Advanced Industrial Economics (second edition): Solutions Manual

Stephen Martin
FEE / F&O
University of Amsterdam
Roetersstraat 11
1018 WB Amsterdam
The Netherlands

July 2001; updated January 2002

Contents

Chapter 1

Introduction

This *Solutions Manual* contains some supplementary material, answers to end-of-chapter problems, and gives supplementary references to the literature that come to my attention.

Typographical errors are noted, as they come to my attention, in a separate document, *aie2typos.pdf*, which may be downloaded from the web page that supports the book.

Chapter 2

Foundations of Oligopoly Theory I

An early perspective on Cournot that is not noted in the text is

Roy, René "L'oeuvre économique d'Augustin Cournot," *Econometrica* 7:2 April 1939, pp. 134–44.

2.1 Answers to Problems

2.1 Let a market have linear inverse demand curve with equation

$$p = a - bQ. \tag{2.1}$$

(a) find Cournot equilibrium price, output, and output per firm if the market is supplied by n firms, each with constant marginal and average cost c per unit.

Firm 1's payoff is

$$\pi_1 = (a - c - bQ)q_1 = [a - c - b(q_1 + Q_{-1})]q_1, \tag{2.2}$$

where

$$Q_{-1} = q_2 + q_2 + \ldots + q_n \tag{2.3}$$

is the combined output of all firms except firm 1.

The first-order condition to maximize (2.2) is

$$\frac{\partial \pi_1}{\partial q_1} = a - c - b(2q_1 + Q_{-1}) = 0. \tag{2.4}$$

The second-order condition for a maximum is satisfied:

$$\frac{d^2 \pi_1}{dq_1^2} = -2b < 0. \tag{2.5}$$

Firms have identical cost functions and identical beliefs; in equilibrium, all firms will produce the same output. Set $q_1 = q$, $Q_{-1} = (n-1)q$ in (2.2) and rearrange terms to obtain an expression for Cournot equilibrium output per firm:

$$q = \frac{1}{n+1}\frac{a-c}{b}. \tag{2.6}$$

Total output in equilibrium is

$$Q = \frac{n}{n+1}\frac{a-c}{b} = \left(1 - \frac{1}{n+1}\right)\frac{a-c}{b}. \tag{2.7}$$

Substituting (2.7) in (2.1), Cournot equilibrium price is

$$p = a - b\left(1 - \frac{1}{n+1}\right)\frac{a-c}{b} = c + \frac{a-c}{n+1} = \frac{a+nc}{n+1}. \tag{2.8}$$

The equilibrium Lerner index of market power is

$$\frac{p-c}{p} = \frac{a-c}{n+1}\frac{n+1}{a+nc} = \frac{a-c}{a+nc}$$

$$= \frac{a-c}{a-c+(n-1)c} = \frac{1}{1+(n-1)\frac{c}{a-c}} < 1. \tag{2.9}$$

Since the product is homogeneous, it is straightforward to compute equilibrium consumers' surplus,

$$CS = \frac{1}{2}(a-p)Q = \frac{1}{2}\left(a - c - \frac{a-c}{n+1}\right)\frac{n}{n+1}\frac{a-c}{b}$$

$$= \frac{1}{2b}\left(\frac{n}{n+1}\right)^2 (a-c)^2. \tag{2.10}$$

Net social welfare is the sum of consumers' surplus and economic profit,

$$NSW = \frac{1}{2b}\left(\frac{n}{n+1}\right)^2 (a-c)^2 + (p-c)Q$$

$$= \frac{1}{2b}\left(\frac{n}{n+1}\right)^2 (a-c)^2 + \frac{a-c}{n+1}\frac{n}{n+1}\frac{a-c}{b}$$

$$= \frac{(n+1)^2 - 1}{(n+1)^2}\frac{(a-c)^2}{2b}. \tag{2.11}$$

(b) find Cournot equilibrium price, output, and output per firm if the constant marginal and average cost of firm i is c_i. (Without loss of generality, let $c_1 \leq c_2 \leq \ldots \leq c_n$ and (hint) write $\overline{c} = (1/n)\sum_1^n c_i$ for the industry-average value of unit cost.)

First solution: firm i's profit is

$$\pi_i = (a - c_i - bQ)q_i = [a - c_i - b(q_i + Q_{-i})]q_i. \tag{2.12}$$

The first-order condition to maximize (2.12) is

$$\frac{\partial \pi_i}{\partial q_i} = a - c_i - b(2q_i + Q_{-i}) = 0 \tag{2.13}$$

Write the system of first-order conditions as

$$\begin{array}{rcl} 2q_1 + q_2 + \ldots + q_n &=& \frac{a-c_1}{b} \\ q_1 + 2q_2 + \ldots + q_n &=& \frac{a-c_2}{b} \\ &\vdots& \\ q_1 + q_2 + \ldots + 2q_n &=& \frac{a-c_n}{b} \end{array} \tag{2.14}$$

Add the first-order conditions to obtain an expression for equilibrium total output:

$$(n+1)(q_1 + q_2 + \ldots + q_n) = \frac{1}{b}\left(na - \sum_1^n c_i\right) = n\frac{a-\overline{c}}{b} \tag{2.15}$$

$$Q = \frac{n}{n+1}\frac{a-\overline{c}}{b}. \tag{2.16}$$

Cournot equilibrium output depends on the average value of marginal cost for firms with positive equilibrium output (Vickrey, 1964, p. 338; Bergstrom and Varian, 1985).

Cournot equilibrium price is

$$p = a - b\left(1 - \frac{1}{n+1}\right)\frac{a-\overline{c}}{b} = \overline{c} + \frac{a-\overline{c}}{n+1}. \tag{2.17}$$

To find the equilibrium output of firm i, use the equation of its first-order condition, (2.13):

$$a - c_i - b(2q_i + Q_{-i}) = a - c_i - b(q_i + Q) = 0 \tag{2.18}$$

$$q_i = \frac{a-c_i}{b} - Q = \frac{a-c_i}{b} - \frac{n}{n+1}\frac{a-\overline{c}}{b}$$

$$q_i = \frac{a-(n+1)c_i + n\overline{c}}{(n+1)b} \tag{2.19}$$

$$q_i = \frac{a - c_i - n\left(c_i - \overline{c}_{-i}\right)}{(n+1)b}, \tag{2.20}$$

where $\overline{c}_{-i} = \frac{1}{n-1}\sum_{j=1,j\neq i}^{n} c_j$ is the average value of marginal cost for all firms except firm i. Firm i has a smaller equilibrium output, the greater is its marginal cost relative to the average marginal cost of all other firms (Cournot, 1838, 1927, pp. 85–6).

Second solution: write the system of equations of first-order conditions (2.14) in matrix form as

$$b\begin{pmatrix} 2 & 1 & \dots & 1 \\ 1 & 2 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \dots & 2 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix} = a\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} - \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \tag{2.21}$$

Let I_n denote an $n \times n$ identity matrix and J_n an $n \times 1$ column vector of 1s, q (with abuse of notation) the $n \times 1$ column vector of outputs, and c

the $n \times 1$ column vector of unit costs. The system of equations (2.21) can be written

$$b(I_n + J_n J_n')q = aJ_n - c. \tag{2.22}$$

Let the inverse of the matrix $I_n + J_n J_n'$ be $I_n + kJ_n J_n'$, where k is a parameter to be determined. If $I_n + kJ_n J_n'$ is the inverse of $I_n + J_n J_n'$, it must be that

$$(I_n + J_n J_n')(I_n + kJ_n J_n') = I_n$$

$$I_n + kJ_n J_n' + J_n J_n' + kJ_n J_n' J_n J_n' = I_n$$

$$I_n + kJ_n J_n' + J_n J_n' + nkJ_n J_n' = I_n$$

$$[1 + (n+1)k]J_n J_n' = 0, \tag{2.23}$$

and for (2.23) to hold it must be that

$$k = -\frac{1}{n+1}. \tag{2.24}$$

Hence

$$(I_n + J_n J_n')^{-1} = I_n - \frac{1}{n+1} J_n J_n' \tag{2.25}$$

and the vector of equilibrium outputs can be found from (2.22)

$$b\left(I_n - \frac{1}{n+1} J_n J_n'\right)(I_n + J_n J_n')q = \left(I_n - \frac{1}{n+1} J_n J_n'\right)(aJ_n - c).$$

$$bq = \left(I_n - \frac{1}{n+1} J_n J_n'\right)(aJ_n - c). \tag{2.26}$$

Now

$$\left(I_n - \frac{1}{n+1} J_n J_n'\right) J_n = J_n - \frac{1}{n+1} J_n J_n' J_n = \left(1 - \frac{n}{n+1}\right) J_n = \frac{1}{n+1} J_n \tag{2.27}$$

and

$$\left(I_n - \frac{1}{n+1} J_n J_n'\right) c = c - \frac{n}{n+1} \overline{c} J_n. \tag{2.28}$$

Hence the vector of equilibrium outputs satisfies

$$bq = \frac{1}{n+1} a J_n - c + \frac{n}{n+1} \overline{c} J_n. \tag{2.29}$$

This gives the same results, firm by firm, as (2.19). The other aspects of equilibrium follow as in the first solution.

2.2 (Stigler, 1940) In a duopoly market, let the equation of the inverse demand curve be

$$p(q_1 + q_2) = 85 - \frac{1}{20}(q_1 + q_2),$$

and let the equations of the cost functions of the two firms be

$$c_1(q_1) = 3000 + 9q_1 + \frac{1}{200}q_1^2$$

and

$$c_2(q_2) = 3500 + 8q_2 + \frac{1}{200}q_2^2$$

respectively.
(a) Find the equations of the best response functions, identifying the levels of rival firm output at which each firm would shut down because its maximum profit would be negative if it produced positive output. Graph the best response functions on the same diagram; indicate the Cournot equilibrium point.

Firm 1's profit is

$$\pi_1 = \left[85 - \frac{1}{20}(q_1 + q_2)\right] q_1 - 3000 - 9q_1 - \frac{1}{200}q_1^2$$

$$= \left(76 - \frac{11}{200}q_1 - \frac{1}{20}q_2\right) q_1 - 3000$$

The first-order condition to maximize π_1 (the equation of firm 1's best response function) is

$$76 - \frac{11}{200}q_1 - \frac{1}{20}q_2 - \frac{11}{200}q_1 = 0.$$

Hence along the best response function

$$76 - \frac{11}{200}q_1 - \frac{1}{20}q_2 = \frac{11}{200}q_1$$

and firm 1's payoff is

$$\pi_1 = \frac{11}{200}q_1^2 - 3000.$$

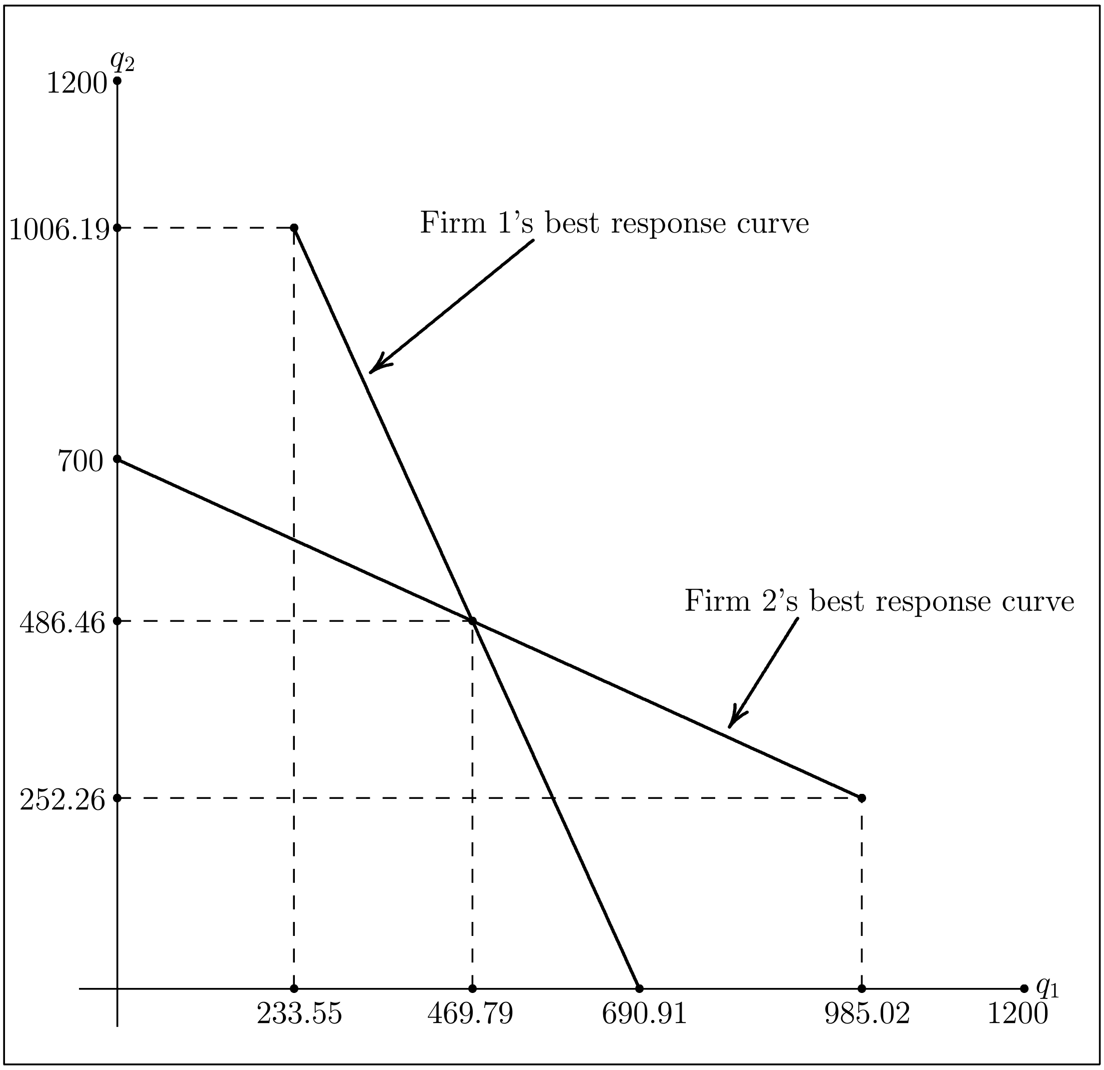


Figure 2.1: Best response curves, Stigler duopoly example

Firm 1 will produce only if its payoff is nonnegative. Hence firm 1 will shut down if

$$q_1 < \sqrt{\frac{200}{11}(3000)} = 233.55.$$

The equation of firm 1's best response function can also be written

$$11q_1 + 5q_2 = 7600,$$

and this is valid for $q_1 \geq 233.55$.

$$\pi_2 = \left[85 - \frac{1}{20}(q_1 + q_2)\right] q_2 - 3500 - 8q_2 - \frac{1}{200}q_2^2$$

$$= \left(77 - \frac{1}{20}q_1 - \frac{11}{200}q_2\right) q_1 - 3500.$$

Firm 2's payoff along its best response function is

$$\pi_2 = \frac{11}{200}q_2^2 - 3500;$$

firm 2 will shut down for

$$q_2 < \sqrt{\frac{200}{11}(3500)} = 252.26.$$

The equation of firm 2's best response function is

$$5q_1 + 11q_2 = 7700,$$

and this is valid for $q_2 \geq 252.26$.

The best response curves are shown in Figure 2.1.

(b) Find the Cournot equilibrium outputs by solving the equations of the best response functions. What is Cournot equilibrium price, consumer surplus, and net social welfare?

Write the system of equations of the best response functions as

$$\begin{pmatrix} 11 & 5 \\ 5 & 11 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = 100 \begin{pmatrix} 76 \\ 77 \end{pmatrix}$$

$$96\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = 100\begin{pmatrix} 11 & -5 \\ -5 & 11 \end{pmatrix}\begin{pmatrix} 76 \\ 77 \end{pmatrix} = 100\begin{pmatrix} 451 \\ 467 \end{pmatrix}$$

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \frac{25}{24}\begin{pmatrix} 45\,1 \\ 46\,7 \end{pmatrix} = \begin{pmatrix} 469\,79 \\ 486.46 \end{pmatrix}.$$

Equilibrium payoffs are

$$\pi_1 = \frac{11}{200}(469.\,79)^2 - 3000 = 9138.6$$

$$\pi_2 = \frac{11}{200}(486.\,46)^2 - 3500 = 9515.4.$$

Cournot equilibrium output and price are

$$Q = 469.\,79 + 486.46 = 956.25$$

$$p = 85 - \frac{1}{20}(956.25) = 37.19.$$

Consumers' surplus, the area of the triangle below the inverse demand curve and above the line $p = 37.19$, is

$$\frac{1}{2}(85 - 37.19)(956.25) = 22859.16.$$

Net social welfare is the sum of firm profits and consumers' surplus:

$$9138.6 + 9515.4 + 22859.16 = 45513.2.$$

2.3 (Constant elasticity of demand) In a duopoly market, let the equation of the demand curve be

$$Q = ap^{-\varepsilon}$$

for $Q = q_1 + q_2$ and a, $\varepsilon > 0$. Let average and marginal cost be $c > 0$.

Discuss the shape of best response curves; find equilibrium outputs.

Shape of the best response curves.

The equation of the inverse demand curve is

$$p = \left(\frac{a}{Q}\right)^{1/\varepsilon} = \left(\frac{a}{q_1 + q_2}\right)^{1/\varepsilon}. \tag{2.30}$$

Firm 1's profit is

$$\pi_1 = \left[\left(\frac{a}{q_1 + q_2}\right)^{1/\varepsilon} - c\right] q_1. \tag{2.31}$$

The first-order condition to maximize π_1 is

$$\frac{\partial \pi_1}{\partial q_1} = \left(\frac{a}{q_1 + q_2}\right)^{1/\varepsilon} - c + q_1 \left[\frac{1}{\varepsilon}\left(\frac{a}{q_1 + q_2}\right)^{\frac{1}{\varepsilon}-1}\left(-\frac{a}{(q_1 + q_2)^2}\right)\right]$$

$$= \left(\frac{a}{q_1 + q_2}\right)^{1/\varepsilon}\left(1 - \frac{1}{\varepsilon}\frac{q_1}{q_1 + q_2}\right) - c = 0 \tag{2.32}$$

or

$$\left(\frac{1}{q_1 + q_2}\right)^{1/\varepsilon}\left(1 - \frac{1}{\varepsilon}\frac{q_1}{q_1 + q_2}\right) = \frac{c}{a^{1/\varepsilon}}. \tag{2.33}$$

The second own-output derivative of π_1 is negative,

$$\frac{1}{a^{\frac{1}{\varepsilon}}}\frac{\partial^2 \pi_1}{\partial q_1^2} =$$

$$\left(\frac{1}{q_1 + q_2}\right)^{1/\varepsilon}\left[-\frac{1}{\varepsilon}\frac{q_2}{(q_1 + q_2)^2}\right] + \left[1 - \frac{1}{\varepsilon}\left(\frac{q_1}{q_1 + q_2}\right)\right]\frac{1}{\varepsilon}\left(\frac{1}{q_1 + q_2}\right)^{\frac{1}{\varepsilon}-1}\left[-\frac{1}{(q_1 + q_2)^2}\right], \tag{2.34}$$

so the second-order condition for profit maximization is satisfied.

Find the slope of the best response function: differentiating (2.32) with respect to q_2, the second cross-output derivative satisfies

$$\frac{1}{a^{\frac{1}{\varepsilon}}}\frac{\partial^2 \pi_1}{\partial q_1 \partial q_2} =$$

$$\frac{1}{\varepsilon}\left(\frac{1}{q_1+q_2}\right)^{\frac{1}{\varepsilon}-1}\left[-\frac{1}{(q_1+q_2)^2}\right]\left(1-\frac{1}{\varepsilon}\frac{q_1}{q_1+q_2}\right)+\left(\frac{1}{q_1+q_2}\right)^{\frac{1}{\varepsilon}}\left\{-\frac{1}{\varepsilon}\left[-\frac{q_1}{(q_1+q_2)^2}\right]\right\} =$$

$$-\frac{1}{\varepsilon}\left(\frac{1}{q_1+q_2}\right)^{\frac{1}{\varepsilon}+1}\left(1-\frac{1}{\varepsilon}\frac{q_1}{q_1+q_2}\right)+\frac{q_1}{\varepsilon}\left(\frac{1}{q_1+q_2}\right)^{\frac{1}{\varepsilon}+2} =$$

$$\frac{1}{\varepsilon}\left(\frac{1}{q_1+q_2}\right)^{\frac{1}{\varepsilon}+1}\left[-\left(1-\frac{1}{\varepsilon}\frac{q_1}{q_1+q_2}\right)+\frac{q_1}{q_1+q_2}\right] =$$

$$-\frac{1}{\varepsilon}\left(\frac{1}{q_1+q_2}\right)^{\frac{1}{\varepsilon}+1}\left[1-\left(1+\frac{1}{\varepsilon}\right)\left(\frac{q_1}{q_1+q_2}\right)\right],$$

so that

$$\frac{\partial^2 \pi_1}{\partial q_1 \partial q_2} = \left(\frac{a}{q_1+q_2}\right)^{\frac{1}{\varepsilon}}\frac{q_1-\varepsilon q_2}{\varepsilon^2\left(q_1+q_2\right)^2}. \tag{2.35}$$

Recall that the slope of the best response function is

$$\left.\frac{dq_1}{dq_2}\right|_{foc} = \frac{\frac{\partial^2 \pi_1(q_1,q_2)}{\partial q_1 \partial q_2}}{\left[-\frac{\partial^2 \pi_1(q_1,q_2)}{\partial q_1^2}\right]}, \tag{2.36}$$

where the denominator on the right is positive by the second-order condition. The slope of the best response function thus has the same sign as the numerator on the right in (2.36), and from (2.35), this is the sign of $q_1 - \varepsilon q_2$. In particular, firm 1's best response function has positive slope at its horizontal axis intercept ($q_2 = 0$), negative slope at its vertical axis intercept ($q_1 = 0$).

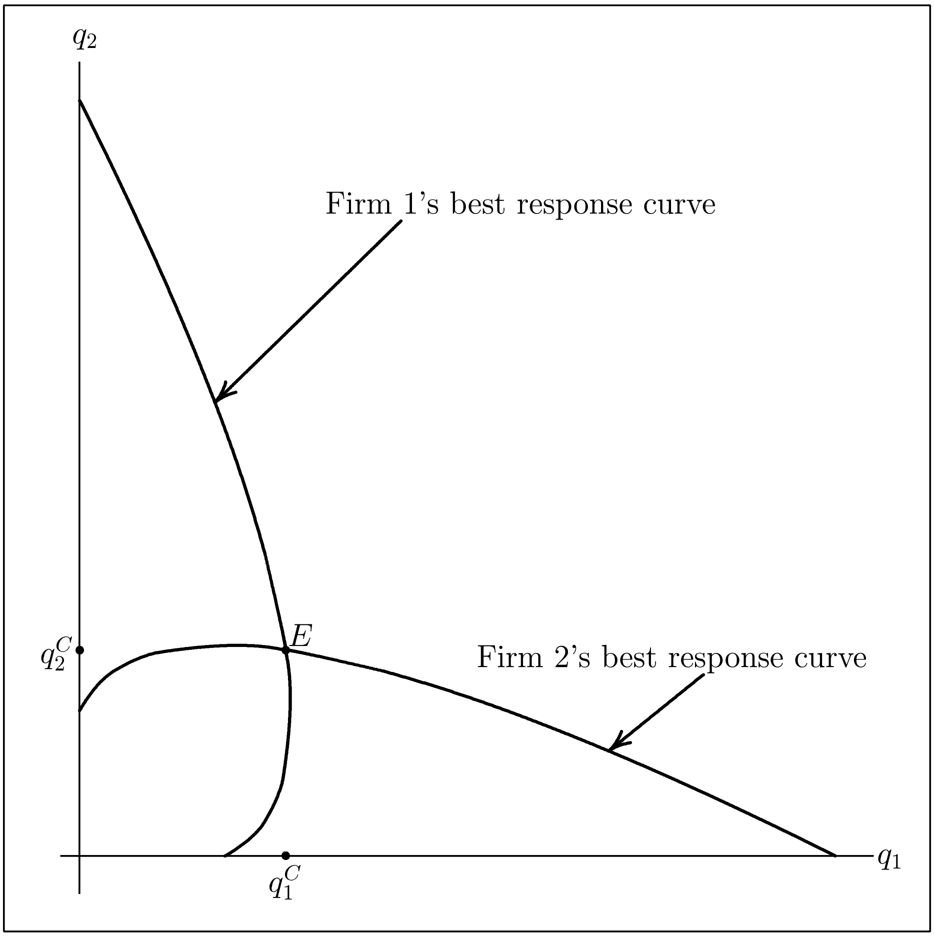


Figure 2.2: Cournot duopoly, constant price elasticity of demand; $a = 10$, $\varepsilon = 1.5$, $c = 1$; superscript C denotes Cournot equilibrium value

Best response curves for typical parameter values are shown in Figure 2.2.

Equilibrium outputs: set $q_1 = q_2 = q$ in (2.33); equilibrium output per firm satisfies

$$\left(\frac{1}{2q}\right)^{1/\varepsilon}\left(1-\frac{1}{2\varepsilon}\right) = \frac{c}{a^{1/\varepsilon}}. \tag{2.37}$$

$$q = \left(\frac{2\varepsilon - 1}{2c\varepsilon}\right)^{\varepsilon}\frac{a}{2}. \tag{2.38}$$

2.4 (Stackelberg, 1934, p. 126). In a duopoly market, let the equation of the inverse demand curve be

$$p = 100e^{-(1/10)\sqrt{Q}} \tag{2.39}$$

for $Q = q_1 + q_2$. Let average and marginal cost be zero.

Discuss the shape of best response curves; find equilibrium outputs.

Figure 2.3 shows the inverse demand curve.

Firm 1's profit is

$$\pi_1(q_1, q_2) = \left[100e^{-(1/10)\sqrt{q_1+q_2}}\right] q_1. \tag{2.40}$$

The first-order condition is

$$\frac{\partial \pi_1(q_1, q_2)}{\partial q_1} = -5\exp\left(-\frac{1}{10}\sqrt{q_1+q_2}\right)\frac{q_1 - 20\sqrt{q_1+q_2}}{\sqrt{q_1+q_2}} = 0, \tag{2.41}$$

or

$$q_1 - 20\sqrt{q_1+q_2} = 0. \tag{2.42}$$

The second-order condition for profit maximization is satisfied over the relevant output range.

The second cross-output is derivative is

$$\frac{\partial^2 \pi_1(q_1, q_2)}{\partial q_1 \partial q_2} - \frac{1}{4}\left(20q_2 + 10q_1 - q_1\sqrt{q_1+q_2}\right)\frac{\exp\left(-\frac{1}{10}\sqrt{q_1+q_2}\right)}{\left(\sqrt{q_1+q_2}\right)^3}, \tag{2.43}$$

and this is positive over the relevant output range.

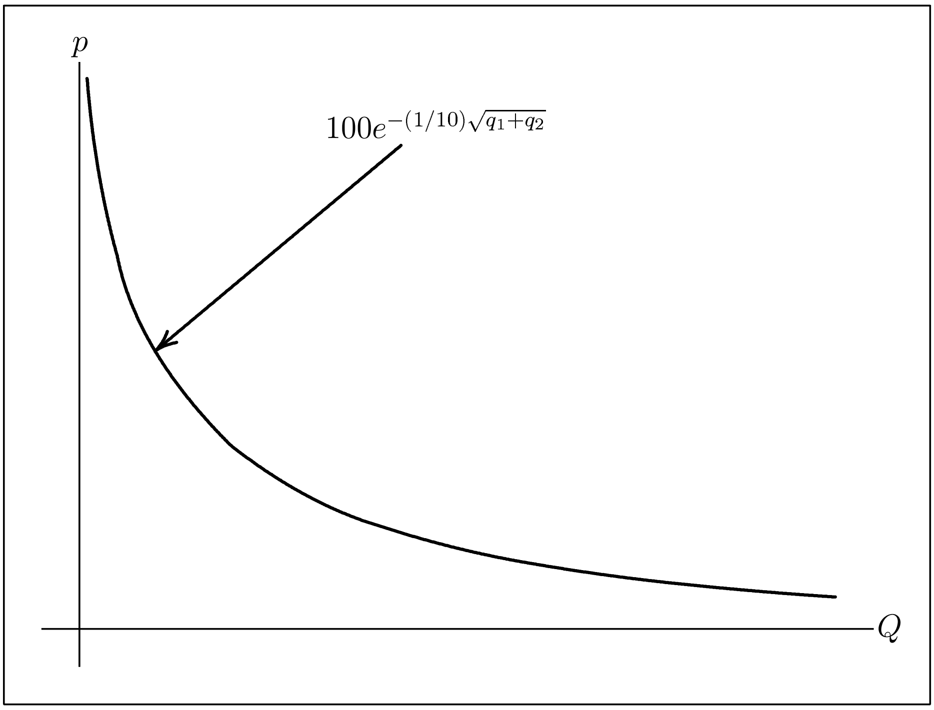


Figure 2.3: Inverse demand curve, Stackelberg inverse exponential example

The equation of firm 1's best response function can be rewritten:

$$q_1 = 20\sqrt{q_1 + q_2}$$

$$q_1^2 = 400(q_1 + q_2)$$

$$q_1^2 - 400q_1 = 400q_2$$

$$q_1^2 - 400q_1 + (200)^2 = (200)^2 + 400q_2$$

$$(q_1 - 200)^2 = 400(100 + q_2)$$

$$q_1 = 200 + 20\sqrt{100 + q_2}. \tag{2.44}$$

The best response curves are shown in Figure 2.4.

Imposing symmetry in equation 2.44, equilibrium outputs are $q_1 = q_2 = 800$.

2.5 In a duopoly market, let the equation of the inverse demand curve be

$$p(Q) = \frac{a}{1 + bQ}$$

for $Q = q_1 + q_2$ and $a, b > 0$. Let average and marginal cost be c, with $0 \leq c < a$.

Discuss the shape of best response curves; find equilibrium outputs.

Firm 1's profit is

$$\pi_1(q_1, q_2) = \frac{aq_1}{1 + b(q_1 + q_2)} - cq_1.$$

The first-order condition to maximize firm 1's profit is

$$\frac{\partial \pi_1(q_1, q_2)}{\partial q_1} = a\frac{(1 + bq_2)}{(1 + bq_1 + bq_2)^2} - c = 0.$$

The second-order condition for a maximum is satisfied.

Rewrite the first-order condition as

$$\frac{a}{c}(1 + bq_2) = (1 + bq_1 + bq_2)^2$$

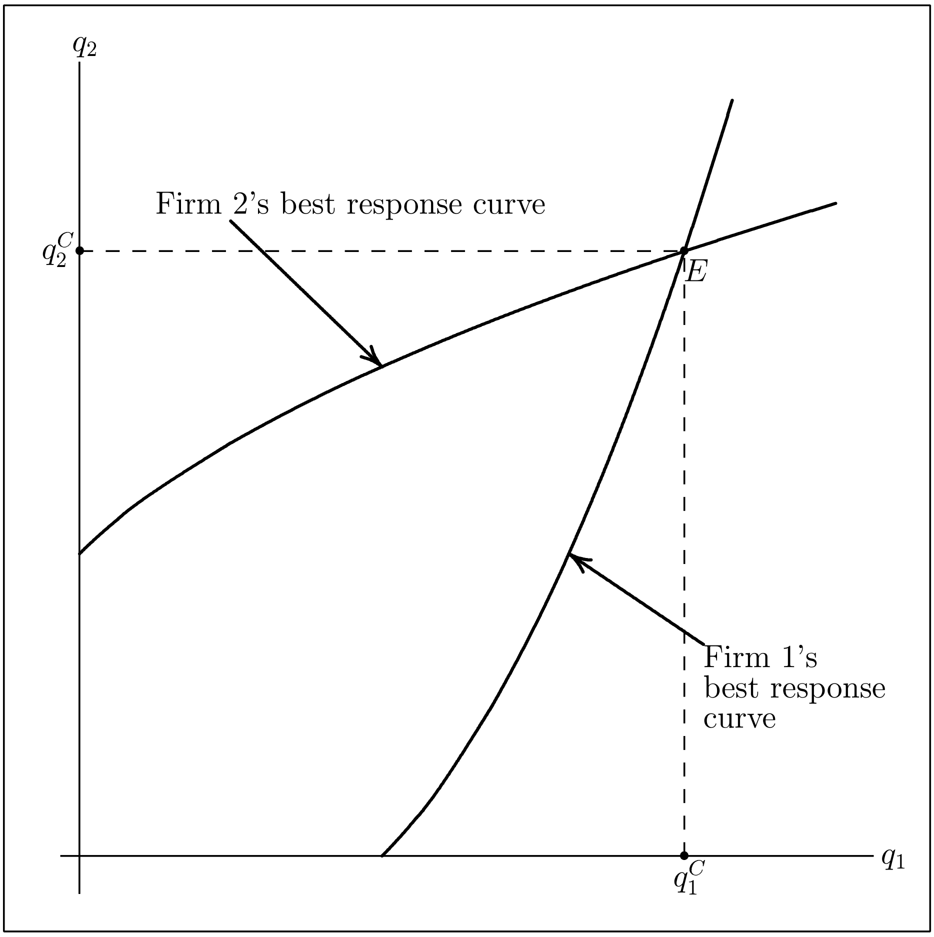


Figure 2.4: Best response curves, Stackelberg inverse exponential demand example

Write $\alpha = a/c$, $z = 1 + bq_2$.

$$\alpha z = (z + bq_1)^2$$

$$\alpha z = z^2 + 2zbq_1 + b^2 q_1^2$$

$$b^2 q_1^2 + 2bzq_1 - (\alpha - z)z = 0$$

$$\begin{aligned} q_1 &= \frac{-2bz + \sqrt{4b^2z^2 + 4b^2(\alpha - z)z}}{2b^2} \\ &= \frac{-2bz + 2b\sqrt{z^2 + (\alpha - z)z}}{2b^2} \\ &= \frac{-z + \sqrt{z^2 + \alpha z - z^2}}{b} \\ &= \frac{-z + \sqrt{\alpha z}}{b} \\ &= \frac{-(1 + bq_2) + \sqrt{\frac{a}{c}(1 + bq_2)}}{b}. \end{aligned}$$

This is the equation of firm 1's best response function. It has slope

$$\begin{aligned} &\frac{\partial}{\partial q_2}\left[\frac{-(1 + bq_2) + \sqrt{\frac{a}{c}(1 + bq_2)}}{b}\right] \\ &= \frac{1}{2}\left[\sqrt{\frac{a/c}{1 + bq_2}} - 2\right]. \end{aligned}$$

The slope of firm 1's best response curve for $q_2 = 0$ is

$$\frac{1}{2}\left(\sqrt{\frac{a}{c}} - 2\right).$$

This is positive if $a/c > 4$, zero if $a/c = 4$, negative if $a/c < 4$.

For what value of q_2 is the slope of 1's best response function equal to zero?

$$\sqrt{\frac{a}{c}} = 2\sqrt{1 + bq_2}$$

$$\frac{a}{4c} = 1 + bq_2$$

$$bq_2 = \frac{a}{4c} - 1 = \frac{a - 4c}{4c}$$

$$q_2 = \frac{a - 4c}{4bc}.$$

If $a - 4c > 0$, the best response curve begins with a positive slope and ends with a negative slope. If $a - 4c \leq 0$, the slope of the best response function is negative throughout.

