functions f and ϕ are partial inverses, i.e. $f(\phi(x)) = x$ for all $x \in [0, x^W]$, but $\phi(f(x)) \neq x$ in general (f is not invertible).

Proof. Fix y. Consider $d_y(x) = D(x,y) = P((N-1)x+y)(y-x)+C(x)-C(y)$. Note that $d'_y(x) = \frac{\partial D(x,y)}{\partial x}$ computed in the proof of Lemma B.4, and

$$d''_y(x) = (N-1)^2 P''((N-1)x+y)(y-x) - 2(N-1)P'((N-1)x+y) + C''(x) > 0$$

i.e. $D(x,y)$ is strictly convex in x for $x > y$.

Let $y < x^W$. Note that $d'_y(y) = -P(N \cdot y) + C'(y)$. Since the function $P(N \cdot z) - C'(z)$ is strictly decreasing in z and has a zero at x^W (see footnote 1 in Lemma A.6), it follows that $d'_y(y) < 0$. Noting that $d_y(Q_{max}) = C(Q_{max}) - C(y) > 0$, it follows that there exists $\phi(y) \in (y, Q_{max})$ such that $d_y(\phi(y)) = 0$ and $d_y(x) > 0 \ \forall \ x > \phi(y)$. Since $D(x^W, y) < 0$ by Lemma 1, necessarily $\phi(y) > x^W$.

Continuity of ϕ follows from convexity of $D(\cdot, y)$ analogously to Lemmata B.3 and B.4.

In particular, consider $x_2 > x_1 > y$ such that $d_y(x_1) = D(x_1, y) \geq 0$. Since $d_y(y) = D(y,y) = 0$, it follows from convexity that $d_y(x)$ is increasing for $x > x_1$ (since, if d_y has a minimum, it must be lower than x_1). Hence, $D(x_2, y) \geq D(x_1, y) \geq 0$. ■

Lemma B.6. *Let $x \in [x^C, x^W]$. Then, $D(h(x), x) > 0$. Moreover, $h(x) > x^W$.*

Proof. By Lemma B.3, either $h(x) \in (x, 2x)$, or $h(x) = 2x$. We distinguish both cases. If $h(x) \in (x, 2x)$, again by Lemma B.3, we have that $H(x, h(x)) = 0$, i.e.

$$P(h(x) + (N-2)x)(h(x) - x) = C(h(x)) - C(x).$$

Then,

$$\begin{aligned} D(h(x), x) = P((N-1)h(x) + x)(x - h(x)) - C(x) + C(h(x)) = \\ [P(h(x) + (N-2)x) - P((N-1)h(x) + x)] \cdot (h(x) - x) \end{aligned}$$

Since $h(x) > x$, it follows that $h(x) + (N-2)x < (N-1)h(x) + x$, thus

$$P(h(x) + (N-2)x) > P((N-1)h(x) + x)$$

implying $D(h(x), x) > 0$.

Suppose now that $h(x) = 2x$. Then,

$$D(2x, x) = P((N-1)2x + x)(x - 2x) - C(x) + C(2x)$$

Let $\overline{p} = P((N-1)2x + x) = P((2N-1)x)$. Since $N > 2$, $\overline{p} < P((N+1)x)$. Taking a Taylor expansion,

$$P((N+1)x) = P(Nx) + P'(Nx)x + \frac{1}{2}P''(\xi_x)x^2 \leq P(Nx) + P'(Nx)x \leq C'(x)$$

where the first inequality follows from $P''(\cdot) \leq 0$, and the second from the fact that $x \geq x^C$.[6] Hence, $\overline{p} < C'(x)$.

Recall that, by Lemma A.7, $g_p(z) = p \cdot z - C(z)$ is strictly concave and attains a maximum at $z(p)$. Since $\overline{p} < C'(x)$, it follows that $g'_{\overline{p}}(x) < 0$ and $z(\overline{p}) \leq x$, implying that $g_{\overline{p}}$ is strictly decreasing for $z \geq x$. In particular,

$$D(2x, x) = \overline{p}(x - 2x) - C(x) + C(2x) = g_{\overline{p}}(x) - g_{\overline{p}}(2x) > 0$$

[6]The function $P(Nx) + P'(Nx)x - C'(x)$ is strictly decreasing and has a unique zero at $x = x^C$.

which proves the claim.

It remains to prove that $h(x) > x^W$. Suppose otherwise. Then, by Lemma B.5, $D(h(x), x) > 0$ implies that $D(x^W, x) > 0$, a contradiction with Lemma 1. ■

Lemma B.7. *For all $x \in [x^C, x^W]$, $\phi(x) < h(x)$.*

Proof. Suppose $\phi(x) \geq h(x)$. Then, let $z \in [h(x), \phi(x)]$. Since $z \leq \phi(x)$ and $z \geq h(x) > x$ (by Lemma B.3), it follows from Lemma B.5 that $D(z, x) \leq 0$ (recall that $D(\cdot, x)$ is strictly convex with $D(x, x) = D(\phi(x), x) = 0$). By Lemma B.6, $D(h(x), x) > 0$. Since $z \geq h(x)$, Lemma B.5 implies that $D(z, x) > 0$, a contradiction. ■