Figure 2.5 shows the best response functions for $a = 225$, $b = 1/2$, $c = 25$; Figure 2.6 shows the best response functions for $a = 225$, $b = 1/2$, $c = 100$.

Equilibrium outputs: impose symmetry on the equation of the best response function and solve:

$$\alpha(1 + bq) = (1 + 2bq)^2$$

$$\alpha + \alpha bq = 1 + 4bq + 4b^2q^2$$

$$4b^2q^2 - (\alpha - 4)bq - (\alpha - 1) = 0$$

$$q = \frac{\frac{a-4c}{c}b + \sqrt{\left(\frac{a-4c}{c}\right)^2 b^2 + 16b^2\frac{a-c}{c}}}{8b^2}$$

$$= \frac{\frac{a-4c}{c} + \sqrt{\left(\frac{a-4c}{c}\right)^2 + 16\frac{a-c}{c}}}{8b}.$$

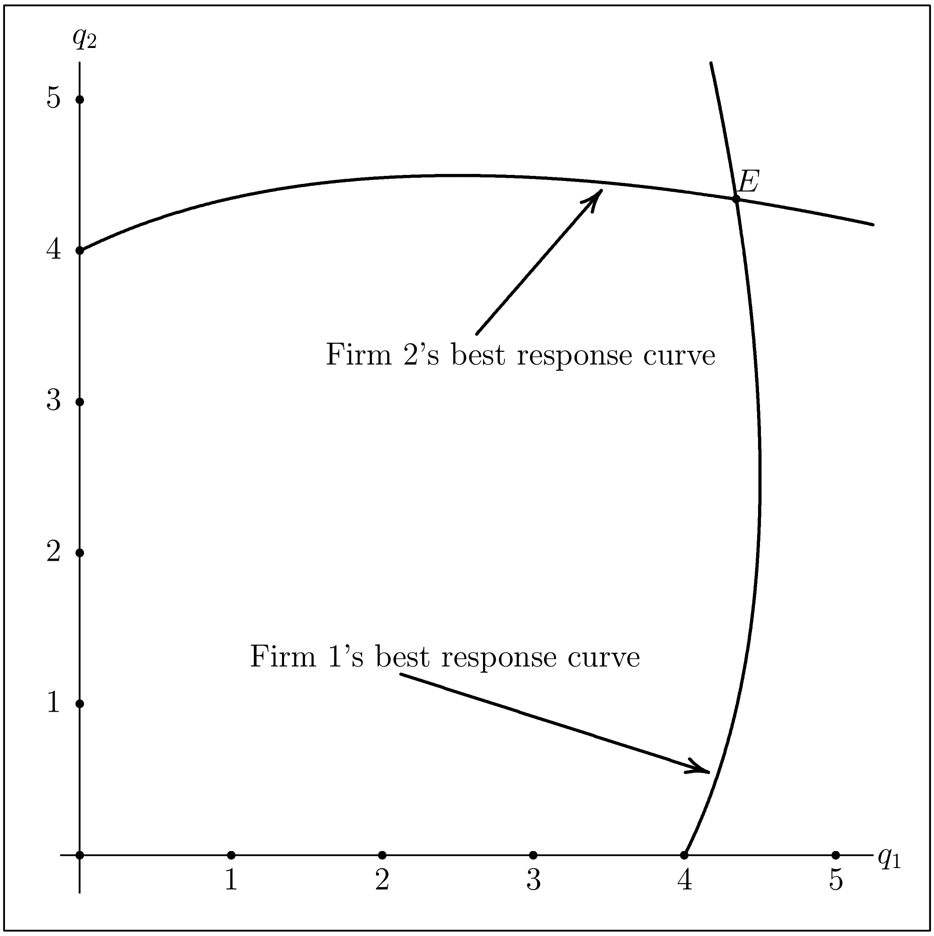


Figure 2.5: Problem 2.5; $a = 225$, $b = 1/2$, $c = 25$

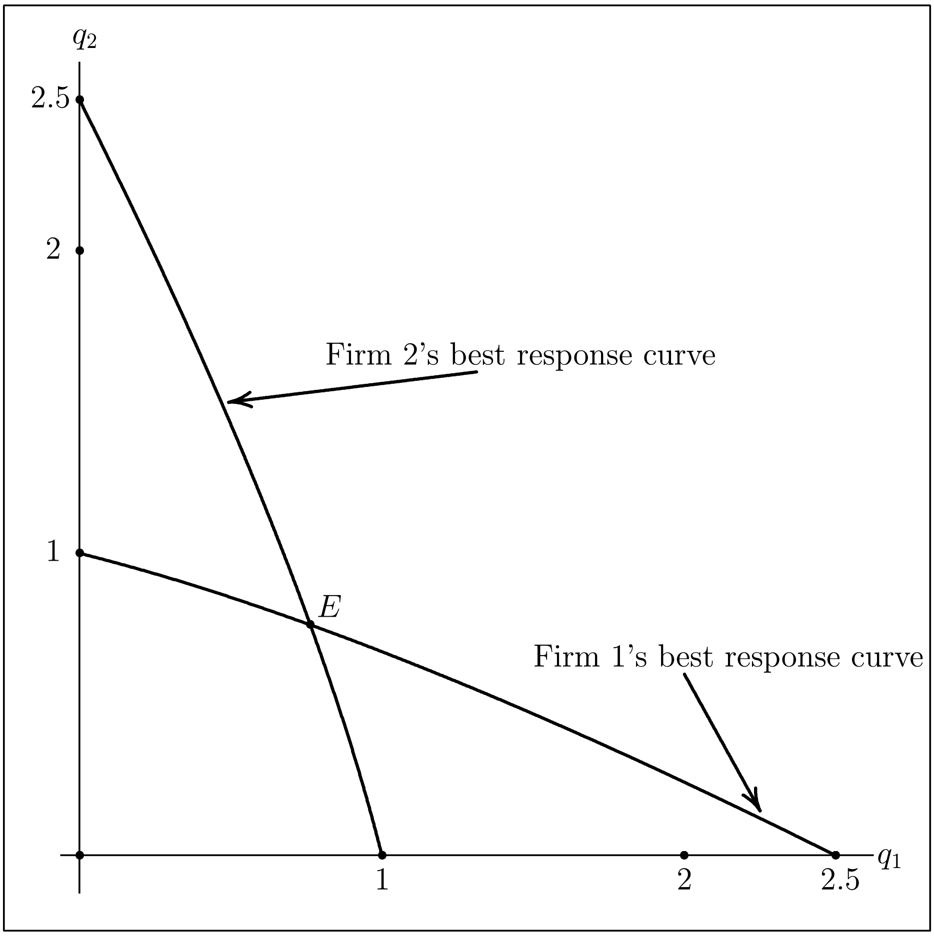


Figure 2.6: Problem 2.5; $a = 225$, $b = 1/2$, $c = 100$

Equilibrium output for the example of Figure 2.5 is

$$\frac{\frac{225-4(25)}{25}+\sqrt{\left(\frac{225-4(25)}{25}\right)^2+16\frac{225-25}{25}}}{8(1/2)}=4.3423.$$

Equilibrium output for the example of Figure 2.6 is

$$\frac{\frac{225-4(100)}{100}+\sqrt{\left(\frac{225-4(100)}{100}\right)^2+16\frac{225-100}{100}}}{8(1/2)}=0.76309.$$

2.1.1 References

Bergstrom, Theodore C. and Varian, Hal R. "Two remarks on Cournot equilibria," *Economics Letters* Volume 19, 1985, pp. 5–8.

Cournot, Augustin *Researches into the Mathematical Principles of the Theory of Wealth.* original Paris: L. Hachette, 1838. English translation by Nathaniel T. Bacon. New York: The Macmillan Company, 1897; reprinted 1927 by The Macmillan Company, New York with notes by Irving Fisher; reprinted 1960, 1964, 1971 by Augustus M. Kelley, New York

Vickrey, William A. *Microstatics.* New York: Harcourt, Brace & World, Inc., 1964.

Chapter 3

Foundations of Oligopoly Theory II

3.1 Best response functions for the Bowley product differentiation model

3.1.1 Demand

The representative consumer utility function is

$$U(q_1, q_2) = a(q_1 + q_2) - \frac{1}{2}b(q_1^2 + 2\theta q_1 q_2 + q_2^2) + m, \tag{3.1}$$

A Lagrangian function to describe the constrained optimization problem is

$$\mathcal{L}_1 =$$

$$U(q_1, q_2) = a(q_1 + q_2) - \frac{1}{2}b(q_1^2 + 2\theta q_1 q_2 + q_2^2) + m \tag{3.2}$$

$$+\lambda_1 \left(Y - m - p_1 q_1 - p_2 q_2\right),$$

where Y is income, m all other goods, and $p_m = 1$ the price of all other goods.

The Kuhn-Tucker conditions are

$$a - b(q_1 + \theta q_2) - \lambda p_1 \leq 0 \qquad q_1 \left[a - b(q_1 + \theta q_2) - \lambda p_1\right] = 0 \qquad q_1 \geq 0 \tag{3.3}$$

$$a - b(\theta q_1 + q_2) - \lambda p_2 \leq 0 \qquad q_2 \left[a - b(\theta q_1 + q_2) - \lambda p_2\right] = 0 \qquad q_2 \geq 0 \tag{3.4}$$

$$1 - \lambda_1 \leq 0 \qquad m(1 - \lambda_1) = 0 \qquad m \geq 0 \tag{3.5}$$

$$Y - m - p_1 q_1 - p_2 q_2 \geq 0 \qquad \lambda_1(Y - m - p_1 q_1 - p_2 q_2) = 0 \qquad \lambda_1 \geq 0. \tag{3.6}$$

Assume Y is sufficiently large so that $m > 0$. Then (3.5) implies that

$$\lambda_1 = 1 \tag{3.7}$$

and (3.6) implies

$$m = Y - (p_1 q_1 + p_2 q_2). \tag{3.8}$$

Substitute (3.7) in (3.3) and (3.4) to obtain

$$a - b(q_1 + \theta q_2) - p_1 \leq 0 \qquad q_1 \left[a - b(q_1 + \theta q_2) - p_1\right] = 0 \qquad q_1 \geq 0 \tag{3.9}$$

$$a - b(\theta q_1 + q_2) - p_2 \leq 0 \qquad q_2 \left[a - b(\theta q_1 + q_2) - p_2\right] = 0 \qquad q_2 \geq 0. \tag{3.10}$$

Case 1: $q_1 > 0$, $q_2 > 0$.
Then (3.9) and (3.10) imply that the inverse demand curves are

$$p_1 = a - b(q_1 + \theta q_2) \tag{3.11}$$

$$p_2 = a - b(\theta q_1 + q_2). \tag{3.12}$$

The equations of the inverse demand curves can be inverted to obtain the equations of the demand curves when consumption of both varieties is positive:

$$q_1 = \frac{a - p_1 - \theta(a - p_2)}{b(1 - \theta^2)} \tag{3.13}$$

$$q_2 = \frac{a - p_2 - \theta(a - p_1)}{b(1 - \theta^2)}. \tag{3.14}$$

Case 2: $q_1 > 0$, $q_2 = 0$.
Then (3.9) implies that the inverse demand curve for variety 1 is

$$p_1 = a - bq_1, \tag{3.15}$$

with corresponding demand curve

$$q_1 = \frac{a - p_1}{b}. \tag{3.16}$$

(3.10) implies

$$p_2 \geq a - b\theta q_1. \tag{3.17}$$

Case 3: $q_1 = 0$, $q_2 > 0$.
(3.10) implies

$$p_2 = a - b\theta q_2, \tag{3.18}$$

with corresponding demand curve

$$q_2 = \frac{a - p_2}{b}. \tag{3.19}$$

(3.9) implies that

$$p_1 \geq a - b\theta q_2. \tag{3.20}$$

3.2 Cournot duopoly

Let $p_1 > 0$, $p_2 > 0$. Firm 1 maximizes

$$\pi_1 = [a - c - b(q_1 + \theta q_2)];$$

the resulting first-order condition is

$$2q_1 + \theta q_2 = \frac{a - c}{2b}$$

or

$$q_1 = \frac{1}{2}\left(\frac{a-c}{b} - \theta q_2\right),$$

so that q_1 falls as q_2 rises.

Along its best-response function,

$$p_1 = c + bq_1,$$

so that $q_1 \geq 0$ implies $p_1 \geq c$.

If $q_2 = 0$, firm 1's best response is to produce its unconstrained monopoly output,

$$q_1 = \frac{a-c}{2b}.$$

Prices are then

$$p_1 = c + b\left(\frac{a-c}{2b}\right) = c + \frac{1}{2}(a-c)$$

and

$$p_2 = a - b\left(\theta\frac{a-c}{2b} + 0\right) = c + \frac{1}{2}(2-\theta)(a-c).$$

As q_2 rises and firm 1 moves along its best-response function,

$$\theta q_1 + q_2 = \frac{1}{2}\theta\left(\frac{a-c}{b} - \theta q_2\right) + q_2 = \frac{1}{2}\left[\theta\frac{a-c}{b} + \left(2-\theta^2\right) q_2\right];$$

firm 2's price is

$$p_2 = c + (a-c) - b\frac{1}{2}\left[\theta\frac{a-c}{b} + \left(2-\theta^2\right) q_2\right] = c + \frac{1}{2}(2-\theta)(a-c) - \frac{1}{2}\left(2-\theta^2\right) bq_2.$$

$p_2 = 0$ for

$$c + \frac{1}{2}(2-\theta)(a-c) - \frac{1}{2}\left(2-\theta^2\right) bq_2 = 0$$

$$q_2 = \frac{(2-\theta)a + c\theta}{(2-\theta^2)b} > 0.$$

For this value of q_2, firm 1's best response output is

$$q_1 = \frac{1}{2}\left[\frac{a-c}{b} - \theta\frac{(2-\theta)a + c\theta}{(2-\theta^2)b}\right] = \frac{(1-\theta)a - c}{(2-\theta^2)b}.$$

$$(q_1, q_2) = \left(\frac{(1-\theta)a - c}{(2-\theta^2)b}, \frac{(2-\theta)a + c\theta}{(2-\theta^2)b}\right).$$

The corresponding price is

$$p_1 = c + b\frac{(1-\theta)a - c}{(2-\theta^2)b} = c + \frac{(1-\theta)a - c}{2-\theta^2}$$

$$(p_1, p_2) = \left(c + \frac{(1-\theta)a - c}{2-\theta^2}, 0\right).$$

$q_1^{br}\left[\frac{(2-\theta)a+c\theta}{(2-\theta^2)b}\right] \geq 0$ for

$$\theta \leq \frac{a-c}{a}.$$

If $\theta > (a-c)/a$, (if products are too close to being homogeneous) firm 1's best-response output falls to zero before firm 2's output becomes so large that $p_2 = 0$.

If $\theta \leq (a-c)/a$, the value of q_2 that makes firm 1's best-response output equal to zero is

$$q_2 = \frac{a-c}{\theta b}.$$

For

$$(q_1, q_2) = \left(0, \frac{a-c}{\theta b}\right),$$

prices are

$$(p_1, p_2) = \left(c, c + \frac{1}{2}(2-\theta)(a-c) - \frac{1}{2}\left(2-\theta^2\right) b\frac{a-c}{\theta b}\right) = \left(c, c - \frac{1-\theta}{\theta}(a-c)\right).$$

If $\theta \leq (a-c)/a$, firm 1's best-response output is positive, and $p_1 > c$, when $q_2 = \frac{(2-\theta)a+c\theta}{(2-\theta^2)b}$ and $p_2 = 0$. If $q_2 > \frac{(2-\theta)a+c\theta}{(2-\theta^2)b}$, firm 1's output is determined along the $p_2 = a - b(\theta q_1 + q_2) = 0$ line,

$$a - b(\theta q_1 + q_2) = 0$$

$$q_1 = \frac{a - bq_2}{\theta b}.$$

Then

$$q_1 + \theta q_2 = \frac{a - bq_2}{\theta b} + \theta q_2 = \frac{a - b(1-\theta^2)q_2}{b\theta}$$

$$p_1 = a - b(q_1 + \theta q_2) = a - b\frac{a - b(1-\theta^2)q_2}{b\theta} =$$

$$= a - \frac{1}{\theta}(a - b(1-\theta^2)q_2)$$

$$= -\frac{1-\theta}{\theta}a + \frac{1-\theta^2}{\theta}bq_2,$$

which rises with q_2. Since (for $\theta \leq (a-c)/a$) $p_1 \geq c$ at the switch point, $p_1 \geq c$ as q_1 falls moving backward along the $q_2 = 0$ line.

The end point of this segment of the quantity best response curve is

$$(q_1, q_2) = \left(0, \frac{a}{b}\right),$$

with corresponding prices

$$(p_1, p_2) = \left(-\frac{1-\theta}{\theta}a + \frac{1-\theta^2}{\theta}b\left(\frac{a}{b}\right), 0\right) = ((1-\theta)a, 0).$$

Hence for $\theta \leq (a-c)/a$, firm 1's quantity best response curve has two linear segments, from

$$\left(\frac{a-c}{2b}, 0\right)$$

(this is point A_1 in Figure 3.2(b) of the text) to

$$\left(\frac{(1-\theta)a-c}{(2-\theta^2)b}, \frac{(2-\theta)a+c\theta}{(2-\theta^2)b}\right)$$

(this is point B_1 in Figure 3.2(b) of the text) to

$$\left(0, \frac{a}{b}\right)$$

(this is point C_1 in Figure 3.2(b) of the text).

The corresponding points in price space are

$$(p_1, p_2) = \left(c + \frac{1}{2}(a-c), c + \frac{1}{2}(2-\theta)(a-c)\right)$$

(this is point A_1' in Figure 3.2(c) of the text);

$$(p_1, p_2) = \left(c + \frac{(1-\theta)a-c}{2-\theta^2}, 0\right)$$

(this is point B_1' in Figure 3.2(c) of the text);

$$(p_1, p_2) = ((1-\theta)a, 0)$$

(this is point C_1' in Figure 3.2(c) of the text).

For $\theta \leq (a-c)/a$, firm 1's quantity best response curve is the straight line connecting

$$\left(\frac{a-c}{2b}, 0\right)$$

and

$$\left(0, \frac{a-c}{\theta b}\right).$$

The corresponding points in price space are

$$(p_1, p_2) = \left(c + \frac{1}{2}(a-c), c + \frac{1}{2}(2-\theta)(a-c)\right)$$

$$(p_1, p_2) = \left(c, c - \frac{1-\theta}{\theta}(a-c)\right).$$

3.2.1 Bertrand duopoly

Case 1: $q_1 > 0$**,** $q_2 > 0$

Firm 1 maximizes

$$\pi_1 = (p_1 - c)\frac{(1-\theta)(a-c) - (p_1 - c) + \theta(p_2 - c)}{b(1-\theta^2)}. \tag{3.21}$$

The first-order condition is

$$2(p_1 - c) - \theta(p_2 - c) = (1-\theta)(a-c) \tag{3.22}$$

or

$$p_1^{br}(p_2) = c + \frac{(1-\theta)(a-c) + \theta(p_2 - c)}{2}. \tag{3.23}$$

Note that along the best response function

$$q_1 = \frac{(1-\theta)(a-c) - (p_1 - c) + \theta(p_2 - c)}{b(1-\theta^2)} = \frac{p_1 - c}{b(1-\theta^2)}$$

and

$$\pi_1 = \frac{(p_1 - c)^2}{b(1-\theta^2)}.$$

For $p_2 = 0$,

$$p_1^{br}(0) = c + \frac{(1-\theta)(a-c) - \theta c}{2}.$$

This is greater than or equal to c for

$$(1-\theta)(a-c) - \theta c \geq 0$$

$$\theta \leq 1 - \frac{c}{a} = \frac{a-c}{a}.$$

If this condition fails (if θ is too close to 1) then firm 1 shuts down for values of p_2 below the value that makes $p_1^{br}(p_2) = c$:

$$p_1^{br}(p_2) = c + \frac{(1-\theta)(a-c) + \theta(p_2 - c)}{2} = c$$

$$p_2 = \frac{c - (1-\theta)a}{\theta}.$$

The quantities demanded at these prices are

$$q_1 = \frac{a - c - \theta\left(a - \frac{c-(1-\theta)a}{\theta}\right)}{b(1-\theta^2)} = 0$$

$$q_2 = \frac{a - \frac{c-(1-\theta)a}{\theta} - \theta(a-c)}{b(1-\theta^2)} = \frac{a-c}{\theta b}.$$

For $p_2 = c$,

$$p_1^{br}(c) = c + \frac{(1-\theta)(a-c)}{2}.$$

Firm 1's best response price rises as p_2 rises.
The quantities demanded for

$$(p_1, p_2) = \left(c + \frac{(1-\theta)(a-c) - \theta c}{2}, 0\right)$$

are

$$q_1 = \frac{a - c - \frac{(1-\theta)(a-c)-\theta c}{2} - \theta(a-0)}{b(1-\theta^2)} = \frac{(1-\theta)a - c}{2(1-\theta^2)b}$$

$$q_2 = \frac{a - 0 - \theta\left(a - c - \frac{(1-\theta)(a-c)-\theta c}{2}\right)}{b(1-\theta^2)} = \frac{(1-\theta)(2+\theta)a + \theta c}{2(1-\theta^2)b}.$$

The condition for

$$q_1\left(c + \frac{(1-\theta)(a-c) - \theta c}{2}, 0\right) \geq 0$$

is

$$(1-\theta)a - c \geq 0$$

$$\theta \leq \frac{a-c}{a}.$$

The lowest point of firm 1's price best response function is

$$(p_1, p_2) = \begin{array}{cc} \left(c + \frac{(1-\theta)(a-c)-\theta c}{2}, 0\right) & \theta \leq \frac{a-c}{a} \\ \left(c, \frac{c-(1-\theta)a}{\theta}\right) & \theta > \frac{a-c}{a} \end{array}.$$

The corresponding quantities are

$$(q_1, q_2) = \begin{array}{cc} \left(\frac{(1-\theta)a-c}{2(1-\theta^2)b}, \frac{(1-\theta)(2+\theta)a+\theta c}{2(1-\theta^2)b}\right) & \theta \leq \frac{a-c}{a} \\ \left(0, \frac{a-c}{\theta b}\right) & \theta > \frac{a-c}{a} \end{array}.$$

The quantities demanded for

$$(p_1, p_2) = \left(c + \frac{(1-\theta)(a-c)}{2}, c\right)$$

are

$$q_1 = \frac{a - c - \frac{(1-\theta)(a-c)}{2} - \theta(a-c)}{b(1-\theta^2)} = \frac{1}{1+\theta}\frac{a-c}{2b}$$

$$q_2 = \frac{a - c - \theta(a - c - \frac{(1-\theta)(a-c)}{2})}{b(1-\theta^2)} = \frac{2+\theta}{1+\theta}\frac{a-c}{2b}.$$

As p_2 rises along (3.22), the quantity demanded of firm 2 falls:

$$q_2 = \frac{1}{b(1-\theta^2)}\left\{a - c - (p_2 - c) - \theta\left[a - c - (p_1^{br}(p_2) - c)\right]\right\}$$

$$= \frac{1}{b(1-\theta^2)}\left\{a - c - (p_2 - c) - \theta\left[a - c - \frac{(1-\theta)(a-c) + \theta(p_2 - c)}{2}\right]\right\}$$

$$= \frac{1}{2(1-\theta^2)b}\left[(1-\theta)(2+\theta)(a-c) - (2-\theta^2)(p_2 - c)\right].$$

$q_2 = 0$ for

$$(1-\theta)(2+\theta)(a-c) - (2-\theta^2)(p_2 - c) = 0$$

$$p_2 = c + \frac{(1-\theta)(2+\theta)}{2-\theta^2}(a-c).$$

For this value of p_2, the value of p_1 along its $q_1 > 0$, $q_2 > 0$ best response function is

$$p_1^{br}\left(c + \frac{(1-\theta)(2+\theta)}{2-\theta^2}(a-c)\right) = c + \frac{1}{2}(1-\theta)(a-c) + \frac{1}{2}\theta\left(\frac{(1-\theta)(2+\theta)}{2-\theta^2}(a-c)\right)$$

$$= c + \frac{1-\theta^2}{2-\theta^2}(a-c).$$

This is less than the unconstrained monopoly price:

$$c + \frac{1-\theta^2}{2-\theta^2}(a-c) - c - \frac{1}{2}(a-c) = \frac{1}{2}\left(\frac{\theta^2}{2-\theta^2}\right)(a-c) > 0.$$

The quantity demanded of firm 1 is

$$q_1 = \frac{a - c - \frac{1-\theta^2}{2-\theta^2}(a-c) - \theta(a - c - \frac{(1-\theta)(2+\theta)}{2-\theta^2}(a-c))}{b(1-\theta^2)} = \frac{1}{2-\theta^2}\frac{a-c}{b}.$$

The lowest point of firm 1's price best response function is

$$(p_1, p_2) = \begin{array}{cc} \left(c + \frac{(1-\theta)(a-c)-\theta c}{2}, 0\right) & \theta \le \frac{a-c}{a} \\ \left(c, \frac{c-(1-\theta)a}{\theta}\right) & \theta > \frac{a-c}{a} \end{array}.$$

Point A_1 in Figure 3.3(c) of the text is

$$(p_1, p_2) = \left(c + \frac{(1-\theta)(a-c)-\theta c}{2}, 0\right).$$

Point B_1 in Figure 3.3(c) of the text is

$$(p_{1B}, p_{2B}) = \left(c + \frac{1-\theta^2}{2-\theta^2}(a-c), c + \frac{(1-\theta)(2+\theta)}{2-\theta^2}(a-c)\right). \qquad (3.24)$$

The corresponding points in quantity space are

$$(q_1, q_2) = \begin{array}{cc} \left(\frac{(1-\theta)a-c}{2(1-\theta^2)b}, \frac{(1-\theta)(2+\theta)a+\theta c}{2(1-\theta^2)b}\right) & \theta \le \frac{a-c}{a} \\ \left(0, \frac{a-c}{\theta b}\right) & \theta > \frac{a-c}{a} \end{array}$$

and

$$(q_1, q_2) = \left(\frac{1}{2-\theta^2}\frac{a-c}{b}, 0\right)$$

(this is point B_1' in Figure 3.3(b)).

$$(q_1, q_2) = \left(\frac{(1-\theta)a-c}{2(1-\theta^2)b}, \frac{(1-\theta)(2+\theta)a+\theta c}{2(1-\theta^2)b}\right)$$

is point A_1' in Figure 3.3(b).

3.2.2 Case 2: $q_1 > 0$, $q_2 = 0$.

The quantity demanded of firm 1 is

$$q_1 = \frac{a-p_1}{b} = \frac{a-c-(p_1-c)}{b},$$

and firm 1 maximizes profit

$$\pi_1 = (p_1-c)q_1 = (p_1-c)\frac{a-c-(p_1-c)}{b}$$

subject to the $q_2 = 0$ constraint that

$$p_2 \geq a - b\theta q_1$$

$$p_2 \geq a - \theta(a-p_1)$$

$$p_2 - (1-\theta)a - \theta p_1 \geq 0.$$

A Lagrangian for firm 1's constrained optimization problem is

$$\mathcal{L}_2 = (p_1-c)\frac{(a-c)-(p_1-c)}{b} + \lambda_2\left[p_2 - (1-\theta)a - \theta p_1\right].$$

The Kuhn-Tucker conditions are

$$\frac{(a-c)-2(p_1-c)}{b} - \theta\lambda_2 \leq 0 \qquad (p_1-c)\left[\frac{(a-c)-2(p_1-c)}{b} - \theta\lambda_2\right] = 0$$

$$(p_1 - c) \geq 0$$

$$p_2 - (1-\theta)a - \theta p_1 \geq 0 \quad \lambda_2 \left[p_2 - (1-\theta)a - \theta p_1\right] = 0 \quad \lambda_2 \geq 0.$$

Case 2(a): $\frac{1}{2}\left(\frac{1-\theta}{2-\theta}\right)(2+\theta+\theta^2)(a-c) < p_1 - c < \frac{1}{2}(a-c)$.
$p_1 - c > 0$ implies

$$\lambda_2 = \frac{2}{\theta b}\left(p_1 - c - \frac{a-c}{2}\right) > 0.$$

$\lambda_2 > 0$ implies

$$p_2 - (1-\theta)a - \theta p_1 = 0$$

or

$$p_1 = \frac{1}{\theta}[p_2 - (1-\theta)a]$$

$$p_1 = c + \frac{1}{\theta}[(p_2 - c) - (1-\theta)(a-c)].$$

From (3.14), this is the equation of the $q_2 = 0$ line:

$$q_2 = \frac{a - p_2 - \theta(a - p_1)}{b(1-\theta^2)} = 0.$$

If we evaluate the expression for p_1 for $p_2 = p_{2B}$, we obtain

$$p_1 = c + \frac{1}{\theta}\left((\frac{(1-\theta)(2+\theta)}{2-\theta^2}(a-c)) - (1-\theta)(a-c)\right)$$

$$= c + \frac{1-\theta^2}{2-\theta^2}(a-c) = p_{1B}.$$

The price best response function is continuous.

Firm 1's best response price runs along the $q_2 = 0$ line until p_1 reaches the unconstrained monopoly price, which occurs for

$$c + \frac{1}{2}(a-c) = c + \frac{1}{\theta}\left((p_2 - c) - (1-\theta)(a-c)\right)$$

$$p_2 = c + \frac{1}{2}(2-\theta)(a-c) = p_{2C}.$$

The second segment of firm 1's Bertrand best response function is the straight line connecting point B_1: (p_{1B}, p_{2B}) and point

$$C_1\colon\ (p_{1C}, p_{2C}) = \left(c + \frac{1}{2}(a-c), c + \frac{1}{2}(2-\theta)(a-c)\right)$$

(see point C_1 in Figure 3.3(c) of the text).

The corresponding point in quantity space is

$$\left(\frac{a-c}{2b}, 0\right).$$

This is point C_1' of Figure 3.3(b) of the text.

Case 2(b): $p_1 - c = \frac{1}{2}(a-c) > 0$.

$$\frac{(a-c) - 2(p_1 - c)}{b} - \theta\lambda_2 \leq 0 \quad (p_1 - c)\left[\frac{(a-c) - 2(p_1 - c)}{b} - \theta\lambda_2\right] = 0$$

$$(p_1 - c) \geq 0$$

$$p_2 - (1-\theta)a - \theta p_1 \geq 0 \quad \lambda_2\left[p_2 - (1-\theta)a - \theta p_1\right] = 0 \quad \lambda_2 \geq 0.$$

$p_1 > c$ implies

$$\lambda_2 = \frac{2}{\theta b}\left(p_1 - c - \frac{a-c}{2}\right) = \frac{2}{\theta b}\left(\frac{a-c}{2} - \frac{a-c}{2}\right) = 0.$$

$p_2 - (1-\theta)a - \theta p_1 \geq 0$ implies

$$p_2 \geq (1-\theta)a + \theta p_1 = c + (1-\theta)(a-c) + \theta(p_1 - c) =$$

$$c + (1-\theta)(a-c) + \frac{1}{2}\theta(a-c) = c + \frac{1}{2}(2-\theta)(a-c) = p_{2H}.$$

For $p_2 \geq p_{2H}$, firm 1's best response price is the unconstrained monopoly price.

In price space, firm 1's Bertrand best response function has three linear segments, from

$$A_1\colon\ (p_1, p_2) = \begin{matrix} \left(c + \frac{(1-\theta)(a-c)-\theta c}{2}, 0\right) & \theta \leq \frac{a-c}{a} \\ \left(c, \frac{c-(1-\theta)a}{\theta}\right) & \theta > \frac{a-c}{a} \end{matrix}.$$

to

$$B_1{:}(p_{1B}, p_{2B}) = \left(c + \frac{1-\theta^2}{2-\theta^2}(a-c), c + \frac{(1-\theta)(2+\theta)}{2-\theta^2}(a-c)\right)$$

to

$$C_1\colon\ (p_{1C}, p_{2C}) = \left(c + \frac{1}{2}(a-c), c + \frac{1}{2}(2-\theta)(a-c)\right)$$

and horizontal for higher values of p_2.

The corresponding points in quantity space are

$$A_1'\colon\ (q_1, q_2) = \begin{matrix} \left(\frac{(1-\theta)a-c}{2(1-\theta^2)b}, \frac{(1-\theta)(2+\theta)a+\theta c}{2(1-\theta^2)b}\right) & \theta \leq \frac{a-c}{a} \\ \left(0, \frac{a-c}{\theta b}\right) & \theta > \frac{a-c}{a} \end{matrix}.$$

and

$$B_1'{:}\left(\frac{1}{2-\theta^2}\frac{a-c}{b}, 0\right).$$

and

$$C_1'\colon\ \left(\frac{a-c}{2b}, 0\right).$$

3.3 Answers to Problems

3.1 In a Cournot duopoly, let the inverse demand curve be linear,

$$p = a - b(q_1 + q_2)$$

and let marginal and average cost per unit be a constant c.

(a) Graph best response curves in the conjectural derivative model if firms hold identical conjectures $\lambda = -1, 0, +1$.

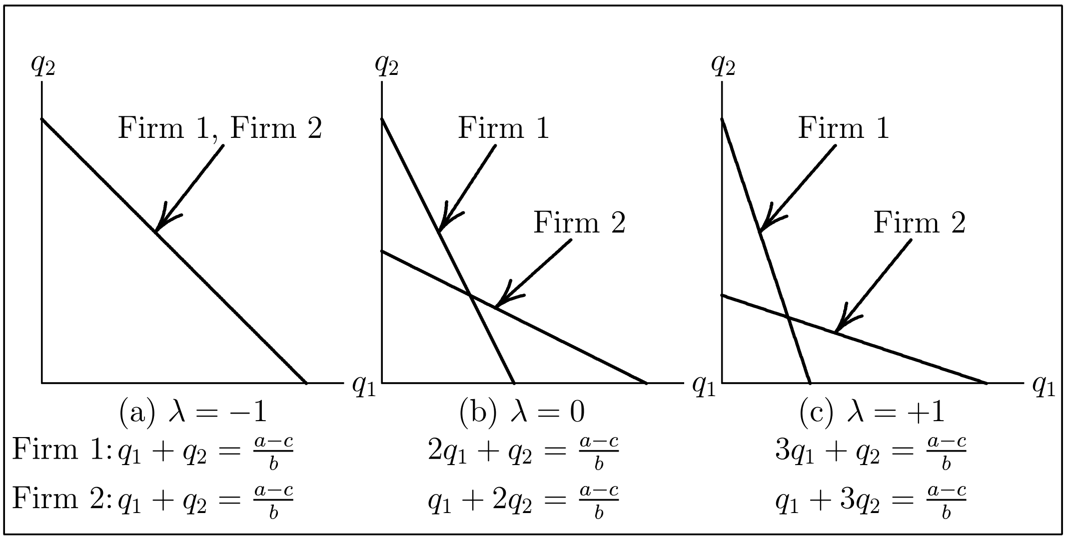


Figure 3.1: Cournot duopoly best response curves with conjectural derivatives

Firm 1's profit is

$$\pi_1 = [a - c - b(q_1 + q_2)]\, q_1.$$

Allowing for the conjectural derivative, the first-order condition for profit maximization is

$$\frac{\partial \pi_1}{\partial q_1} = a - c - b(q_1 + q_2) - b(1 + \lambda_1)q_1 = 0,$$

which can be rewritten

$$(2 + \lambda_1)q_1 + q_2 = \frac{a - c}{b}$$

or

$$q_1 = \frac{1}{2+\lambda_1}\left(\frac{a-c}{b} - q_2\right).$$

As λ rises from -1 to $+1$, the slope of the best response function changes from -1 to $-1/3$: best response curves become flatter as the conjectural derivative increases.

In the same way, the equation of firm 2's best response function with conjectural derivative λ_2 is

$$q_1 + (2+\lambda_2)q_2 = \frac{a-c}{b}.$$

Best response curves are drawn in Figure 3.1 for the three cases $\lambda_1 = \lambda_2 = -1, 0, +1$.

Symmetric equilibrium outputs are

$$q = \frac{1}{3+\lambda}\frac{a-c}{b}.$$

Equilibrium outputs fall as the conjectural derivative rises.

(b) Graph best response curves in the conjectural elasticity model if firms hold identical conjectures $\alpha = -1, 0, +1$.

Firm 1's profit is

$$\pi_1 = [a - c - b(q_1 + q_2)]\, q_1$$

Allowing for the conjectural elasticity, the first-order condition for profit maximization is

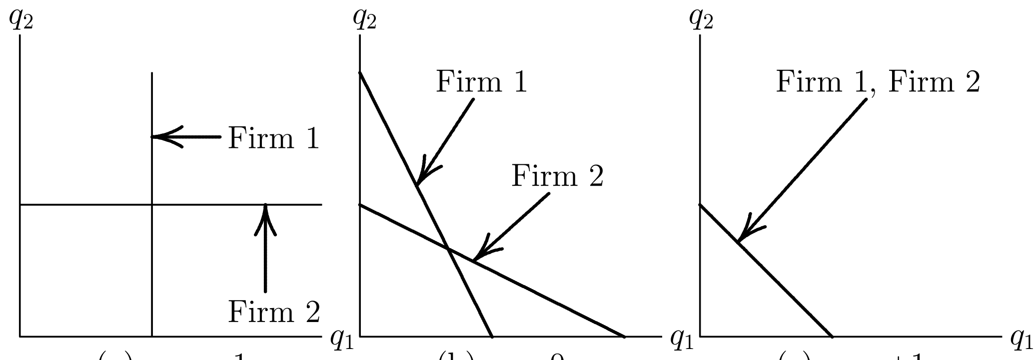
$$\frac{\partial \pi_1}{\partial q_1} = a - c - b(q_1+q_2) - b\left(1 + \alpha_1 \frac{q_2}{q_1}\right) q_1 = 0,$$

which can be rewritten

$$2q_1 + (1+\alpha_1)q_2 = \frac{a-c}{b}$$

or

$$q_1 = \frac{1}{2}\left[\frac{a-c}{b} - (1+\alpha_1)q_2\right].$$



(a) $\alpha = -1$	(b) $\alpha = 0$	(c) $\alpha = +1$
Firm 1: $2q_1 = \frac{a-c}{b}$	$2q_1 + q_2 = \frac{a-c}{b}$	$q_1 + q_2 = \frac{a-c}{2b}$
Firm 2: $2q_2 = \frac{a-c}{b}$	$q_1 + 2q_2 = \frac{a-c}{b}$	$q_1 + q_2 = \frac{a-c}{2b}$

Figure 3.2: Cournot duopoly best response curves with conjectural elasticities

As α rises from -1 to $+1$, the slope of the best response function changes from 0 to -1: best response curves become steeper as the conjectural elasticity increases.

In the same way, the equation of firm 2's best response function with conjectural elasticity α_2 is

$$(1 + \alpha_2)q_1 + 2q_2 = \frac{a - c}{b}.$$

Best response curves are drawn in Figure 3.2 for the three cases $\alpha_1 = \alpha_2 = -1, 0, +1$.

Symmetric equilibrium outputs are

$$q = \frac{1}{3 + \alpha} \frac{a - c}{b}.$$

Equilibrium outputs fall as α rises.

3.2 There are two varieties, 1 and 2. There are two groups of n consumers each, A and B, uniformly distributed over $[0,1]$.

Consumers in group A regard variety 1 as being of higher quality than variety 2. A consumer in group A located at $i \in [0,1]$ gets utility $e - fi - p_1$ from purchasing variety 1 and utility $g - i - p_2$ for purchasing variety 2, where $e > g$, $f > 1$.

Consumers in group B have preferences of the same functional forms as consumers of group A, but group B consumers regard variety 2 as being of higher quality than variety 1. Producers cannot price discriminate between members of the two groups.

Show that the equations of the demand curves and the inverse demand curves for the two varieties are

$$q_1 = \frac{n}{f-1}\left[(f-1)g - (1+f)p_1 + 2p_2\right]$$

$$q_2 = \frac{n}{f-1}\left[(f-1)g + 2p_1 - (1+f)p_2\right]$$

and

$$p_1 = g - \frac{1}{n}\frac{(1+f)q_1 + 2q_2}{3+f} = g - \frac{1}{n}\frac{1+f}{3+f}\left(q_1 + \frac{2}{1+f}q_2\right)$$

$$p_2 = g - \frac{1}{n}\frac{2q_1 + (1+f)q_2}{3+f} = g - \frac{1}{n}\frac{1+f}{3+f}\left(\frac{2}{1+f}q_1 + q_2\right)$$

respectively.

> *Note: subject to certain restrictions on the parameters, there is a region of price space in which the quantities demanded have the indicated form. Noncooperative equilibrium with price-setting firms does not, in general, occur in this region.*

Compare the equations of the inverse demand curves with inverse demand curves from the Bowley model of product differentiation.

At equal prices, a group A consumer located at 0 gets higher utility from purchasing variety 1 than variety 2. Moving from 0 toward 1, utility from variety 1 falls more rapidly than utility from variety 2.

The group A consumer that is indifferent between the two varieties is located at i^* defined by

$$e - fi - p_1 = g - i - p_2$$

$$e - p_1 - fi = g - p_2 - i$$

$$e - p_1 - (g - p_2) = (f - 1)i$$

$$i^* = \frac{e - p_1 - (g - p_2)}{f - 1}.$$

The demand for variety 1 from group A consumers is

$$q_{1A} = n\frac{e - p_1 - (g - p_2)}{f - 1},$$

since consumers located from 0 to i^* will buy variety 1.

Group A consumers located from i^* to i^{**} defined by $g - i^{**} - p_2 = 0$ will buy variety 2.

$$g - i^{**} - p_2 = 0$$

$$i^{**} = g - p_2.$$

The length of the interval on which group A consumers purchase variety 2 is

$$i^{**} - i^* = (g - p_2) - \frac{e - p_1 - (g - p_2)}{f - 1}$$

$$= \frac{f(g - p_2) - (e - p_1)}{f - 1}$$

and the demand for variety 2 from group A consumers is

$$q_{2A} = n\frac{f(g - p_2) - (e - p_1)}{f - 1}.$$

Assume that $q_{2A} > 0$.

Group A consumers located from i^{**} to 1 do not purchase the good.

Group B consumers have preferences of the same functional form as group A consumers, but group B consumers regard variety 2 as being of higher quality than variety 1. Group B demands are

$$q_{1B} = n\frac{f(g - p_1) - (e - p_2)}{f - 1},$$

$$q_{2B} = n\frac{e - p_2 - (g - p_1)}{f - 1}.$$

Firms cannot discriminate between consumers in the two groups. Demand equations for the two varieties are

$$q_1 = n\frac{e - p_1 - (g - p_2)}{f - 1} + n\frac{f(g - p_1) - (e - p_2)}{f - 1}$$

$$q_2 = n\frac{f(g - p_2) - (e - p_1)}{f - 1} + n\frac{e - p_2 - (g - p_1)}{f - 1}.$$

Combining terms, the equations of the demand curves are

$$q_1 = \frac{n}{f - 1}\left[e - p_1 - (g - p_2) + f(g - p_1) - (e - p_2)\right]$$

$$= \frac{n}{f - 1}\left[(f - 1)g - (1 + f)p_1 + 2p_2\right].$$

$$q_2 = \frac{n}{f - 1}\left[f(g - p_2) - (e - p_1) + e - p_2 - (g - p_1)\right]$$

$$= \frac{n}{f - 1}\left[(f - 1)g + 2p_1 - (1 + f)p_2\right].$$

Now invert the equations of the demand curves to obtain the equations of the inverse demand curves.

$$(1 + f)p_1 - 2p_2 = (f - 1)g - \frac{f - 1}{n}q_1$$

$$-2p_1 + (1+f)p_2 = (f-1)g - \frac{f-1}{n}q_2$$

$$\begin{pmatrix} 1+f & -2 \\ -2 & 1+f \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = (f-1)g \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{f-1}{n} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$

$$\frac{(f-1)(3+f)}{f-1} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

$$= g \begin{pmatrix} 1+f & 2 \\ 2 & 1+f \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{n} \begin{pmatrix} 1+f & 2 \\ 2 & 1+f \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$

$$(3+f) \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = (3+f)g \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{n} \begin{pmatrix} (1+f)q_1 + 2q_2 \\ 2q_1 + (1+f)q_2 \end{pmatrix}$$

$$p_1 = g - \frac{1}{n}\frac{(1+f)q_1 + 2q_2}{3+f} = g - \frac{1}{n}\frac{1+f}{3+f}\left(q_1 + \frac{2}{1+f}q_2\right)$$

$$p_2 = g - \frac{1}{n}\frac{2q_1 + (1+f)q_2}{3+f} = g - \frac{1}{n}\frac{1+f}{3+f}\left(\frac{2}{1+f}q_1 + q_2\right)$$

Compare with the Bowley formulation:

$$p_1 = a - b(q_1 + \theta q_2).$$

If we set

$$a = g$$

$$b = \frac{1}{n}\frac{1+f}{3+f}$$

$$\theta = \frac{2}{1+f},$$

the two formulations are equivalent.
3.3 Derive (a) the demand function for the Bowley linear model of product differentiation,

$$q_i = \frac{(1-\theta)\,a - [1+(n-2)\theta]p_i + \theta\sum_{j\neq i} p_j}{b(1-\theta)\,[1+(n-1)\theta]}$$

and (b) the inverse demand function implied by the Shubik-Levitan linear model of product differentiation:

$$p_i = \frac{\alpha}{\beta} - \frac{1}{\beta}\frac{n+\gamma}{1+\gamma}\left(\frac{\gamma}{n+\gamma}q_1 + \frac{\gamma}{n+\gamma}q_2 + \ldots + q_i \ldots + \frac{\gamma}{n+\gamma}q_n\right).$$

(a) The n-firm version of the Bowley inverse demand function for variety 1 is

$$p_1 = a - b(q_1 + \theta q_2 + \theta q_3 + \ldots + \theta q_n).$$

Written in matrix form, the system of equations formed by the inverse demand functions is

$$\begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix} = a\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} - b\left[(1-\theta)\begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix} + \theta\begin{pmatrix} 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \ldots. & 1 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix}\right].$$

Let J_n be a column vector of n ones. The system of equations of the inverse demand curves can be written

$$p = aJ_n - b(1-\theta)\left(I_n + \frac{\theta}{1-\theta}J_nJ_n'\right)q$$

$$b(1-\theta)\left(I_n + \frac{\theta}{1-\theta}J_nJ_n'\right)q = aJ_n - p.$$

Let the coefficient matrix

$$I_n + \frac{\theta}{1-\theta}J_nJ_n'$$

have inverse matrix of the form

$$I_n + kJ_nJ_n'$$

for a constant k the value of which is to be determined.

Then it must be that

$$I_n = \left(I_n + \frac{\theta}{1-\theta}J_nJ_n'\right)(I_n + kJ_nJ_n')$$

$$= I_n + kJ_nJ_n' + \frac{\theta}{1-\theta}J_nJ_n' + k\frac{\theta}{1-\theta}J_nJ_n'J_nJ_n'$$

$$= I_n + kJ_nJ_n' + \frac{\theta}{1-\theta}J_nJ_n' + k\frac{n\theta}{1-\theta}J_nJ_n'$$

(using $J_n'J_n = n$)

$$= I_n + \left[k\left(1 + \frac{n\theta}{1-\theta}\right) + \frac{\theta}{1-\theta}\right]J_nJ_n',$$

from which we must have

$$k = -\frac{\theta}{1+(n-1)\theta}.$$

Then the inverse of the coefficient matrix is

$$\left(I_n + \frac{\theta}{1-\theta}J_nJ_n'\right)^{-1} = I_n - \frac{\theta}{1+(n-1)\theta}J_nJ_n'.$$

Use this expression to find the equations of the demand functions:

$$b(1-\theta)\left(I_n + \frac{\theta}{1-\theta}J_nJ_n'\right)q = aJ_n - p.$$

$$b(1-\theta)q = \left(I_n - \frac{\theta}{1+(n-1)\theta}J_nJ_n'\right)(aJ_n - p).$$

Examine the constant terms:

$$\left(I_n - \frac{\theta}{1+(n-1)\theta}J_nJ_n'\right)J_n =$$

$$J_n - \frac{n\theta}{1+(n-1)\theta} J_n =$$

$$\left(1 - \frac{n\theta}{1+(n-1)\theta}\right) J_n =$$

$$\left(\frac{1+(n-1)\theta - n\theta}{1+(n-1)\theta}\right) J_n =$$

$$\frac{1-\theta}{1+(n-1)\theta} J_n.$$

The system of equations of the demand functions satisfies

$$b(1-\theta)q = \frac{1-\theta}{1+(n-1)\theta} aJ_n - \left[I_n - \frac{\theta}{1+(n-1)\theta} J_n J_n'\right] p$$

$$b(1-\theta)\left[1+(n-1)\theta\right] q = (1-\theta)\, aJ_n - \{\left[1+(n-1)\theta\right] I_n - \theta J_n J_n'\}\, p.$$

The demand function for variety 1, for example, satisfies

$$b(1-\theta)\left[1+(n-1)\theta\right] q_1 = (1-\theta)\, a - [1+(n-1)\theta]p_1 + \theta \sum_1^n p_j$$

$$b(1-\theta)\left[1+(n-1)\theta\right] q_1 = (1-\theta)\, a - [1+(n-2)\theta]p_1 + \theta \sum_2^n p_j.$$

(b) The n-firm version of the Shubik-Levitan demand function for variety 1 can be written

$$nq_1 = \alpha - \beta(1+\gamma)p_1 + \frac{1}{n}\beta\gamma \sum_1^n p_i.$$

In matrix notation, the system of equations can be written

$$n \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} - \beta(1+\gamma) \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix} + \frac{1}{n}\beta\gamma \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \dots. & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix}.$$

Let J_n be a column vector of n ones. The system of equations of the demand curves can be written

$$nq = \alpha J_n - \beta(1+\gamma)p + \frac{1}{n}\beta\gamma J_n J_n' p$$

$$nq = \alpha J_n - \beta\left[(1+\gamma)I_n - \frac{1}{n}\gamma J_n J_n'\right]p$$

$$\beta(1+\gamma)\left(I_n - \frac{1}{n}\frac{\gamma}{1+\gamma}J_n J_n'\right)p = \alpha J_n - nq.$$

Now consider the matrix

$$I_n - \frac{1}{n}\frac{\gamma}{1+\gamma}J_n J_n'.$$

Let this matrix have an inverse of the form

$$I_n + kJ_n J_n',$$

where k is a constant the value of which is to be determined.

Then it must be that

$$I_n = \left(I_n - \frac{1}{n}\frac{\gamma}{1+\gamma}J_n J_n'\right)(I_n + kJ_n J_n')$$

$$= I_n + kJ_n J_n' - \frac{1}{n}\frac{\gamma}{1+\gamma}J_n J_n' - \frac{k}{n}\frac{\gamma}{1+\gamma}J_n J_n' J_n J_n'$$

$$= I_n + kJ_n J_n' - \frac{1}{n}\frac{\gamma}{1+\gamma}J_n J_n' - k\frac{\gamma}{1+\gamma}J_n J_n'$$

(using $J_n' J_n = n$)

$$= I_n + \left(k - k\frac{\gamma}{1+\gamma} - \frac{1}{n}\frac{\gamma}{1+\gamma}\right)J_n J_n'$$

$$= I_n + \left[\left(1 - \frac{\gamma}{1+\gamma}\right) k - \frac{1}{n}\frac{\gamma}{1+\gamma}\right] J_n J_n'$$

$$= I_n + \left(\frac{1}{1+\gamma} k - \frac{1}{n}\frac{\gamma}{1+\gamma}\right) J_n J_n'$$

$$= I_n + \frac{1}{1+\gamma}\left(k - \frac{1}{n}\gamma\right) J_n J_n',$$

from which we must have

$$k = \frac{1}{n}\gamma$$

and the inverse of the coefficient matrix is

$$\left(I_n - \frac{1}{n}\frac{\gamma}{1+\gamma} J_n J_n'\right)^{-1} = I_n + \frac{\gamma}{n} J_n J_n'.$$

Use this expression for the inverse of the coefficient matrix to find the equations of the inverse demand curves:

$$\beta(1+\gamma)\left(I_n - \frac{1}{n}\frac{\gamma}{1+\gamma} J_n J_n'\right) p = \alpha J_n - nq$$

$$\beta(1+\gamma) p = \left(I_n + \frac{1}{n}\gamma J_n J_n'\right)(\alpha J_n - nq)$$

$$\beta(1+\gamma) p = \alpha\left(I_n + \frac{1}{n}\gamma J_n J_n'\right) J_n - n\left(I_n + \frac{1}{n}\gamma J_n J_n'\right) q$$

$$\beta(1+\gamma) p = \alpha(1+\gamma) J_n - (nI_n + \gamma J_n J_n') q.$$

The inverse demand function for variety 1, for example, is

$$p_1 = \frac{\alpha}{\beta} - \frac{1}{\beta}\frac{n+\gamma}{1+\gamma}\left(q_1 + \frac{\gamma}{n+\gamma} q_2 + \ldots + \frac{\gamma}{n+\gamma} q_n\right).$$

3.4 Find quantity-setting oligopoly equilibrium outputs and prices with the Shubik-Levitan linear model of product differentiation if firms have identical constant marginal and average cost c per unit.

From the equation of firm 1's inverse demand function

$$p_1 = \frac{\alpha}{\beta} - \frac{1}{\beta}\frac{n+\gamma}{1+\gamma}\left(q_1 + \frac{\gamma}{n+\gamma}q_2 + \ldots + \frac{\gamma}{n+\gamma}q_n\right)$$

its payoff is

$$\pi_1 = \left[\frac{\alpha}{\beta} - c - \frac{1}{\beta}\frac{n+\gamma}{1+\gamma}\left(q_1 + \frac{\gamma}{n+\gamma}q_2 + \ldots + \frac{\gamma}{n+\gamma}q_n\right)\right] q_1.$$

The first-order condition to maximize π_1 is

$$2q_1 + \frac{\gamma}{n+\gamma}q_2 + \ldots + \frac{\gamma}{n+\gamma}q_n = \beta\frac{1+\gamma}{n+\gamma}\left(\frac{\alpha}{\beta} - c\right).$$

The first-order condition implies that in equilibrium

$$\frac{\alpha}{\beta} - c - \frac{1}{\beta}\frac{n+\gamma}{1+\gamma}\left(q_1 + \frac{\gamma}{n+\gamma}q_2 + \ldots + \frac{\gamma}{n+\gamma}q_n\right) = \frac{1}{\beta}\frac{n+\gamma}{1+\gamma}q_1,$$

so that in equilibrium

$$p_1 = c + \frac{1}{\beta}\frac{n+\gamma}{1+\gamma}q_1$$

and

$$\pi_1 = \frac{1}{\beta}\frac{n+\gamma}{1+\gamma}q_1^2.$$

Imposing symmetry on the first-order condition, equilibrium output satisfies

$$2q_1 + \frac{\gamma}{n+\gamma}q_2 + \ldots + \frac{\gamma}{n+\gamma}q_n = \beta\frac{1+\gamma}{n+\gamma}\left(\frac{\alpha}{\beta} - c\right)$$

$$\left[2 + (n-1)\frac{\gamma}{n+\gamma}\right] q = \beta\frac{1+\gamma}{n+\gamma}\left(\frac{\alpha}{\beta} - c\right)$$

$$\frac{2(n+\gamma)+(n-1)\gamma}{n+\gamma}q = \beta\frac{1+\gamma}{n+\gamma}\left(\frac{\alpha}{\beta}-c\right)$$

$$q = \frac{1+\gamma}{2n+(n+1)\gamma}(\alpha-\beta c).$$

Equilibrium price is

$$p_1 = c + \frac{n+\gamma}{2n+(n+1)\gamma}\left(\frac{\alpha}{\beta}-c\right).$$

Equilibrium profit per firm is

$$\pi_1 = \frac{1}{\beta}\frac{n+\gamma}{1+\gamma}\left[\frac{1+\gamma}{2n+(n+1)\gamma}(\alpha-\beta c)\right]^2.$$

3.5 Find price-setting oligopoly prices and outputs with the Bowley linear model of product differentiation if firms have identical constant marginal and average cost c per unit.

The demand equation for variety 1 is

$$q_i = \frac{(1-\theta)\,a - [1+(n-2)\theta]\,p_i + \theta\sum_{j\neq i}p_j}{b(1-\theta)\,[1+(n-1)\theta]}.$$

Rewrite this in terms of deviations from marginal cost:

$$q_i = \frac{(1-\theta)\,(a-c) - [1+(n-2)\theta]\,(p_i-c) + \theta\sum_{j\neq i}(p_j-c)}{b(1-\theta)\,[1+(n-1)\theta]}.$$

Firm 1's profit is

$$\pi_1 = (p_1-c)\,\frac{(1-\theta)\,(a-c) - [1+(n-2)\theta]\,(p_1-c) + \theta\sum_{j=2}^{n}(p_j-c)}{b(1-\theta)\,[1+(n-1)\theta]}.$$

The first-order condition to maximize π_1 with respect to p_1 is

$$(1-\theta)\,(a-c) - 2[1+(n-2)\theta]\,(p_1-c) + \theta\sum_{j=2}^{n}(p_j-c) \equiv 0.$$

The first-order condition implies

$$(1-\theta)\,(a-c) - [1+(n-2)\theta]\,(p_1-c) + \theta\sum_{j=2}^{n}(p_j-c) = [1+(n-2)\theta]\,(p_1-c),$$

so that firm 1's equilibrium quantity demanded and payoff are

$$q_1 = \frac{1+(n-2)\theta}{b(1-\theta)\left[1+(n-1)\theta\right]}(p_1 - c)$$

and

$$\pi_1 = \frac{1+(n-2)\theta}{b(1-\theta)\left[1+(n-1)\theta\right]}(p_1 - c)^2,$$

respectively.

Rewrite firm 1's first-order condition as

$$2[1+(n-2)\theta](p_1 - c) - \theta\sum_{j=2}^{n}(p_j - c) = (1-\theta)(a-c).$$

Given the assumed symmetry of firms, all firms will set the same price in equilibrium. Set $p_1 = p_2 = ... = p_n = p$ and collect terms to find equilibrium price:

$$\{2[1+(n-2)\theta] - (n-1)\,\theta\}(p-c) = (1-\theta)(a-c)$$

$$[2+(n-3)\,\theta](p-c) = (1-\theta)(a-c)$$

$$p = c + \frac{1-\theta}{2+(n-3)\,\theta}(a-c).$$

Equilibrium output per firm is then

$$q_1 = \frac{1+(n-2)\theta}{b(1-\theta)\left[1+(n-1)\theta\right]}\frac{1-\theta}{2+(n-3)\,\theta}(a-c)$$

$$q_1 = \frac{1+(n-2)\theta}{\left[1+(n-1)\theta\right]\left[2+(n-3)\,\theta\right]}\frac{a-c}{b}.$$

3.6 Find noncooperative equilibrium prices, quantities, and payoffs for a duopoly if demand follows the Bowley linear model of product differentiation, if firms have identical constant marginal and average cost c per unit, and if firm 1 sets quantity while firm 2 sets price.

The equations of the inverse demand curves and demand curves are

$$p_1 - c = a - c - b(q_1 + \theta q_2) \tag{3.25}$$

$$p_2 - c = a - c - b(\theta q_1 + q_2) \tag{3.26}$$

and

$$q_1 = \frac{(1-\theta)(a-c) - (p_1 - c) + \theta(p_2 - c)}{(1-\theta^2)b} \tag{3.27}$$

$$q_2 = \frac{(1-\theta)(a-c) - (p_2 - c) + \theta(p_1 - c)}{(1-\theta^2)b} \tag{3.28}$$

respectively.

Firm 2 sets price; substitute (3.28) in (3.25) and solve to obtain an expression for firm 1's net price $p_1 - c$ as a function of q_1 and $p_2 - c$:

$$p_1 - c = a - c - b\left[q_1 + \theta\frac{(1-\theta)(a-c) - (p_2 - c) + \theta(p_1 - c)}{(1-\theta^2)b}\right]$$

$$p_1 - c = a - c - bq_1 - \theta\frac{(1-\theta)(a-c) - (p_2 - c) + \theta(p_1 - c)}{(1-\theta^2)}$$

$$p_1 - c = a - c - bq_1 - \frac{\theta}{1+\theta}(a-c) + \frac{\theta}{1-\theta^2}(p_2 - c) - \frac{\theta^2}{1-\theta^2}(p_1 - c)$$

$$\left(1 + \frac{\theta^2}{1-\theta^2}\right)(p_1 - c) = (a-c) - \frac{\theta}{1+\theta}(a-c) - bq_1 + \frac{\theta}{1-\theta^2}(p_2 - c)$$

$$p_1 - c = (1-\theta^2)\left[\frac{1}{1+\theta}(a-c) - bq_1 + \frac{\theta}{1-\theta^2}(p_2 - c)\right]$$

$$p_1 - c = (1-\theta)(a-c) - (1-\theta^2)bq_1 + \theta(p_2 - c).$$

Firm 1's profit is

$$\pi_1 = (p_1 - c)q_1 = \left[(1-\theta)(a-c) - (1-\theta^2)bq_1 + \theta(p_2 - c)\right] q_1.$$

The first-order condition to maximize π_1 with respect to q_1 is

$$(1-\theta)(a-c) - (1-\theta^2)bq_1 + \theta(p_2 - c) + q_1\left[-(1-\theta^2)b\right] = 0.$$

Thus in equilibrium

$$p_1 - c = (1-\theta^2)bq_1.$$

Combining terms, the equation of firm 1's first-order condition can be written

$$2(1-\theta^2)bq_1 - \theta(p_2 - c) = (1-\theta)(a-c).$$

Firm 1 sets quantity; substitute (3.25) in (3.28) and collect terms to obtain an expression for q_2 in terms of q_1 and $p_2 - c$:

$$q_2 = \frac{(1-\theta)(a-c) - (p_2 - c) + \theta\left[a - c - b(q_1 + \theta q_2)\right]}{(1-\theta^2)b}$$

$$(1-\theta^2)bq_2 = (1-\theta)(a-c) - (p_2 - c) + \theta(a-c) - \theta bq_1 - \theta^2 bq_2$$

$$bq_2 = a - c - (p_2 - c) - \theta bq_1.$$

Firm 2's profit is

$$\pi_2 = (p_2 - c)\frac{a - c - (p_2 - c) - \theta bq_1}{b}.$$

The first-order condition to maximize π_2 with respect to p_2 is

$$b\frac{\partial \pi_2}{\partial p_2} = a - c - (p_2 - c) - \theta bq_1 + (p_2 - c)(-1) = 0.$$

Hence in equilibrium

$$q_2 = \frac{p_2 - c}{b}$$

and

$$\pi_2 = \frac{1}{b}(p_2 - c)^2.$$

Rewrite the first-order condition as

$$\theta b q_1 + 2(p_2 - c) = a - c.$$

The system of equations formed by the first-order conditions of the two firms is

$$\begin{pmatrix} 2(1-\theta^2)b & -\theta \\ \theta b & 2 \end{pmatrix} \begin{pmatrix} q_1 \\ p_2 - c \end{pmatrix} = \begin{pmatrix} 1-\theta \\ 1 \end{pmatrix} (a-c).$$

This has solution:

$$\left[4(1-\theta^2)b + \theta^2 b\right] \begin{pmatrix} q_1 \\ p_2 - c \end{pmatrix} = \begin{pmatrix} 2 & \theta \\ -\theta b & 2(1-\theta^2)b \end{pmatrix} \begin{pmatrix} 1-\theta \\ 1 \end{pmatrix} (a-c)$$

$$(4-3\theta^2)b \begin{pmatrix} q_1 \\ p_2 - c \end{pmatrix} = \begin{pmatrix} 2-\theta \\ (1-\theta)(2+\theta)b \end{pmatrix} (a-c)$$

Firm 1's equilibrium output is

$$q_1 = \frac{2-\theta}{4-3\theta^2}\frac{a-c}{b}.$$

Firm 2's equilibrium price is

$$p_2 - c = \frac{(1-\theta)(2+\theta)}{4-3\theta^2}(a-c).$$

Then firm 1's equilibrium price is

$$p_1 - c = (1-\theta^2)\frac{2-\theta}{4-3\theta^2}(a-c).$$

Firm 2's equilibrium quantity is

$$q_2 = \frac{(1-\theta)(2+\theta)}{4-3\theta^2}\frac{a-c}{b}.$$

3.7 For duopoly with the Bowley product differentiation specification and constant marginal cost c per unit for both firms:

(a) Find equilibrium prices and quantities in quantity-setting duopoly if firms have symmetric conjectural quantity elasticities $\alpha = \frac{q_i}{q_j}\frac{\partial q_j}{\partial q_i}$, for i, $j = 1$, 2 and $j \neq i$.

The equations of the inverse demand curves are:

$$p_1 = a - b(q_1 + \theta q_2)$$

$$p_2 = a - b(\theta q_1 + q_2).$$

Firm 1's profit is

$$\pi_1 = [a - c - b(q_1 + \theta q_2)]\, q_1.$$

1's first-order condition with conjectural elasticities is:

$$\frac{\partial \pi_1}{\partial q_1} = a - c - b(q_1 + \theta q_2) - b\left(1 + \alpha\theta\frac{q_2}{q_1}\right) q_1 =$$

$$a - c - b(q_1 + \theta q_2) - b\,(q_1 + \alpha\theta q_2) = 0.$$

Note: this implies that along the best response function

$$p_1 - c = a - c - b(q_1 + \theta q_2) = b\,(q_1 + \alpha\theta q_2)$$

and

$$\pi_1 = b\,(q_1 + \alpha\theta q_2)\, q_1.$$

Rewrite the equation of the best response function:

$$2q_1 + \theta(1 + \alpha)q_2 = \frac{a - c}{b}.$$

In the same way, the equation of firm 2's best response function is

$$\theta(1 + \alpha)q_1 + 2q_2 = \frac{a - c}{b}.$$

By symmetry, equilibrium outputs are:

$$[2+\theta(1+\alpha)]q_C = \frac{a-c}{b}$$

$$q_{C(\alpha)} = \frac{1}{2+\theta(1+\alpha)}\frac{a-c}{b}.$$

Then Cournot equilibrium prices are

$$p_{C(\alpha)} = c + \frac{1+\alpha\theta}{2+\theta(1+\alpha)}(a-c).$$

(b) Find equilibrium prices and quantities in price-setting duopoly if firms have symmetric conjectural quantity elasticities $\beta = \frac{p_i}{p_j}\frac{\partial p_j}{\partial p_i}$, for i, $j = 1$, 2 and $j \neq i$;

The conjectural elasticity is

$$\beta = \frac{p_1}{p_2}\frac{\partial p_2}{\partial p_1} \rightarrow \frac{\partial p_2}{\partial p_1} = \frac{p_2}{p_1}\beta.$$

The equations of the demand curves are:

$$q_1 = \frac{(1-\theta)(a-c)-(p_1-c)+\theta(p_2-c)}{(1-\theta^2)b}$$

$$q_2 = \frac{(1-\theta)(a-c)-(p_2-c)+\theta(p_1-c)}{(1-\theta^2)b}.$$

Firm 1's profit is

$$\pi_1 = (p_1-c)\frac{(1-\theta)(a-c)-(p_1-c)+\theta(p_2-c)}{(1-\theta^2)b}$$

$$(1-\theta^2)b\pi_1 = (p_1-c)[(1-\theta)(a-c)-(p_1-c)+\theta(p_2-c)].$$

The first-order condition to maximize π_1 with respect to p_1 is:

$$(1-\theta^2)b\frac{\partial \pi_1}{\partial p_1} = (p_1-c)[(1-\theta)(a-c)-(p_1-c)+\theta(p_2-c)] =$$

$$(1-\theta)(a-c)-(p_1-c)+\theta(p_2-c)+(p_1-c)\left[-1+\theta\frac{\partial p_2}{\partial p_1}\right]=$$

$$(1-\theta)(a-c)-(p_1-c)+\theta(p_2-c)+(p_1-c)\left[-1+\theta\frac{p_2}{p_1}\beta\right]=$$

$$(1-\theta)(a-c)-(p_1-c)+\theta(p_2-c)+(p_1-c)\left[-1+\theta\frac{p_2}{p_1}\beta\right]=0.$$

Remark: along the best response function

$$(1-\theta^2)bq_1=(1-\theta)(a-c)-(p_1-c)+\theta(p_2-c)$$

$$=(p_1-c)\left[1-\theta\frac{p_2}{p_1}\beta\right]$$

and

$$\pi_1=\frac{(p_1-c)^2}{(1-\theta^2)b}\left[1-\theta\frac{p_2}{p_1}\beta\right].$$

In symmetric equilibrium

$$q=\frac{1-\beta\theta}{1-\theta^2}\left(\frac{p-c}{b}\right)$$

$$\pi=\frac{1}{b}\frac{1-\beta\theta}{1-\theta^2}(p-c)^2.$$

Go back to equation of price best response function:

$$(1-\theta)(a-c)-(p_1-c)+\theta(p_2-c)+(p_1-c)\left(-1+\theta\frac{p_2}{p_1}\beta\right)=0$$

$$(1-\theta)(a-c)-(p_1-c)+\theta(p_2-c)-(p_1-c)+\beta\theta\frac{p_2}{p_1}(p_1-c)=0.$$

In symmetric equilibrium

$$(1-\theta)(a-c)-(p-c)+\theta(p-c)-(p-c)+\beta\theta(p-c)=0$$

$$(1-\theta)(a-c) = 2(p-c) - \theta(p-c) - \beta\theta(p-c)$$

$$[2-(1+\beta)\theta](p-c) = (1-\theta)(a-c)$$

$$p_{B(\beta)} - c = \frac{1-\theta}{2-(1+\beta)\theta}(a-c)$$

$$q_{B(\beta)} = \frac{1-\beta\theta}{1-\theta^2}\left(\frac{p-c}{b}\right)$$

$$= \frac{1-\beta\theta}{1-\theta^2}\frac{1-\theta}{2-(1+\beta)\theta}\left(\frac{a-c}{b}\right)$$

$$= \frac{1}{(1+\theta)}\frac{1-\beta\theta}{2-(1+\beta)\theta}\left(\frac{a-c}{b}\right).$$

(c) Show that the equilibrium prices and quantities from (a), (b) are the same for

$$\beta = \frac{\alpha+\theta}{1+\alpha\theta}.$$

Compare the Bertrand outcomes with the Cournot outcomes:

$$p_{C(\alpha)} - c = \frac{1+\alpha\theta}{2+\theta(1+\alpha)}(a-c)$$

$$q_{C(\alpha)} = \frac{1}{2+\theta(1+\alpha)}\frac{a-c}{b}.$$

For equilibrium prices to be the same

$$\frac{1-\theta}{2-(1+\beta)\theta} = \frac{1+\alpha\theta}{2+\theta(1+\alpha)};$$

solve this for

$$\beta = \frac{\alpha+\theta}{1+\alpha\theta}$$

Then Bertrand quantities are proportional to

$$\frac{1}{(1+\theta)}\frac{1-\frac{\alpha+\theta}{1+\alpha\theta}\theta}{2-(1+\frac{\alpha+\theta}{1+\alpha\theta})\theta}=\frac{1}{2+(1+\alpha)\theta},$$

which checks.

3.8 Derive expressions for supply function duopoly price and output if the inverse demand curve is

$$p = a - b(Q - \varepsilon)$$

and firms have the cost function

$$C(q) = fq + gq^2.$$

Compare these results with those for Cournot and Bertrand duopoly.

First we will derive the Cournot and Bertrand "supply functions."

The Cournot supply function is a form of the first-order condition for a single firm.

For a known value of ε, firm 1's profit is

$$\pi_1 = (a + b\varepsilon - bq_1 - bq_2)q_1 - fq_1 - gq_1^2$$

$$= [a + b\varepsilon - f - (b + g)q_1 - bq_2]\, q_1.$$

The first-order condition to maximize π_1 is

$$a + b\varepsilon - f - 2(b + g)q_1 - bq_2 = 0.$$

so that when the first-order condition is satisfied,

$$a + b\varepsilon - f - (b + g)q_1 - bq_2 = (b + g)q_1$$

and along the best response function

$$\pi_1 = (b + g)q_1^2.$$

Furthermore, substituting

$$a + b\varepsilon - b(q_1 + q_2) = f + (b + 2g)q_1$$

in the equation of the inverse demand curve

$$p = a + b\varepsilon - b(q_1 + q_2)$$

gives

$$p = f + (b + 2g)q,$$

where for simplicity we drop the subscript on output since in equilibrium both firms will produce the same output. This is the equation of the Cournot equilibrium supply curve. It is the equation of a straight line with price-axis intercept equal to $C'(0) = f$ and slope $b + 2g$.

From the condensed first-order condition we can also obtain expressions Cournot equilibrium output and price:

$$a - f - 2(b + g)q - bq = 0$$

$$a - f - (3b + 2g)q = 0$$

$$q_C = \frac{a - f}{3b + 2g}$$

$$p_C = f + \frac{b + 2g}{3b + 2g}(a - f).$$

In Bertrand duopoly equilibrium with a standardized product, firms produce an output that makes marginal cost equal to price, so that

$$p = f + 2gq.$$

This is the equation of the Bertrand equilibrium supply curve. Once again, it is linear, with the same price-axis intercept as the Cournot equilibrium supply curve but a smaller slope.

To find Bertrand equilibrium outputs, express price along the inverse demand curve in terms of output

$$a - b(2q) = f + 2gq$$

and solve for

$$q_B = \frac{1}{2}\frac{a-f}{b+g}.$$

Bertrand equilibrium price is then

$$p_B = f + 2g\left(\frac{1}{2}\frac{a-f}{b+g}\right) = f + \frac{g}{b+g}(a-f) = \frac{ag+bf}{b+g}.$$

Now turn to supply-function oligopoly. The equations of the inverse demand curve and the demand curve are

$$p = a - b(Q - \varepsilon)$$

and

$$Q = \frac{a}{b} - \frac{1}{b}p + \varepsilon$$

respectively. The cost function is

$$C(q) = fq + gq^2.$$

For this demand function and this cost function, the differential equation that must be satisfied by the equilibrium supply curve,

$$S'(p) = \frac{S(p)}{p - C'[S(p)]} + D_p(p)$$

is

$$S'(p) = \frac{S}{p - f - 2gS} - \frac{1}{b}.$$

For notational simplicity, measure price in terms of deviations from $C'(0) = f$:

$$P = p - f;$$

then the differential equation that must be satisfied by the equilibrium supply curve is

$$\frac{dS}{dP} = \frac{S}{P - 2gS} - \frac{1}{b}.$$

Suppose the solution takes the form

$$S = KP,$$

where K is a constant the value of which is to be determined.

Substituting in the differential equation gives

$$K = \frac{KP}{P - 2gKP} - \frac{1}{b}$$

$$K = \frac{K}{1 - 2gK} - \frac{1}{b},$$

which can be rearranged to obtain the quadratic equation

$$2bgK^2 + 2gK - 1 = 0.$$

It is the positive root that is of interest:

$$K(b, g) = \frac{\sqrt{g^2 + 2bg} - g}{2bg} > 0.$$

The equation of the equilibrium supply curve is

$$S(p - f) = \frac{\sqrt{g^2 + 2bg} - g}{2bg}(p - f)$$

or (for comparison with the Cournot and Bertrand results)

$$p = f + \frac{2bg}{\sqrt{g^2 + 2bg} - g} q$$

$K(b, g)$ falls as either b or g rise:

$$\frac{\partial}{\partial b}\left(\frac{\sqrt{g^2 + 2bg} - g}{2bg}\right) = -\frac{1}{2}\frac{b + g - \sqrt{g^2 + 2bg}}{b^2\sqrt{g^2 + 2bg}} < 0$$

(since the numerator on the right is positive).

$$\frac{\partial}{\partial g}\left(\frac{\sqrt{g^2 + 2bg} - g}{2bg}\right) = -\frac{1}{2g\sqrt{g^2 + 2bg}} < 0.$$

To find equilibrium outputs, turn to the requirement that the market clear:

$$a - b(2q) = f + \frac{2bg}{\sqrt{g^2 + 2bg} - g} q$$

$$q_S = \frac{\sqrt{g^2 + 2bg} - g}{\sqrt{g^2 + 2bg}} \left(\frac{a - f}{2b} \right).$$

The supply-function oligopoly equilibrium price is then

$$p_S = f + \frac{g}{\sqrt{g^2 + 2bg}} (a - f).$$

Comparing the Bertrand and supply-function oligopoly equilibrium outputs per firm

$$2\frac{q_B - q_S}{a - f} = \frac{1}{b + g} - \frac{\sqrt{g^2 + 2bg} - g}{b\sqrt{g^2 + 2bg}}$$

$$= \sqrt{g}\frac{b + g - \sqrt{g^2 + 2bg}}{(b + g)\, b\sqrt{g + 2b}} > 0.$$

Comparing the Cournot and supply-function oligopoly equilibrium outputs per firm,

$$2\frac{q_S - q_C}{a - f} = \frac{\sqrt{g^2 + 2bg} - g}{b\sqrt{g^2 + 2bg}} - \frac{2}{3b + 2g}$$

$$= \frac{(b + 2g)\sqrt{g^2 + 2bg} - g(3b + 2g)}{b\,(3b + 2g)\,\sqrt{g^2 + 2bg}}$$

The numerator is positive

$$(b + 2g)\sqrt{g^2 + 2bg} > g(3b + 2g) > 0 \leftrightarrow$$

$$(b + 2g)^2(g^2 + 2bg) > g^2(3b + 2g)^2 > 0 \leftrightarrow$$

$$(b+2g)^2(g+2b) > g(3b+2g)^2 > 0 \leftrightarrow$$

$$(b+2g)^2(g+2b) - g(3b+2g)^2 = 2b^3 > 0.$$

Hence

$$q_C < q_S < q_B.$$

3.9 If the inverse demand curve is

$$p = a - b(Q - \varepsilon)$$

and each firm has its own cost function

$$C_i(q) = f_i q + g_i q^2$$

for $i = 1, 2$, show that the equations of the equilibrium supply functions (written in inverse form) are

$$p = f_1 + \frac{b\left(bg_1 + 2g_2g_1 + g_2b\right)}{\sqrt{(b+g_1)(b+g_2)\left(bg_1 + g_1g_2 + bg_2\right)} - g_1\left(b+g_2\right)} q_1$$

and

$$p = f_2 + \frac{b\left(bg_1 + 2g_2g_1 + g_2b\right)}{\sqrt{(b+g_1)(b+g_2)\left(bg_1 + g_1g_2 + bg_2\right)} - g_1\left(b+g_1\right)} q_2.$$

With different cost functions, we lose symmetry. In the general case, firm 1's first-order condition is

$$S_1(p) + \{p - C'[S_1(p)]\}[D_p(p) - S_2'(p)] = 0,$$

which can be rewritten as

$$\frac{S_1(p)}{p - C'[S_1(p)]} + D_p(p) - S_2'(p) = 0$$

or

$$S_2'(p) = \frac{S_1(p)}{p - C_1'[S_1(p)]} + D_p(p).$$

In the same way, from firm 2's first-order condition we have

$$S_1'(p) = \frac{S_1(p)}{p - C_2'[S_1(p)]} + D_p(p).$$

With linear inverse demand and quadratic cost functions, the pair of differential equations are

$$S_2'(p) = \frac{S_1(p)}{p - f_1 - 2g_1 S_1(p)} - \frac{1}{b}$$

$$S_1'(p) = \frac{S_2(p)}{p - f_2 - 2g_2 S_2(p)} - \frac{1}{b}.$$

Suppose that the solutions have the form

$$S_1(p) = K_1(p - f_1)$$

$$S_2(p) = K_2(p - f_2).$$

Substituting in the two differential equations yields the pair of simultaneous equations

$$K_1 = \frac{K_2(p - f_2)}{(p - f_2) - 2g_2 K_2 (p - f_2)} - \frac{1}{b}$$

$$K_1 = \frac{K_2}{1 - 2g_2 K_2} - \frac{1}{b}$$

and in the same way

$$K_2 = \frac{K_1}{1 - 2g_1 K_1} - \frac{1}{b}.$$

By substitution, this system of simultaneous equations can be reduced to two quadratic equations, one in K_1 and one in K_2:

$$\left(b^2 g_1 + 2g_2 g_1 b + g_2 b^2\right) K_1^2 + 2g_1 \left(b + g_2\right) K_1 - \left(b + g_2\right) = 0.$$

$$\left(b^2 g_1 + 2g_2 g_1 b + g_2 b^2\right) K_2^2 + 2g_2 \left(b + g_1\right) K_2 - \left(b + g_1\right) = 0.$$

The solutions are

$$K_1 = \frac{\sqrt{4g_1^2\left(b+g_2\right)^2 + 4\left(b^2 g_1 + 2g_2 g_1 b + g_2 b^2\right)\left(b+g_2\right)} - 2g_1\left(b+g_2\right)}{2\left(b^2 g_1 + 2g_2 g_1 b + g_2 b^2\right)}$$

$$K_1 = \frac{\sqrt{(b+g_1)(b+g_2)\left(bg_1+g_1g_2+bg_2\right)} - g_1\left(b+g_2\right)}{b\left(bg_1+2g_2g_1+g_2b\right)}$$

and

$$K_2 = \frac{\sqrt{(b+g_1)(b+g_2)\left(bg_1+g_1g_2+bg_2\right)} - g_2\left(b+g_1\right)}{b\left(bg_1+2g_2g_1+g_2b\right)}.$$

The solutions of the differential equations are then

$$S_1(p) = \frac{\sqrt{(b+g_1)(b+g_2)\left(bg_1+g_1g_2+bg_2\right)} - g_1\left(b+g_2\right)}{b\left(bg_1+2g_2g_1+g_2b\right)}(p-f_1)$$

$$p = f_1 + \frac{b\left(bg_1+2g_2g_1+g_2b\right)}{\sqrt{(b+g_1)(b+g_2)\left(bg_1+g_1g_2+bg_2\right)} - g_1\left(b+g_2\right)}q_1$$

$$p = f_2 + \frac{b\left(bg_1+2g_2g_1+g_2b\right)}{\sqrt{(b+g_1)(b+g_2)\left(bg_1+g_1g_2+bg_2\right)} - g_1\left(b+g_1\right)}q_2.$$

Note that

$$K_1 - K_2 =$$

$$\frac{g_2\left(b+g_1\right) - g_1\left(b+g_2\right)}{b\left(bg_1+2g_2g_1+g_2b\right)} =$$

$$\frac{g_2-g_1}{bg_1+2g_2g_1+g_2b}$$

Hence the firm with the larger coefficient of the quadratic term in its cost function — the firm with the flatter marginal cost curve — has the steepest equilibrium supply function.

For completeness, one should compute expressions for equilibrium outputs and work out conditions for both outputs to be nonnegative.

3.10 If products are differentiated with inverse demand curves following the Bowley specification

$$p_1 = a - b(q_1 + \theta q_2)$$

$$p_2 = a - b(\theta q_1 + q_2)$$

for $i = 1, 2$ and firms have the same cost function,

$$C(q) = fq + gq^2,$$

show that the equilibrium supply functions (written in inverse form) are

$$p = f + \frac{\sqrt{g^2 + 2bg + (1-\theta^2)b^2} - g}{b(1-\theta^2)b + 2g} q.$$

First consider the general case of differentiated products:

$$q_1 = G(p_1, p_2) + \varepsilon$$

$$q_2 = G(p_2, p_1) + \varepsilon$$

Let firm 2 have the supply function

$$q_2 = S_2(p_2),$$

so that market clearing implies

$$S_2(p_2) = G(p_2, p_1) + \varepsilon.$$

This implicitly determines 2's market-clearing price as a function of p_1 and ε:

$$p_2 = \phi_2(p_1, \varepsilon).$$

By implicit differentiation, the derivative of ϕ_2 with respect to its first argument is

$$S_2'(p_2)\frac{\partial \phi_2}{\partial p_1} = G_1(p_2, p_1)\frac{\partial \phi_2}{\partial p_1} + G_2(p_2, p_1)$$

$$[S_2'(p_2) - G_1(p_2,p_1)]\,\frac{\partial \phi_2}{\partial p_1} = G_1(p_2,p_1)$$

$$\frac{\partial \phi_2}{\partial p_1} = \frac{G_1(p_2,p_1)}{S_2'(p_2) - G_1(p_2,p_1)}.$$

In the same way, the derivative of ϕ_2 with respect to its second argument is

$$S_2'(p_2)\frac{\partial \phi_2}{\partial \varepsilon} = G_1(p_2,p_1)\frac{\partial \phi_2}{\partial \varepsilon} + 1$$

$$\frac{\partial \phi_2}{\partial \varepsilon} = \frac{1}{S_2'(p_2) - G_1(p_2,p_1)}.$$

Firm 1's residual demand curve is

$$q_1 = G\left(p_1, \phi_2(p_1,\varepsilon)\right) + \varepsilon$$

and its derivative with respect to p_1, the slope of the residual demand curve, is

$$\frac{\partial q_1}{\partial p_1} = G_1(p_1,p_2) + G_2(p_1,p_2)\frac{\partial \phi_2}{\partial p_1}$$

$$= G_1(p_1,p_2) + \frac{G_1(p_2,p_1)G_2(p_1,p_2)}{S_2'(p_2) - G_1(p_2,p_1)}.$$

Firm 1 maximizes

$$p_1\left[G(p_1,\phi_2(p_1,\varepsilon)) + \varepsilon\right] - C\left[G(p_1,\phi_2(p_1,\varepsilon)) + \varepsilon\right].$$

The first-order condition is

$$G(p_1,\phi_2(p_1,\varepsilon)) + \varepsilon + (p_1 - C')\left[G_1(p_1,p_2) + G_2(p_1,p_2)\frac{\partial \phi_2}{\partial p_1}\right] = 0$$

$$G(p_1,p_2) + \varepsilon + (p_1 - C'(G(p_1,p_2)+\varepsilon))\left[G_1(p_1,p_2) + \frac{G_1(p_2,p_1)G_2(p_1,p_2)}{S_2'(p_2) - G_1(p_2,p_1)}\right] = 0.$$

Require this to hold exactly for each ε:

$$\frac{G(p_1,p_2)}{p_1 - C'(G(p_1,p_2))} + G_1(p_1,p_2) + \frac{G_1(p_2,p_1)G_2(p_1,p_2)}{S_2'(p_2) - G_1(p_2,p_1)} = 0$$

$$\frac{G(p_1,p_2) + (p_1 - C'(G(p_1,p_2)))\, G_1(p_1,p_2)}{p_1 - C'(G(p_1,p_2))} + \frac{G_1(p_2,p_1)G_2(p_1,p_2)}{S_2'(p_2) - G_1(p_2,p_1)} = 0$$

$$-\frac{G(p_1,p_2) + (p_1 - C'(G(p_1,p_2)))\, G_1(p_1,p_2)}{p_1 - C'(G(p_1,p_2))} = \frac{G_1(p_2,p_1)G_2(p_1,p_2)}{S_2'(p_2) - G_1(p_2,p_1)}$$

$$\frac{S_2'(p_2) - G_1(p_2,p_1)}{G_1(p_2,p_1)G_2(p_1,p_2)} = -\frac{p_1 - C'(G(p_1,p_2))}{G(p_1,p_2) + (p_1 - C'(G(p_1,p_2)))\, G_1(p_1,p_2)}$$

$$S_2'(p_2) - G_1(p_2,p_1) = -\frac{p_1 - C'(G(p_1,p_2))}{G(p_1,p_2) + (p_1 - C'(G(p_1,p_2)))\, G_1(p_1,p_2)} G_1(p_2,p_1)G_2(p_1,p_2)$$

$$S_2'(p_2) = G_1(p_2,p_1) - \frac{p_1 - C'(G(p_1,p_2))}{G(p_1,p_2) + (p_1 - C'(G(p_1,p_2)))\, G_1(p_1,p_2)} G_1(p_2,p_1)G_2(p_1,p_2)$$

$$S_2'(p_2) = G_1(p_2,p_1)\left[1 - \frac{[p_1 - C'(G(p_1,p_2))]\, G_2(p_1,p_2)}{G(p_1,p_2) + (p_1 - C'(G(p_1,p_2)))\, G_1(p_1,p_2)}\right].$$

Now impose symmetry and market clearing:

$$S'(p) = G_1(p,p)\left[1 - \frac{[p - C'(S))]\, G_2(p,p)}{S(p) + (p - C'(S))\, G_1(p,p)}\right]$$

$$S'(p) = G_1(p,p) - \frac{[p - C'(S))]\, G_1(p,p)G_2(p,p)}{S(p) + (p - C'(S))\, G_1(p,p)}.$$

Symmetry implies that

$$G_1(p_2,p_1) = G_2(p_1,p_2)$$

when $p_1 = p_2$;

$$S'(p) = G_1(p,p) - \frac{(p - C'(S))G_2^2(p,p)}{S(p) + G_1(p,p)\,(p - C'(S))}.$$

Now evaluate this for the linear inverse demand, quadratic cost case.

Solving the inverse demand curves, the equations of the demand curves are

$$q_1 = G(p_1,p_2) = \frac{(1-\theta)a - p_1 + \theta p_2}{b(1-\theta^2)}$$

$$q_2 = G(p_2,p_1) = \frac{(1-\theta)a + \theta p_1 - p_2}{b(1-\theta^2)}.$$

$$G_1(p_1,p_2) = -\frac{1}{b(1-\theta^2)}$$

$$G_2(p_1,p_2) = \frac{\theta}{b(1-\theta^2)}.$$

Substitute these expressions in the equation of the differential equation:

$$S'(p) = -\frac{1}{b(1-\theta^2)} - \frac{[p - C'(S)]\left[\frac{\theta}{b(1-\theta^2)}\right]^2}{S(p) - (p - C'(S))\frac{1}{b(1-\theta^2)}}.$$

Now substitute for the cost function:

$$C'(S) = f + 2gS$$

$$S'(p) = -\frac{(p - f - 2gS)\left(\frac{\theta}{b(1-\theta^2)}\right)^2}{S(p) - (p - f - 2gS)\frac{1}{b(1-\theta^2)}} - \frac{1}{b(1-\theta^2)}.$$

Now suppose that the solution takes the form

$$S = K(p - f)$$

where K is a constant the value of which is to be determined:

$$K = -\frac{(1-2gK)\left(\frac{\theta}{b(1-\theta^2)}\right)^2}{K-(1-2gK)\frac{1}{b(1-\theta^2)}} - \frac{1}{b(1-\theta^2)}.$$

This is a quadratic equation, with solution

$$K = \frac{\sqrt{g^2+2bg+(1-\theta^2)b^2}-g}{b\left[(1-\theta^2)b+2g\right]}.$$

$$S = \frac{\sqrt{g^2+2bg+(1-\theta^2)b^2}-g}{b\left[(1-\theta^2)b+2g\right]}(p-f)$$

$$p = f + \frac{b\left[(1-\theta^2)b+2g\right]}{\sqrt{g^2+2bg+(1-\theta^2)b^2}-g}q.$$

Note that if $\theta = 1$, K becomes

$$K = \frac{\sqrt{g^2+2bg}-g}{2bg},$$

which is the expression derived for the homogeneous-product case.

Chapter 4

Foundations of Oligopoly Theory III

4.1 Additional References

Lahmandi-Ayed, Rim "Natural oligopolies: a vertical differentiation model," *International Economic Review* 41(4), November 2000, pp. 971–87.

4.2 Answers to end-of-chapter problems

4.1 In the Hotelling linear duopoly model, analyze the demand facing firm A if the quantity demanded at any point on the line is

$$q = \max\left(0, s\rho - sp_D\right),$$

for delivered price p_D.

Individual demand is positive for $p_D < \rho$, zero for $p_D \geq \rho$.

Left

On its left, the quantity demanded of firm A at a particular location falls to zero where

$$0 = s\rho - s\left(p_A + cd_A\right)$$

that is, at a distance d_A from A's location, provided $d_A \leq a$, where

$$d_A = \frac{\rho - p_A}{c}.$$

Provided firm A keeps its mill price below firm B's delivered price at A's location, $p_A \leq p_A^H(p_B) = p_B + c(l - a - b)$, the quantity demanded of firm A on the left is

$$q_{AL} = s\int_0^{\min(a,d_A)} (\rho - p_A - cz)dz.$$

$$= s\left[\left(\rho - p_A - \frac{1}{2}cz\right)z\right]_0^{\min(a,d_A)}$$

$a \leq d_A$ for

$$a \leq \frac{\rho - p_A}{c} \text{ or } p_A + ac \leq \rho,$$

that is, if A's delivered price at the left end of the line is less than the reservation price ρ, and in this case

$$q_{AL} = s\left[(\rho - p_A) - \frac{1}{2}ca\right]a.$$

For $p_A = \rho - ac$, this is

$$s\left[(\rho - \rho + ac) - \frac{1}{2}ca\right]a = \frac{1}{2}sca^2.$$

If A's delivered price at the left end of the line exceeds the reservation price, for

$$a \geq \frac{\rho - p_A}{c} \text{ or } p_A + ac \geq \rho,$$

firm A sells a distance d_A to its left, and

$$q_{AL} = s\left[(\rho - p_A) - \frac{1}{2}c\frac{\rho - p_A}{c}\right]\frac{\rho - p_A}{c} = \frac{s}{2c}(\rho - p_A)^2.$$

For $p_A = \rho - ac$, this is

$$\frac{s}{2c}(\rho - \rho + ac)^2 = \frac{1}{2}sca^2.$$

These results hold provided A does not raise its price above B's delivered price at A's location; there are two subcases to consider.

(a) $\rho \leq p_A^H(p_B) = p_B + c(l-a-b)$: the reservation price is less than B's delivered price at A's location:

$$q_{AL} = \begin{cases} s\left[(\rho - p_A) - \frac{1}{2}ca\right]a & p_A \leq \rho - ca \\ \frac{s}{2c}(\rho - p_A)^2 & \rho - ca \leq p_A \leq \rho \leq p_A^H(p_B) \\ 0 & \rho \leq p_A \end{cases}.$$

$p_A \leq \rho - ca$ or $p_A + ca \leq \rho$ means that A's delivered price at the left end of the line is less than the reservation price; then A sells to the left end of the line.

For p_A in the range $\rho - ca \leq p_A \leq \rho$, firm A sells to a distance d_A from its plant; such a range always exists.

(b) $p_A^H(p_B) = p_B + c(l-a-b) \leq \rho$: B's delivered price at A's location is less than the reservation price; then

$$q_{AL} = \begin{cases} s\left[(\rho - p_A) - \frac{1}{2}ca\right]a & p_A \leq \rho - ca \\ \frac{s}{2c}(\rho - p_A)^2 & \rho - ca \leq p_A \leq p_A^H(p_B) \leq \rho \\ 0 & p_B + c(l-a-b) \leq p_A \end{cases}.$$

As before, if A's delivered price at the left end of the line is less than the reservation price, A sells to the left end of the line.

The condition for the second range of prices to exist is

$$p_B + c(l-a-b) - (\rho - ca) > 0$$

$$p_B + c(l-b) > \rho$$

or that B's delivered price at the left end of the line exceed the reservation price.

Right

The quantity demanded of firm A on its right: if A does not undercut B's mill price, A sells out to d_A or out to the indifferent consumer, which comes first.

The indifferent consumer, as before, is located at:

$$x = \frac{1}{2}\left(l - a - b + \frac{p_B - p_A}{c}\right).$$

For $p_A \geq p_B - c(l-a-b)$,

$$q_{AR} = s\int_0^{\min(x,d_A)} (\rho - p_A - cz)dz = s\left[\left(\rho - p_A - \frac{1}{2}cz\right)z\right]_0^{\min(x,d_A)}.$$

The most straightforward way to write the condition for $x \leq d_A$ is

$$\frac{1}{2}\left(l-a-b+\frac{p_B-p_A}{c}\right) \leq \frac{\rho-p_A}{c}$$

$$c(l-a-b)+p_B-p_A \leq 2(\rho-p_A)$$

$$p_A \leq 2\rho - p_B - c(l-a-b) = 2\rho - p_A^H(p_B);$$

this has no immediate interpretation.

Alternatively, one can write the condition for $x \leq d_A$ as

$$p_A - \rho \leq \rho - p_A^H(p_B)$$

$$\rho - p_A \geq p_A^H(p_B) - \rho.$$

This condition is that A's mill price be farther below the reservation price than B's delivered price at A's location is above the reservation price.

Alternatively, one can write the condition for $x \leq d_A$ as

$$\frac{1}{2}\left[p_A + p_B + c(l-a-b)\right] \leq \rho.$$

Written in this way, $x \leq d_A$ if the average delivered price facing a consumer located midway between the two plants is less than the reservation price.

If $x \leq d_A$, then the quantity demanded of firm A on its right is

$$q_{AR} = s\left(\rho - p_A - \frac{1}{2}cx\right)x =$$

$$s\left[\rho - p_A - \frac{1}{2}c\frac{1}{2}\left(l-a-b+\frac{p_B-p_A}{c}\right)\right]\frac{1}{2}\left(l-a-b+\frac{p_B-p_A}{c}\right) =$$

$$Z = \frac{1}{8}\frac{s}{c}\left[4\rho - 3p_A - c(l-a-b) - p_B\right]\left[p_B + c(l-a-b) - p_A\right]$$

If $p_A = 2\rho - c(l-a-b) - p_B$,

$$q_{AR} = \frac{s}{2c}\left[p_B + c\,(l-a-b) - \rho\right]^2.$$

If individual demand falls to zero before reaching the indifferent consumer,

$$q_{AR} = \frac{s}{2c}\left(\rho - p_A\right)^2,$$

and it is clear that q_{AR} is continuous at $p_A = 2\rho - c(l-a-b) - p_B$.

$$q_{AR} = \frac{s}{2c}\left(\rho - p_A\right)^2.$$

If firm A does not undercut B's mill price, there are two cases to consider:

(c) $2\rho - p_A^H(p_B) \leq \rho$ or $\rho \leq p_A^H(p_B)$: for low prices, A sells out to the indifferent consumer; for higher prices, A sells a distance d_A to the right; for $p_A \geq \rho$, firm A sells nothing on its right:

$$q_{AR} = \begin{cases} Z & p_A^L(p_B) \leq p_A \leq 2\rho - p_A^H(p_B) \leq \rho \\ \frac{s}{2c}\left(\rho - p_A\right)^2 & 2\rho - p_A^H(p_B) \leq p_A \leq \rho \\ 0 & \rho \leq p_A \end{cases}.$$

(d) $2\rho - p_A^H(p_B) \geq \rho$ or $\rho \geq p_A^H(p_B)$: for $p_A \leq \rho$, A sells out to the indifferent consumer; for $p_A \geq \rho$, firm A sells nothing on its right:

$$q_{AR} = \begin{cases} Z & p_A^L(p_B) \leq p_A \leq \rho \leq 2\rho - p_A^H(p_B) \\ 0 & \rho \leq p_A \end{cases}.$$

If firm A undercuts firm B's mill price,

$$p_A \leq p_B - c(l-a-b) = p_A^L(p_B),$$

then firm A sells on its right a distance d_A or until the right end of the line, whichever is less.

For

$$d_A \leq l - a,$$

firm A sells a distance d_A to its right.

For

$$d_A \geq l - a,$$

firm A sells to the right end of the line. The condition for

$$d_A \leq l - a$$

is

$$\frac{\rho - p_A}{c} \leq l - a$$

$$\rho \leq p_A + c(l - a).$$

If firm A's delivered price at the right end of the line is greater than the reservation price, and firm undercuts firm B's mill price, firm A sells a distance d_A to its right.

There are two cases to consider.

(b) for[1]

$$\rho - c(l - a) \leq p_A \leq p_B - c(l - a - b),$$

firm A sells d_A units to its right.

$$q_{AR} = \frac{s}{2c}(\rho - p_A)^2.$$

The condition for such an interval to exist is

$$p_B - c(l - a - b) - \rho + c(l - a) > 0$$

$$p_B + cb - \rho > 0$$

$$p_B + bc > \rho,$$

[1] I do not examine the case $R \leq p_B - c(l - a - b)$ or equivalently $p_B \geq R + c(l - a - b)$; firm B could in principle set such a price, but would never do so in equilibrium.

where $p_B + bc$ is B's delivered price at the right end of the line. If B's delivered price at the right end of the line exceeds the reservation price, there is a range of prices p_A for which A sells a distance d_A to its right.

(a)

$$p_A \leq \rho - c(l-a) \leq p_B - c(l-a-b),$$

firm A sells to the right end of the line, and

$$q_{AR} = s\left[\rho - p_A - \frac{1}{2}c(l-a)\right](l-a)$$

Combine results for A's right side:

(a) $p_A \leq \rho - c(l-a) \leq p_A^L(p_B)$: firm A undercuts firm B and sells to the right end of the line,

$$q_{AR} = s\left[\rho - p_A - \frac{1}{2}c(l-a)\right](l-a).$$

(b) $\rho - c(l-a) \leq p_A \leq p_A^L(p_B)$: firm A undercuts firm B and sells d_A units to its right,

$$q_{AR} = \frac{s}{2c}(\rho - p_A)^2.$$

(c) $p_A^L(p_B) \leq p_A \leq 2\rho - p_A^H(p_B) \leq \rho$: firm A sells to the indifferent consumer,

$$q_{AR} = Z$$

(d) $2\rho - p_A^H(p_B) \leq p_A \leq \rho$: firm A sells d_A units to its right,

$$q_{AR} = \frac{s}{2c}\left(\rho - p_A\right)^2.$$

(e) $\rho \leq p_A$:

$$q_{AR} = 0.$$

4.2 Analyze the Hotelling model with no mill-price undercutting conjectures.

With no mill-price undercutting conjectures, the demand function becomes

$$q_A = \begin{cases} l-b & p_A < p_A^L(p_B) = p_B - c(l-a-b) \\ \frac{1}{2}\left(l+a-b+\frac{p_B-p_A}{c}\right) & p_A^L(p_B) \leq p_A \leq p_A^H(p_B) \\ 0 & p_B + c(l-a-b) = p_A^H(p_B) < p_A \end{cases}.$$

If A undercuts B's mill price, A expects B to match the price cut, so that B will continue to supply its own hinterland.

Evaluate the middle demand function at the endpoints of middle region:

$$\frac{1}{2}\left\{l+a-b+\frac{p_B-[p_B-c(l-a-b)]}{c}\right\} = l-b$$

$$\frac{1}{2}\left\{l+a-b+\frac{p_B-[p_B+c(l-a-b)]}{c}\right\} = a.$$

The profit function is

$$\pi_A = \begin{cases} p_A(l-b) & p_A < p_A^L(p_B) \\ \frac{1}{2}p_A\left(l+a-b+\frac{p_B-p_A}{c}\right) & p_A^L(p_B) \leq p_A \leq p_A^H(p_B) \\ 0 & p_A^H(p_B) < p_A \end{cases}.$$

Evaluate the low-p_A profit function at $p_A = p_A^L(p_B)$:

$$\left(p_B - c(l-a-b)\right)(l-b).$$

Evaluate the middle profit function at boundary between the low-p_A and middle-p_A regions:

$$\frac{1}{2}\left[p_B - c(l-a-b)\right]\left\{l+a-b+\frac{p_B-[p_B-c(l-a-b)]}{c}\right\} =$$

$$\left[p_B - c(l-a-b)\right](l-b).$$

The profit function is continuous at the boundary between the low-p_A and middle-p_A regions.

Firm A's profit function on the middle range is a parabola. If the maximum of this profit function occurs on the middle range, it is the global maximum and the price at which the maximum occurs is the best-response price. If the maximum of the middle-range profit function lies in the range $0 \leq p_A < p_A^L(p_B)$, the global maximum of A's profit function is at $p_A = p_A^L(p_B)$. If the maximum of the middle-range profit function lies in the range $p_A^H(p_B)$, the global maximum of A's profit function is at $p_A = p_A^H(p_B)$.

For what values of p_B does the maximum of the middle-range profit function lie above the middle range of p_B?

$$p_B + c(l - a - b) < \frac{1}{2}\left(p_B + c(l - b + a)\right)$$

$$2p_B + 2c(l - a - b) < p_B + c(l - b + a)$$

$$p_B < c(l - b + a) - 2c(l - a - b)$$

$$p_B < c\,(3a + b - l) \equiv \widetilde{p}_B^L$$

(where we now rename what was called p_B^L in the text as $\widetilde{p}_B^L$).

For what values of p_B does the maximum of the middle-range profit function lie below the middle range of p_B?

$$p_B - c(l - a - b) > \frac{1}{2}\left(p_B + c(l - b + a)\right)$$

$$2p_B - 2c(l - a - b) > p_B + c(l - b + a)$$

$$p_B > c(l - b + a) + 2c(l - a - b)$$

$$p_B > c\,(3l - 3b - a) \equiv \widetilde{p}_B^H.$$

The maximum of the middle-region profit function is internal to the middle region for

$$\widetilde{p}_B^L \leq p_B \leq \widetilde{p}_B^H.$$

Note that the middle price range always exists:

$$\frac{\widetilde{p}_B^H - \widetilde{p}_B^L}{c} = 3l - 3b - a - (3a + b - l) = 4(l - a - b) > 0.$$

A's best response function is

$$p_A(p_B) = \begin{cases} p_B + c(l - a - b) & 0 \le p_B < c(3a + b - l) = \widetilde{p}_B^L \\ \frac{1}{2}(p_B + c(l - b + a)) & \widetilde{p}_B^L \le p_B \le \widetilde{p}_B^H \\ p_B - c(l - a - b) & \widetilde{p}_B^H = c(3l - 3b - a) < p_B \end{cases}.$$

If p_B is very low, A maximizes its profit by matching B's delivered price at A's mill and selling a units to customers located in its hinterland. If p_B is very high, A maximizes its profit by having its delivered price at B's mill match B's mill price, and supplying all customers except those in B's hinterland. For intermediate values of p_B, firm A's profit-maximizing strategy is to share the middle of the market with firm B.

The corresponding quantities demanded and payoffs are

$$q_A^* = \begin{cases} a & 0 \le p_B < c(3a + b - l) = \widetilde{p}_B^L \\ \frac{1}{2c} p_A^* & \widetilde{p}_B^L \le p_B \le \widetilde{p}_B^H \\ l - b & \widetilde{p}_B^H = c(3l - 3b - a) < \widetilde{p}_B \end{cases}$$

$$\pi_A^* = \begin{cases} (p_B + c(l - a - b))\, a & 0 \le p_B < c(3a + b - l) = \widetilde{p}_B^L \\ \frac{1}{8c}(p_B + c(l - b + a))^2 & \widetilde{p}_B^L \le p_B \le \widetilde{p}_B^H \\ (p_B - c(l - a - b))(l - b) & \widetilde{p}_B^H = c(3l - 3b - a) < \widetilde{p}_B \end{cases}.$$

In order for the lower price range $0 \le p_B < c(3a + b - l) = \widetilde{p}_B^L$ to exist, we must have

$$3a + b - l > 0$$

$$a > \frac{1}{3}(l - b)$$

$$\alpha > \frac{1}{3}(1 - \beta).$$

For $\alpha \le \frac{1}{3}(1 - \beta)$, firm A's best-response function has only two segments,

$$p_A^* = \begin{cases} \frac{1}{2}(p_B + c(l - b + a)) & 0 \le p_B \le \widetilde{p}_B^H \\ p_B - c(l - a - b) & \widetilde{p}_B^H = c(3l - 3b - a) < p_B \end{cases}.$$

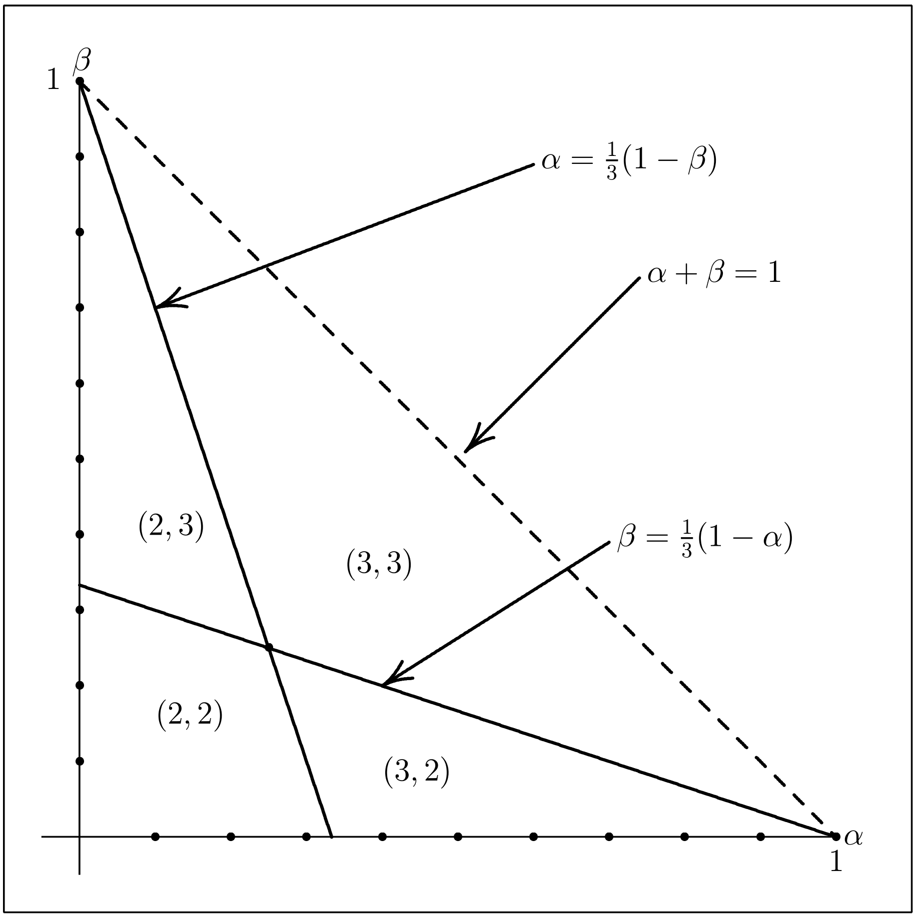


Figure 4.1: Types of best response functions, by regions in location space, Hotelling linear duopoly model, no mill-price undercutting conjectures; (n_A, n_B) indicates that in this region firm A's best-response function has n_A segments, firm B's best-response function has n_B segments, for $n_A, n_B = 2, 3$; $\alpha = a/l$, $\beta = b/l$.

Location space can thus be divided into four regions, as shown in Figure 4.1:

- $\alpha \leq \frac{1}{3}(1-\beta)$, $\beta \leq \frac{1}{3}(1-\alpha)$: best response functions of both firms have two segments;
- $\alpha \leq \frac{1}{3}(1-\beta)$, $\beta > \frac{1}{3}(1-\alpha)$: firm A's best response function has two segments, firm B's best response function has three segments;
- $\alpha > \frac{1}{3}(1-\beta)$, $\beta \leq \frac{1}{3}(1-\alpha)$: firm A's best response function has three segments, firm B's best response function has two segments;
- $\alpha > \frac{1}{3}(1-\beta)$, $\beta > \frac{1}{3}(1-\alpha)$: best response functions of both firms have three segments.

The middle-range price reaction functions intersect at

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} p_A \\ p_B \end{pmatrix} = c \begin{pmatrix} l-b+a \\ l-a+b \end{pmatrix}$$

$$3 \begin{pmatrix} p_A \\ p_B \end{pmatrix} = c \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} l-b+a \\ l-a+b \end{pmatrix}$$

$$\begin{pmatrix} p_A^* \\ p_B^* \end{pmatrix} = \frac{1}{3} c \begin{pmatrix} 3l+a-b \\ 3l+b-a \end{pmatrix}.$$

When does the value of p_B^* defined above fall in the range $(\widetilde{p}_B^L, \widetilde{p}_B^H)$: when is

$$\widetilde{p}_B^L = c\,(3a+b-l) \leq \frac{1}{3} c\,(3l+b-a) \leq c\,(3l-3b-a) = \widetilde{p}_B^H \ ?$$

Right-hand side inequality:

$$c\,(3l-3b-a) - \frac{1}{3} c\,(3l+b-a) \geq 0$$

$$3l-3b-a-\frac{1}{3}(3l+b-a) \geq 0$$

$$9l - 9b - 3a - (3l + b - a) \geq 0$$

$$6l - 2a - 10b \geq 0$$

$$a + 5b \leq 3l$$

$$\alpha + 5\beta \leq 3.$$

Left-hand side inequality:

$$c\,(3a + b - l) \leq \frac{1}{3}c\,(3l + b - a)$$

$$9a + 3b - 3l \leq 3l + b - a$$

$$10a + 2b \leq 6l$$

$$5a + b \leq 3l$$

$$5\alpha + \beta \leq 3.$$

When both inequalities $\alpha + 5\beta \leq 3$, $5\alpha + \beta \leq 3$ are met, firm B's equilibrium price falls in the range that makes the middle, "sharing," range of firm A's best response function apply, and vice-versa. This region is shown in Figure 4.2.

Suppose one of the inequalities is violated, for example $\alpha + 5\beta \leq 3$, $5\alpha + \beta \leq 3$. For concreteness, let $l = 10$, $a = 1$, $b = 8$, $c = 1$. Then the best response functions are

$$p_A(p_B) = \begin{cases} p_B + 1 & 0 \leq p_B < 1 = \widetilde{p}_B^L \\ \frac{1}{2}\,(p_B + 3) & 1 \leq p_B \leq 5 \\ p_B - 1 & \widetilde{p}_B^H = 5 < p_B \end{cases}.$$

$$p_B(p_A) = \begin{cases} p_A + 1 & 0 \leq p_A < 15 = \widetilde{p}_A^L \\ \frac{1}{2}\,(p_A + 3) & 15 \leq p_A \leq 19 \\ p_A - 1 & \widetilde{p}_A^H = 19 < p_A \end{cases}.$$

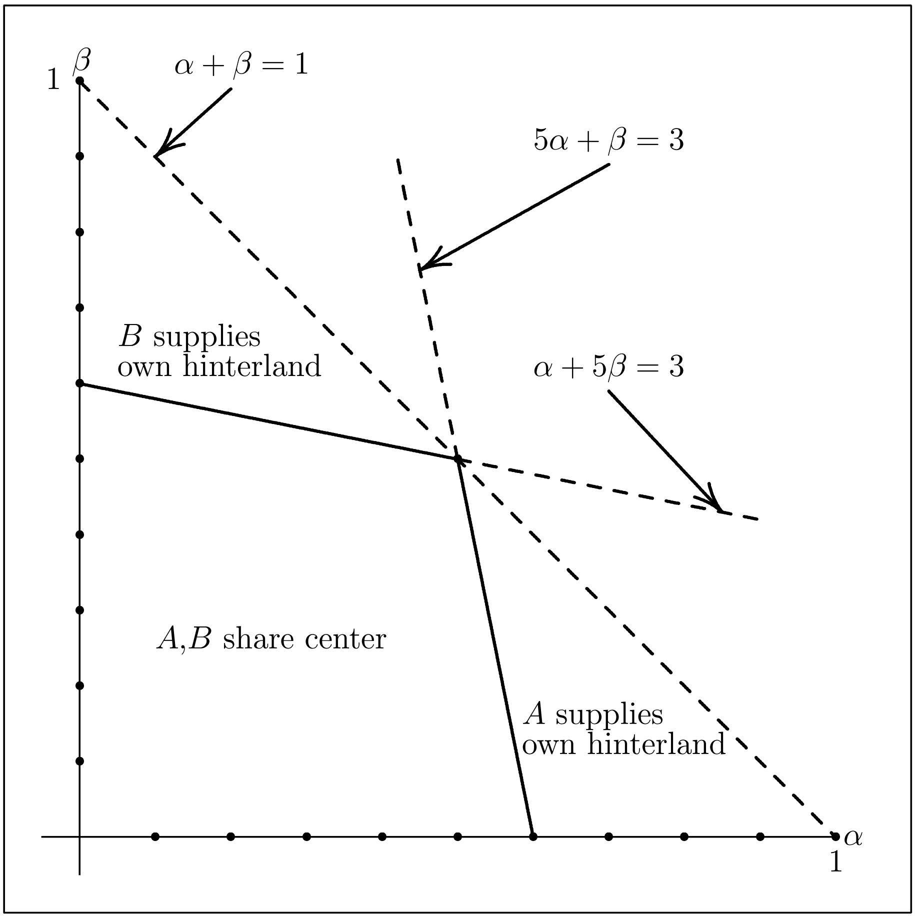


Figure 4.2: Types of equilibrium, by regions in location space, Hotelling linear duopoly model; no mill-price undercutting conjectures; $\alpha = a/l$, $\beta = b/l$.

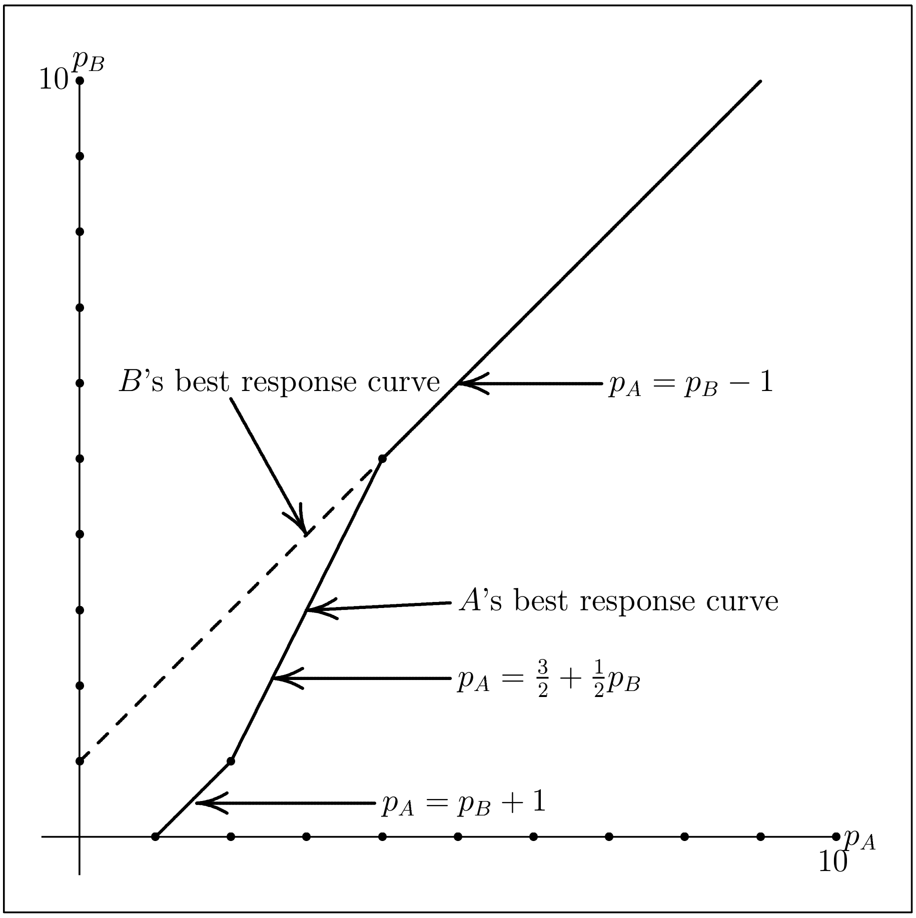


Figure 4.3: No mill-price undercutting conjectures example; $l = 10$, $a = 1$, $b = 8$, $c = 1$.

(see Figure 4.3).

The segments of the best-response functions $p_A(p_B) = p_B - 1$ and $p_B(p_A) = p_A + 1$ coincide on the interval $4 \leq p_A \leq 15$ (equivalently, $5 \leq p_B \leq 16$). More generally, for locations in the region $\alpha + 5\beta \leq 3$, $5\alpha + \beta \leq 3$, best response functions coincide over the range

$$\widetilde{p}_B^H - 1 \leq p_A \leq \widetilde{p}_A^L$$

$$\widetilde{p}_B^H \leq p_B \leq \widetilde{p}_A^L + 1.$$

At any point in this range on the line $p_A = p_B - 1$, firm A sells quantity $l - b$ at price $p_B - 1$, firm B sells quantity b at price $p_A + 1$; payoffs are

$$\pi_A = (p_B - 1)(l - b)$$

$$\pi_B = (p_A + 1)b.$$

A's payoff depends only on B's control variables. B's payoff depends on b; B would wish to make b as large as possible, taking a as given (and subject to the constraint $a + b \leq l$).

4.3 Answer **Problem 4.1** if firms maintain no mill-price undercutting conjectures.

The results of **Problem 4.1** for A's right side are

(a) $p_A \leq \rho - c(l - a) \leq p_A^L(p_B)$: firm A undercuts firm B and sells to the right end of the line,

$$q_{AR} = s\left[\rho - p_A - \frac{1}{2}c(l - a)\right](l - a).$$

(b) $\rho - c(l - a) \leq p_A \leq p_A^L(p_B)$: firm A undercuts firm B and sells d_A units to its right,

$$q_{AR} = \frac{s}{2c}(\rho - p_A)^2.$$

(c) $p_A^L(p_B) \leq p_A \leq 2\rho - p_A^H(p_B) \leq \rho$: firm A sells to the indifferent consumer,

$$q_{AR} = Z.$$

(d) $2\rho - p_A^H(p_B) \leq p_A \leq \rho$: firm A sells d_A units to its right,

$$q_{AR} = \frac{s}{2c}\left(\rho - p_A\right)^2.$$

(e) $\rho \leq p_A$:

$$q_{AR} = 0.$$

It is (a) and (b) that are affected by no mill-price undercutting conjectures; they become

(a)' $p_A \leq \rho - c(l-a) \leq p_A^L(p_B)$: firm A undercuts firm B and expects to sell to B's location, a distance $l-a-b$ to A's right:

$$q_{AR} = s\left[\rho - p_A - \frac{1}{2}c(l-a-b)\right](l-a-b).$$

(b)' $\rho - c(l-a) \leq p_A \leq p_A^L(p_B)$: for A to undercut B's mill price and for the reservation price to be binding closer to A's location than at B's plant, we would need to have

$$p_A + cd_A \leq p_A + c(l-a-b) \leq p_B$$

$$p_A + c\frac{\rho - p_A}{c} \leq p_A + c(l-a-b) \leq p_B;$$

$$\rho \leq p_A + c(l-a-b) \leq p_B.$$

Firm B would never set so high a price in equilibrium.

Hence for $\rho - c(l-a) \leq p_A \leq p_A^L(p_B)$ as well, firm A undercuts firm B and expects to sell to B's location, a distance $l-a-b$ to A's right:

$$q_{AR} = s\left[R - p_A - \frac{1}{2}c(l-a-b)\right](l-a-b).$$

Demand on firm A's right becomes

(a)-(b) $p_A \leq p_A^L(p_B)$: firm A undercuts firm B and expects to sell to B's location, a distance $l-a-b$ to A's right:

$$q_{AR} = s\left[\rho - p_A - \frac{1}{2}c(l-a-b)\right](l-a-b).$$

(c) $p_A^L(p_B) \leq p_A \leq 2\rho - p_A^H(p_B) \leq \rho$: firm A sells to the indifferent consumer,

$$q_{AR} = Z.$$

(d) $2\rho - p_A^H(p_B) \leq p_A \leq \rho$: firm A sells d_A units to its right,

$$q_{AR} = \frac{s}{2c}\left(\rho - p_A\right)^2.$$

(e) $\rho \leq p_A$:

$$q_{AR} = 0.$$

A's demand on the left:

(a) $\rho \leq p_A^H(p_B) = p_B + c(l - a - b)$: the reservation price is less than B's delivered price at A's location:

$$q_{AL} = \begin{cases} s\left[(\rho - p_A) - \frac{1}{2}ca\right]a & p_A \leq \rho - ca \\ \frac{s}{2c}\left(\rho - p_A\right)^2 & \rho - ca \leq p_A \leq \rho \leq p_A^H(p_B) \\ 0 & \rho \leq p_A \end{cases}.$$

$p_A \leq \rho - ca$ or $p_A + ca \leq \rho$ means that A's delivered price at the left end of the line is less than the reservation price; then A sells to the left end of the line.

For p_A in the range $\rho - ca \leq p_A \leq \rho$, firm A sells to a distance d_A from its plant; such a range always exists.

(b) $p_A^H(p_B) = p_B + c(l - a - b) \leq \rho$: B's delivered price at A's location is less than the reservation price; then

$$q_{AL} = \begin{cases} s\left[(\rho - p_A) - \frac{1}{2}ca\right]a & p_A \leq \rho - ca \\ \frac{s}{2c}\left(\rho - p_A\right)^2 & \rho - ca \leq p_A \leq p_A^H(p_B) \leq \rho \\ 0 & p_B + c(l - a - b) \leq p_A \end{cases}.$$

As before, if A's delivered price at the left end of the line is less than the reservation price, A sells to the left end of the line.

The condition for the second range of prices to exist is

$$p_B + c(l - a - b) - (\rho - ca) > 0$$

$$p_B + c(l - b) > \rho$$

or that B's delivered price at the left end of the line exceed the reservation price.

The critical value on A's left is $\rho - ca$; the critical values on the right are $p_B - c(l - a - b)$ and $2\rho - [p_B + c(l - a - b)]$.

Compare $\rho - ca$ and $p_B - c(l - a - b)$:

$$\rho - ca - p_B + c(l - a - b) = \rho - p_B + c(l - b)$$

and this will be positive for $p_B \leq \rho$.

Compare $2\rho - [p_B + c(l - a - b)]$ and $\rho - ca$:

$$2\rho - [p_B + c(l - a - b)] - (\rho - ca) =$$

$$\rho + ca - [p_B + c(l - a - b)]$$

which may be positive or negative.

4.4 Assume quadratic transportation cost. Maintaining all other assumptions of the Hotelling model but with $b > a > l$, find equilibrium prices for the case shown in Figure 4.4.

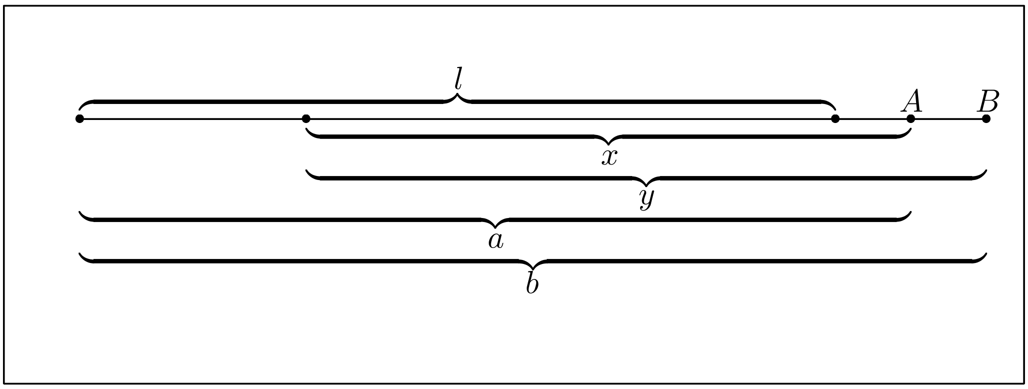


Figure 4.4: Linear vertical product differentiation duopoly

Let x denote the distance from A, y the distance from B. Note that

$$y = x + b - a.$$

Consider the case in which both firms have positive sales. The location of the marginal consumer satisfies

$$p_A + cx^2 = p_B + cy^2,$$

from which

$$p_A - p_B = c\left(y^2 - cx^2\right).$$

Since $y > x$, $p_A > p_B$ if both firms have positive sales.

$$p_A + cx^2 = p_B + c\,(x + b - a)^2$$

$$x = \frac{p_A - p_B - c(b-a)^2}{2c\,(b-a)} = \frac{p_A - p_B}{2c\,(b-a)} - \frac{1}{2}(b-a).$$

and a consistency condition is that

$$0 \leq \frac{p_A - p_B}{2c\,(b-a)} - \frac{1}{2}(b-a) \leq l.$$

The quantity demanded of firm A is

$$a - x = a + \frac{1}{2}(b-a) - \frac{p_A - p_B}{2c\,(b-a)} = \frac{1}{2}\left[b + a - \frac{p_A - p_B}{2c\,(b-a)}\right].$$

The quantity demanded of firm B is

$$l - (a - x) = \frac{1}{2}\left[2l - (b+a) - \frac{p_B - p_A}{2c\,(b-a)}\right].$$

Firm A's profit is

$$\pi_A = \frac{1}{2}p_A\left[b + a - \frac{p_A - p_B}{2c\,(b-a)}\right].$$

The first-order condition to maximize π_A, which is the equation of firm A's price best response function, is

$$b + a - \frac{2p_A - p_B}{2c\,(b-a)} \equiv 0.$$

$$2p_A - p_B = 2c(b^2 - a^2).$$

Firm B's profit is

$$\pi_B = \frac{1}{2}p_B\left[2l - (b+a) - \frac{p_B - p_A}{2c\,(b-a)}\right].$$

The first-order condition to maximize π_B is

$$2l - (b + a) - \frac{2p_B - p_A}{2c(b - a)} \equiv 0$$

$$2l - (b + a) - \frac{2p_B - p_A}{2c(b - a)} \equiv 0$$

$$-p_A + 2p_B = 2c(b - a)[2l - (b + a)].$$

The system of equations formed by the first-order conditions is

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} p_A \\ p_B \end{pmatrix} = 2c(b - a) \begin{pmatrix} a + b \\ 2l - (a + b) \end{pmatrix}$$

$$3 \begin{pmatrix} p_A \\ p_B \end{pmatrix} = 2c(b - a) \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} a + b \\ 2l - (a + b) \end{pmatrix}$$

$$3 \begin{pmatrix} p_A \\ p_B \end{pmatrix} = 2c(b - a) \begin{pmatrix} 2l + (a + b) \\ 4l - (a + b) \end{pmatrix}$$

$$\begin{pmatrix} p_A \\ p_B \end{pmatrix} = \frac{2}{3}c(b - a) \begin{pmatrix} 2l + (a + b) \\ 4l - (a + b) \end{pmatrix}$$

$$p_A - p_B =$$

$$(1, -1) \begin{pmatrix} p_A \\ p_B \end{pmatrix} = \frac{2}{3}c(b - a)(1, -1) \begin{pmatrix} 2l + (a + b) \\ 4l - (a + b) \end{pmatrix}$$

$$= \frac{4}{3}c(b - a)(a + b - l).$$

$$\frac{p_A - p_B}{2c(b - a)} - \frac{1}{2}(b - a) =$$

$$\frac{\frac{4}{3}c\,(b-a)\,(a+b-l)}{2c\,(b-a)} - \frac{1}{2}(b-a) =$$

$$\frac{2}{3}(a+b-l) - \frac{1}{2}(b-a)$$

$$= \frac{1}{6}\,(7a+b-4l)\,,$$

and the consistency condition is

$$0 \leq \frac{1}{6}\,(7a+b-4l) \leq l$$

$$0 \leq 7a+b-4l \leq 6l.$$

The left-hand inequality becomes

$$4l \leq 7a+b$$

$$l \leq \frac{1}{4}\,(7a+b)\,.$$

The right-hand inequality becomes

$$7a+b \leq 10l$$

$$\frac{1}{10}\,(7a+b) \leq l.$$

The consistency condition — a restriction on a, b, and l that must be satisfied for both firms to have positive sales in equilibrium — can then be written

$$\frac{1}{10} \leq \frac{l}{7a+b} \leq \frac{1}{4}.$$

Chapter 5

Early Empirical Studies of Structure–conduct–performance Relationships

This chapter intentionally left blank.

Chapter 6

Debates Over Interpretation and Specification

6.1 (Differential efficiency) Let the market demand function be

$$p = 10 - Q,$$

and suppose the market is potentially supplied by two firms, firm L (marginal cost $c_L = 1$ and firm H (marginal cost $c_H = 2$). Each firm must cover a fixed cost $F > 0$ if it has positive output.

Compare net social welfare if the firms act as price takers with net social welfare if the firms act as Cournot duopolists.

With price-taking behavior, firm L supplies the entire market at price $p = 2$. Consumers surplus is

$$\frac{1}{2}(10-2)^2 = 32.$$

Deadweight welfare loss is

$$\frac{1}{2}(2-1)(9-8) = \frac{1}{2},$$

lost consumers surplus on the one unit of output that would be produced if price were equal to the marginal cost of the low-cost firm but is not produced when price is the marginal cost of the high-cost firm.

In Cournot duopoly equilibrium, the equations of the best response functions of the two firms are

$$2q_L + q_H = 9$$

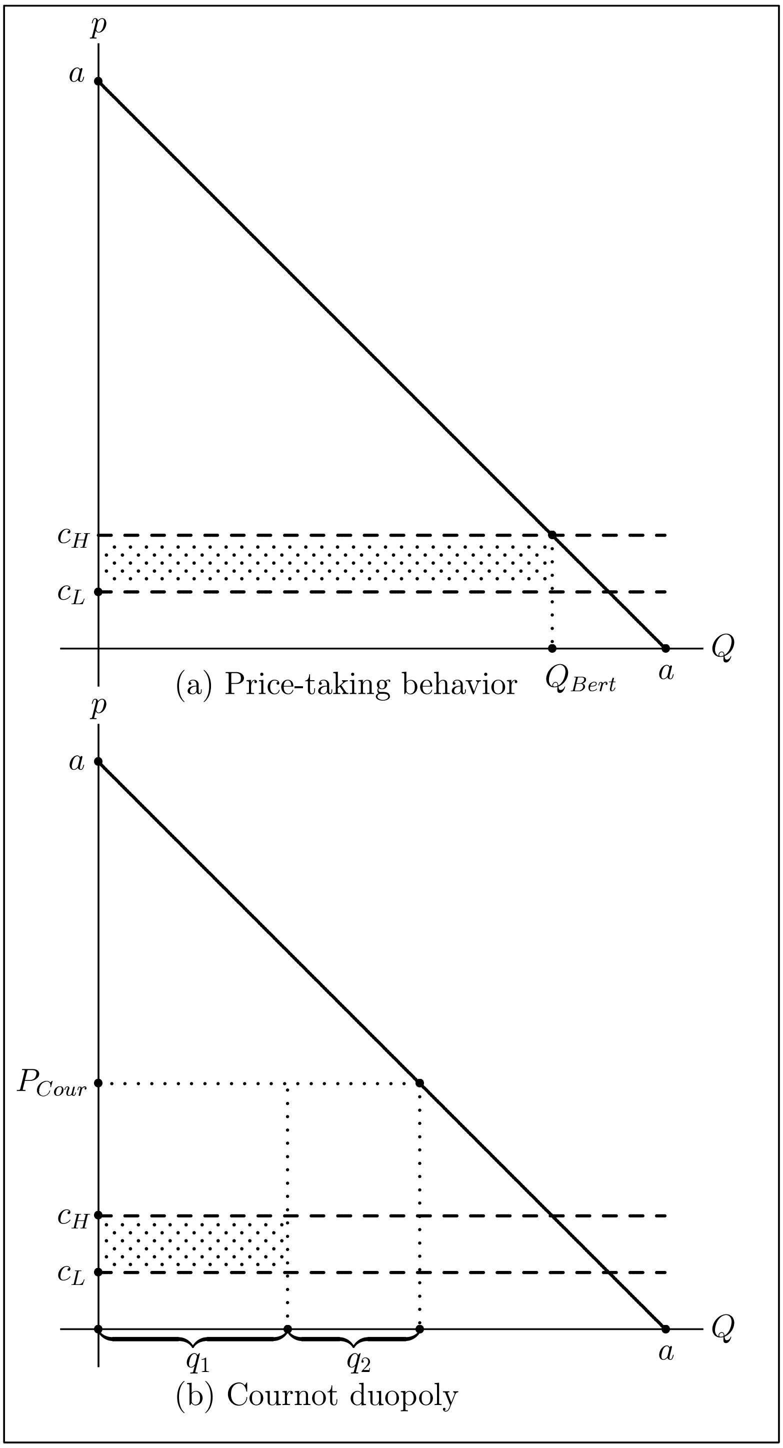


Figure 6.1: Market power and welfare losses with efficiency differentials; shaded areas are efficiency rents

$$q_L + 2q_H = 8$$

To find equilibrium outputs, solve the system formed by the equations of the best response functions:

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} q_L \\ q_H \end{pmatrix} = \begin{pmatrix} 9 \\ 8 \end{pmatrix}$$

$$3 \begin{pmatrix} q_L \\ q_H \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 9 \\ 8 \end{pmatrix}$$

$$\begin{pmatrix} q_L \\ q_H \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 10 \\ 7 \end{pmatrix}.$$

Total output is

$$\frac{17}{3};$$

Cournot equilibrium price is

$$10 - \frac{17}{3} = \frac{13}{3}.$$

Economic profits are

$$\pi_L = \left(\frac{17}{3} - 1\right) \frac{10}{3} = \frac{140}{9}$$

$$\pi_H = \left(\frac{17}{3} - 2\right) \frac{7}{3} = \frac{77}{9}.$$

Consumer surplus is

$$\frac{1}{2} \left(\frac{17}{3}\right)^2 = \frac{289}{18}.$$

Deadweight welfare loss is

$$\frac{1}{2} \left(9 - \frac{17}{3}\right)^2 = \frac{50}{9},$$

lost consumer surplus on output that would be produced if price were equal to the marginal cost of the lost cost firm but is not produced in Cournot equilibrium.

The welfare loss due to production by the higher-cost firm is

$$(2-1)\frac{7}{3} = \frac{7}{3}.$$

6.2 Considering the model of a firm that raises funds for investment by the sale of bonds, suppose K_1, L_1, and B_1 are chosen to maximize

$$M = -(1-\delta)p_1^k K_0 \pi_< + \frac{\left[p_1 q_1 - w_1 L_1 - \lambda_1 p_1^k K_1 - (r_1 - r^*)B_1\right]\pi_> + E_>}{1+r^*}$$

$$+\gamma\left[B_1 + (1-\delta)p_1^k K_0 - p_1^k K_1\right]$$

where γ the Lagrangian multiplier associated with the constraint

$$B_1 + (1-\delta)p_1^k K_0 - p_1^k K_1 \equiv 0.$$

Write out the first-order conditions for B_1, K_1, and L_1. Derive equation (6.66) in the text.

Derivation of the first-order conditions is complicated by the fact that L, K, and B appear in the definition of ψ^*, equation (6.62) in the text, and ψ^* appears in the limits of the integrals that define $\pi_<$, $\pi_>$, and $E_>$.

The first-order condition for B is

$$\frac{\partial M}{\partial B_1} = -(1-\delta)p_1^k K_0 f(\psi^*)\frac{\partial \psi^*}{\partial B_1} - \frac{r_1 - r^* + B_1\frac{\partial(r_1-r^*)}{\partial B_1}}{1+r^*}\pi_>$$

$$+\frac{p_1 q_1 - w_1 L_1 - \lambda_1 p_1^k K_0 - (r_1 - r^*)B_1}{1+r^*}[-f(\psi^*)]\frac{\partial \psi^*}{\partial B_1}$$

$$+\frac{1}{1+r^*}\left[-\psi^* f(\psi^*)\frac{\partial \psi^*}{\partial B_1}\right] + \gamma = 0.$$

In view of the definition of ψ^*, the coefficients of the terms in $f(\psi^*)\partial\psi^*/\partial B_1$ add up to zero. The first order condition for B can therefore be solved for γ:

$$\gamma = \left[r_1 + B_1\frac{\partial(r_1 - r^*)}{\partial B_1}\right]\frac{\pi_>}{1+r^*}.$$

γ is the expected present discounted value of the excess of the firm's marginal interest cost over the rate of return on a safe asset.

The first-order conditions for L and K likewise involve terms reflecting marginal changes in ψ^*. These terms add up to zero, and are omitted here. The first-order conditions for L and K can be written

$$\left(MR - \frac{rB_1}{q_1}\varepsilon_{r_1 q_1} \right) MP_L = w_1$$

and

$$\left(MR - \frac{rB_1}{q_1}\varepsilon_{r_1 q_1} \right) MP_K = \lambda_B p_1^k,$$

respectively, where the first-order condition for B has been used to eliminate γ from the first-order condition for K, MR is marginal revenue, and λ_B is given by (6.67) in the text,

$$\lambda_B = r_B + \delta - (1-\delta)\frac{p_2^k - p_1^k}{p_1^k}, \tag{6.1}$$

where

$$r_B = r_1 + B_1 \frac{\partial r_1}{\partial B_1} = (1 + \varepsilon_{r_1 q_1}) r_1. \tag{6.2}$$

To derive the price-cost margin equation (6.66) from the first-order conditions for L and K, multiply the first-order condition for L by L, multiply the first-order condition for K by K, add, and substitute the definition of the function coefficient to eliminate $K(MP_K) + L(MP_L)$. The resulting expression can be rearranged to yield (6.66).

6.3 Considering the model of a firm that raises funds for investment by the sale of stock and bonds, suppose K_1, L_1, and $B_1$and S_N are chosen to maximize

$$M = -\frac{S_E}{S_E + S_N}(1-\delta)p_1^k K_0 + \frac{S_E}{S_E + S_N}\left\{ -(1-\delta)p_1^k K_0 \pi_< \right.$$

$$\left. + \frac{\left[p_1 q_1 - w_1 L_1 - \lambda_1 p_1^k K_1 + (1+r^*) m_1 S_N - (r_1 - r^*) B_1 \right] \pi_> + E_>}{1 + r^*} \right\}$$

$$+\gamma\left[B_1 + mS_N + (1-\delta)p_1^k K_0 - p_1^k K_1\right].$$

Write out the first-order conditions for B_1, S_N, K_1, and L_1.

The sale of stock brings in funds at the start of the period. But the sale of stock does not entail end-of-period interest payments. Rather, it means that beginning of period stockholders give up part of their ownership of the firm.

The firm sells S_N shares of stock at the beginning of the period, at price m_1 per share. Investment must be financed by the sale of stocks and bonds:

$$B + m_1 S_N = p_1^k I_1 = p_1^k[K_1 - (1-\delta)K_0]. \tag{6.4}$$

With the sale of stock, the realized present discounted sell-off value of the firm is

$$PDS = (1-\delta)p_1^k K_0 + m_1 S_N + \frac{p_1 q_1 - w_1 L_1 - \lambda_1 p_1^k K_1 + \psi}{1+r^*}. \tag{6.5}$$

If PDS is negative the firm is bankrupt, since it is unable to redeem its borrowings. The critical value of ψ is now

$$\psi^* =$$

$$(r_1 - r^*)B_1 - (p_1 q_1 - w_1 L_1 - \lambda_1 p_1^k K_1) - (1+r^*)\left[(1-\delta)p_1^k K_0 + m_1 S_N\right] \tag{6.6}$$

The firm is bankrupt for $\psi < \psi^*$. Funds raised from the sale of stock help avoid bankruptcy, since they reduce the critical value ψ^*.

If the firm is bankrupt, the original owners of the firm receive nothing at the end of the period. If the firm is not bankrupt, they receive only a fraction of the end-of-period value of the firm. The sale of stock dilutes their ownership of the firm. The expected discounted return of the original stockholders is

$$B_1 + m_1 S_N - p_1^k K_1 +$$

$$\frac{S_E}{S_E + S_N}\frac{\left[p_1 q_1 - w_1 L_1 - (1+r_1)B_1 + (1-\delta)p_2^k K_1\right]\pi_> + E_>}{1+r^*}, \tag{6.7}$$

which can be rewritten

$$-\frac{S_E}{S_E+S_N}(1-\delta)p_1^k K_0 + \frac{S_E}{S_E+S_N}\times$$

$$\left\{-(1-\delta)p_1^k K_0\pi_< + \frac{\left[p_1q_1 - w_1L_1 - \lambda_1 p_1^k K_1 + (1+r^*)m_1S_N - (r_1-r^*)B_1\right]\pi_> + E_>}{1+r^*}\right\} \tag{6.8a}$$

$\pi_<$, $\pi_>$, and $E_>$ are defined as for Problem **6.2**, using (6.6) as the definition of ψ^*.

When new shares are sold at the start of the period, the original shareholders give up part of their ownership of the beginning-of-period capital of the firm. This is the first term in (6.8a). The second term is the original shareholders' portion of the expected end-of-period return. If the firm is managed to maximize the expected return of the original shareholders, then B_1, S_N, L_1, and K_1 are chosen to maximize (6.8a), subject to the financing constraint (6.4).

As with the response to Problem **6.2**, each first-order condition involves a number of terms reflecting marginal changes in the limits of the integrals that define $\pi_<$, $\pi_>$, and $E_>$. These terms always add up to zero, and are omitted here.

The first-order condition for bonds yields a value for γ:

$$\gamma = \frac{S_E}{S_E+S_N}\frac{\pi_>}{1+r^*}\frac{r_1 + B_1\frac{\partial r_1}{\partial B_1} - r^* - (1+r^*)S_N\frac{\partial m}{\partial B_1}}{1+S_N\frac{\partial m}{\partial B_1}}$$

The first-order condition for new shares of stock can be written

$$m + S_N\frac{\partial m}{\partial S_N} - \frac{B_1}{1+r^*}\frac{\partial r_1}{\partial S_N} =$$

$$\frac{1}{S_E+S_N}\left[\frac{p_1q_1 - w_1L_1 - (1+r_1)B_1 + (1-\delta)p_2^k K_1 + \frac{E_>}{\pi_>}}{1+r^*}\right]$$

$$+\gamma\left(1+\frac{\partial m}{\partial S_N}\right)\left(\frac{S_N}{S_E}\right)\left(\frac{S_E+S_N}{\pi_>}\right).$$

This can be solved to obtain an alternative expression for γ; the two expressions can be set equal to each other, to eliminate γ from the set of equations that characterize the solution.

The first-order conditions for labor and capital are

$$\left[MR + \frac{1 + r_1(1+\varepsilon_{r_1 B_1})}{1 + \frac{mS_N}{B_1}\varepsilon_{mB_1}} \varepsilon_{mq_1} \frac{mS_N}{q_1} - \varepsilon_{r_1 q_1} \frac{r_1 B_1}{q_1} \right] MP_L = w_1$$

$$\left[MR + \frac{1 + r_1(1+\varepsilon_{r_1 B_1})}{1 + \frac{mS_N}{B_1}\varepsilon_{mB_1}} \varepsilon_{mq_1} \frac{mS_N}{q_1} - \varepsilon_{r_1 q_1} \frac{r_1 B_1}{q_1} \right] MP_K = \lambda_{SB} p_1^k,$$

where the expression for γ that comes from the first-order condition for B is used to simplify the first-order condition for capital.

Equation (6.69) in the text follows from the first-order conditions for labor and capital in the usual way: multiply the first-order condition for L by L, multiply the first-order condition for K by K, add, and substitute the definition of the function coefficient to eliminate $K(MP_K) + L(MP_L)$. (6.69) assumes constant returns to scale; the corresponding expression when returns to scale are not constant is

$$\frac{p_1 q_1 - w_1 L_1}{p_1 q_1} = 1 - FC_1 + FC_1 \frac{\alpha_1 + (1-\alpha_1)s_1}{\varepsilon_{Qp}} + FC_1 \varepsilon_{r_1 q_1} \frac{r_1 B_1}{p_1 q_1}$$

$$-FC_1(1 + r_{SB})\varepsilon_{m_1 q_1} \frac{m_1 S_N}{p_1 q_1} + \lambda_B \frac{p^k K_1}{p_1 q_1}. \tag{6.9}$$

Chapter 7

Empirical Studies of Market Performance

This chapter intentionally left blank.

Chapter 8

Strategic Behavior

8.1 Answers to Problems

13.1 (Stackelberg quantity leadership) Let the inverse demand curve of a market supplied by two firms be

$$p = a - b(q_1 + q_2).$$

Both firms have cost function

$$c(q) = cq + F.$$

Firm 2, the follower, acts as a Cournot quantity-setter. Firm 1 knows this.

Find equilibrium outputs, price, and payoffs.

See Dowrick (1986); Hamilton and Slutsky (1990); Daughety (1990).

Firm 2, a Cournot quantity setter, operates on best response function

$$q_2 = \frac{1}{2}\left(\frac{a-c}{b} - q_1\right),$$

provided the resulting profit is nonnegative. Knowing this, firm 1 understands that its residual demand curve has equation

$$p = a - b\left[q_1 + \frac{1}{2}\left(\frac{a-c}{b} - q_1\right)\right]$$

$$p = a - bq_1 - \frac{1}{2}\left(a - c - bq_1\right)$$

$$p = c + (a - c) - bq_1 - \frac{1}{2}(a - c) + \frac{1}{2}bq_1$$

$$p = c + \frac{1}{2}(a - c) - \frac{1}{2}bq_1.$$

Firm 1 maximizes its payoff by picking an output that makes marginal revenue along this residual demand curve equal to marginal cost,

$$c + \frac{1}{2}(a - c) - bq_1 = c$$

$$q_1 = \frac{a - c}{2b}$$

(which is the output that would be produced by an otherwise identical monopolist in the same industry).

Firm 1's output is

$$q_2 = \frac{1}{2}\left(\frac{a - c}{b} - \frac{a - c}{2b}\right) = \frac{a - c}{4b},$$

provided that the resulting payoff

$$\pi_2 = b\left(\frac{a - c}{4b}\right) - F,$$

is nonnegative.

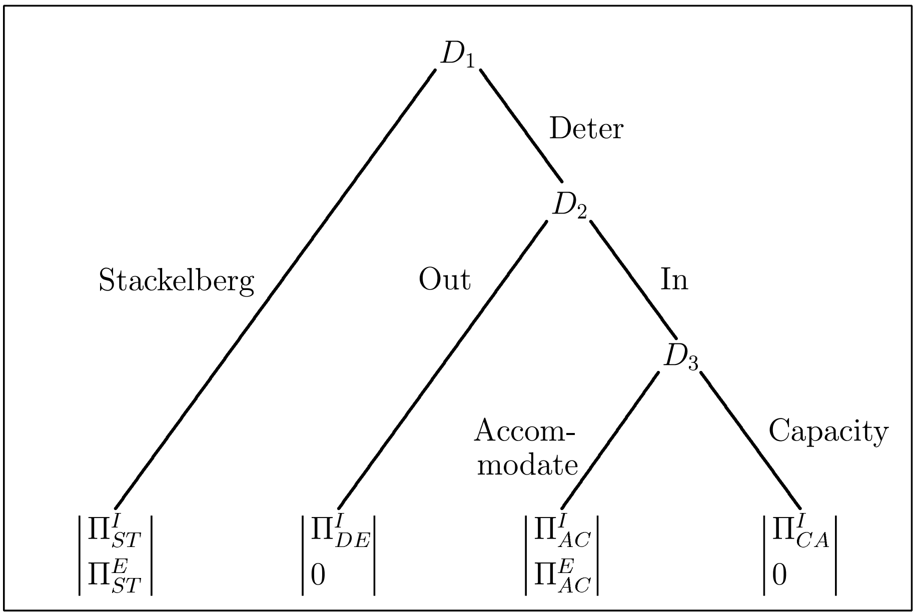


Figure 8.1: Game tree for Spence entry deterrence game; ST denotes Stackelberg leadership payoffs, DE denotes deterrence payoffs; AC denotes accommodation payoffs; CA denotes capacity output payoffs.

13.2 Show that in the game beginning from decision node D_1 of Figure 8.1, the incumbent's equilibrium choice is to act as a Stackelberg quantity leader.

Let fixed cost F be in the range that places the solution to (8.8) in case 3. Consider the situation of the incumbent at decision node D_1 in Figure 8.1. The incumbent must choose between installing the entry-deterring level of capacity $\overline{k}$ or acting as a Stackelberg leader from the beginning of the game. As a Stackelberg leader with marginal cost $c+r$ facing a Cournot follower with marginal cost $c+r$, the incumbent's output and profit are

$$q_{ST} = \frac{1}{2}S_{c+r} \text{ and } \Pi^I_{ST} = \frac{1}{8}bS^2_{c+r} - F \tag{8.1}$$

respectively. The Stackelberg leader produces the monopoly output.

The incumbent choose to carry capacity $\overline{k}$ only if

$$\Pi^I_{AC} > \Pi^I_{ST}. \tag{8.2}$$

Using (8.3)

$$\Pi^I_{AC} = \frac{b}{2}\left(\frac{1}{2}S_{c+r} + \frac{r}{b}\right)^2 - r\overline{k} - F \tag{8.3}$$

and (8.1), a necessary condition for condition (8.2) to be met is that

$$\frac{b}{16}\left(S_{c+r} - \frac{r}{b}\right)^2 < F. \tag{8.4}$$

But (8.4) is never satisfied in case 3 (see (8.19) in the text). Hence if entry deterrence would require carrying excess capacity, the incumbent would always prefer to act as a Stackelberg leader from the beginning of the game. By acting as a Stackelberg leader from the beginning of the game, the incumbent avoids the cost of excess capacity that he would in any event prefer not to use.

13.3 Let the inverse demand curve be

$$p = a - b(q_1 + q_2).$$

Suppose the potential entrant can produce with cost function

$$c(q_2) = cq_2 + F.$$

The incumbent's long-run cost function exhibits constant average and marginal cost c per unit. But the incumbent's short-run cost exceeds the long-run cost by a factor which depends on short-run deviations of output from capacity k:

$$c(q_1) = cq_1 + d(q_1 - k)^2$$

where $d > 0$ measures the extent to which cost rises if output differs from capacity.

Suppose the entrant believes that the incumbent will expand output to capacity if entry occurs. What capacity level must the incumbent maintain to make the entrant's expected profit equal to zero? What output will the incumbent produce if it maintains this entry-precluding level of capacity?

Compare Wenders (1971); for a similar approach with careful attention to micro foundations, Dixon (1986); and Lapham and Ware (1994, p. 581).

The capacity which the incumbent needs to maintain to deter entry is the same as in the Spence model,

$$\overline{k} = \frac{a-c-r}{b} - 2\sqrt{\frac{F}{b}}.$$

If the incumbent maintains this capacity level, its profit is

$$\pi_1 = (p-c)q_1 - d(q_1 - \overline{k})^2 = b\left(\frac{a-c}{b} - q_1\right)q_1 - d\left(q_1 - \overline{k}\right)^2.$$

Maximizing π_1 with respect to q_1 gives

$$q_1 = \frac{\frac{a-c}{b} + 2\frac{d}{b}\overline{k}}{2\left(1+\frac{d}{b}\right)} = \frac{\left(1+2\frac{d}{b}\right)\frac{a-c}{b} - 4\frac{d}{b}\sqrt{\frac{F}{b}}}{2\left(1+\frac{d}{b}\right)}$$

Excess capacity is then

$$\overline{k} - q_1 = \overline{k} - \frac{\frac{a-c}{b} + 2\frac{d}{b}\overline{k}}{2\left(1+\frac{d}{b}\right)}$$

$$= \overline{k} - \frac{2\frac{d}{b}\overline{k}}{2\left(1+\frac{d}{b}\right)} - \frac{\frac{a-c}{b}}{2\left(1+\frac{d}{b}\right)}$$

$$= \left(1 - \frac{\frac{d}{b}}{1+\frac{d}{b}}\right)\overline{k} - \frac{\frac{a-c}{b}}{2\left(1+\frac{d}{b}\right)}$$

$$= \frac{1}{1+\frac{d}{b}}\overline{k} - \frac{\frac{a-c}{b}}{2\left(1+\frac{d}{b}\right)}$$

$$= \frac{1}{2\left(1+\frac{d}{b}\right)}\left[2\overline{k} - \frac{a-c}{b}\right]$$

$$= \frac{1}{2\left(1+\frac{d}{b}\right)}\left[2\left(\frac{a-c-r}{b} - 2\sqrt{\frac{F}{b}}\right) - \frac{a-c}{b}\right]$$

$$= \frac{1}{2\left(1+\frac{d}{b}\right)}\left(\frac{a-c-2r}{b} - 4\sqrt{\frac{F}{b}}\right)$$

where the numerator of the fraction on the right must be positive for the constrained optimization problem to make sense. (If the numerator of the fraction on the right is negative, than the capacity that the incumbent would choose to hold, acting as a monopolist, is greater than $\overline{k}$.)

13.4 Suppose a price-setting incumbent firm operates with inverse demand curve

$$p_1 = a - b\left(q_1 + \theta q_2\right)$$

and that if a price-setting entrant comes into the market it operates on the inverse demand curve

$$p_2 = a - b\left(\theta b q_i + q_2\right).$$

If the cost function is

$$C(q) = F + rk + cq,$$

if capacity must be installed in advance of production, and the entrant believes that the incumbent will expand output to capacity, what is the entrant's best response function? (You may ignore the boundary conditions imposed by the requirement that quantities be nonnegative). What level of capacity must the incumbent choose to drive the entrant's expected profit to zero?

If the entrant believes that the incumbent will produce k_1 in the post-entry market, then the entrant believes it faces the demand curve

$$q_2 = \frac{a-c-r}{b} - \theta k_1 - \frac{p_2 - (c+r)}{b}.$$

The entrant's expected profit is then

$$\pi_2 = [p_2 - (c+r)]q_2 - F = [p_2 - (c+r)]\left[\frac{a-c-r}{b} - \theta k_1 - \frac{p_2-(c+r)}{b}\right] - F.$$

Taking the derivative of the entrant's profit with respect to p_2, we obtain the equation of the entrant's price-quantity (or rather, price-capacity) best response function,

$$p_2 - (c+r) = \frac{b}{2}[S_{c+r} - \theta k_1]$$

Substituting, the entrant's maximum expected profit is

$$\pi_2 = \frac{b}{4}\left(\frac{a-c-r}{b} - \theta k_1\right)^2 - F,$$

and this will be zero for

$$k_1 = \frac{1}{\theta}\left(\frac{a-c-r}{b} - 2\sqrt{\frac{F}{b}}\right).$$

13.5 Suppose there are two technologies available to serve a market, the α technology and the β technology. The cost functions of the two technologies are

$$c_\alpha(q) = c_\alpha q + F_\alpha \text{ and } c_\beta(q) = c_\beta q + F_\beta,$$

where

$$F_\beta > F_\alpha \text{ and } c_\beta < c_\alpha.$$

That is, the α technology has low fixed cost but high marginal cost, while the β has high fixed cost but low marginal cost.

(a) For what range of output is the α technology preferred?

As shown in Figure 8.2, average cost is less using the α technology for low outputs, where average cost is determined mainly by average fixed cost. Average cost is less using the β technology for higher outputs, where average cost is determined mainly by average variable cost. The critical output level, below which the α technology is preferred, is

$$c_\alpha q + F_\alpha = c_\beta q + F_\beta$$

$$(c_\alpha - c_\beta)\, q = F_\beta - F_\alpha$$

$$q = \frac{F_\beta - F_\alpha}{c_\alpha - c_\beta}.$$

(b) Assume equilibrium outputs occur in the range for which the α technology has the lowest average cost. Suppose up to three firms may enter a

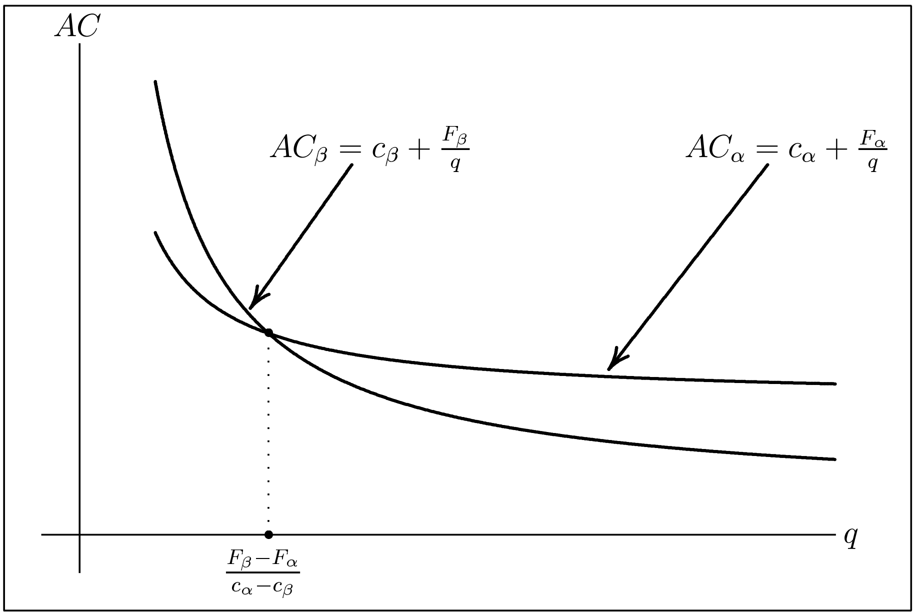


Figure 8.2: Average cost curves, $c_\alpha(q) = c_\alpha q + F_\alpha$, $c_\beta(q) = c_\beta q + F_\beta$; $F_\beta > F_\alpha$, $c_\beta < c_\alpha$.

market, and that if more than one firm is in the market they compete as Cournot quantity-setting oligopolists.

In what circumstances will the first firm enter and choose the α technology, the second firm enter and choose the β technology, and the third firm decline to enter?

If the first firm enters at all, it will choose the α technology, since we assume equilibrium outputs lie in the output range in which the α technology gives lowest average cost. Thus for the first firm to enter, monopoly profit using the α technology must be positive.

If the second firm comes in, it will compete as a Cournot quantity-setting duopolist. If it chooses the α technology, its profit will be

$$\pi_2(\alpha, \alpha, N) = \frac{b}{9}\left(\frac{a - c_\alpha}{b}\right)^2 - F_\alpha,$$

where the triple (α, α, N) indicates that the first firm has chosen the α technology, the second firm has chosen the α technology, and the third firm has not entered the market.

If the second firm chooses the β technology, its profit will be

$$\pi_2(\alpha, \beta, N) = \frac{b}{9}\left(\frac{a - c_\alpha}{b} + 2\frac{c_\alpha - c_\beta}{b}\right)^2 - F_\beta.$$

For the second firm to enter and choose the β technology, it must be the case that $\pi_2(\alpha, \beta, N) > \pi_2(\alpha, \alpha, N)$ and $\pi_2(\alpha, \beta, N) > 0$. On some manipulation, the condition $\pi_2(\alpha, \beta, N) > \pi_2(\alpha, \alpha, N)$ becomes

$$\frac{2}{9}\left(\frac{a - c_\alpha}{b} + \frac{c_\alpha - c_\beta}{b}\right) > \frac{F_\beta - F_\alpha}{c_\alpha - c_\beta}.$$

The third firm will decline to enter if its profits, using the α technology, are negative, given that the first firm chooses the α technology and the second firm chooses the β technology. The relevant model is Cournot triopoly with cost differences. The third firm's profit would be

$$\pi_3(\alpha, \beta, \alpha) = \frac{b}{16}\left(\frac{a - c_\alpha}{b} - \frac{c_\alpha - c_\beta}{b}\right)^2 - F_\alpha,$$

and this will be negative if

$$\frac{a - c_\alpha}{b} - \frac{c_\alpha - c_\beta}{b} < 4\sqrt{\frac{F_\alpha}{b}}.$$

See McLean and Riordan (1989).

13.6 In the model of dominant firm behavior with switching costs discussed in the text, suppose that both firms know that if the entrant comes into the market second-period rivalry will be as in a Cournot quantity-setting duopoly. What is the incumbent's best response curve? What is the range of possible equilibria, and what first-period output would the incumbent choose to maximize the present-discounted value of its profit?

This model is reminiscent of Dixit's model of entry deterrence by capacity choice. The incumbent's second-period best response curve has two segments, separated by a discontinuity. If in period 2 the incumbent produces less than in period 1, its profit function is

$$\pi_{12} = b\left[\frac{a - c}{b} - (q_{12} + q_{22})\right] q_{12},$$

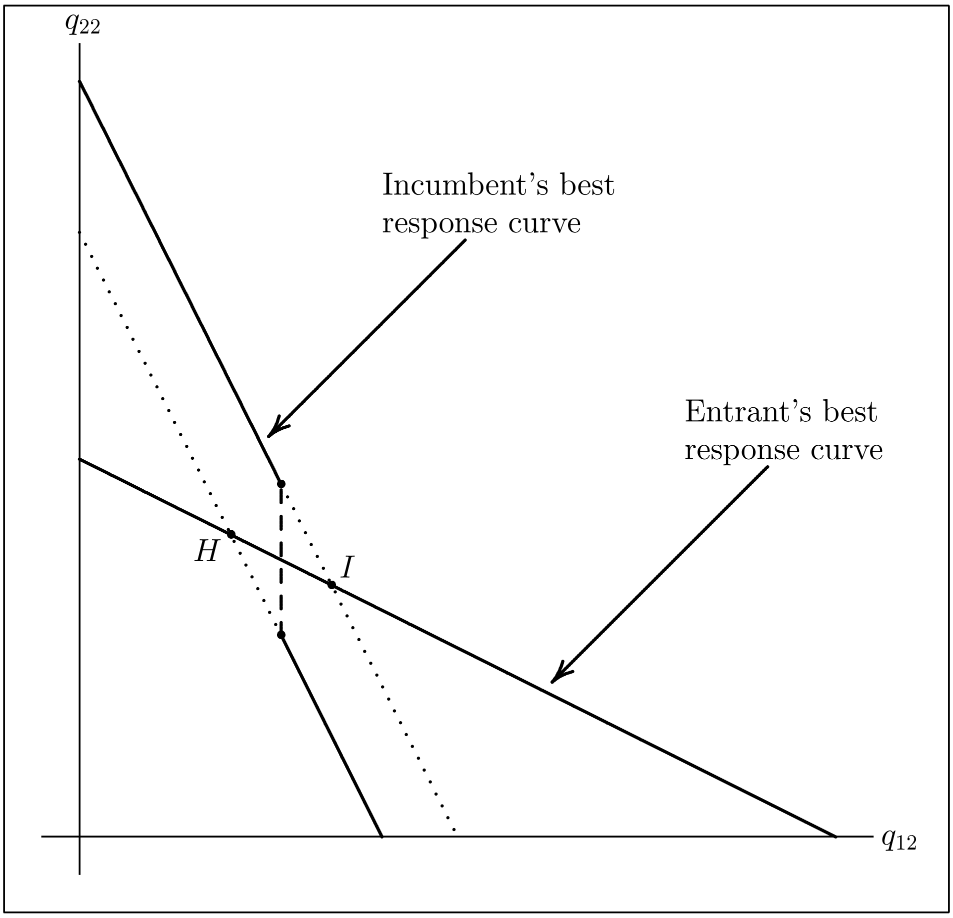


Figure 8.3: Second period response functions, Problem 8.6.

since all second-period consumers are experienced. The corresponding best response function has equation

$$q_{12} = \frac{1}{2}\left(\frac{a-c}{b} - q_{22}\right).$$

If in period 2 the incumbent sells more than in period 1, it faces the profit function

$$\pi_{12} = b\left[\frac{a-s-c}{b} - (q_{12}+q_{22})\right] q_{12},$$

and the equation of its best response function is

$$q_{12} = \frac{1}{2}\left(\frac{a-s-c}{b} - q_{22}\right).$$

The equation of the incumbent's period 2 best response function is therefore

$$q_{12} = \begin{cases} \frac{1}{2}\left(\frac{a-c}{b} - q_{22}\right) & q_{12} \leq q_{11} \\ \frac{1}{2}\left(\frac{a-s-c}{b} - q_{22}\right) & q_{12} > q_{11} \end{cases}$$

There is a vertical segment in firm 1's best response curve, at the output level $q_{12} = q_{11}$. By its choice of period 1 output, the incumbent determines the location of the vertical segment in its best response curve. In a model of dominant firm behavior with switching costs, selection of first-period output serves much the same role as capacity choice in the Dixit model: it determines the location of the vertical segment in the incumbent's best response curve.

The vertical line $q_{12} = q_{11}$ crosses the upper best response curve where

$$q_{22} = \frac{a-c}{b} - 2q_{11}$$

and crosses the lower best response curve where

$$q_{22} = \frac{a-s-c}{b} - 2q_{11}.$$

If the incumbent produces the same output in period 2 as in period 1, it operates on the higher demand curve. Its profit is

$$\pi_{12} = b\left[\frac{a-c}{b} - (q_{11}+q_{22})\right] q_{11}.$$

If, on the other hand, firm 1 produces on the lower segment of the best response curve, its second-period profit is

$$\pi_{12} = \frac{b}{4}\left(\frac{a-s-c}{b} - q_{22}\right)^2.$$

Let H denote the point at which the lower segment of the incumbent's best response curve

$$q_{12} = \frac{1}{2}\left(\frac{a-s-c}{b} - q_{22}\right)$$

intersects entrant's best response curve.

Solve the equation of the entrant's best response curve for q_{12}:

$$q_{22} = \frac{1}{2}\left(\frac{a-c}{b} - q_{12}\right)$$

$$q_{12} = \frac{a-c}{b} - 2q_{22}$$

$$q_{12} = \frac{1}{2}\left(\frac{a-s-c}{b} - q_{22}\right) = \frac{a-c}{b} - 2q_{22}$$

$$\frac{1}{2}\left(\frac{a-s-c}{b} - q_{22}\right) = \frac{a-c}{b} - 2q_{22}$$

$$q_{22} = \frac{a-c+s}{3b}$$

$$q_{12} = \frac{1}{2}\left(\frac{a-s-c}{b} - \frac{a-c+s}{3b}\right)$$

$$q_{12} = \frac{a-2s-c}{3b}$$

$$\left(\frac{a-2s-c}{3b}, \frac{a-c+s}{3b}\right)$$

Let I denote the point at which the upper segment of the incumbent's best response curve

$$q_{12} = \frac{1}{2}\left(\frac{a-c}{b} - q_{22}\right)$$

intersects the entrant's best response curve.

$$q_{12} = \frac{1}{2}\left(\frac{a-c}{b} - q_{22}\right) = \frac{a-c}{b} - 2q_{22}$$

$$\frac{1}{2}\left(\frac{a-c}{b} - q_{22}\right) = \frac{a-c}{b} - 2q_{22}$$

$$q_{22} = \frac{a-c}{3b}$$

$$q_{12} = \frac{1}{2}\left(\frac{a-c}{b} - \frac{a-c}{3b}\right) = \frac{a-c}{3b}$$

$$\left(\frac{a-c}{3b}, \frac{a-c}{3b}\right).$$

There is a vertical segment in firm 1's second-period best response function, at the level of its first-period output, connecting the upper and lower segments of its best response function.

If firm 1 produces less than

$$q_{11} < q_{1H} = \frac{a-2s-c}{3b},$$

second-period equilibrium is at H, and firm 1's payoff is

$$(a-s-c-bq_{11})\,q_{11} + \frac{1}{1+r}\left[a-s-c-b\left(q_{1H}+q_{2H}\right)\right]q_{1H}.$$

If firm 1 produces

$$q_{11} \geq q_{1I} = \frac{a-c}{3b},$$

second-period equilibrium is at I and firm 1's payoff is

$$(a - s - c - bq_{11})\, q_{11} + \frac{1}{1+r}\left[a - c - b\,(q_{1I} + q_{2I})\right] q_{1I}.$$

If firm 1 produces

$$q_{1H} \leq q_{11} \leq q_{1I}$$

in the first period, second period equilibrium is on firm 2's best response curve at

$$\left(q_{11}, \frac{1}{2}\,(a - s - c - bq_{11})\right)$$

and firm 1's payoff is

$$\pi_{11} = (a - s - c - bq_{11})\, q_{11}$$

$$+\frac{1}{1+r}\left\{a - c - b\left[q_{11} + \frac{1}{2}\,(a - s - c - bq_{11})\right]\right\}.$$

The optimal first-period output is found by comparing firm 1's payoff at H and at I with its maximum payoff along firm 2's best response function between H and I.

13.7 Given (8.59) and (8.60) in the text, show that the equation of the incumbent's residual demand curve is

$$p = c + \frac{1}{\frac{1}{b} + \frac{1}{f}}\left[S + \frac{1}{f}(e - c) - Q_D\right],$$

where $S = (a - c)/b$.

(a) Calculate dominant firm and fringe output, dominant firm profit, the price – cost margin, and deadweight welfare loss.

(b) Verify that the incumbent's profit-maximizing value of α is as given by (8.63) in the text. Compare the higher-cost equilibrium with the initial values.

The equation of the residual demand curve is obtained by subtraction. Setting marginal revenue equal to marginal cost gives the dominant firm's output and price (along the residual demand curve). Fringe output is then determined by the fringe supply curve. Features of the initial equilibrium are:

$$Q_D = \frac{1}{2}\left(\frac{a-c}{b} + \frac{e-c}{f}\right)$$

$$Q_f = \frac{1}{2f}\frac{\frac{a-c}{b} + \frac{e-c}{f}}{\frac{1}{b}+\frac{1}{f}} - \frac{e-c}{f}$$

$$\pi_D = \frac{1}{\frac{1}{b}+\frac{1}{f}}\left[\left(\frac{1}{2}\right)\left(\frac{a-c}{b} + \frac{e-c}{f}\right)\right]^2$$

$$p - c = \frac{1}{2}\frac{\frac{a-c}{b} + \frac{e-c}{f}}{\frac{1}{b}+\frac{1}{f}}$$

$$DWL = \frac{1}{b}\left(\frac{1}{2}\frac{S + \frac{e-c}{f}}{\frac{1}{b}+\frac{1}{f}}\right)^2 = \frac{1}{b}(p-c)^2$$

where π_D is the profit of the dominant firm and DWL is deadweight welfare loss.

If the incumbent accepts an increase in its own marginal cost and average cost from c to $c+\alpha$, the fringe supply curve becomes

$$p = e + (1+\rho_1)\alpha - \rho_2\alpha^2,$$

where $\rho_1, \rho_2 > 0$.

After engaging in the cost-raising activity, the incumbent's residual demand curve becomes

$$\left(\frac{1}{b}+\frac{1}{f}\right)(p-c-\alpha) = \frac{a-c}{b} + \frac{e-c}{f} + \alpha\left(\frac{\rho_1 - \rho_2\alpha}{f} - \frac{1}{b}\right) - Q_D.$$

For notational simplicity, let X denote the first three terms on the right. Except for the scale factor $\left(\frac{1}{b}+\frac{1}{f}\right)$, X is the difference between the price axis intercepts of the residual demand curve and the incumbent's marginal cost. Note that

$$\left.\frac{\partial X}{\partial \alpha}\right|_{\alpha=0} = \frac{\rho_1}{f} - \frac{1}{b}.$$

If the expression on the right is positive, increases in α increase the distance between the intercept of the residual demand curve and the intercept of the fringe supply curve. The incumbent's profit is

$$\left(\frac{1}{b} + \frac{1}{f}\right)\pi_D = \frac{1}{4}\left(\frac{a-c}{b} + \frac{e-c}{f} + \frac{\rho_1\alpha - \rho_2\alpha^2}{f} - \frac{\alpha}{b}\right)^2$$

Differentiating with respect to α, we obtain the profit-maximizing value of α:

$$\alpha = \frac{f}{2\rho_2}\left(\frac{\rho_1}{f} - \frac{1}{b}\right).$$

The optimal value of α is positive if and only if the initial investment in raising rivals' cost raises the residual demand curve relative to the incumbent's marginal cost. The relation between the initial and higher-cost equilibria is

$$Q_D = Q_{D,0} + \frac{\rho_2\alpha^2}{f}$$

$$Q_F = Q_{F,0} - \frac{\alpha}{4}\left(\frac{1}{b} + \frac{\rho_1}{f}\right)\left(\frac{b+2f}{b+f}\right)$$

$$\pi_D = \frac{1}{\frac{1}{b}+\frac{1}{f}}\frac{1}{4}\left(\frac{a-c}{b} + \frac{e-c}{f} + \frac{\rho_1\alpha - \rho_2\alpha^2}{f}\right)^2$$

$$p = p_0 + \alpha + \frac{\rho_2}{f}\frac{\alpha^2}{\frac{1}{b}+\frac{1}{f}}$$

13.8 Consider a duopoly of quantity-setting firms. The market demand curve is

$$p = a - b(q_1 + q_2).$$

The dominant firm's marginal cost (c_1) is less than the rival's marginal cost (c_2). If the incumbent accepts an increase in marginal cost to $c_1 + \alpha$, the rival's marginal cost rises to

$$c_2 + (1 + \rho_1)\alpha - \rho_2\alpha^2,$$

with ρ_1, $\rho_2 > 0$.

Under what circumstances, if any, will the dominant firm set a positive value for α? Explain.

Reference may be made to Problem 2.1, which deals with Cournot quantity-setting duopoly when firms have different marginal costs. If the incumbent accepts higher costs, we can write

$$S_2 = \frac{a - c_2 - (1 + \rho_1)\alpha + \rho_2\alpha^2}{b}$$

and the equilibrium outputs of the two firms are

$$q_1 = \frac{1}{3}\left(S_2 + 2\frac{c_2 - c_1 + \rho_1\alpha - \rho_2\alpha^2}{b}\right)$$

$$q_2 = \frac{1}{3}\left(S_2 - \frac{c_2 - c_1 + \rho_1\alpha - \rho_2\alpha^2}{b}\right)$$

Firm 1's profit is

$$\pi_1 = \frac{b}{9}\left(S_2 + 2\frac{c_2 - c_1 + \rho_1\alpha - \rho_2\alpha^2}{b}\right)$$

which falls as α rises.

It follows that the dominant firm will set $\alpha = 0$. Increases in α have two effects. First, an increase in α makes the market smaller (reduces the quantity which would be demanded if price were set equal to the dominant firm's marginal cost). This tends to reduce the dominant firm's profit. Second, an increase in α increases the incumbent's relative cost advantage over the rival, and tends to increase the dominant firm's share of the smaller market.

In this model, the second effect overwhelms the first, and it is on balance unprofitable for the dominant firm to engage in cost-raising strategies.

Chapter 9

Advertising

9.1 Let the equation of the inverse demand curve be

$$p_{it} = f(q_{it} + \theta Q_{-it}, a_{it} + \phi a_{-it}). \tag{9.1}$$

If a single firm maximizes

$$V_i = \sum_{t=0}^{\infty} (1+r)^{-t} \left\{ (p_{it} - c) q_{it} - p_t^A \left[a_{it} - (1-\gamma) a_{i,t-1} \right] \right\}, \tag{9.2}$$

derive expressions for the partial derivatives of q_{it} and a_{it} with respect to θ and ϕ.

The first-order conditions are

$$f(q_{it} + \theta Q_{-it}, a_{it} + \phi a_{-it}) - c \equiv -q_{it} f_1 \tag{9.3}$$

$$f_2 q_{it} - p_t^A + \frac{1-\gamma}{1+r} p_{t+1}^A \equiv 0. \tag{9.4}$$

Totally differentiating the first-order conditions, one obtains

$$\begin{pmatrix} 2f_1 + q_{it} f_{11} & f_2 + q_{it} f_{12} \\ f_2 + q_{it} f_{12} & q_{it} f_{22} \end{pmatrix} \begin{pmatrix} dq_{it} \\ da_{it} \end{pmatrix} =$$

$$- \begin{pmatrix} f_1 + q_{it} f_{11} & f_2 + q_{it} f_{12} \\ q_{it} f_{12} & q_{it} f_{22} \end{pmatrix} \begin{pmatrix} Q_{-it} d\theta \\ a_{-it} d\phi \end{pmatrix}. \tag{9.5}$$

Call the determinant of the matrix on the left

$$D = (2f_1 + q_{it} f_{11})\, q_{it} f_{22} - (f_2 + q_{it} f_{12})^2 . \tag{9.6}$$

D is negative by second-order conditions.

Then

$$D \begin{pmatrix} 2f_1 + q_{it} f_{11} & f_2 + q_{it} f_{12} \\ f_2 + q_{it} f_{12} & q_{it} f_{22} \end{pmatrix}^{-1} = \begin{pmatrix} q_{it} f_{22} & -(f_2 + q_{it} f_{12}) \\ -(f_2 + q_{it} f_{12}) & 2f_1 + q_{it} f_{11} \end{pmatrix}. \tag{9.7}$$

$$D \begin{pmatrix} 2f_1 + q_{it} f_{11} & f_2 + q_{it} f_{12} \\ f_2 + q_{it} f_{12} & q_{it} f_{22} \end{pmatrix}^{-1} \begin{pmatrix} f_1 + q_{it} f_{11} & f_2 + q_{it} f_{12} \\ q_{it} f_{12} & q_{it} f_{22} \end{pmatrix} =$$

$$\begin{pmatrix} q_{it} f_{22} & -(f_2 + q_{it} f_{12}) \\ -(f_2 + q_{it} f_{12}) & 2f_1 + q_{it} f_{11} \end{pmatrix} \begin{pmatrix} f_1 + q_{it} f_{11} & f_2 + q_{it} f_{12} \\ q_{it} f_{12} & q_{it} f_{22} \end{pmatrix} =$$

$$\begin{pmatrix} -q_{it} \left(f_{22} (f_1 + q_{it} f_{11}) - f_{12} (f_2 + q_{it} f_{12}) \right) & 0 \\ (2f_1 + q_{it} f_{11})\, q_{it} f_{12} - (f_1 + q_{it} f_{11})(f_2 + q_{it} f_{12}) & (2f_1 + q_{it} f_{11})\, q_{it} f_{22} - (f_2 + q_{it} f_{12})^2 \end{pmatrix} \tag{9.8}$$

$$D \begin{pmatrix} dq_{it} \\ da_{it} \end{pmatrix} =$$

$$\begin{pmatrix} q_{it} \left(f_{12} (f_2 + q_{it} f_{12}) - f_{22} (f_1 + q_{it} f_{11}) \right) & 0 \\ (f_1 + q_{it} f_{11})(f_2 + q_{it} f_{12}) - (2f_1 + q_{it} f_{11})\, q_{it} f_{12} & (f_2 + q_{it} f_{12})^2 - (2f_1 + q_{it} f_{11})\, q_{it} f_{22} \end{pmatrix} \cdot$$

$$\times \begin{pmatrix} Q_{-it} d\theta \\ a_{-it} d\phi \end{pmatrix} \tag{9.9}$$

$$\frac{dq_{it}}{d\theta} = Q_{-it} q_{it} \frac{f_{12} (f_2 + q_{it} f_{12}) - f_{22} (f_1 + q_{it} f_{11})}{D}. \tag{9.10}$$

$$\frac{dq_{it}}{d\phi} = 0. \tag{9.11}$$

$$\frac{da_{it}}{d\theta} = Q_{-it}\frac{(f_1 + q_{it}f_{11})(f_2 + q_{it}f_{12}) - (2f_1 + q_{it}f_{11})\, q_{it}f_{12}}{D}. \tag{9.12}$$

$$\frac{da_{it}}{d\phi} = a_{-it}\frac{(f_2 + q_{it}f_{12})^2 - (2f_1 + q_{it}f_{11})\, q_{it}f_{22}}{D}. \tag{9.13}$$

9.2 Answer **Problem 9.1** if the equation of the inverse demand curve is

$$p_{it} = \alpha + u(a_{it} + \phi a_{-it}) - v(a_{it} + \phi a_{-it})^2 - \beta(q_{it} + \theta Q_{-it}). \tag{9.14}$$

Then

$$f(q_{it} + \theta Q_{-it}, a_{it} + \phi a_{-it}) \tag{9.15}$$

$$f_1 = -\beta \tag{9.16}$$

$$f_{11} = 0 \tag{9.17}$$

$$f_{12} = 0 \tag{9.18}$$

$$f_2 = u - 2v(a_{it} + \phi a_{-it}), \tag{9.19}$$

which we assume to be positive (the firm will not operate where the marginal effect of goodwill on demand is negative).

$$f_{22} = -2v < 0. \tag{9.20}$$

Then substituting in the expressions obtained in the answer to **9.1**,

$$\frac{dq_{it}}{d\theta} = -Q_{-it}q_{it}\frac{f_{22}f_1}{D} = -Q_{-it}q_{it}\frac{2v\beta}{D} > 0. \tag{9.21}$$

$$\frac{dq_{it}}{d\phi} = 0. \tag{9.22}$$

$$\frac{da_{it}}{d\theta} = -\beta Q_{-it}\frac{f_2}{D} > 0 \tag{9.23}$$

$$\frac{da_{it}}{d\phi} = a_{-it}\frac{f_2^2 - 4\beta v q_{it}}{D} = -a_{-it}. \tag{9.24}$$

Directly from

$$\begin{pmatrix} 2f_1 + q_{it}f_{11} & f_2 + q_{it}f_{12} \\ f_2 + q_{it}f_{12} & q_{it}f_{22} \end{pmatrix} \begin{pmatrix} dq_{it} \\ da_{it} \end{pmatrix} =$$

$$-\begin{pmatrix} f_1 + q_{it}f_{11} & f_2 + q_{it}f_{12} \\ q_{it}f_{12} & q_{it}f_{22} \end{pmatrix} \begin{pmatrix} Q_{-it}d\theta \\ a_{-it}d\phi \end{pmatrix}$$

one obtains

$$\begin{pmatrix} 2f_1 & f_2 \\ f_2 & q_{it}f_{22} \end{pmatrix} \begin{pmatrix} dq_{it} \\ da_{it} \end{pmatrix} = -\begin{pmatrix} f_1 & f_2 \\ 0 & q_{it}f_{22} \end{pmatrix} \begin{pmatrix} Q_{-it}d\theta \\ a_{-it}d\phi \end{pmatrix}. \tag{9.25}$$

$$\begin{pmatrix} -2\beta & f_2 \\ f_2 & -2vq_{it} \end{pmatrix} \begin{pmatrix} dq_{it} \\ da_{it} \end{pmatrix} = -\begin{pmatrix} -\beta & f_2 \\ 0 & q_{it}f_{22} \end{pmatrix} \begin{pmatrix} Q_{-it}d\theta \\ a_{-it}d\phi \end{pmatrix}. \tag{9.26}$$

$$D = \det\begin{pmatrix} -2\beta & f_2 \\ f_2 & -2vq_{it} \end{pmatrix} = 4\beta v q_{it} - f_2^2 < 0. \tag{9.27}$$

$$D\begin{pmatrix} -2\beta & f_2 \\ f_2 & -2vq_{it} \end{pmatrix}^{-1} = -\begin{pmatrix} 2vq_{it} & f_2 \\ f_2 & 2\beta \end{pmatrix} \tag{9.28}$$

$$D\begin{pmatrix} dq_{it} \\ da_{it} \end{pmatrix} = \begin{pmatrix} 2vq_{it} & f_2 \\ f_2 & 2\beta \end{pmatrix} \begin{pmatrix} -\beta & f_2 \\ 0 & -2vq_{it} \end{pmatrix} \begin{pmatrix} Q_{-it}d\theta \\ a_{-it}d\phi \end{pmatrix}. \tag{9.29}$$

$$\begin{pmatrix} 2vq_{it} & f_2 \\ f_2 & 2\beta \end{pmatrix} \begin{pmatrix} -\beta & f_2 \\ 0 & -2vq_{it} \end{pmatrix}$$

$$D\begin{pmatrix} dq_{it} \\ da_{it} \end{pmatrix} = \begin{pmatrix} -2\beta v q_{it} & 0 \\ -f_2\beta & -D \end{pmatrix} \begin{pmatrix} Q_{-it}d\theta \\ a_{-it}d\phi \end{pmatrix}. \tag{9.30}$$

$$\frac{dq_{it}}{d\theta} = -Q_{-it}\frac{2vq_{it}\beta}{D} > 0. \tag{9.31}$$

$$\frac{dq_{it}}{d\phi} = 0. \tag{9.32}$$

$$\frac{da_{it}}{d\theta} = -Q_{-it}\frac{\beta f_2}{D} > 0 \tag{9.33}$$

$$\frac{da_{it}}{d\phi} = a_{-it}\frac{-D}{D} = -a_{-it}. \tag{9.34}$$

9.3 (Cubbin, 1981) Let inverse demand curves be

$$p_i = a + \alpha(A_i + \phi A_j) - b(q_i + \theta q_j),$$

where i, $j = 1, 2$ and $\phi < 0$. Assume that the marginal and average cost of production is a constant, c; and that the cost of advertising is

$$c(A_i) = A_i + dA_i^2.$$

(a) If firm 1 is an incumbent monopolist, what are the profit-maximizing levels of output and advertising?

The incumbent picks q_1 and A_1 to maximize

$$\pi_1 = (a - c + \alpha A_1 - bq_1)\, q_1 - A_1 - dA_1^2.$$

Maximizing first with respect to q_1, taking A_1 as given, yields

$$q_1 = \frac{a - c + \alpha A_1}{2b}.$$

Substituting back in the expression for π_1, the incumbent's profit as a function of advertising alone is

$$\pi_1 = bq_1^2 - A_1 - dA_1^2 = \frac{1}{4b}\left(a - c + \alpha A_1\right)^2 - A_1 - dA_1^2.$$

The profit-maximizing value of A_1 is

$$A_1 = \frac{\alpha b}{4bd - \alpha^2}\left(\frac{a-c}{b} - \frac{2}{\alpha}\right),$$

which implies

$$q_1 = \frac{1}{2 - \frac{\alpha^2}{2bd}} \left(\frac{a-c}{b} - \frac{\alpha}{2bd} \right).$$

Maximum profit can be written in different ways; one is

$$\pi_1 = \frac{1}{4b} \left[(a-c)^2 - (\alpha A_1)^2 \right] + dA_1^2 = b \left(\frac{a-c}{b} \right)^2 + \frac{4bd - \alpha^2}{4b} A_1^2.$$

(b) Analyze the shape of the incumbent's isoprofit curves.

Write the equation of an isoprofit curve as

$$\overline{\pi}_1 = b \left(\frac{a-c}{b} + \frac{\alpha}{b} A_1 - q_1 \right) q_1 - A_1 - dA_1^2.$$

This is the equation of a conic section; the presence of the cross-product term $q_1 A_1$ guarantees that it is a rotated conic section.

(c) If the incumbent knows that a potential entrant maximizes profit taking q_1 and A_1 as given, what levels of q_1 and A_1 deter entry? Under what circumstances would the incumbent find it profitable to deter entry?

The maximum profit of an entrant that takes q_1 and A_1 as given is

$$\pi_2 = \frac{1}{4b} \left(a - c + \alpha\phi A_1 - b\theta q_1 \right)^2 + \frac{4bd - \alpha^2}{4b} A_2^2,$$

where

$$A_2 = \frac{\alpha}{4bd - \alpha^2} \left(a - c + \alpha\phi A_1 - b\theta q_1 - 2\frac{b}{\alpha} \right).$$

Values of q_1 and A_1 that would make $\pi_2 = 0$ would deter entry. Whether or not the incumbent would want to deter entry would require a specification of the nature of post-entry competition and a comparison of duopoly profit with profit at the entry-deterring level.

9.4 Derive the prices and payoffs in of the Milgrom and Roberts linear advertising signaling model.

Consider first $\pi(p, H, H)$, the payoff of the producer of a high-quality product that first-period consumers believe to be a high-quality product.

With a first-period price p_1, the firm sells $q_1 = (Ha - p_1)/Hb$ in the first period and $q_2 = Hq_1$ in the second period, at price $p_2 = p_1/H$. Hence

$$\pi(p_1, H, H) = (p_1 - c_H)q_1 + \alpha\left(\frac{p_1}{H} - c_H\right)Hq_1 \tag{9.35}$$

$$= \frac{1+\alpha}{Hb}\left(p_1 - \frac{1+\alpha H}{1+\alpha}c_H\right)(Ha - p_1), \tag{9.36}$$

where α is the discount factor.

The first-order condition is

$$-\left(p_1 - \frac{1+\alpha H}{1+\alpha}c_H\right) + Ha - p_1 \equiv 0,$$

which implies that

$$\pi(p_1, H, H) = \frac{1+\alpha}{Hb}(Ha - p_1)^2$$

along the first-order condition.

Solving the first-order condition

$$p_{HH} = \frac{1}{2}\left(Ha + \frac{1+\alpha H}{1+\alpha}c_H\right),$$

so that

$$\pi(p_1, H, H) = \frac{1+\alpha}{4Hb}\left(Ha - \frac{1+\alpha H}{1+\alpha}c_H\right)^2.$$

By the same sort of argument,

$$\pi(p_1, L, L) = (p_1 - c_L)q_1 + \alpha\left(\frac{p_1}{L} - c_L\right)Lq_1 \tag{9.37}$$

$$= \frac{1+\alpha}{Lb}\left(p_1 - \frac{1+\alpha L}{1+\alpha}c_L\right)(La - p_1). \tag{9.38}$$

$$p_{LL} = \frac{1}{2}\left(La + \frac{1+\alpha L}{1+\alpha}c_L\right). \tag{9.39}$$

$$\pi(p_{LL}, L, L) = \frac{(1+\alpha)L}{4b}\left[a - \frac{1+\alpha L}{(1+\alpha)L}c_L\right]^2. \tag{9.40}$$

Now turn to $\pi(p, L, H)$, the payoff of a producer of a low-quality product that first-period consumers believe to be a high-quality good. With a price p_1 the firm sells $q_1 = (Ha - p_1)/Hb$ in the first period. In the second period, it is able to sell $q_2 = Lq_1$ at price

$$p_2 = a - \frac{b}{L}Lq_1 = a - bq_1 = a - b\left(\frac{a}{b} - \frac{p_1}{Hb}\right) = \frac{p_1}{H}. \tag{9.41}$$

Hence

$$\pi(p_1, L, H) = (p_1 - c_L)q_1 + \alpha\left(\frac{p_1}{H} - c_L\right)Lq_1 \tag{9.42}$$

$$= \frac{H+\alpha L}{H^2 b}\left[p_1 - \frac{H(1+\alpha L)}{H+\alpha L}c_L\right](Ha - p_1), \tag{9.43}$$

which is maximized for

$$p_{LH} = \frac{1}{2}H\left(a + \frac{1+\alpha L}{H+\alpha L}c_L\right) \tag{9.44}$$

and

$$\pi(p_{LH}, L, H) = \frac{H+\alpha L}{4b}\left(a - \frac{1+\alpha L}{H+\alpha L}c_L\right)^2. \tag{9.45}$$

By the same sort of argument,

$$\pi(p_1, H, L) = \frac{L+\alpha H}{L^2 b}\left(p_1 - L\frac{1+\alpha H}{L+\alpha H}c_H\right)(La - p_1). \tag{9.46}$$

This is maximized for

$$p_{HL} = \frac{1}{2}L\left[a + \frac{1+\alpha H}{L+\alpha H}c_H\right], \tag{9.47}$$

which yields payoff

$$\pi(p_{HL}, H, L) = \frac{L+\alpha H}{4b}\left(a - \frac{1+\alpha H}{L+\alpha H}c_L\right)^2.$$

Chapter 10

Collusion and Noncooperative Collusion

See

Lanning, Steven G. "Costs of maintaining a cartel," *Journal of Industrial Economics* 36(2), December 1987, pp. 157–74

for a model of noncooperative collusion that takes into account the cost of enforcing a cartel agreement.

10.1 Show that a single firm can profit by defecting from a joint-profit-maximizing cartel in an n-firm price-setting oligopoly with product differentiation. Assume inverse demand functions and demand functions follow the Bowley model of product differentiation:

$$p_i = a - b(\theta q_1 + \theta q_2 + ... + q_i + ... + \theta q_2)$$

$$q_i = \frac{(1-\theta)\, a - [1 + (n-2)\theta] p_i + \theta \sum_{j \neq i} p_j}{b(1-\theta)\, [1 + (n-1)\theta]},$$

constant marginal cost, identical for all firms, and no fixed cost.

The demand curve facing a single firm is

$$q_i = \frac{1}{1 + (n-1)\theta} \frac{1}{b} \left\{ a - \left[p_i + \frac{n\theta}{1-\theta} (p_i - \overline{p}) \right] \right\}$$

or

$$q_i = \frac{(1-\theta)(a-c) + \theta \sum_{j \neq i}(p_j - c) - [1+(n-2)\theta](p_i - c)}{b(1-\theta)[1+(n-1)\theta]}$$

The joint-profit-maximizing price is

$$p_{j\pi m} = c + \frac{1}{2}(a-c),$$

which is also the monopoly price for a linear demand/constant marginal cost standardized product.

If remaining firms set the cartel price, firm 1 faces a residual demand curve

$$q_1 = \frac{[2+(n-3)\theta](a-c) - 2[1+(n-2)\theta](p_1 - c)}{2b(1-\theta)[1+(n-1)\theta]}$$

From the first-order condition for maximization of

$$\pi_1 = (p_1 - c)q_1,$$

firm 1's individual profit-maximizing price satisfies

$$p_{defect} - c = \frac{2+(n-3)\theta}{1+(n-2)\theta}\frac{(a-c)}{4},$$

and

$$p_{j\pi m} - p_{defect} = \frac{(n-1)\theta}{1+(n-2)\theta}\frac{(a-c)}{4} \geq 0,$$

with equality holding only if $n = 1$. Thus a single firm in a price-setting cartel will profit by cutting the joint-profit-maximizing price, if it believes other firms will adhere to the cartel price.

10.2 Adopt the linear demand curve (10.1) of the text, and let each firm operate with the quadratic cost function

$$c(q_1) = cq_i + dq_i^2,$$

where $d > 0$ indexes diseconomies of scale. Let there be N firms, K of which form a cartel. The remaining F firms form a price-taking fringe. The K cartel members maximize joint profit, given cartel behavior. What restrictions on K and F are required for internal cartel stability? What restrictions on K and F are required for external cartel stability?

Each fringe firm, acting as a price-taker, picks an output that equates marginal cost and price. Thus the fringe firm and fringe supply functions are

$$q_f = \frac{p-c}{2d} \quad Q_F = F\frac{p-c}{2d}$$

respectively. Substituting for Q_F into the equation of the market demand curve and collecting terms, the cartel residual demand curve, written in inverse form, is

$$p = c + \frac{2b\delta}{F+2\delta}(S - Q_K)$$

$\delta = d/b$ is the diseconomies of scale parameter measured in proportion to the slope of the demand curve.

As all firms in the cartel operate with the same cost function, under profit maximization each firm in the cartel will produce the same output. The cartel cost function for output Q_K is then

$$C(Q_K) = Kc\left(\frac{Q_K}{K}\right) = cQ_K + d\frac{Q_K^2}{K}.$$

Equating cartel marginal cost to marginal revenue along the cartel residual demand curve, profit-maximizing output per cartel member is

$$q_k = \frac{S}{N+2\delta+K}.$$

Substituting back into the equation of the cartel residual demand curve, the price that maximizes cartel profit is

$$p = c + \frac{N+2\delta}{(N+2\delta)^2 - K^2}2\delta(a-c)$$

Profit per cartel firm and fringe firm are

$$\pi_k(K,N) = \frac{d\left(\frac{a-c}{b}\right)^2}{(N+2\delta)^2 - K^2}$$

and

$$\pi_f(K,N) = b\delta S^2 \left[\frac{N+2\delta}{(N+2\delta)^2 - K^2}\right]^2.$$

The cartel is internally stable if

$$\pi_k(K,N) \geq \pi_f(K-1,N)$$

The cartel is externally stable if $F \geq W_1$ (otherwise, there are no firms in the fringe) and

$$\pi_f(K,N) \geq \pi_k(K+1,N)$$

If all firms join the cartel, external stability is not an issue. For a complete cartel, we need only examine the condition for internal stability. A complete cartel is internally stable if

$$\pi_k(N,N) \geq \pi_f(N-1,N)$$

or if

$$(N-1)^4 \geq (N+2\delta)^2[2(N-1)^2 - N^2]$$

This inequality is satisfied for all values of δ if $N = 1, 2,$ *or* 3. A complete cartel is thus internally stable for $N = 1$, 2, and 3. If $N \geq 4$, the second right-hand-side term in above is positive. By division, if $N \geq 4$, a complete cartel will be stable provided

$$\delta \leq \frac{1}{2}\frac{(N-1)^2}{\sqrt{N^2 - 4N + 2}} - N.$$

The term on the right is an upper bound for values of δ that are consistent with internal stability for the cartel of the whole when there are 4 or more firms. The upper bound converges to zero as N rises. For 4 or more firms, a complete cartel is unstable if diseconomies of scale are too great (if d is too large).

Now suppose the fringe is non-empty. Consider first internal stability. When there is at least one firm in the fringe, the internal stability condition is

$$(K-1)^4 \geq (N+2\delta)^2[2(K-1)^2 - K^2].$$

For $K = 1$, 2, and 3, this condition is satisfied for all N and δ. Combined with the results for a complete cartel, it follows that a one-, two- or three-firm cartel is internally stable for all values of δ and for all values of F.

For $K \geq 4$, the second term on the right in the above condition is positive. By division, the cartel will be internally stable if

$$F + 2\delta \leq \frac{(K-1)^2}{\sqrt{2(K-1)^2 - K^2}} - K.$$

The term on the left is an upper bound on the size of the fringe that is consistent with internal cartel stability.

For $K = 4$, a cartel will be internally stable for $F + 2\delta \ \leq\ 2.36$. The maximum number of fringe members consistent with internal stability of a four-member cartel is 2; the exact upper bound on F depends on δ. If there are 3 or more firms in the fringe, then the profit split by the cartel is so small that a cartel member will find it profitable to join the fringe, and the cartel is internally unstable.

For $K = 5$, a cartel will be internally stable for $F + 2\delta \ \leq\ 1.05$. The maximum number of fringe firms consistent with internal stability of a five-firm cartel is 1.

For $K = 6$, the upper limit on $F + 2\delta$ is 0.68, which is less than one. Further, the upper limit on $F + 2\delta$ falls as K rises. It follows that cartels of 6 or more firms are internally unstable if there are any firms in the fringe.

Now turn to external stability. The external stability condition is

$$0 \geq (N+2\delta)^2[(K+1)^2 - 2K^2] + K^4.$$

This inequality is violated for all values of N and δ if $K = 1$ or 2. A fringe firm will always want to join a one- or two-firm cartel, and such cartels are externally unstable unless the fringe is empty.

For $K \geq 3$, the external stability condition will be satisfied for values of F sufficiently large:

$$F + 2\delta \ \geq \frac{K^2}{\sqrt{2K^2 - (K+1)^2}} - K.$$

The intuition behind this result is straightforward. The more firms there are in the fringe, the smaller the profit collected by each cartel member. If the number of firms in the fringe is large enough, fringe firms will not be tempted to join the cartel.

For $K = 3$, a cartel will be externally stable provided $F + 2\delta \geq 3.36$. We know from our earlier discussion that a 3-firm cartel is internally stable for all values of F and δ. We can conclude that a 3-firm cartel that does not include all firms will be horizontally stable — internally stable and externally stable — if there are at least 4 firms in the fringe, and possibly with fewer fringe firms, depending on the value of d.

For $K = 4$, a cartel will be externally stable provided $F + 2\delta \geq 2.05$. Combined with the condition for internal stability, a 4-firm cartel that does not include all firms will be horizontally stable provided $2.36 \geq F + 2\delta \geq 2.05$. If $F = 2$, there is a narrow range for δ ($0.18 \geq \delta \geq 0.05$) over which this condition is met. If $F = 1$, there is an implausible range for δ ($0.68 \geq \delta \geq 0.53$) over which this condition is met. But for $F \geq 3$, a 4-firm cartel is not horizontally stable.

A 5-firm cartel will be externally stable provided $F + 2\delta \geq 1.68$. But for $K = 5$, internal stability requires $F + 2\delta \leq 1.05$. It follows that the presence of any fringe at all will destabilize a cartel of 5 or more firms. If the number of fringe firms is too small, firms in the fringe will want to join the cartel. If the number of firms in the cartel is too great, then cartel members will want to join the fringe. The same result holds for all $K \geq 5$.

10.3 Suppose the demand curve is as in the Bowley model of product differentiation, that firms produce under conditions of constant returns to scale, and that firms set prices rather than quantities.

(a) For what values of the discount factor α will a trigger strategy sustain noncooperative joint profit maximization?

(b) How do the limits on these values change as product differentiation increases?

Altering equation (10.21) of the text slightly to allow for price-setting behavior, the trigger-strategy adherence condition is

$$\frac{1}{r} \geq \frac{\pi_{i,defect} - \pi_{i,ncc}}{\pi_{i,ncc} - \pi_{i,Bertrand}} = \frac{1}{r^T}.$$

Let firm i's inverse demand curve be

$$p_i = a - \left(q_i + \theta \sum_{j \neq i} q_j \right).$$

The system of inverse demand curves can be written

$$p = aJ_n - [(1-\theta)I_n + \theta J_n J_n']q,$$

where J_n is a column vector of n 1s, I_n is an $n \times n$ identify matrix, and p and q are column vectors of prices and quantities, respectively.

Taking advantage of the fact that

$$[(1-\theta)I_n + \theta J_n J_n']^{-1} = \frac{1}{1-\theta} \left[I_n - \frac{\theta}{1+(n-1)\theta} J_n J_n' \right],$$

we can invert the system of equations of obtain the system of inverse demand curves. One way to write the solution is

$$[1+(n-1)\theta]q = aJ_n - p - \frac{\theta}{1-\theta} n \left(p - \overline{p} J_n \right),$$

where $\overline{p}$ is average price. It thus appears that demand for the variety of a single falls as its own price rises and as its own price rises above the industry average price.

It is useful, however, to collect all terms in own price. The equation of the demand curve for a single variety is

$$(1-\theta)[1+(n-1)\theta]q_1 = (1-\theta)a - [1+(n-2)\theta]p_1 + \theta \sum_{j=2}^{n} p_j$$

$$= (1-\theta)(a-c) - [1+(n-2)\theta](p_1 - c) + \theta \sum_{j=2}^{n} (p_j - c).$$

Maximizing firm 1's profit,

$$1-\theta)[1+(n-1)\theta]\pi_1 =$$

$$((p_1 - c) \left\{ (1-\theta)(a-c) - [1+(n-2)\theta](p_1 - c) + \theta \sum_{j=2}^{n} (p_j - c) + \ \theta \sum_{j=2}^{n} (p_j - c) \right\},$$

we obtain the equation of firm 1's price best response function,

$$2[1+(n-2)\theta](p_1-c)=(1-\theta)(a-c)+\theta\sum_{j=2}^{n}(p_j-c).$$

Note that along firm 1's best response function

$$(1-\theta)[1+(n-1)\theta]q_1=$$

$$(1-\theta)(a-c)-[1+(n-2)\theta](p_1-c)+\theta\sum_{j=2}^{n}(p_j-c)=[1+(n-2)\theta](p_1-c),$$

so that along firm 1's best response function its profit is

$$\pi_1=\frac{[1+(n-2)\theta]}{(1-\theta)[1+(n-1)\theta]}(p_1-c)^2.$$

Since this holds anywhere along firm 1's best response function, it holds in equilibrium.

Because firms are symmetric, all firms will charge the same price in equilibrium. Imposing symmetry in the equation of firm 1's best response function, we obtain expressions for noncooperative equilibrium price and firm profit:

$$\pi_{i,Bertrand}=\frac{(1-\theta)[1+(n-2)\theta]}{[1+(n-1)\theta][2+(n-3)\theta]^2}(a-c)^2.$$

On the other hand, if firms maximize joint profit, each firm will set a price

$$p_{i,ncc}-c=\frac{1}{2}(a-c)$$

and earn a payoff

$$\pi_{i,ncc}=\frac{1}{1+(n-1)\theta}\left(\frac{a-c}{2}\right)^2.$$

Comparing collusive and noncooperative payoffs, one obtains

$$\pi_{i,ncc} - \pi_{i,Bertrand} = \frac{1}{1+(n-1)\theta}\left(\frac{a-c}{2}\right)^2 \frac{(n-1)^2\theta^2}{[2+(n-3)\theta]^2}.$$

It remains to calculate the numerator of the fraction that defines the adherence value r^T.

If all other firms set the collusive price, firm 1 faces residual demand

$$(1-\theta)[1+(n-1)\theta]q_1 = (1-\theta)(a-c) - [1+(n-2)\theta](p_1-c) + \theta(n-1)\left(\frac{a-c}{2}\right)$$

$$= \frac{2+(n-3)\theta}{2}(a-c) - [1+(n-2)\theta](p_1-c)$$

Firm 1's profit-maximizing defection price and payoff are

$$p_{1,defect} - c = \frac{2+(n-3)\theta}{1+(n-1)\theta}\frac{a-c}{4}$$

and

$$\pi_{1,defect} = \frac{1+(n-2)\theta}{(1-\theta)[1+(n-1)\theta]}\left[\frac{2+(n-3)\theta}{1+(n-1)\theta}\frac{a-c}{4}\right]^2$$

respectively.

$$\pi_{i,defect} - \pi_{i,ncc} = \frac{1}{1+(n-1)\theta}\left(\frac{a-c}{2}\right)^2 \frac{(n-1)^2\theta^2}{4(1-\theta)[1+(n-2)\theta]}$$

Substituting in the expression for the critical value of the interest rate, a trigger strategy will sustain joint profit maximization if

$$\frac{1}{r} \geq \frac{\pi_{i,defect} - \pi_{i,ncc}}{\pi_{i,ncc} - \pi_{i,Bertrand}} = \frac{[2+(n-3)\theta]^2}{4(1-\theta)[1+(n-2)\theta]} = \frac{1}{r^T}.$$

The derivative of the right-hand side with respect to θ is equation (10.34) in the text.

For what values of α will a stick-and-carrot strategy sustain noncooperative joint profit maximization?

Only the beginning of a solution is given here. If firms set prices, the "stick" portion of the strategy means setting low prices, while the "carrot" is high prices. The adherence conditions (10.26) and (10.29) become

$$\alpha = \frac{\pi^*(p_l) - \pi(p_l)}{\pi(p_h) - \pi(p_l)} = \alpha_h$$

and

$$\alpha \geq \frac{\pi^*(p_h) - \pi(p_h)}{\pi(p_h) - \pi(p_l)} = \alpha_l$$

respectively (where for consistency with the text and the discussion of quantity-setting oligopoly, α_h denotes the critical value for the high-output, low-price phase and α_l the critical value for the low-output, high-price phase). Observe that since $\pi(p_h) - \pi(p_l)$ appears in the denominator in both equations, by substituting from the α_h condition one can rewrite the α_l condition as

$$\pi^*(p_l) - \pi(p_l) \geq \pi^*(p_h) - \pi(p_h).$$

Suppose the high price p_h is the joint-profit-maximizing price. Then we know from the first part of the problem that

$$\pi^*(p_h) = \frac{1}{1 + (n-1)\theta}\left(\frac{a-c}{2}\right)^2$$

and

$$\pi^*(p_h) - \pi(p_h) = \frac{1}{1 + (n-1)\theta}\left(\frac{a-c}{2}\right)^2 \frac{(n-1)^2\theta^2}{4(1-\theta)[1 + (n-2)\theta]}.$$

If all firms charge a price p_l,

$$\pi(p_l) = \frac{1}{1 + (n-1)\theta}\left\{\left(\frac{a-c}{2}\right)^2 - \left[\frac{a-c}{2} - (p_l - c)\right]^2\right\},$$

and the defection profit of a single firm is

$$\pi^*(p_l) = \frac{1 + (n-2)\theta}{(1-\theta)[1 + (n-1)\theta]}(p_l^* - c)^2,$$

where the defection price p_l^* satisfies

$$p_l^* - c = \frac{1}{2}\frac{(1-\theta)(a-c) + (n-1)\theta(p_l - c)}{1 + (n-2)\theta}.$$

Substituting for $\pi^*(p_l)$, $\pi(p_l)$, and $\pi(p_h)$ in the α_h condition gives an equation that determines $p_l - c$. Differentiation determines $\partial p_l/\partial\theta$. Differentiation of the α_l condition then determines $\partial\alpha_l/\partial\theta$.

10.4 Let the inverse demand function be

$$p_t = a + \varepsilon_t - bQ_t$$

and the firm cost function

$$c(q_i) = cq_i + dq_i^2.$$

In an n-firm quantity-setting oligopoly, show that a single firm's one-period payoff from defecting from a joint-profit-maximizing collusive equilibrium rises as ε_t rises.

A cartel will maximize expected joint profit

$$E(\Pi) = (a - bQ) - n\left[c\left(\frac{Q}{n}\right) + d\left(\frac{Q}{n}\right)^2\right] = b(S - Q) - \frac{d}{n}Q^2$$

(where use has been made of the fact that symmetry implies equal output from each firm in collusive equilibrium).

Industry and firm output are

$$Q_\chi = \frac{n}{n + (d/b)}\frac{a-c}{2b}$$

and

$$q_\chi = \frac{1}{n + (d/b)}\frac{a-c}{2b}.$$

If all other firms produce their collusive outputs, firm 1 faces the residual demand curve

$$p_t = c + \varepsilon_t + \frac{n + 1 + 2(d/b)}{2[n + (d/b)]}bS - bq_{1t}.$$

If firm 1 defects (after ε_t is known), it maximizes its profit along the residual demand curve,

$$(p_t - c)q_{1t} - dq_{1t}^2.$$

Firm 1's defection output and profit are

$$q_{1t}^{def} = \frac{\varepsilon_t}{2b[1 + (d/b)]} + \left[\frac{1}{1 + (d/b)} + \frac{1}{n + (d/b)}\right] \frac{a - c}{4b}$$

and

$$\pi_{1t}^{def} = \frac{1}{4b} \frac{1}{1 + (d/b)} \left[\varepsilon_t + \frac{n + 1 + 2(d/b)}{n + (d/b)} \frac{a - c}{2}\right]^2.$$

Evidently, these both rise as ε_t rises.

Chapter 11

Market Structure, Entry, and Exit

11.1 In the Farrell and Shapiro model of Cournot oligopoly with industry-specific assets, let the inverse demand curve be linear

$$p = a - bQ$$

and let firm i's variable cost function be quadratic,

$$c^i(q_i, k_i) = cq + d\frac{q_i^2}{k_i}, \ i = 1, 2, \ldots, n.$$

Show that equilibrium firm outputs, industry output, and price are

$$q_i = \frac{\kappa_i}{1+\kappa}\frac{a-c}{b}, \ i = 1, 2, \ldots, n;$$

$$Q = \frac{\kappa}{1+\kappa}\frac{a-c}{b}$$

and

$$p = c + \frac{a-c}{1+\kappa},$$

respectively, where $\kappa_i = k_i/(k_i+\delta)$, $\delta = d/b$, and $\kappa = \kappa_1 + \kappa_2 + \ldots + \kappa_n$.

Leaving aside fixed cost, firm i's profit is

$$\pi_i = (p-c)q_i - d\frac{q_i^2}{k_i} = b(S-Q)q_i - \frac{q_i^2}{k_i}.$$

The first-order condition for maximization of π_i is

$$S - Q - q_i - 2\frac{\delta}{k_i}q_i = 0.$$

On the one hand, this can be solved for the equation of firm i's reaction function:

$$q_i = \frac{S - Q_{-i}}{2\left(1 + \frac{\delta}{k_i}\right)}$$

But it is more useful to rewrite the first order condition as

$$q_i = (S - Q)\frac{k_i}{k_i + 2\delta} = (S - Q)\kappa_i.$$

Adding over all i gives

$$Q = (S - Q)\sum_{i=1}^{n}\kappa_i = (S - Q)\kappa,$$

which in turn can be solved for industry output

$$Q = \frac{\kappa}{1 + \kappa}S.$$

Substituting this expression for Q in gives

$$q_i = \frac{\kappa_i}{1 + \kappa}S.$$

Substituting this expression for Q in the market demand curve gives the indicated expression for equilibrium price.

11.**2** With the inverse demand curve and cost functions of Problem 11.1, show that equation (11.51) in the text becomes

$$\chi_i = \frac{c^i_{qk_i}}{c^i_{qq} - p'} = \frac{p - c}{b} = \frac{2\delta}{(k_i + 2\delta)^2.}$$

Direct evaluation gives

$$\chi_i = \frac{2\delta}{k_i + 2\delta}\frac{q_i}{k_i}.$$

But from firm i's first-order condition for profit maximization,

$$\frac{q_i}{k_i} = \frac{S - Q}{k_i + 2\delta},$$

where $S = (a - c)/b$.

Since $p = c + b(S - Q)$, the result follows.

Chapter 12

Firm Structure, Mergers, and Joint Ventures

12.1 Using the fact that, for a linear inverse demand curve $p = a - bQ$, the price elasticity of demand is $\varepsilon_{Qp} = p/bQ$, show that (12.87) and (12.88) can be written

$$\frac{p-c}{p} = \frac{s^P + \sum_{j=1}^{J} \phi_{ij} s_j^J}{\varepsilon_{Qp}}, i = 1, 2, \ldots, n$$

and

$$\frac{p-c}{p} = \frac{s_j^J}{\varepsilon_{Qp}}.$$

respectively, where s denotes market share. Multiply each equation by the market share of the firm to which it applies and add the resulting expressions over all firms to obtain an expression for the price-cost margin in terms of market shares of all firms and joint venture shares.

Equation (12.87) is

$$2q_i^P + \sum_{k \neq i}^{n} q_k^P + \sum_{j=1}^{J} (1 + \phi_{ij}) q_j^J = \frac{a-c}{b}$$

from which

$$\frac{a-c}{b} - Q = \frac{a-c}{b} - \left(\sum_{k=i}^{n} q_k^P + \sum_{j=1}^{J} q_j^J \right) = q_i^P + \sum_{j=1}^{J} \phi_{ij} q_j^J$$

From the equation of the inverse demand curve

$$p = a - bQ$$

we have

$$p - c = a - c - bQ = b\left(\frac{a-c}{b} - Q\right)$$

$$\frac{p-c}{p} = \frac{bQ}{p}\left(\frac{\frac{a-c}{b} - Q}{Q}\right).$$

But using the expression for $\frac{a-c}{b} - Q$ derived from the parent firm first-order condition, this is

$$\frac{p-c}{p} = \frac{1}{\varepsilon_{Qp}}\left(\frac{q_i^P + \sum_{j=1}^J \phi_{ij} q_j^J}{Q}\right),$$

and this is the first of the two desired expressions.

Equation (12.88) is

$$2q_j^J + \sum_{i=1}^n q_i^P + \sum_{k \neq j}^J q_k^J = \frac{a-c}{b},$$

which implies

$$\frac{a-c}{b} - Q = q_j^J.$$

This leads to the second desired expression in a straightforward way.

Chapter 13

Vertical Restraints

	Non-integration
Retail price	$\frac{1}{2}\left[E(d-\gamma)+c\right]+\gamma$
Wholesale price	$\frac{1}{2}\left[E(d-\gamma)+c\right]$
Franchise fee	0
Manufacturer's profit	$\frac{1}{4}\left[E(d-\gamma)-c\right]^2$
Consumers' surplus	$\frac{1}{8}\left[E(d)-c-E(\gamma)\right]^2+\frac{1}{2}\left(\sigma_d^2+\sigma_\gamma^2\right)$
Net social welfare	$\frac{3}{8}\left[E(d-\gamma)-c\right]^2+\frac{1}{2}\left(\sigma_d^2+\sigma_\gamma^2\right)$

Table 13.1: Equilibrium values, Rey and Tirole vertical restraints model: nonintegration

	Exclusive territories
Retail price	$\frac{1}{2}\left[d+\gamma+c+E(d)-d+\overline{\gamma}-E(\gamma)\right]$
Wholesale price	$c+\left[E(d)-\underline{d}\right]+\left[\underline{\gamma}-E(\gamma)\right]$
Franchise fee	$\frac{1}{8}\left\{\underline{d}-c-\overline{\gamma}-\left[E(d)-\underline{d}\right]-\left[\underline{\gamma}-E(\gamma)\right]\right\}$
Manufacturer's profit	$\frac{1}{4}\left\{\left[\underline{d}-c-\overline{\gamma}\right]^2+\left[E(d-\gamma)-(\underline{d}-\overline{\gamma})\right]\right\}$
Consumers' surplus	$\frac{1}{8}\left\{\left[\underline{d}-c-\overline{\gamma}\right]^2+\left(\sigma_d^2+\sigma_\gamma^2\right)\right\}$
Net social welfare	$\frac{3}{8}\left[\underline{d}-c-\overline{\gamma}\right]^2+\frac{1}{8}\left(\sigma_d^2+\sigma_\gamma^2\right)+\frac{1}{4}\left[E(d-\gamma)-(\underline{d}-\overline{\gamma})\right]$

Table 13.2: Equilibrium values, Rey and Tirole vertical restraints model: exclusive territories

	Resale price maintenance
Retail price	$\frac{1}{2}\left[E(d)+\overline{\gamma}+c\right]$
Wholesale price	$\frac{1}{2}\left[E(d)-\overline{\gamma}+c\right]$
Franchise fee	0
Manufacturer's profit	$\frac{1}{4}\left[E(d)-\overline{\gamma}+c\right]^2$
Consumers' surplus	$\frac{1}{8}\left\{\left[E(d)-\overline{\gamma}+c\right]+2\left[d-E(d)\right]\right\}^2$
Net social welfare	$\frac{3}{8}\left[E(d)-\overline{\gamma}+c\right]^2+\frac{1}{2}\sigma_d^2$

Table 13.3: Equilibrium values, Rey and Tirole vertical restraints model: resale price maintenance

Chapter 14

Research & Development

14.1 Answers to Problems

14.1 (Exogenously determined prize, one-time expenditure on research and development; Loury, 1979)

Let n firms undertake rival research projects aiming at successful development of a new product. The first firm to bring its research project to conclusion receives a lump-sum payoff V. Other firms receive nothing.

Let the random variable τ_i denote the uncertain time of discovery, and assume that the relationship between research effort h_i and the probability of success is exponential,

$$G(t) = \Pr(\tau_i \leq t) = 1 - e^{-h_i t}, \tag{14.1}$$

where to set up a research project at intensity h a firm must make a lump-sum payment $F(h)$ at time $t = 0$. The fixed cost function $F(h)$ satisfies

$$F'(h) > 0 \qquad (\overline{h} - h)F''(h) < 0. \tag{14.2}$$

(a) write out the expression for the expected present-discounted value of a single firm, $E(\Pi_1)$;
(b) find first- and second-order necessary conditions to maximize $E(\Pi_1)$;
(c) differentiating the equation of the first-order condition with respect to h_i, find the equation of the slope of firm i's R&D intensity best response function;
(d) taking advantage of the fact that firms are identical, find the equation that is satisfied by equilibrium R&D intensity;

	d'A-J
Payoff function π_1	$[a - c + x_1 + \sigma x_2 - b(q_1 + q_2)]q_1 - \frac{1}{2}\gamma x_1^2$
2nd period output q_1	$\frac{a-c+(2-\sigma)x_1+(2\sigma-1)x_2}{3b}$
Best response function	$x_1 = \frac{2(2-\sigma)}{9b} \frac{a-c+(2\sigma-1)x_2}{\gamma-\frac{2}{9b}(2-\sigma)^2}$
Second-order condition	$b\gamma > \frac{2}{9}(2-\sigma)^2$
Stability condition	$b\gamma > \frac{2}{9}\left(1+\sigma\right)\left(2-\sigma\right)$
	$(1+\sigma)x_{NN} = \frac{2}{9}\frac{(1+\sigma)(2-\sigma)}{b\gamma-\frac{2}{9}(1+\sigma)(2-\sigma)}(a-c)$
q_1	$\frac{b\gamma}{b\gamma-\frac{2}{9}(1+\sigma)(2-\sigma)}\frac{a-c}{3b}$
Per-firm equilibrium payoffs	$\frac{1}{9}\gamma\frac{b\gamma-\frac{2}{9}(2-\sigma)^2}{\left(b\gamma-\frac{2}{9}(1+\sigma)(2-\sigma)\right)^2}(a-c)^2$
Consumers' surplus	$2b\left(\frac{b\gamma}{b\gamma-\frac{2}{9}(1+\sigma)(2-\sigma)}\frac{a-c}{3b}\right)^2$
Net social welfare	$\frac{4}{9}\gamma\frac{b\gamma-\frac{1}{9}(2-\sigma)^2}{\left(b\gamma-\frac{2}{9}(1+\sigma)(2-\sigma)\right)^2}(a-c)^2$

Table 14.1: d'Aspremont & Jacquemin model results, Noncooperative R&D, Cournot product market

	KMZ
Payoff function π_1	$\left(a - c + \sqrt{\frac{2}{\gamma}\left(y_1 + sy_2\right)} - b(q_1+q_2)\right)q_1 - y_1$
2nd period output q_1	$\frac{a-c+\sqrt{\frac{2}{\gamma}}\left(2\sqrt{(y_1+sy_2)}-\sqrt{(sy_1+y_2)}\right)}{3b}$
Best response function	no closed form solution
Second-order condition	$b\gamma > \frac{2(2-s)^3}{9(2-s^2)}$
Stability condition	
	$\sqrt{\frac{2}{\gamma}(1+s)y_{NN}} = \frac{2}{9}\frac{(2-s)(a-c)}{b\gamma-\frac{2}{9}(2-s)}$
q_1	$\frac{b\gamma}{b\gamma-\frac{2}{9}(2-s)}\frac{a-c}{3b}$
Per-firm equilibrium payoffs	$\frac{1}{9}\gamma\left(\frac{b\gamma-\frac{2}{9}\frac{(2-s)^2}{(1+s)}}{\left(b\gamma-\frac{2}{9}(2-s)\right)^2}\right)(a-c)^2$
Consumers' surplus	$2b\left(\frac{b\gamma}{b\gamma-\frac{2}{9}(2-s)}\frac{a-c}{3b}\right)^2$
Net social welfare	$\frac{4}{9}\gamma\frac{b\gamma-\frac{1}{9}\frac{(2-s)^2}{1+s}}{\left(b\gamma-\frac{2}{9}(2-s)\right)^2}(a-c)^2$

Table 14.2: Kamien-Muller-Zang model results, Noncooperative R&D, Cournot product market

(e) (ignoring the fact that the number of firms is an integer) how is equilibrium R&D intensity affected by a change in the number of firms?
(f) (ignoring the fact that the number of firms is an integer) how is expected time to discovery affected by a change in the number of firms?

The corresponding density function is

$$g(t) = G'(t) = h_i e^{-h_i t}, \tag{14.3}$$

and the probability that discovery has not occurred at time t is

$$\Pr(\tau \geq t) = 1 - G(t) = e^{-ht}. \tag{14.4}$$

For this distribution function, the probability of discovery in a short interval of time dt, given that discovery has not occurred at the start of the interval, is proportional to the length of the time interval,

$$\frac{G'(t)dt}{1 - G(t)} = h_i dt, \tag{14.5}$$

with constant of proportionality h_i.

Integrating by parts, the expected time of completion of a project run at intensity h_i is

$$E(\tau) = \int_0^\infty \tau h_i e^{-h_i \tau} d\tau = \frac{1}{h_i} \tag{14.6}$$

The greater the firm's research effort, the sooner the expected time of completion of the research project.

The marginal cost of carrying out a more intense research effort is always positive. The marginal cost of research effort, although positive, falls as h rises for $h < \overline{h}$ and rises as h rises for $h > \overline{h}$. There is thus an initial range of increasing returns to scale in research and development, followed by eventual decreasing returns to scale.

Consider first the case of duopoly. Firm 1 gets the prize V if it is the first firm to complete a research project successfully. Otherwise it receives nothing. If firm 1 discovers first at time t, its payoff, discounted back to time zero, is

$$Ve^{-rt}. \tag{14.7}$$

The instantaneous probability of this event is the product of the instantaneous probabilities that firm 1 discovers at time t and that firm 2 has not yet discovered at time t,

$$e^{-h_2 t}\left[h_1 e^{-h_1 t}\right] = h_1 e^{-(h_1+h_2)t}. \tag{14.8}$$

Firm 1's gross expected payoff, before allowing for the cost of R&D, is therefore

$$E(Payoff) = \int_{t=0}^{t=\infty}\left[h_1 e^{-(h_1+h_2)t} V e^{-rt}\right] dt = \frac{h_1}{r+h_1+h_2}V. \tag{14.9}$$

Firm 1's optimization problem is therefore

$$\max_{h_1}\; E(\Pi_1) = \frac{h_1}{r+h_1+h_2}V - F(h_1). \tag{14.10}$$

The expected-profit maximizing value of h_1 is defined by the first-order condition

$$\frac{dE(\Pi_1)}{dh_1} = V\frac{r+h_2}{(r+h_1+h_2)^2} - F'(h_1) \equiv 0. \tag{14.11}$$

This has a straightforward interpretation: to maximize expected profit from R&D, firm 1 should pick the value for h_1 that makes its expected marginal profit, given h_2, equal to the marginal cost of additional research effort.

The first-order condition (14.11) implicitly defines the profit-maximizing value of h_1 as a function of $r + h_2$ and V:

$$h_1 = h_1(r + h_2, V). \tag{14.12}$$

This is the equation of firm 1's R&D–effort best response function.

The second-order condition, which must hold if (14.11) is to define a maximum, is that

$$\left.\frac{d^2E(\Pi_1)}{dh_1^2}\right|_{\text{1's rf}} = -2V\frac{r+h_2}{(r+h_1+h_2)^3} - F''(h_1) < 0. \tag{14.13}$$

For later notational convenience, define

$$D = 2V\frac{r+h_2}{(r+h_1+h_2)^3} + F''(h_1) > 0. \tag{14.14}$$

We assume that $h_1(r+0,V) > 0$. If this condition is not met, R&D is so costly that it is not profitable for a single firm to undertake a research project even if the firm is certain to win the prize that results from first completion. R&D will not take place in such industries.

Differentiating equation (14.11) with respect to h_1 gives an expression for the slope of firm 1's R&D best response function,

$$\left.\frac{dh_1}{dh_2}\right|_{\text{1's rf}} = \frac{V}{D}\frac{h_1-(r+h_2)}{(r+h_1+h_2)^3}. \tag{14.15}$$

Since the second-order condition guarantees that $D > 0$, the sign of the slope of firm 1's best response curve depends on whether h_1 is greater or less than $r+h_2$. Two cases are possible.

If $h_1(r,V) < r$, the slope of firm 1's best response curve is negative throughout. If $h_1(r,V) > r$, firm 1's best response curve has an initial positive segment. In the latter case, firm 1's best response curve has a positive slope until h_2 rises enough to make $r+h_2 = h_1$. For greater values of h_2, firm 1's best response curve has a negative slope.

The best response curve of firm 2 has similar properties. The noncooperative equilibrium in research and development effort occurs where the best response curves intersect. Since the two firms are identical, $h_1 = h_2$ in equilibrium; the two best response curves intersect along a 45^o line from the origin. This means that the best response curves slope downward in the neighborhood of equilibrium, whether or not they have an initial segment with positive slope. In the neighborhood of equilibrium, the R&D efforts of the two firms are strategic substitutes.

In n-firm oligopoly, h_1 is implicitly defined by the first-order condition

$$\frac{dE(\Pi_1)}{dh_1} = V\frac{r+h_2+\ldots+h_n}{(r+h_1+h_2+\ldots+h_n)^2} - F'(h_1) = 0. \tag{14.16}$$

The corresponding second-order condition is

$$\frac{d^2E(\Pi_1)}{dh_1^2} = -2V\frac{r+h_2+\ldots+h_n}{(r+h_1+h_2+\ldots+h_n)^3} - F''(h_1) < 0. \tag{14.17}$$

In equilibrium, $h_1 = h_2 = \ldots = h_n =$ (say) h, defined from (14.16) by

$$V\frac{r+(n-1)h}{(r+nh)^2} - F'(h) = 0. \tag{14.18}$$

Differentiating (14.18) with respect to n, we obtain

$$\left.\frac{dh}{dn}\right|_{eq} = -\frac{r+(n-2)h}{(r+nh)^3}\frac{Vh}{E} < 0. \tag{14.19}$$

The first term on the right hand side in

$$E = \{r+(n-2)[r+(n-1)h]\}\frac{V}{(r+nh)^3} + \left[2V\frac{r+(n-1)h}{(r+nh)^3} + F''(h)\right] > 0 \tag{14.20}$$

is positive and the second term on the right is positive by the second-order condition (14.19).

In view of (14.20), the equilibrium value of h falls as the number of firms rises. The greater the number of firms that compete for an exogenous R&D prize, the less likely it is that any one of them will win, and the less any one of them will invest, in equilibrium, in R&D.

From the point of view of technological performance, what is of interest is the impact of rivalry (the number of firms) on the expected time to discovery. To examine this, note that if (14.6) gives the probability that firm i completes its research project after time t, then the probability that all firms complete their research projects after time t is

$$\Pr(\tau_i > t \ \forall \ i) = e^{-(\sum h_i)t}. \tag{14.21}$$

It follows that the probability that at least one firm has completed its research project before time t is

$$\Pr(min \ \tau_i < t) = 1 - e^{-(\sum h_i)t}. \tag{14.22}$$

The expected time of completion of the first research project is a random variable with distribution function (14.22). But this is simply an exponential distribution. The expected time of first discovery is

$$E(min \ \tau_i) = \frac{1}{\sum h_i} = \frac{1}{nh}, \tag{14.23}$$

where the final equality holds only in equilibrium.

Differentiating (14.23) and using (14.20), we find that $\partial E(min \ \tau_i)/\partial n$ has sign opposite to that of

$$(n-2)[r+(n-2)h]V + 2V[r+(n-1)h] + (r+nh)^3F''(h) \tag{14.24}$$

which is positive by the second-order condition (14.17). An increase in the number of firms reduces expected time to discovery.

14.2 (Continuous expenditure on research and development; Lee and Wilde, 1980)

Keeping all other aspects of **Problem 14.1** unchanged, suppose that running an R&D project at intensity h requires an expenditure $z(h)$ per unit time as long as the project is under way.[1] Let the properties of the cost function $z(h)$ be

$$z'(h) > 0 \qquad (\overline{h} - h)z''(h) < 0. \tag{14.25}$$

Answer parts (a)–(f) of **Problem 14.1** with this altered specification.

In a duopoly, firm 1's expected profit is

$$E(\Pi_1) = \frac{Vh_1 - z(h_1)}{r + h_1 + h_2} = V - z'(h_1), \tag{14.26}$$

where the first equality is the definition of expected profit and the second is implied by the first-order condition for profit maximization. Thus if R&D profitable in equilibrium

$$V - z'(h) \geq 0, \tag{14.27}$$

which is henceforth assumed.

The second-order condition for profit maximization is

$$\frac{d^2E(\Pi_1)}{dh_1^2} = -\frac{z''(h_1)}{(r + h_1 + h_2)^2} < 0, \tag{14.28}$$

which implies that in equilibrium $z'' > 0$.

The first-order condition implicitly defines firm 1's optimal R&D effort, $h_1(r + h_2, V)$. Differentiating (14.26) yields an expression for the slope of firm 1's R&D best response function:

$$\left.\frac{dh_1}{dh_2}\right|_{\text{1's rf}} = \frac{V - z'(h_1)}{(r + h_1 + h_2)z''(h_1)} \geq 0. \tag{14.29}$$

Thus best response curves slope upward if an R&D project entails continuous expenditure.

[1] Of course, one may consider the case in which an R&D project requires both an initial (set-up) cost and continuous expenditure over the period of operation.

Once again, we generalize the duopoly model to the case of n firms, and for the same purpose: to analyze the impact of a change in the number of firms on firm investment in R&D and on the expected time of discovery. If there are n firms, equilibrium R&D effort h satisfies the equation

$$(r+nh)[V-z'(h)]-[Vh-z(h)]=0. \qquad (14.30)$$

This implies that

$$\left.\frac{dh}{dn}\right|_{eq}=\frac{h[V-z'(h)]}{(r+nh)z''(h)-(n-1)[V-z'(h)]}>0. \qquad (14.31)$$

The numerator on the right is positive if R&D is expected to be profitable. That the denominator is positive follows from a stability argument. Increases in the number of firms increase equilibrium per-firm investment in R&D.

As the number of firms increases there are more firms and each makes a greater investment in R&D. Hence the expected time to first successful completion of a research project falls as the number of firms rises. Rivalry improves technological market performance.

14.3 The second-stage payoff function in the Section 14.3.1 version of the Kamien-Muller-Zang model is

$$\pi_i=\left(a-c+\sqrt{\frac{2}{\gamma}(y_i+sy_j)}-b(q_1+q_2)\right)q_i-y_i.$$

Show that

(b) second period output as a function of R&D spending levels is

$$q_i=\frac{a-c+\sqrt{\frac{2}{\gamma}}\left(2\sqrt{(y_i+sy_j)}-\sqrt{(sy_i+y_j)}\right)}{3b};$$

(c) the first-stage first-order condition for profit maximization can be rewritten

$$\frac{1}{9b}\left(a-c+\sqrt{\frac{2}{\gamma}}\left(2\sqrt{(y_1+sy_2)}-\sqrt{(sy_1+y_2)}\right)\right)\times$$

$$\sqrt{\frac{2}{\gamma}}\left(\frac{2\sqrt{(sy_1+y_2)}-s\sqrt{(y_1+sy_2)}}{\sqrt{(y_1+sy_2)}\sqrt{(sy_1+y_2)}}\right)=1;$$

(d) the first-stage second-order condition for profit maximization, evaluated at the equilibrium, is

$$b\gamma > \frac{2\,(2-s)^3}{9\,(2-s^2)};$$

(e) the equilibrium realized cost saving is

$$\sqrt{\frac{2}{\gamma}(1+s)y_{NN}} = \frac{2}{9}\frac{(2-s)(a-c)}{b\gamma - \frac{2}{9}(2-s)};$$

(f) equilibrium output per firm is

$$q_i = \frac{b\gamma}{b\gamma - \frac{2}{9}(2-s)}\frac{a-c}{3b}$$

(g) equilibrium profit per firm is

$$\pi_i = \frac{1}{9}\gamma\frac{b\gamma - \frac{2}{9}\frac{(2-s)^2}{(1+s)}}{\left[b\gamma - \frac{2}{9}(2-s)\right]^2}(a-c)^2$$

(h) equilibrium consumers' surplus

$$2b\left[\frac{b\gamma}{b\gamma - \frac{2}{9}(2-s)}\frac{a-c}{3b}\right]^2$$

(i) equilibrium net social welfare is

$$\frac{4}{9}\gamma\frac{b\gamma - \frac{1}{9}\frac{(2-s)^2}{1+s}}{\left[b\gamma - \frac{2}{9}(2-s)\right]^2}(a-c)^2.$$

$$\pi_1 = \left\{a - \left[c - \sqrt{\frac{2}{\gamma}(y_1 + sy_2)}\right] - b(q_1+q_2)\right\}q_1 - y_1$$

Second stage: 1's first-order condition is

$$2q_1 + q_2 = \frac{a - c + \sqrt{\frac{2}{\gamma}(y_1+sy_2)}}{b}$$

The system of equations formed by the first-order conditions is

$$b\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = (a-c)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \sqrt{\frac{2}{\gamma}(y_1+sy_2)} \\ \sqrt{\frac{2}{\gamma}(sy_1+y_2)} \end{pmatrix}$$

and this has solutions

$$q_1 = \frac{a-c+\sqrt{\frac{2}{\gamma}}\left(2\sqrt{(y_1+sy_2)}-\sqrt{(sy_1+y_2)}\right)}{3b}$$

$$q_2 = \frac{a-c+\sqrt{\frac{2}{\gamma}}\left(2\sqrt{(sy_1+y_2)}-\sqrt{(y_1+sy_2)}\right)}{3b}.$$

Firm 1's first stage payoff function is

$$\pi_1 = b\left(\frac{a-c+\sqrt{\frac{2}{\gamma}}\left(2\sqrt{(y_1+sy_2)}-\sqrt{(sy_1+y_2)}\right)}{3b}\right)^2 - y_1$$

$$= \frac{1}{9b}\left(a-c+\sqrt{\frac{2}{\gamma}}\left(2\sqrt{(y_1+sy_2)}-\sqrt{(sy_1+y_2)}\right)\right)^2 - y_1$$

The first-order condition is

$$\frac{2}{9b}\left(a-c+\sqrt{\frac{2}{\gamma}}\left(2\sqrt{(y_1+sy_2)}-\sqrt{(sy_1+y_2)}\right)\right)\times$$

$$\frac{\partial}{\partial y_1}\left(2\sqrt{(y_1+sy_2)}-\sqrt{(sy_1+y_2)}\right) - 1 = 0$$

Evaluate the partial derivative:

$$\frac{\partial}{\partial y_1}\left(2\sqrt{(y_1+sy_2)}-\sqrt{(sy_1+y_2)}\right) = \frac{1}{2}\frac{2\sqrt{(sy_1+y_2)}-s\sqrt{(y_1+sy_2)}}{\sqrt{(y_1+sy_2)}\sqrt{(sy_1+y_2)}}$$

Substitute back in the expression for 1's first-order condition:

$$\frac{1}{9b}\left(a-c+\sqrt{\frac{2}{\gamma}}\left(2\sqrt{(y_1+sy_2)}-\sqrt{(sy_1+y_2)}\right)\right)\sqrt{\frac{2}{\gamma}}\left(\frac{2\sqrt{(sy_1+y_2)}-s\sqrt{(y_1+sy_2)}}{\sqrt{(y_1+sy_2)}\sqrt{(sy_1+y_2)}}\right)=1$$

This is (implicitly) the equation of firm 1's R&D best response function. Impose symmetry to obtain

$$\frac{1}{9b}\left(a-c+\sqrt{\frac{2}{\gamma}}\sqrt{(1+s)y}\right)\sqrt{\frac{2}{\gamma}}\left(\frac{(2-s)}{\sqrt{(1+s)y}}\right)=1$$

Solve this for effective R&D, $\sqrt{\frac{2}{\gamma}(1+s)y}$:

$$\sqrt{\frac{2}{\gamma}(1+s)y}=\frac{(2-s)(a-c)}{\frac{9}{2}b\gamma-(2-s)}$$

Equilibrium outputs:

$$q_1=\frac{a-c+\frac{(2-s)(a-c)}{\frac{9}{2}b\gamma-(2-s)}}{3b}$$

$$q_1=\frac{b\gamma}{b\gamma-\frac{2}{9}(2-s)}\left(\frac{a-c}{3b}\right)$$

Second-order condition: write $X=a-c$. Using Maple to evaluate the second derivative of the payoff function and imposing symmetry leads to this expression for the second derivative:

$$\frac{1}{18}\frac{4ys^3+\sqrt{2}\sqrt{\gamma}Xs^2\sqrt{(s+1)}\sqrt{y}-4s^2y-4ys-2\sqrt{2}\sqrt{\gamma}X\sqrt{(s+1)}\sqrt{y}+4y}{y^2\left(s+1\right)^2b\gamma}$$

Now substitute the equilibrium value of y:

$$\sqrt{y}=\frac{(2-s)X}{\sqrt{\frac{2}{\gamma}(1+s)}\left(\frac{9}{2}b\gamma-(2-s)\right)}$$

$$y = \left(\frac{(2-s)X}{\sqrt{\frac{2}{\gamma}(1+s)} \left(\frac{9}{2} b\gamma - (2-s) \right)} \right)^2$$

and simplify to obtain

$$-\frac{1}{36} \left(9b\gamma - 2(2-s)\right)^2 \frac{9\left(2-s^2\right)b\gamma - 2\left(2-s\right)^3}{\left(2-s\right)^3 \gamma^2 b X^2}$$

The second-order condition is satisfied for

$$9\left(2-s^2\right)b\gamma - 2\left(2-s\right)^3 > 0$$

$$b\gamma - \frac{2\left(2-s\right)^3}{9\left(2-s^2\right)} > 0$$

Per-firm equilibrium payoffs:

$$\pi_1 = bq_1^2 - y_1$$

$$b\left(\frac{1}{3} \frac{\gamma}{b\gamma - \frac{2}{9}(2-s)} (a-c) \right)^2 - \left(\frac{2}{9} \frac{(2-s)(a-c)}{\sqrt{\frac{2}{\gamma}(1+s)} \left(b\gamma - \frac{2}{9}(2-s) \right)} \right)^2$$

Collect terms to obtain

$$\pi_1 = \frac{1}{9} \gamma \frac{b\gamma - \frac{2}{9} \frac{(2-s)^2}{(1+s)}}{\left(b\gamma - \frac{2}{9}(2-s) \right)^2} (a-c)^2$$

Consumers' surplus: $\frac{1}{2} bQ^2$

$$\frac{1}{2} b \left(2 \frac{1}{3} \frac{\gamma}{b\gamma - \frac{2}{9}(2-s)} (a-c) \right)^2$$

$$2b \left(\frac{b\gamma}{b\gamma - \frac{2}{9}(2-s)} \frac{a-c}{3b} \right)^2$$

Net social welfare:

$$2b\left(\frac{b\gamma}{b\gamma-\frac{2}{9}(2-s)}\frac{a-c}{3b}\right)^2+\frac{2}{9}\gamma\frac{b\gamma-\frac{2}{9}\frac{(2-s)^2}{(1+s)}}{\left(b\gamma-\frac{2}{9}(2-s)\right)^2}(a-c)^2$$

$$NSW=\frac{4}{81}\gamma\frac{9b\gamma\,(1+s)-(2-s)^2}{1+s}\left(\frac{a-c}{b\gamma-\frac{2}{9}(2-s)}\right)^2$$

$$\frac{4}{9}\gamma\frac{b\gamma-\frac{1}{9}\frac{(2-s)^2}{1+s}}{\left(b\gamma-\frac{2}{9}(2-s)\right)^2}(a-c)^2$$

14.4 Suppose that a firm can carry out more than one research project, the probability of success of each of which obeys the exponential distribution,

$$F(t)=\Pr(\tau\leq t)=1-e^{-ht}.$$

If firm i carries out m_i such projects, what is distribution function for the event "success by the firm"? (For simplicity, assume that the probability of success of individual experiments is independent, although this is an assumption that one might wish to abandon.) Reformulate the patent race models of the text to allow for multiple research projects by individual firms.

See Chapter 8 of

Scott, John T. *Purposive Diversification and Economic Performance.* Cambridge: Cambridge University Press, 1993.